



## Masses of hadrons containing heavy quarks and their QED corrections

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Jan. 31th, 2026



CLQCD



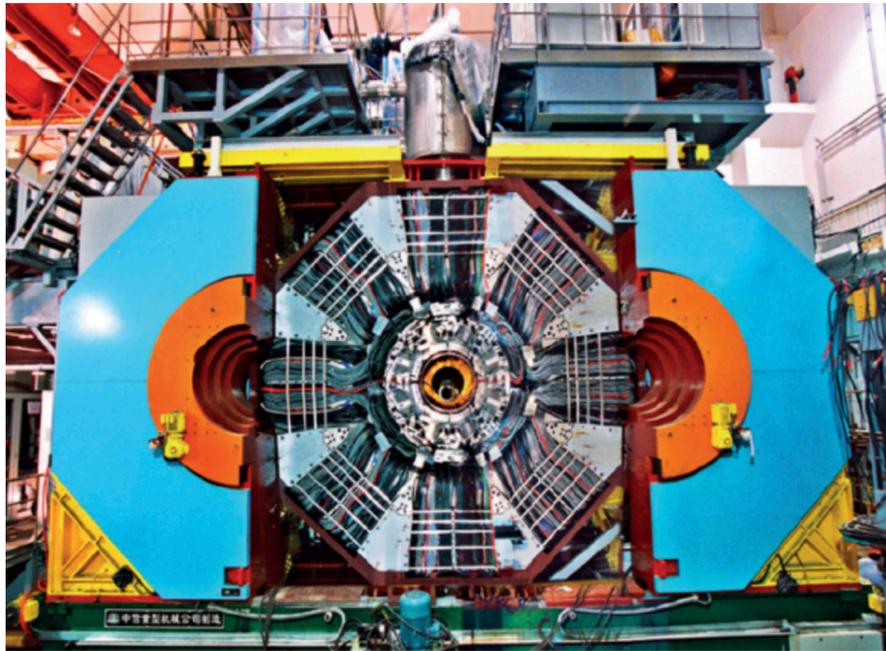
中国科学院大学  
University of Chinese Academy of Sciences



ICTP-AP  
International Centre  
for Theoretical Physics Asia-Pacific  
国际理论物理中心-亚太地区

# SM precision test

# and their strong interaction inputs



BESIII leads the international charm physics research

**BESIII**

$\approx 1.275 \text{ GeV}/c^2$   
 $2/3$   
 $1/2$   
  
 charm



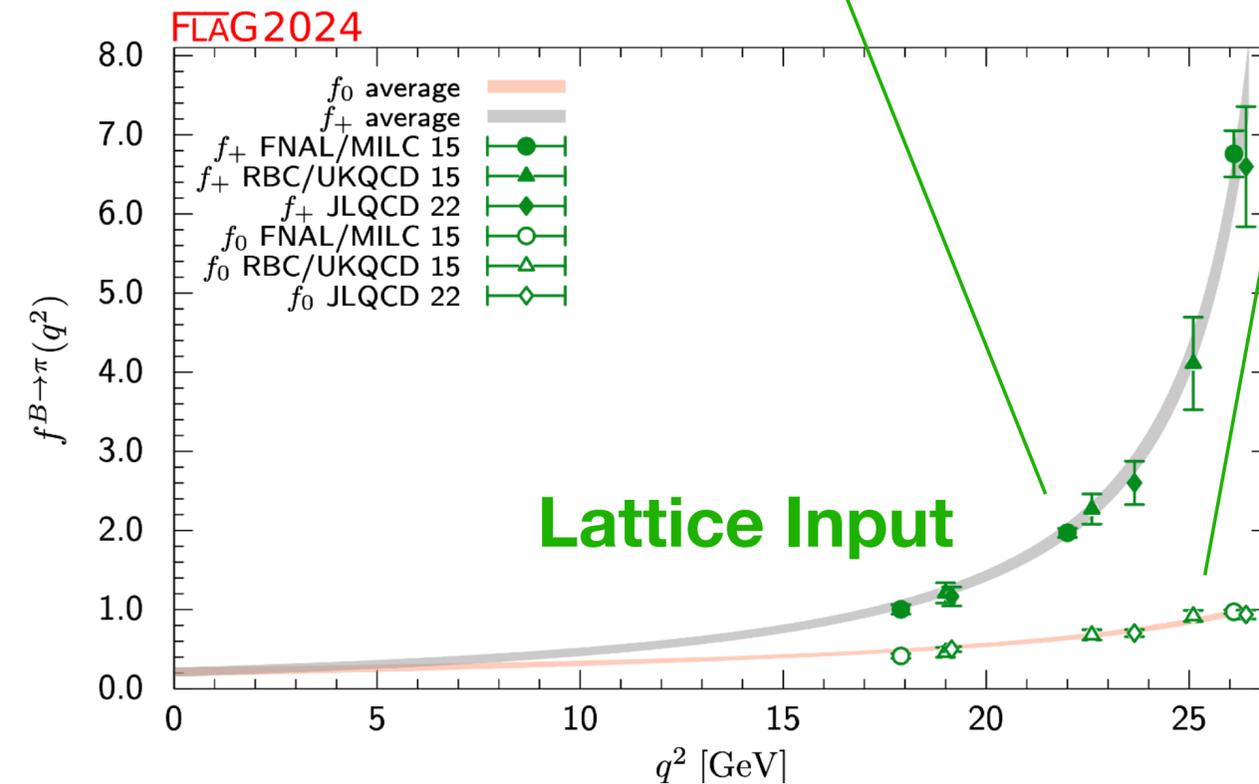
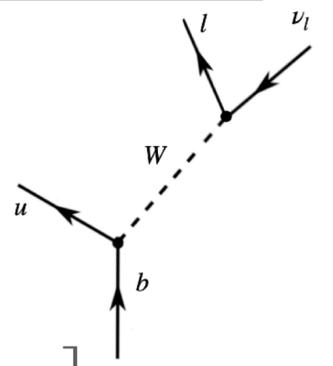
Belle2 completed upgrades and starts data collection.

**Belle II**

$\approx 4.18 \text{ GeV}/c^2$   
 $-1/3$   
 $1/2$   
  
 bottom

## Experimental results

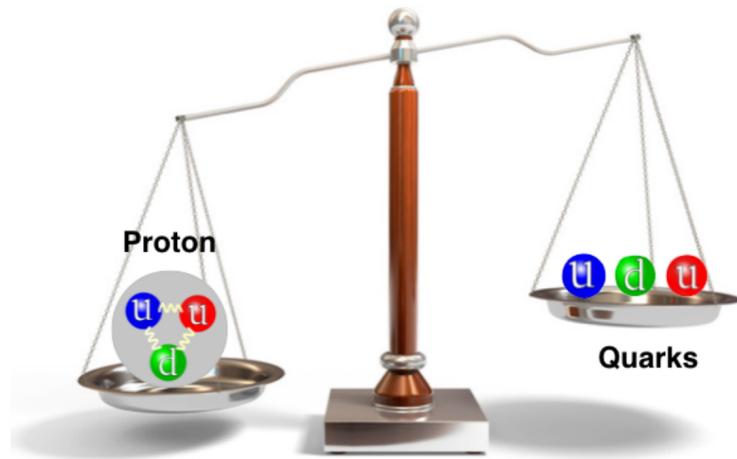
$$\frac{d\Gamma(B_{(s)} \rightarrow P\ell\nu)}{dq^2} = \frac{G_F^2 |\eta_{EW}|^2 |V_{ub}|^2 (q^2 - m_\ell^2)^2 \sqrt{E_P^2 - m_P^2}}{24\pi^3 q^4 m_{B_{(s)}}^2} \times \left[ \left(1 + \frac{m_\ell^2}{2q^2}\right) m_{B_{(s)}}^2 (E_P^2 - m_P^2) |f_+(q^2)|^2 + \frac{3m_\ell^2}{8q^2} (m_{B_{(s)}}^2 - m_P^2)^2 |f_0(q^2)|^2 \right]$$



- Precision determination of CKM ME is the foundation of the standard model precision test;
- Extract all the CKM ME except  $V_{tb}$  requires inputs from the strong interaction predictions.

# Baryon mass and spin

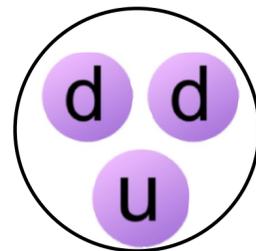
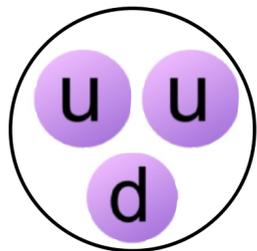
and their strong interaction origin



QUARKS	mass →	≈2.3 MeV/c <sup>2</sup>	≈4.8 MeV/c <sup>2</sup>
	charge →	2/3	-1/3
	spin →	1/2	1/2
		<b>u</b>	<b>d</b>
		up	down
		≈1.275 GeV/c <sup>2</sup>	≈95 MeV/c <sup>2</sup>
	2/3	-1/3	
	1/2	1/2	
	<b>c</b>	<b>s</b>	
	charm	strange	
	≈173.07 GeV/c <sup>2</sup>	≈4.18 GeV/c <sup>2</sup>	
	2/3	-1/3	
	1/2	1/2	
	<b>t</b>	<b>b</b>	
	top	bottom	

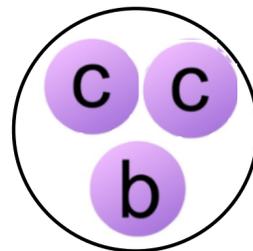
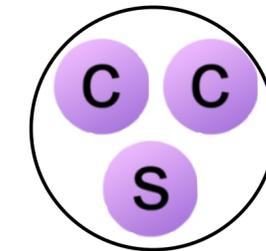
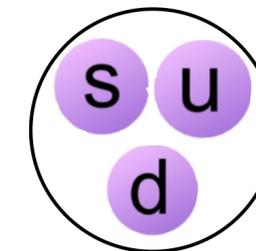


$$m_p = 0.9383 \text{ GeV} \quad m_n = 0.9396 \text{ GeV}$$



$$m_H = \langle \sum_q m_q \bar{q}q \rangle_H + \langle H_{en} \rangle_H + \frac{1}{4} \langle H_a \rangle_H$$

$$m_\Lambda = 1.116 \text{ GeV}, \quad m_{\Omega_{cc}} = 3.519 \text{ GeV}, \quad m_{\Omega_{ccb}} = ?$$

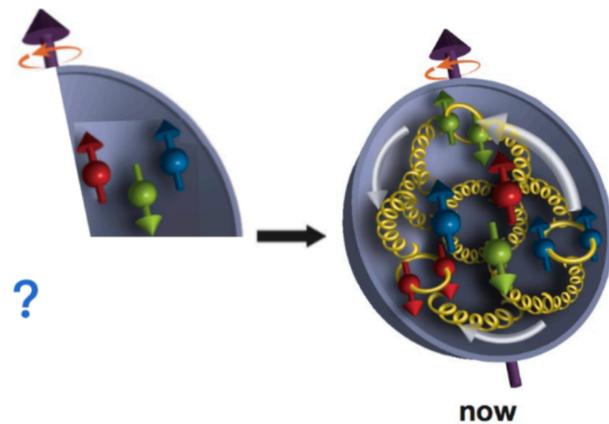
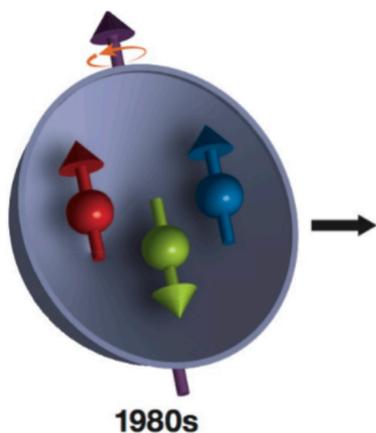


$S_\Lambda = ?$

$S_{\Omega_{cc}} = ?$

$S_{\Omega_{ccb}} = ?$

$$S_H = \frac{1}{2} \sum_q \Delta q + \Delta G + \sum_{i=q,g} L_i$$



- The hadron mass and matrix elements in the real world require the full QCD+QED calculation. But since the lattice calculation can only reach an  $\mathcal{O}(1\%)$  ( $\mathcal{O}(0.1\%)$  in the heavy quark case) precision, one can expand the prediction in term of the polynomial of  $\alpha$  and also  $\delta_{\text{ISB}} \equiv (m_d - m_u)/\Lambda_{\text{QCD}}$  :

$$\mathcal{M}^{\text{QCD+QED}} = \mathcal{M}^{\text{isoQCD}} + \alpha \mathcal{M}^{(0,1)} + \delta_{\text{ISB}} \mathcal{M}^{(1,0)} + \mathcal{O}(\alpha^2, \alpha \delta_{\text{ISB}}, \delta_{\text{ISB}}^2)$$

- Naive power counting suggests that both ISB and QED corrections are 1%;
- There are kinds of known results for the ISB and QED corrections:

$$m_{\pi^+} - m_{\pi^0} = \sim 0_{\text{ISB}} + 4.53_{\text{QED}}(6) \text{ MeV},$$

$$m_{K^+} - m_{K^0} = -6.00_{\text{ISB}}(15) \text{ MeV} + 2.07_{\text{QED}}(15) \text{ MeV},$$

$$m_n - m_p = 2.52_{\text{ISB}}(29) \text{ MeV} - 1.00_{\text{QED}}(16) \text{ MeV},$$

$$m_{D^+} - m_{D^0} = 2.54_{\text{ISB}}(13) \text{ MeV} + 2.14_{\text{QED}}(13) \text{ MeV},$$

$$m_{B^+} - m_{B^0} = -1.88_{\text{ISB}}(60) \text{ MeV} + 1.58_{\text{QED}}(24) \text{ MeV}.$$

- The QED correction for  $m_{B^+}$  itself ?

BMWc, Science 347(2015)1452

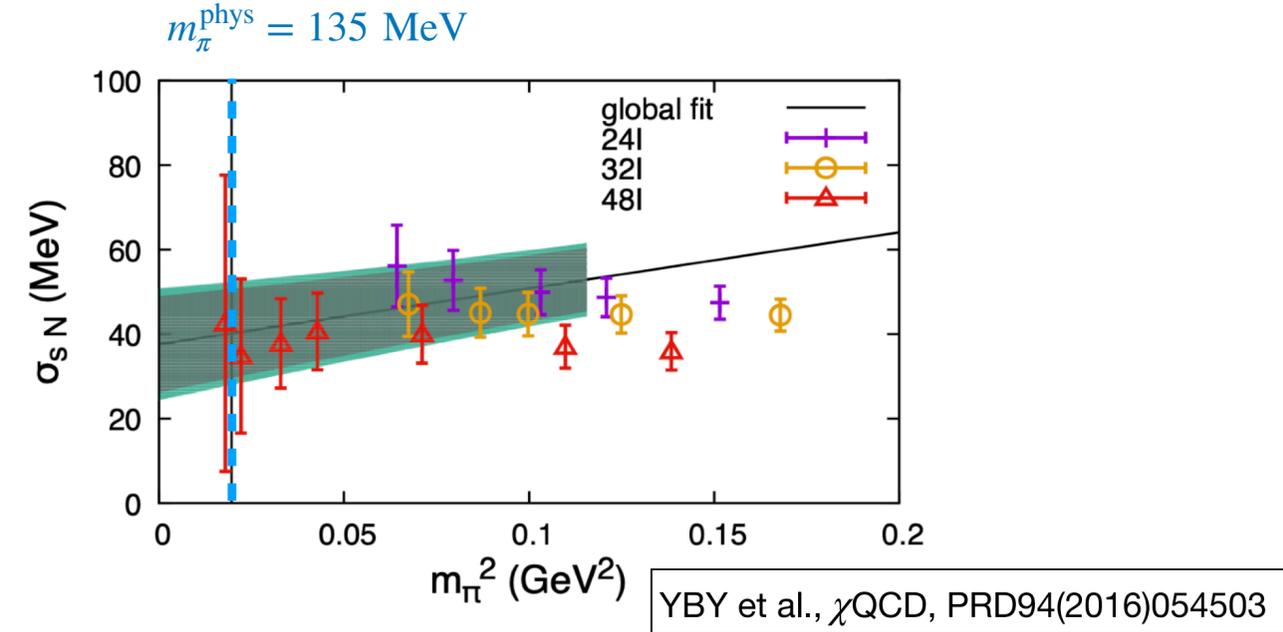
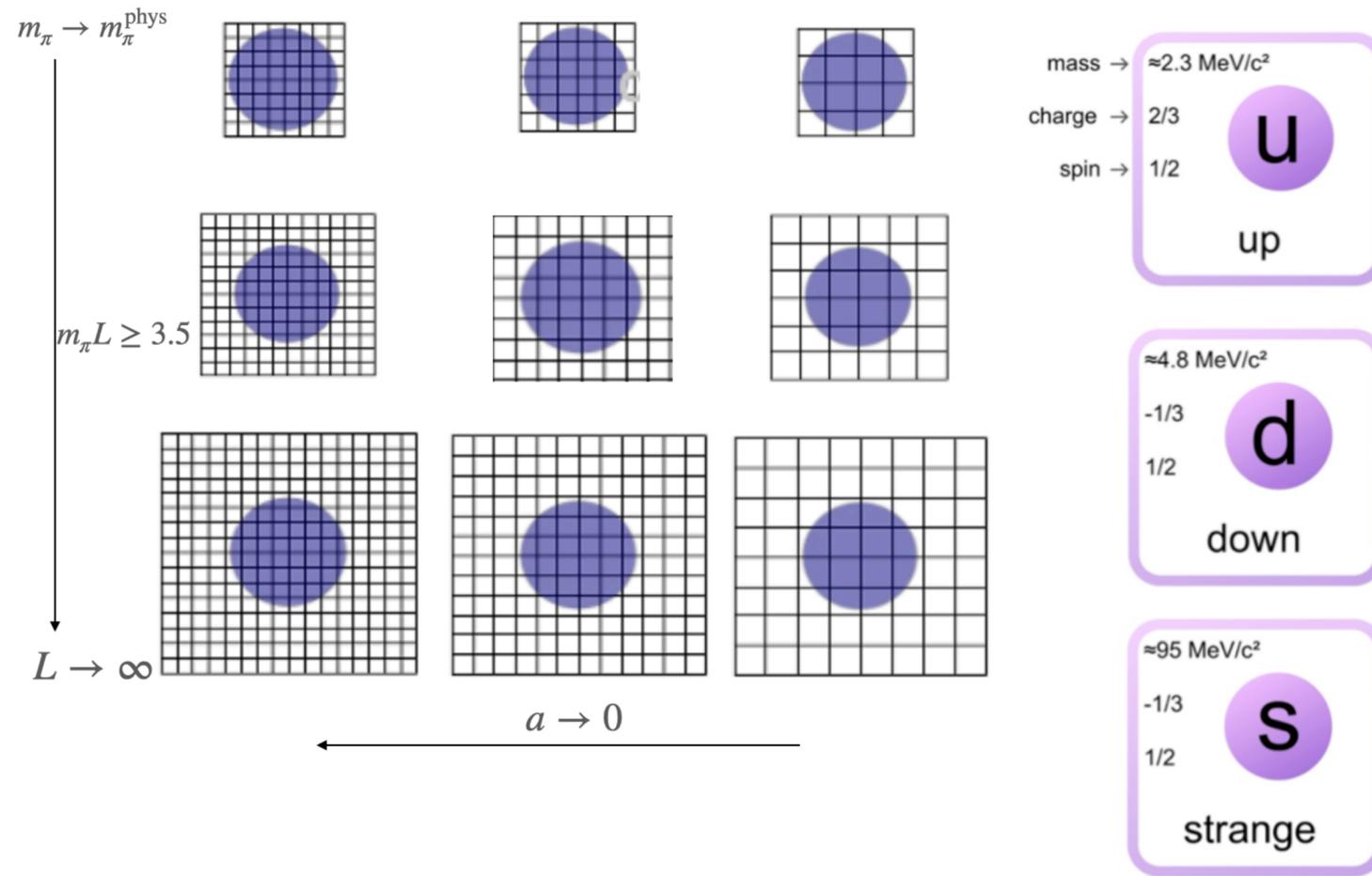
M. Rowe, R. Zwicky, JHEP(2023)089

D. Giusti et al., PRD95(2017)114504

X. Feng et al., PRL128(2022)052003

# Flavor physics

and their challenges on the lattice



To reach the same statistical precision for a physical light quark as we achieve for the strange quark:

- Computational cost would increase by a factor of **700**.

$\approx 1.275 \text{ GeV}/c^2$   
 $2/3$   
 $1/2$   
**C**  
 charm

- Naive discretization error estimate  $m_c^2 a^2 \leq 0.1$  requires  $a \leq 0.06$  fm and then the computational cost would increase by a factor of **7**, compared to the  $a \sim 0.1$  fm case.

$\approx 4.18 \text{ GeV}/c^2$   
 $-1/3$   
 $1/2$   
**b**  
 bottom

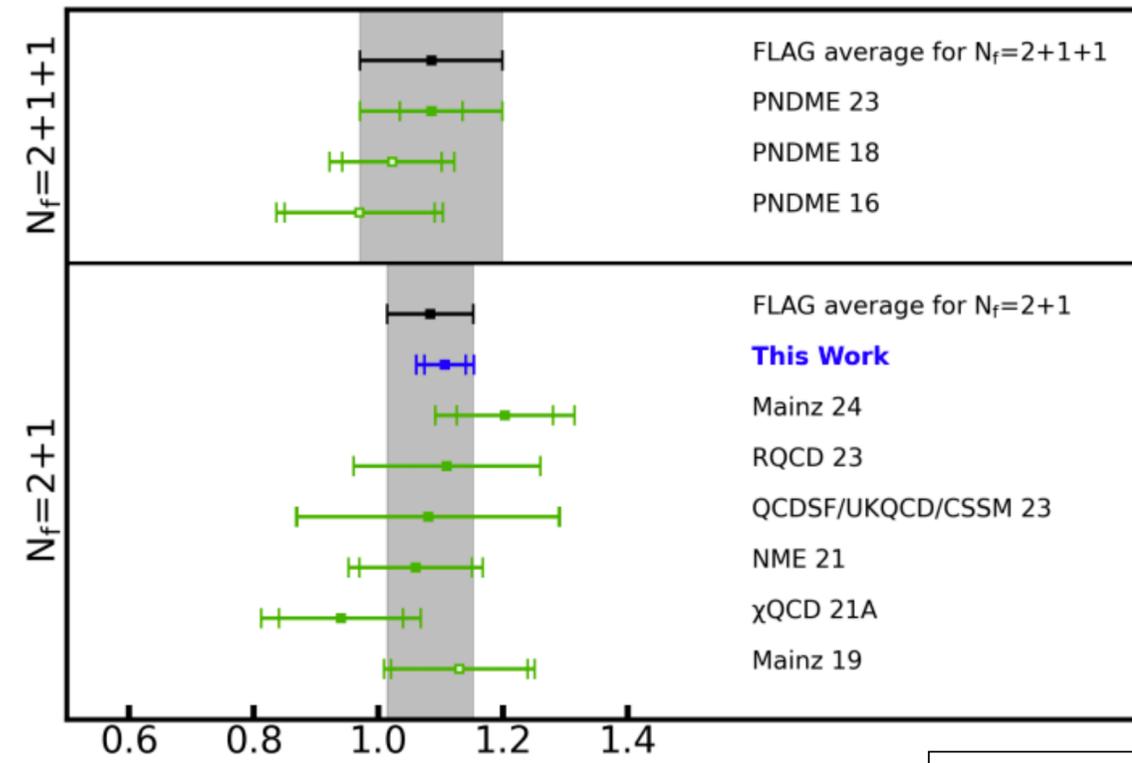
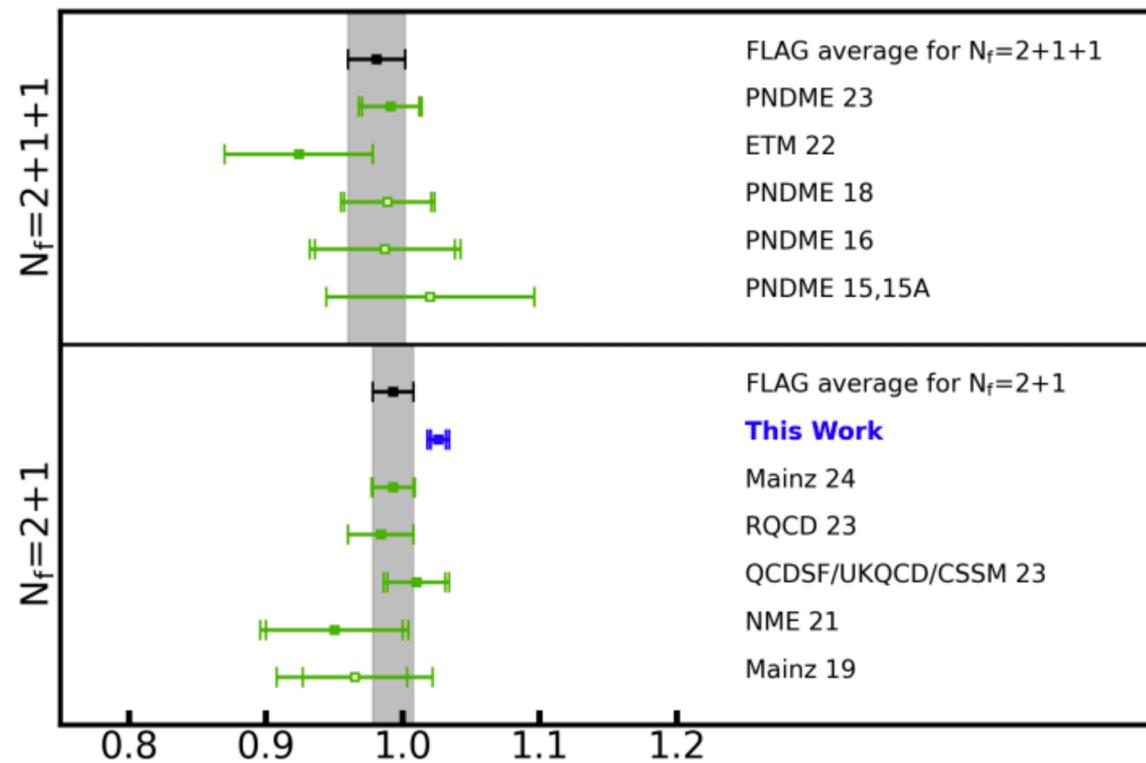
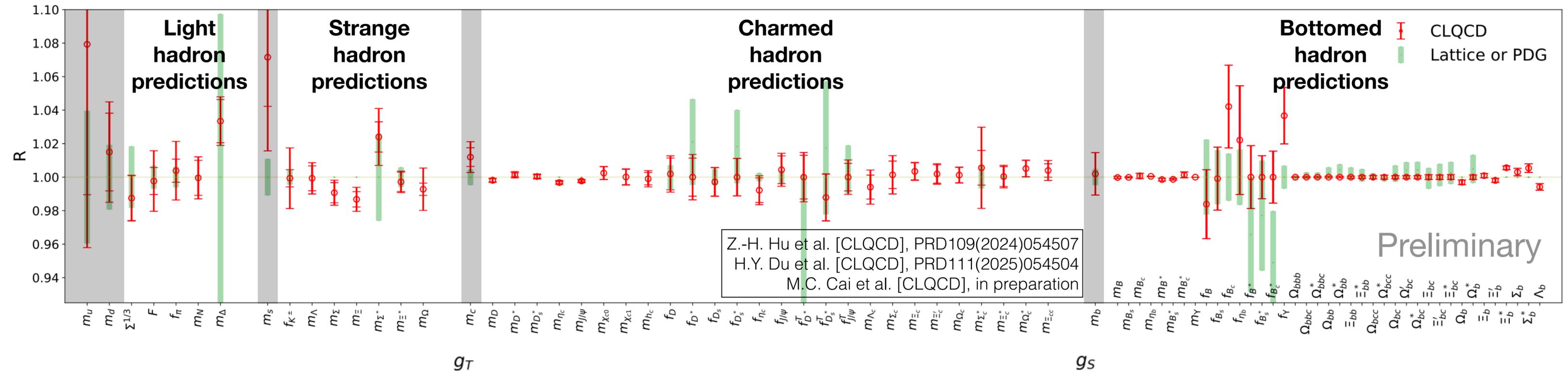
- Similar estimate  $m_b^2 a^2 \leq 0.1$  requires  $a \leq 0.013$  fm and then the factor is **4823**.

- The top quark information can only be accessed through the heavy quark extrapolations.

$\approx 173.07 \text{ GeV}/c^2$   
 $2/3$   
 $1/2$   
**t**  
 top

# CLQCD ensembles

# Predictions

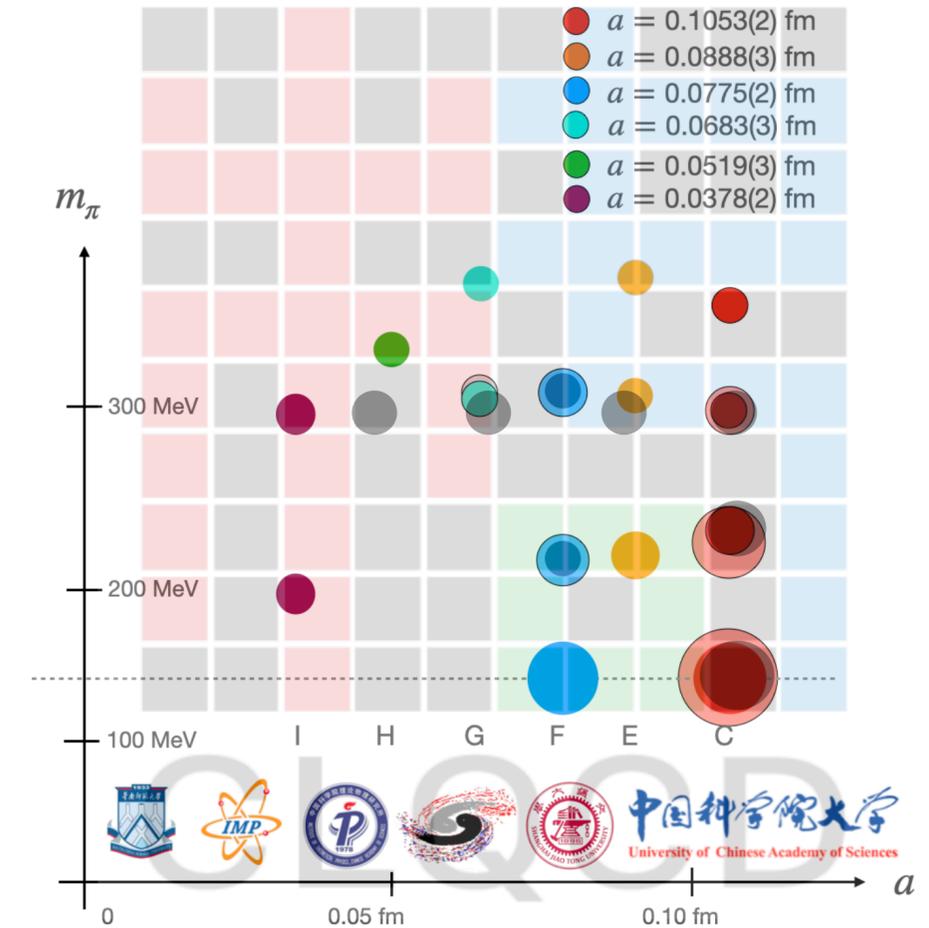




# Light and strange quarks

## CLQCD ensembles

	Country/ Region	Smallest lattice spacing	No. of physical point ensembles	Largest spacial size	No. of fermion discretization
MILC	US	0.03 fm	5	5.8 fm	1
RBC	US	0.06 fm	3	5.5 fm	1
BMW	EN	0.05 fm	15	10 fm	2
CLS	EN	0.04 fm	2	5.5 fm	1
ETM	EN	0.05 fm	5	6.3 fm	1
PACS	JP	0.06 fm	3	10 fm	1
<b>CLQCD</b>	<b>CN</b>	<b>0.04 fm</b>	<b>4</b>	<b>6.7 fm</b>	<b>2</b>



Fermion Discretization	No. of ensembles	Sea fermion flavors	Lattice spacing range (fm)	No. of ensembles at physical pion mass	Device for data generation	Device for data analysis
Clover	20	2+1	0.038-0.105	3	CUDA GPU	Sugon DCU
HISQ	10	2+1+1	0.048-0.108	1	Sugon DCU	Sugon DCU

# Light and strange quarks

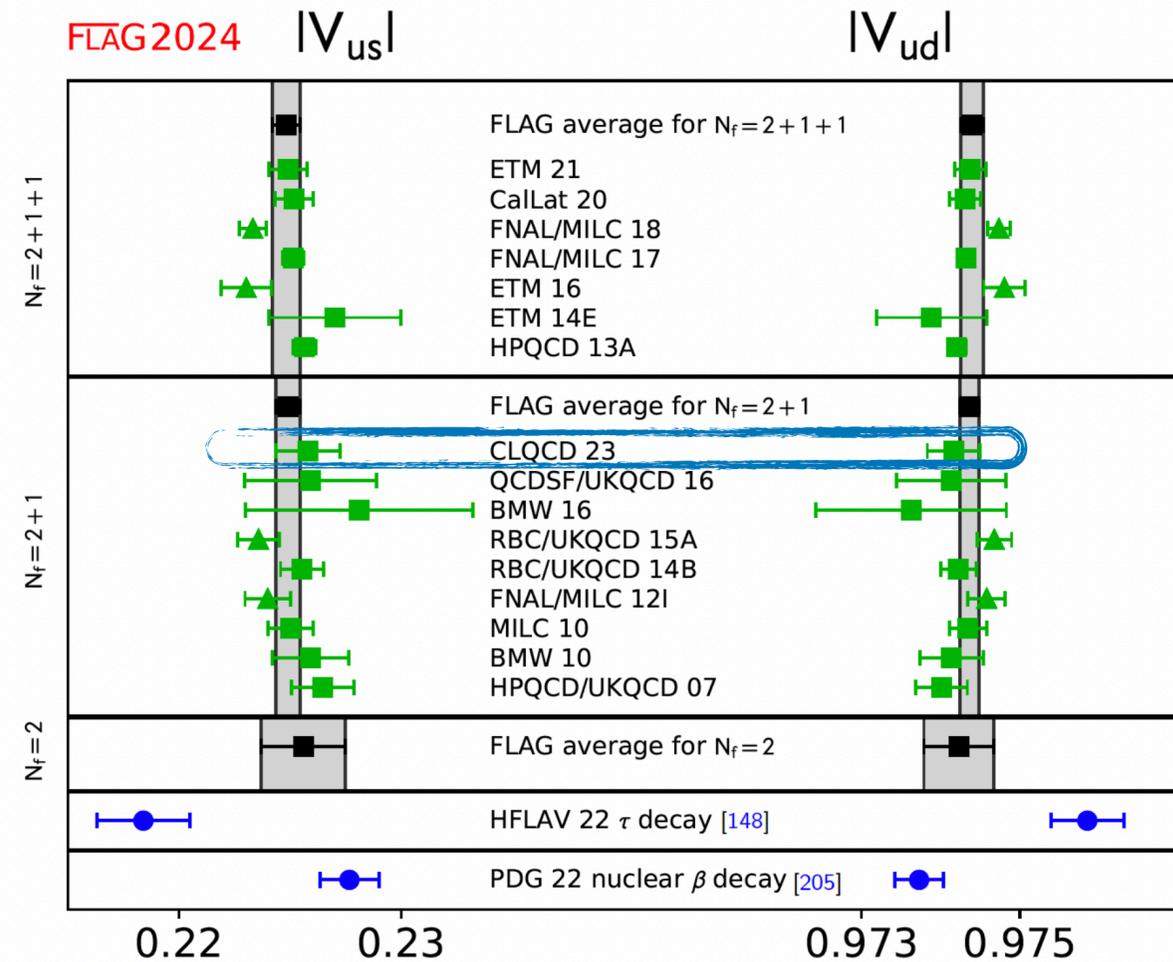
## CLQCD inputs for FLAG

Y. Aoki et al.

FLAG Review 2024  
 publication status  
 chiral extrapolation  
 continuum extrapolation  
 finite volume extrapolation  
 renormalization  
 running

2411.04268

Collaboration	Ref.	publication status	chiral extrapolation	continuum extrapolation	finite volume extrapolation	renormalization	running	$m_{ud}$	$m_s$
CLQCD 23	[10]	A	★	★	★	★	e	3.60(11)(15)	98.8(2.9)(4.7)
ALPHA 19	[11]	A	○	★	★	★	e	3.54(12)(9)	95.7(2.5)(2.4)
Maezawa 16	[12]	A	■	★	★	★	d	—	92.0(1.7)
RBC/UKQCD 14B <sup>⊖</sup>	[13]	A	★	★	★	★	d	3.31(4)(4)	90.3(0.9)(1.0)
RBC/UKQCD 12 <sup>⊖</sup>	[9]	A	★	○	★	★	d	3.37(9)(7)(1)(2)	92.3(1.9)(0.9)(0.4)(0.8)
PACS-CS 12*	[14]	A	★	■	■	★	b	3.12(24)(8)	83.60(0.58)(2.23)
Laiho 11	[15]	C	○	★	★	○	—	3.31(7)(20)(17)	94.2(1.4)(3.2)(4.7)
BMW 10A, 10B <sup>+</sup>	[16, 17]	A	★	★	★	★	c	3.469(47)(48)	95.5(1.1)(1.5)
PACS-CS 10	[18]	A	★	■	■	★	b	2.78(27)	86.7(2.3)
MILC 10A	[19]	C	○	★	★	○	—	3.19(4)(5)(16)	—
HPQCD 10**	[20]	A	○	★	★	—	—	3.39(6)	92.2(1.3)
RBC/UKQCD 10A	[21]	A	○	○	★	★	a	3.59(13)(14)(8)	96.2(1.6)(0.2)(2.1)
Blum 10 <sup>†</sup>	[22]	A	○	■	○	★	—	3.44(12)(22)	97.6(2.9)(5.5)
PACS-CS 09	[23]	A	★	■	■	★	b	2.97(28)(3)	92.75(58)(95)
HPQCD 09A <sup>⊕</sup>	[24]	A	○	★	★	—	—	3.40(7)	92.4(1.5)
MILC 09A	[25]	C	○	★	★	○	—	3.25 (1)(7)(16)(0)	89.0(0.2)(1.6)(4.5)(0.1)
MILC 09	[26]	A	○	★	★	○	—	3.2(0)(1)(2)(0)	88(0)(3)(4)(0)
PACS-CS 08	[27]	A	★	■	■	■	—	2.527(47)	72.72(78)
RBC/UKQCD 08	[28]	A	○	■	★	★	—	3.72(16)(33)(18)	107.3(4.4)(9.7)(4.9)
CP-PACS/ JLQCD 07	[29]	A	■	★	★	■	—	3.55(19)( <sup>+56</sup> <sub>-20</sub> )	90.1(4.3)( <sup>+16.7</sup> <sub>-4.3</sub> )
HPQCD 05	[30]	A	○	○	○	○	—	3.2(0)(2)(2)(0) <sup>‡</sup>	87(0)(4)(4)(0) <sup>‡</sup>
MILC 04, HPQCD/ MILC/UKQCD 04	[31, 32]	A	○	○	○	■	—	2.8(0)(1)(3)(0)	76(0)(3)(7)(0)



Flavor lattice average group, 2411.04268

- First domestic light quark masses and CKM matrix elements determination which enters the lattice QCD world averages;

Z.-H. Hu, B.-L. Hu, J.-H. Wang, et. al., CLQCD, PRD109(2024) 054507

- Reach the highest “green star” grade in the chiral, continuum, infinite volume extrapolation and also renormalization.

- Most precise prediction on  $g_{S,T}^{u-d}$  so far, among three new results appears 2025;

$$m_n^{\text{exp}} - m_p^{\text{exp}} = 1.293 \text{ MeV}$$

$$\begin{aligned} m_n - m_p &= m_u \left( \frac{\partial m_n}{\partial m_u} - \frac{\partial m_p}{\partial m_u} \right) + m_d \left( \frac{\partial m_n}{\partial m_d} - \frac{\partial m_p}{\partial m_d} \right) + \delta^{\text{QED}} m_p^{\text{isoQCD}} = (m_d - m_u) g_S^{u-d} + \delta^{\text{QED}} m_p^{\text{isoQCD}} \\ &= (2.35(12) \text{ MeV})_{m_d-m_u} * 1.11(5)_{g_S} - 1.00(16) \text{ MeV}_{\text{QED}} \\ &= 1.60[0.23]_{\text{tot}}(11)_{g_S}(0.13)_{\text{ISB}}(0.16)_{\text{QED}} \text{ MeV}. \end{aligned}$$

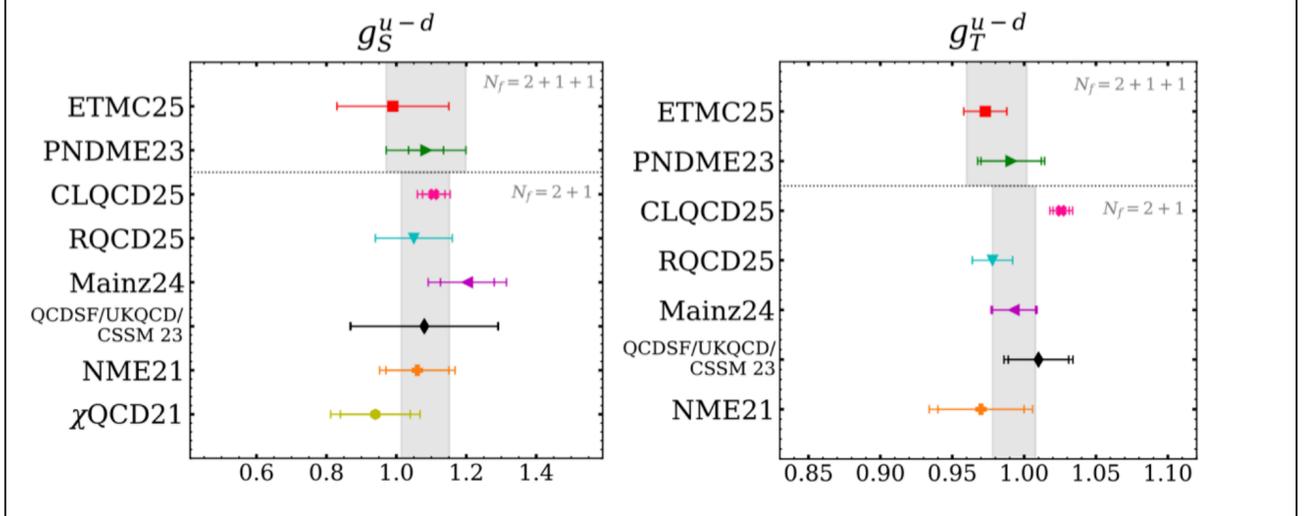
ISB : Flavor lattice average group, 2411.04268

QED: BMWc, Science 347(2015)1452

- Better control on all kinds of systematic uncertainties thanks to the large data set of CLQCD ensembles, and also high precision data using newly proposed “blending method”.

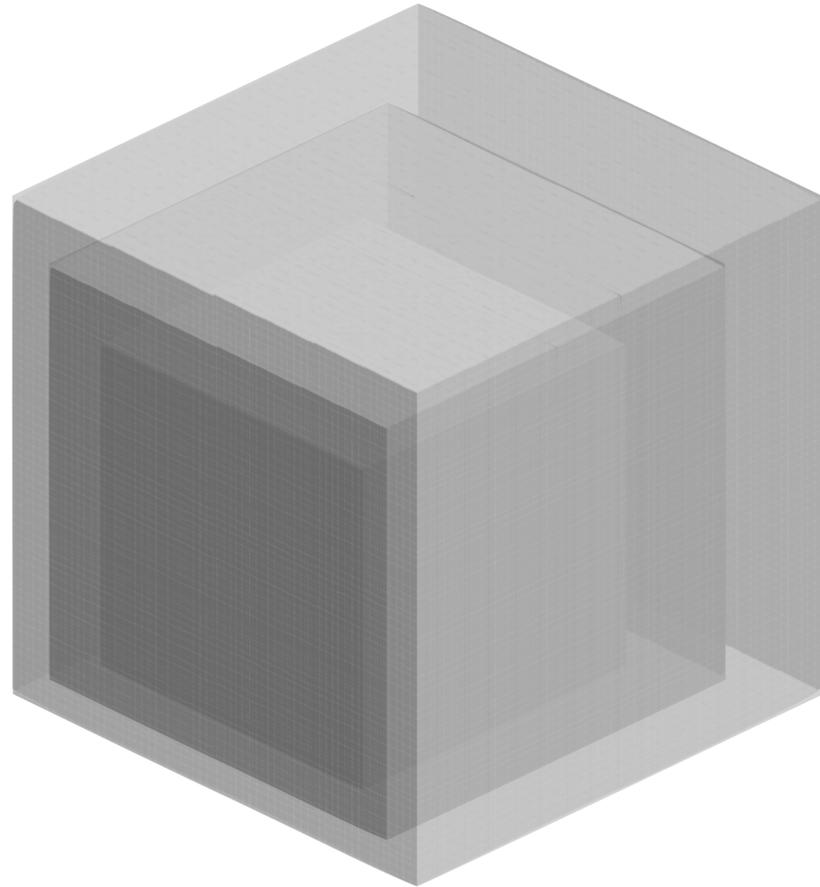
✱ Nucleon isovector charges are well-studied by many collaborations: Recent results by

- RQCD: 47 CLS ensembles at  $6 a \sim (0.038-0.098) \text{ fm}$ ,  $m_\pi \sim (480-130) \text{ MeV}$  and multiple L
- ETMC: 4 ensembles at  $4 a \sim (0.08-0.05) \text{ fm}$  and  $m_\pi \sim 140 \text{ MeV}$
- CLQCD: 16 Clover ensembles at  $4 a \sim (0.105-0.052) \text{ fm}$ ,  $m_\pi \sim (340-134) \text{ MeV}$  and multiple L



# Light and strange quarks

## Fermion propagator on the lattice



Only the bosonic two-point function of the fermion, namely propagator, can be calculated on the lattice, but the all-to-all fermion propagator can be extremely costly:

- A modern  $24^3 \times 64$  lattice will require  $\sim 15\text{K}\$^*$  and 820 TB to calculate and storage the all-to-all propagator for a given quark mass and configuration.

\*Assuming A100 GPU can generate a 12-column propagator within 1min

- State-of-the-arts  $48^3 \times 64$  lattice will requires **8x** and **64x** on the resource and storage, respectively.
- Realistic calculation will require  $\mathcal{O}(100)$  configuration and repeat the calculation on multiple lattices.

	Cost efficiency	Hadron state optimization	(N>2)-point functions
Tranditional	Low	Yes	Yes
Distillation	High	Yes	No
Sparsening	High	No	Yes

# Light and strange quarks

## Blending method

- The “blending” method projects the identity matrix into exact low momentum modes plus stochastic samples of high momentum modes;

$$\hat{I} = \sum_{k=1}^{N_e} |\phi_k\rangle\langle\phi_k| + \sum_{k=N_e+1}^{N_e+N_{st}} \frac{N_c N_L^3 - N_e}{N_{st}} |\phi_k\rangle\langle\phi_k| + \mathcal{O}\left(\frac{1}{N_{st}^n}\right)$$

- The correlation function can be projected to the subspace of those modes:

$$\sum_{x,y,z,w} \langle \dots S(x,y) \mathcal{O}(y,z) S(z,w) \dots \rangle = \sum_{i,j,k,l} \langle \dots S_{ij} \mathcal{O}_{ik} S_{kl} \dots \rangle + \mathcal{O}\left(\frac{1}{N_{st}}\right)$$

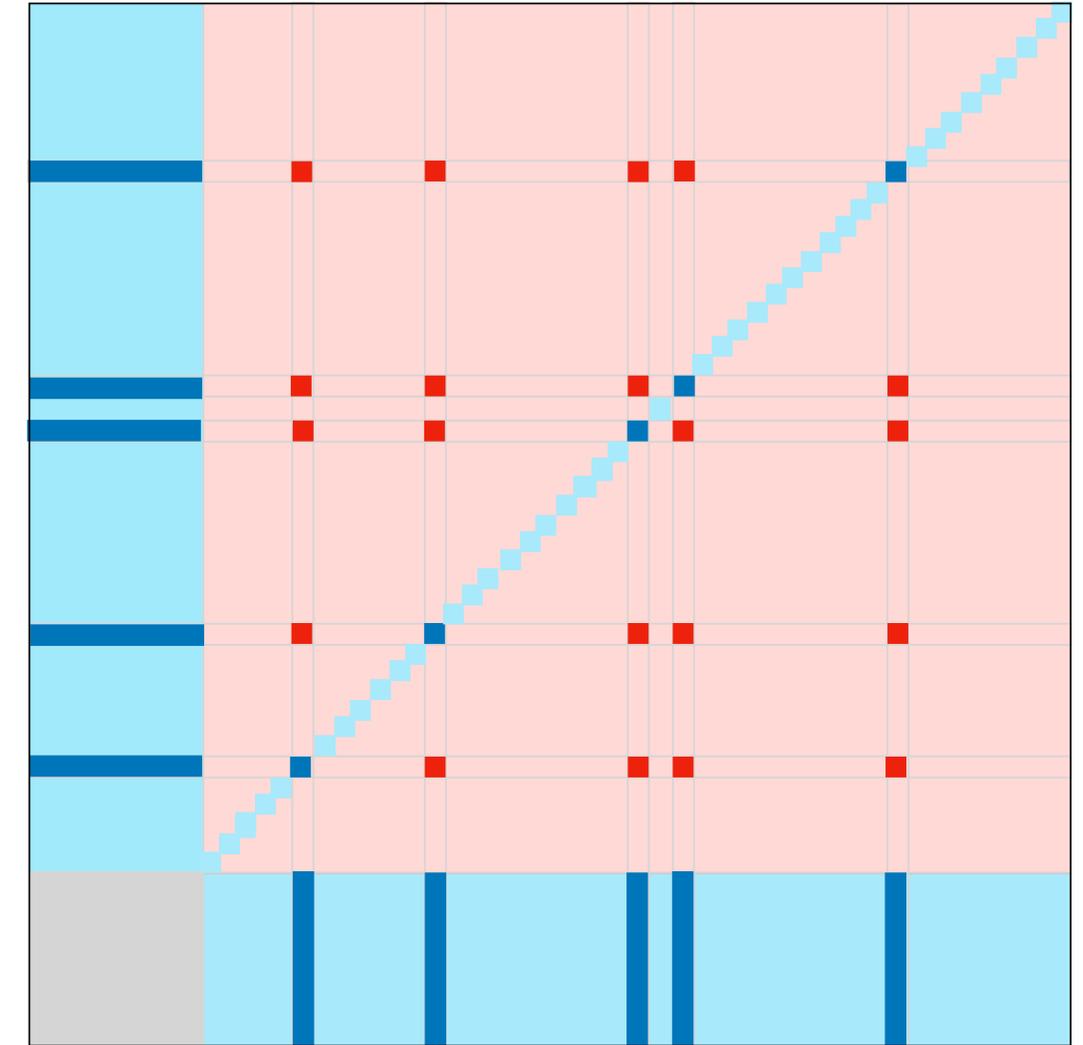
with the projected propagators

$$S_{ij} \equiv \int d^3z d^3w \langle \phi_i(z) | S(z,w) | \phi_j(w) \rangle$$

and projected and reweighed operators

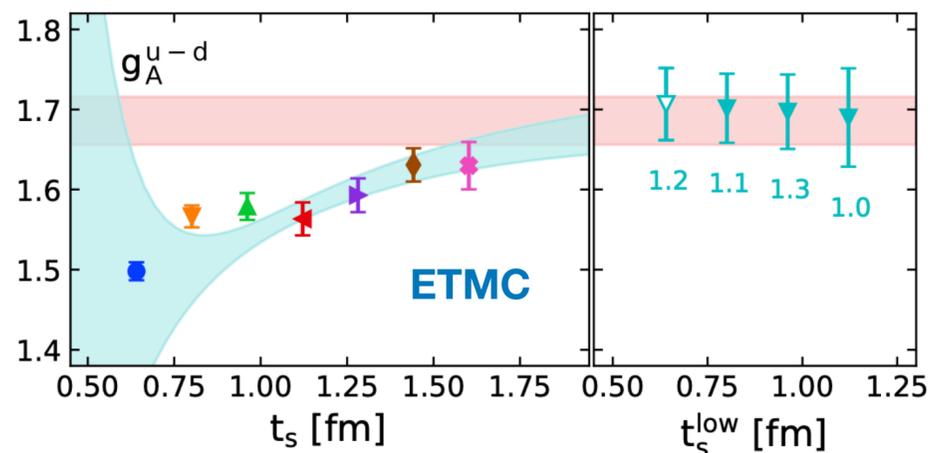
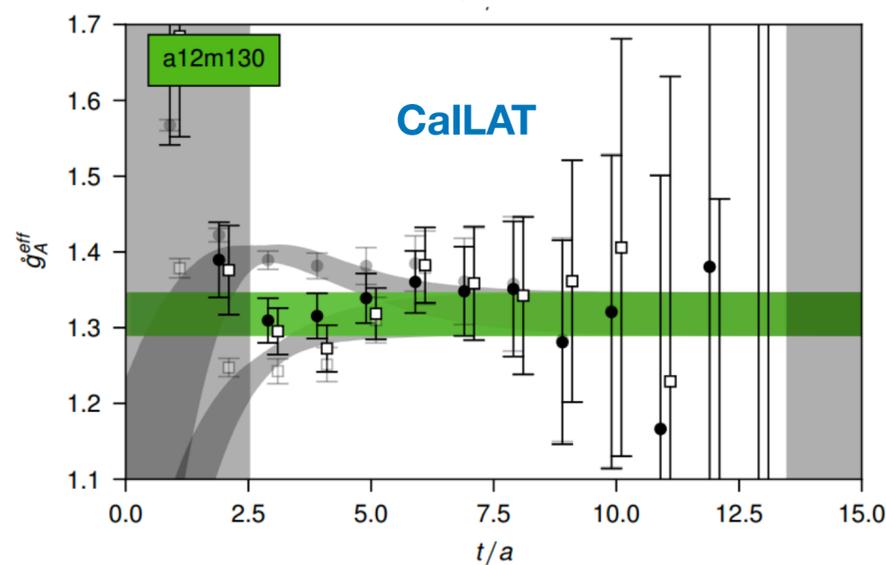
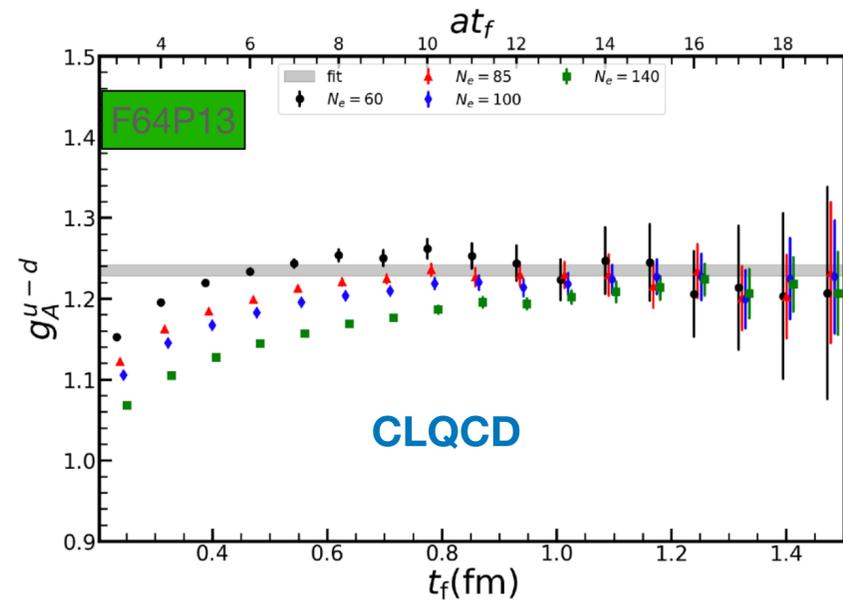
$$\mathcal{O}_{ij} \equiv \Omega_{ij}^{(2)} \int d^3z d^3w \langle \phi_i(z) | M_{\mathcal{O}}(z,w) | \phi_j(w) \rangle, \quad M_{\mathcal{O}}(x,y) = \int d^3z d^3w \hat{I}_{x,z} \mathcal{O}(z,w) \hat{I}_{w,y} = \sum_{i=1}^{N_e+N_{st}} \sum_{j=1}^{N_e+N_{st}} |\phi_i(x)\rangle \mathcal{O}_{ij} \langle \phi_j(y)| + \mathcal{O}\left(\frac{1}{N_{st}}\right),$$

$$\Omega_{ij}^{(2)} = \begin{cases} 1 & \text{for } i, j \leq N_e, \\ \prod_{i=0}^1 \frac{V - N_e - i}{N_{st} - i} & \text{for } i, j > N_e, i \neq j \\ \frac{V - N_e}{N_{st}} & \text{for the other cases,} \end{cases}$$



# Light and strange quarks

## Cost comparison at physical pion mass



	Ensemble	L	T	a(fm)	$m_\pi$	$n_{\text{cfg}}$	$g_A^{u-d}$	Propagators	Propagators for 1% error
CaLLAT	a12m130	48	64	0.121	131	1000	1.29(03)	0.03M	0.15M
CLQCD	F64P13	64	128	0.078	134	40	1.24(01)	0.34M	0.11M
ETMC	cB211.072.64	64	128	0.080	139	750	1.29(02)	1.71M	5.5M
RQCD	D452	64	128	0.076	156	1000	1.19(25)	0.01M	5.2M
PNDME	a09m130	64	96	0.090	138	1290	1.32(03)	1.69M	11.2M

C.C.Chang et. al., [CaLLAT], Nature 558(2018)91

Z.C.Hu, et. al., [CLQCD] in preparation

C.Alexandrou et. al., [ETMC], PRD102(2020)054517

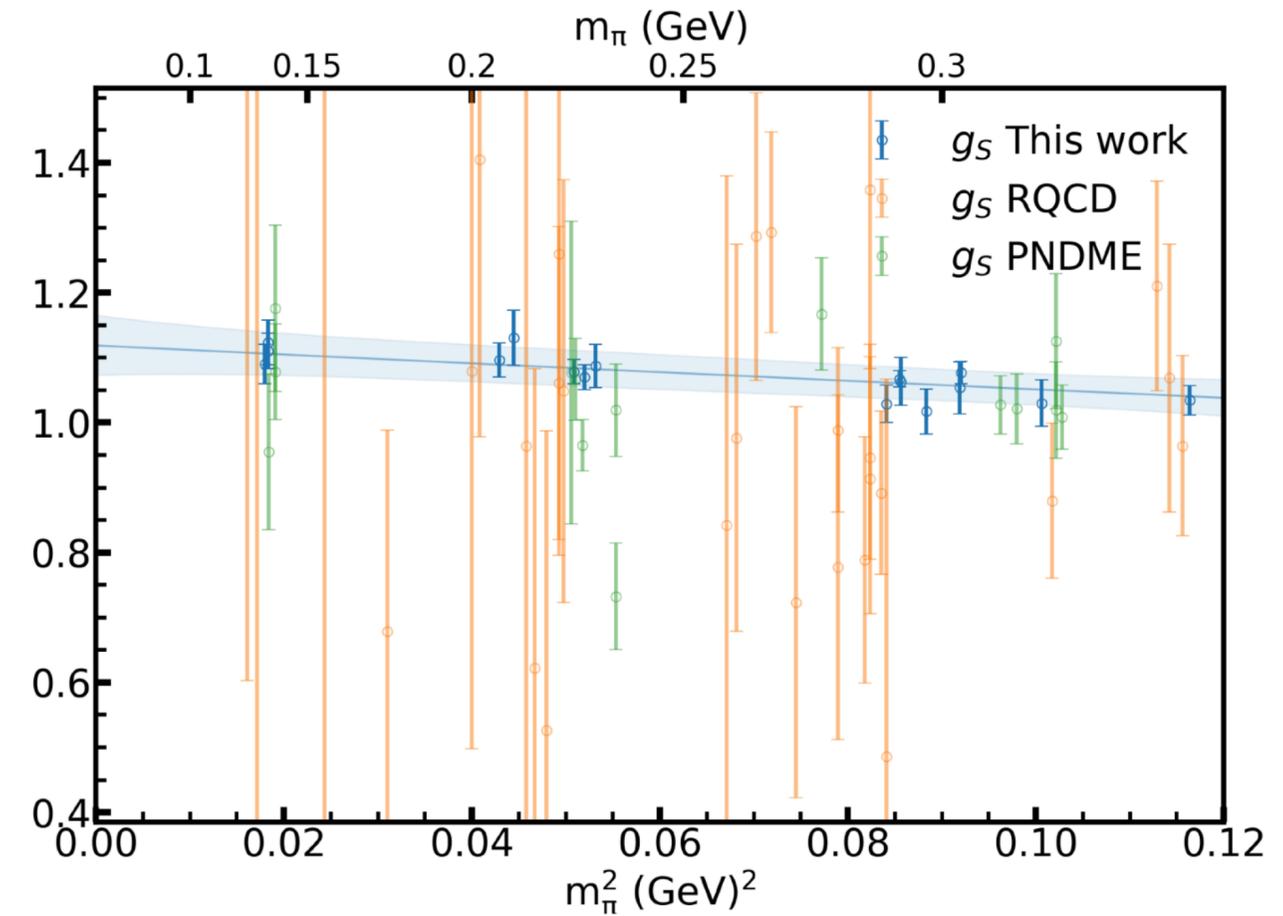
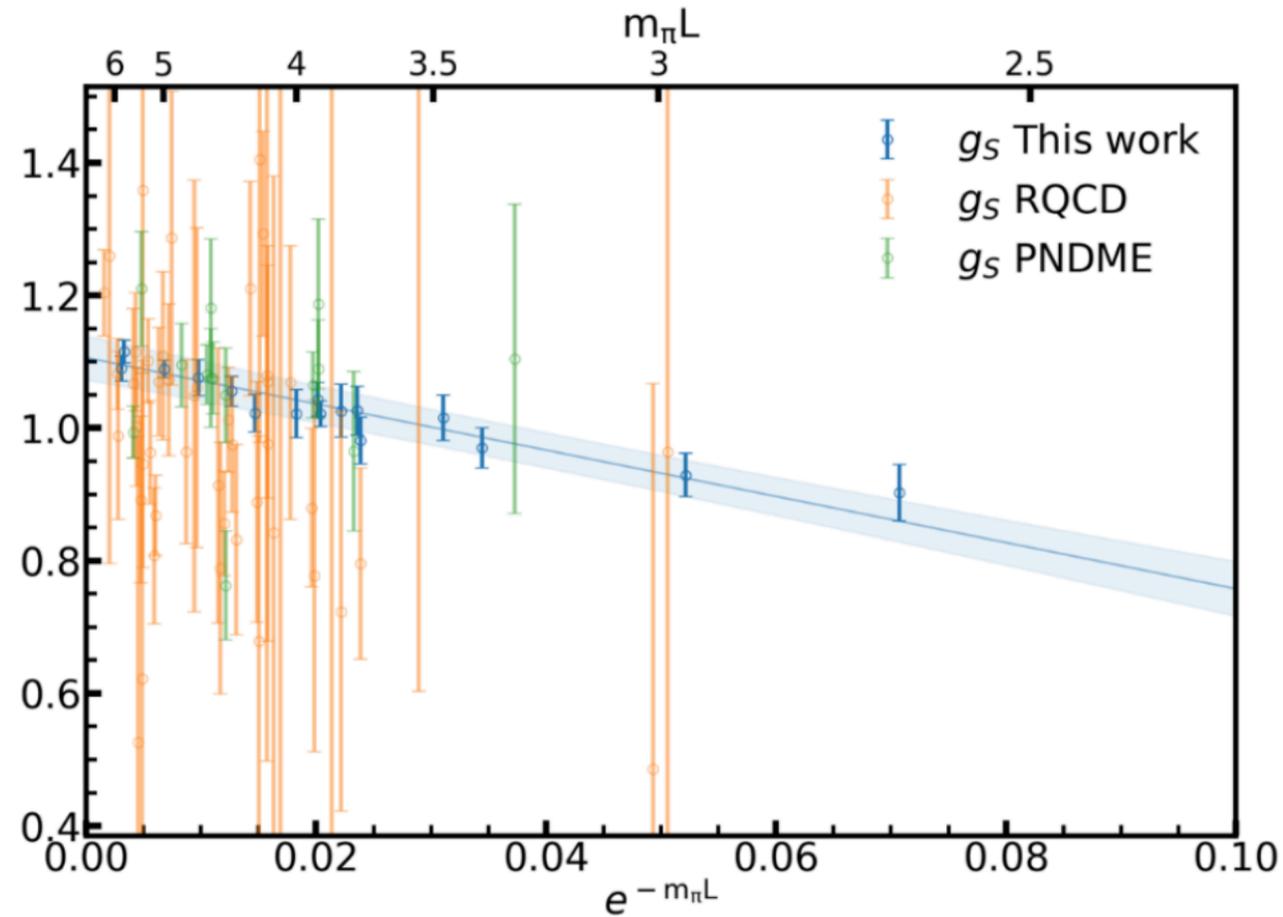
G.S.Bali et. al., [RQCD], PRD108(2023)094512

Y.C.Jang et. al., [PNDME], PRD109(2023)014503

- Advantage becomes much more significant at the physical pion mass, except the CaLLAT results which is only available at much larger lattice spacing;
- And also provide much more information on different source-current-sink separations and nucleon interpolation fields, which allow us to have much better control on the excited state contaminations.

# Light and strange quarks

## Control on systematic uncertainties



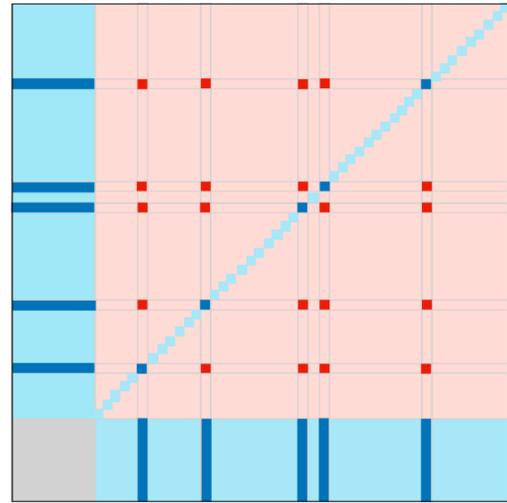
J.H. Wang et al. [CLQCD], arXiv:2511.02326

Ensemble	L	T	a(fm)	
C48P14/C64P14	48/64	96	0.105	135
C24P23/C32P23/C48P23	24/32/48	64/96	0.105	227
C24P29/C32P29/C48P29	24/32/48	64/72	0.105	290
F32P21/F48P21	32/48	96	0.077	210
F32P30/F48P30	32/48	64/96	0.077	300

- **Best parameter combinations** to constraint the finite volume effects at different lattice spacing and light quark masses;
- **Much better signal at physical pion mass** compared to all the previous results.

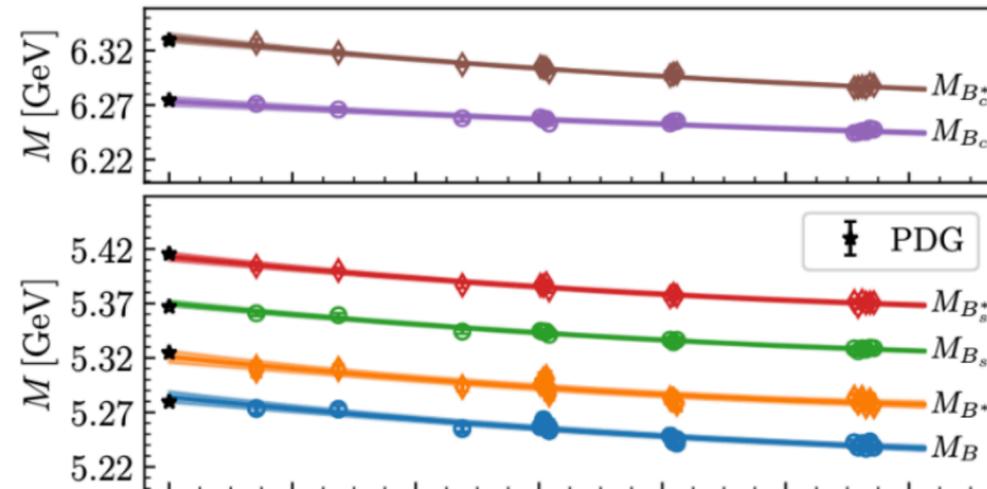
# Outline

- Light and strange quark

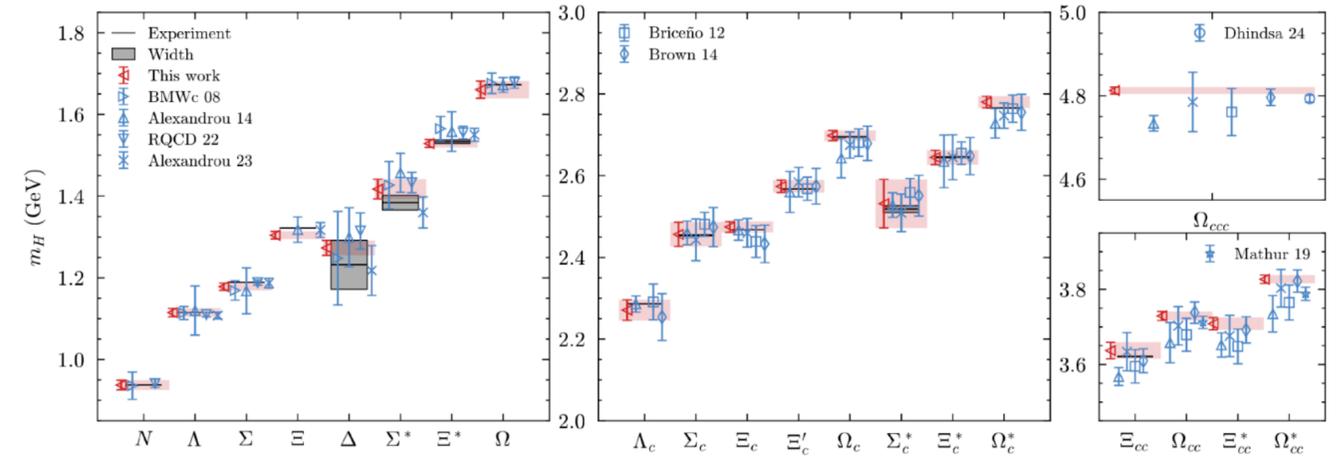


$$\Omega_{ij}^{(2)} = \begin{cases} 1 & \text{for } i, j \leq N_e, \\ \prod_{i=0}^1 \frac{v - N_e - i}{N_{st} - i} & \text{for } i, j > N_e, i \neq j \\ \frac{v - N_e}{N_{st}} & \text{for the other cases,} \end{cases}$$

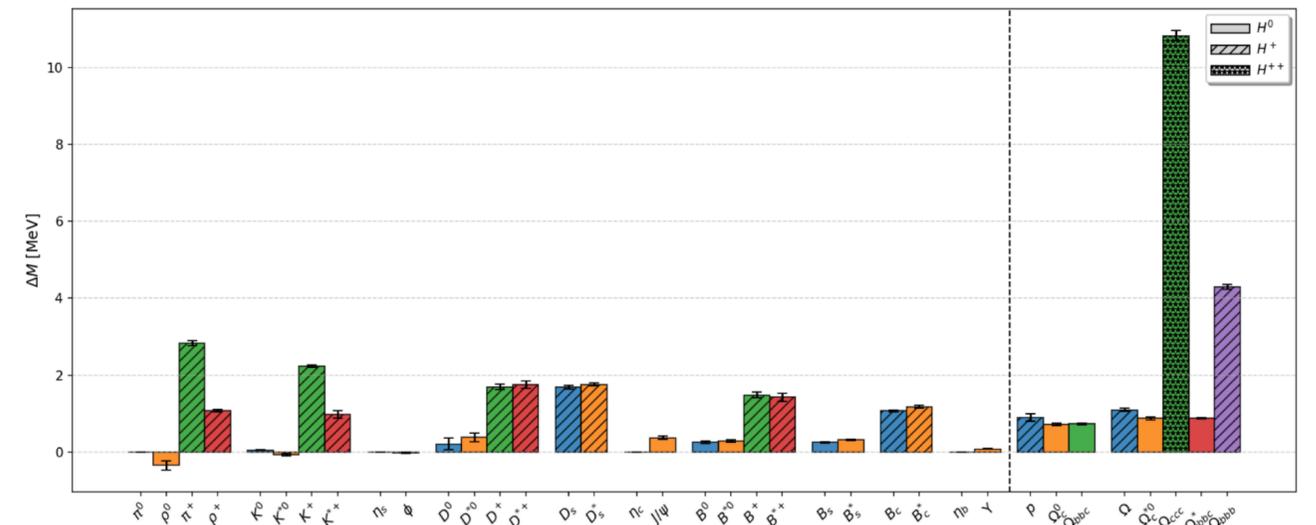
- Bottom quark



- Charm quark



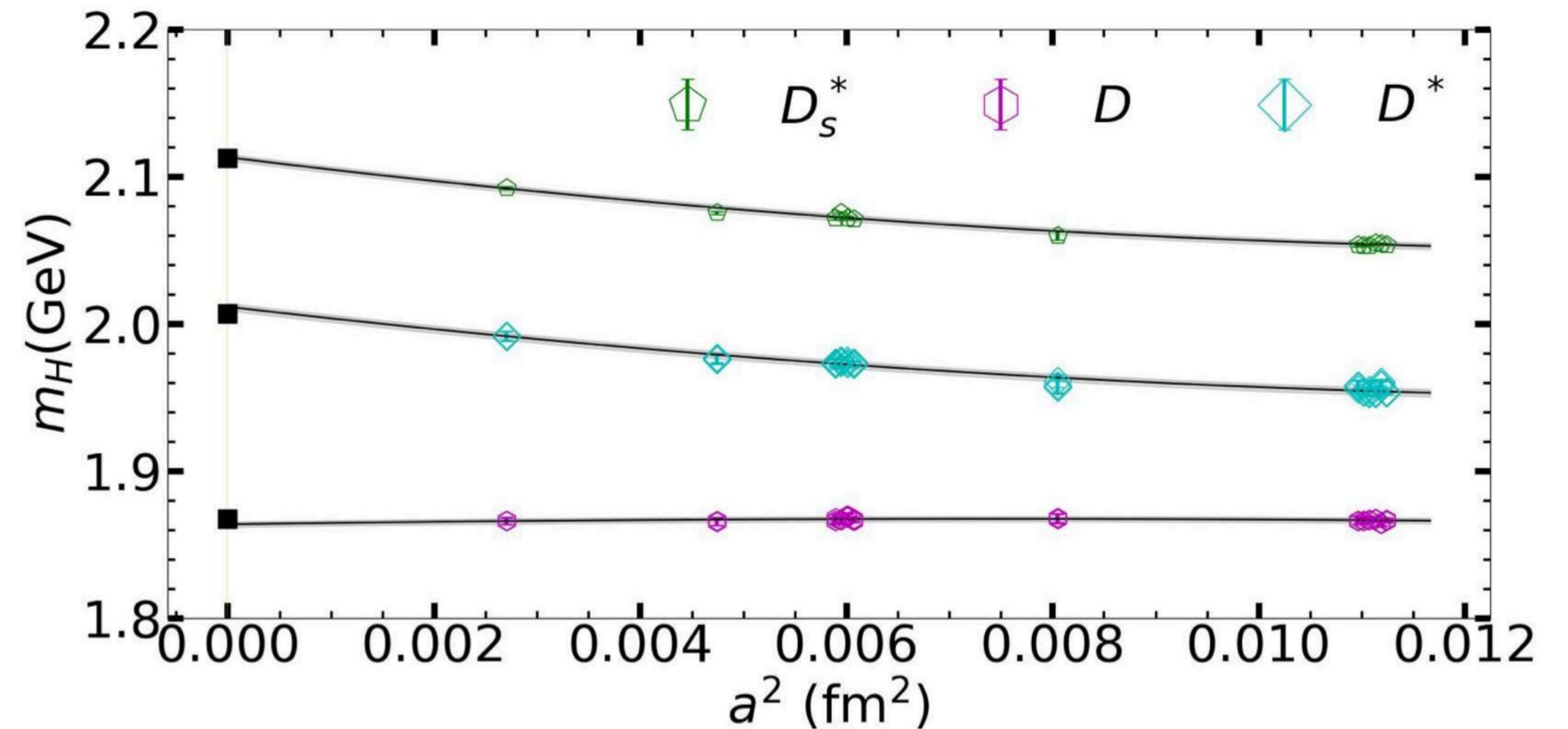
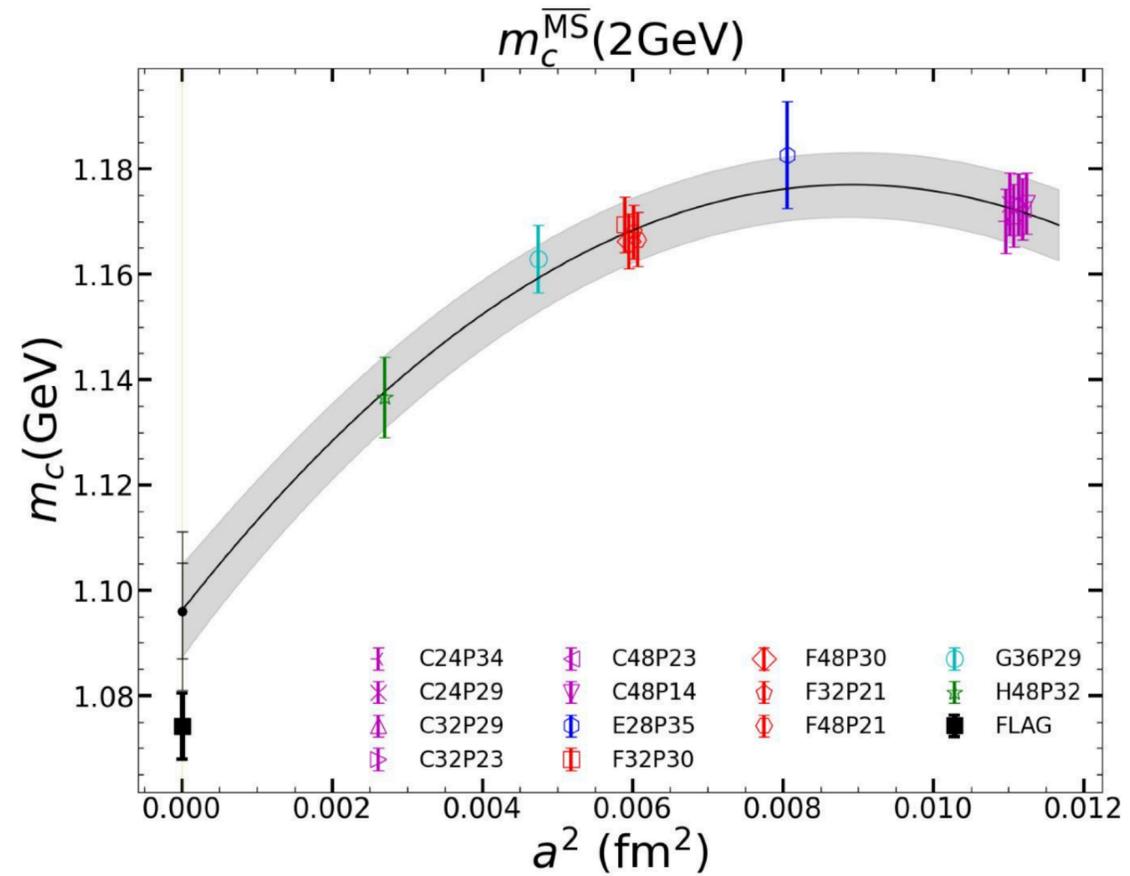
- QED corrections



- Summary

# Charm quark

## Quark mass v.s Hadron masses



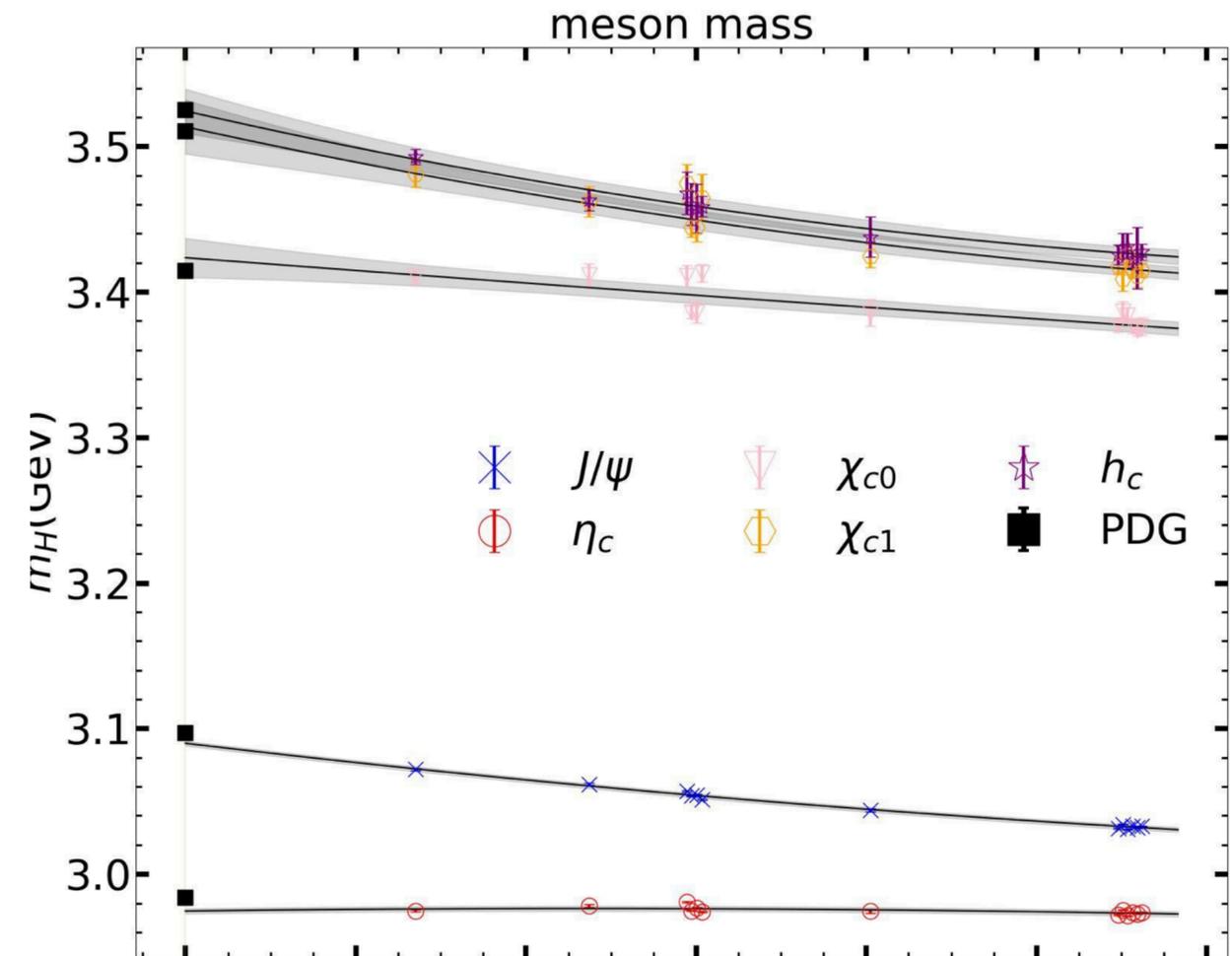
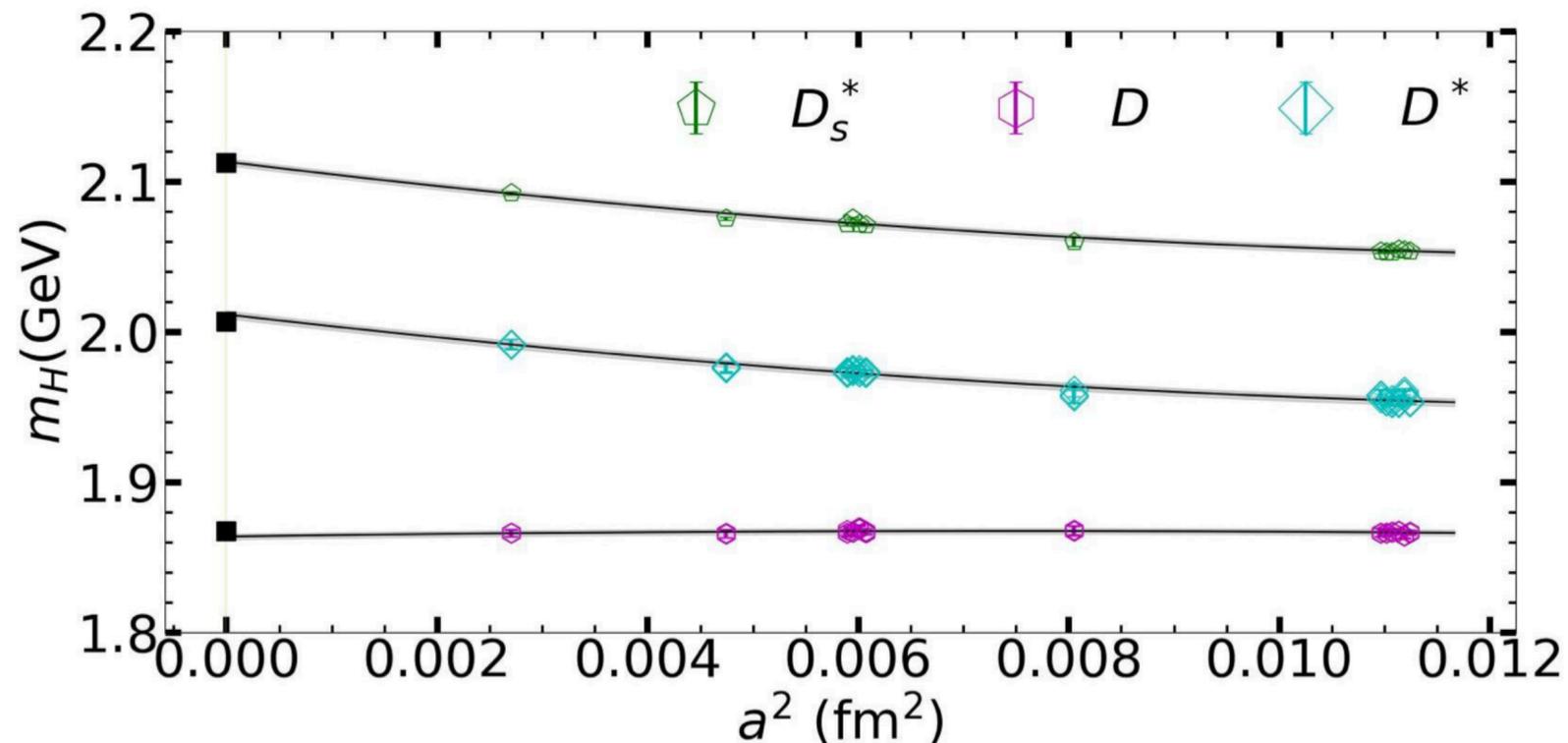
Hai-Yang Du, B.L. Hu, et. al., CLQCD, PRD111(2025) 054504

- Using  $m_{D_s}^{\text{QCD}} = m_{D_s}^{\text{phys}} - \Delta^{\text{QED}} m_{D_s} = 1.9667(15)$  GeV as input to determine the quark mass on each sample at each lattice spacing;
- Most of the  $\mathcal{O}(m_c^2 a^2)$  errors in the hadron mass are absorbed into the definition of  $m_c(a)$ ;
- The systematic uncertainty of the scale setting ( $\sim 0.5\%$ ) is also majorly cancelled for the masses of other hadron containing charm quark, e.g., we predict  $m_D = 1.864(2)(2)$  GeV.

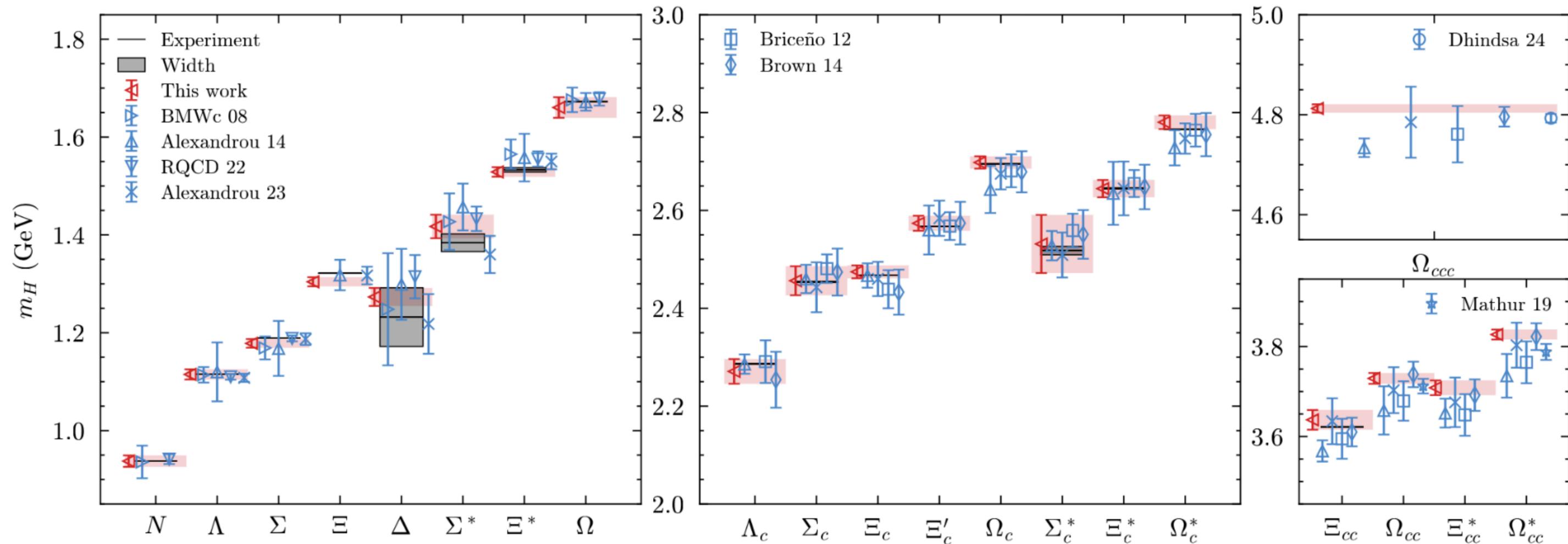
# Charm quark

## Meson spectrum

- $m_D^\pm - m_D^0 = 2.9(2)_{\text{QCD}} + 2.4(5)_{\text{QED}} = 5.3(2)(5)$  MeV agrees with the PDG value  $4.8(1)$  MeV well;
- $m_{J/\psi} - m_{\eta_c} = 116(3)$  MeV agree with previous HPQCD pure QCD prediction  $119(1)$  MeV.

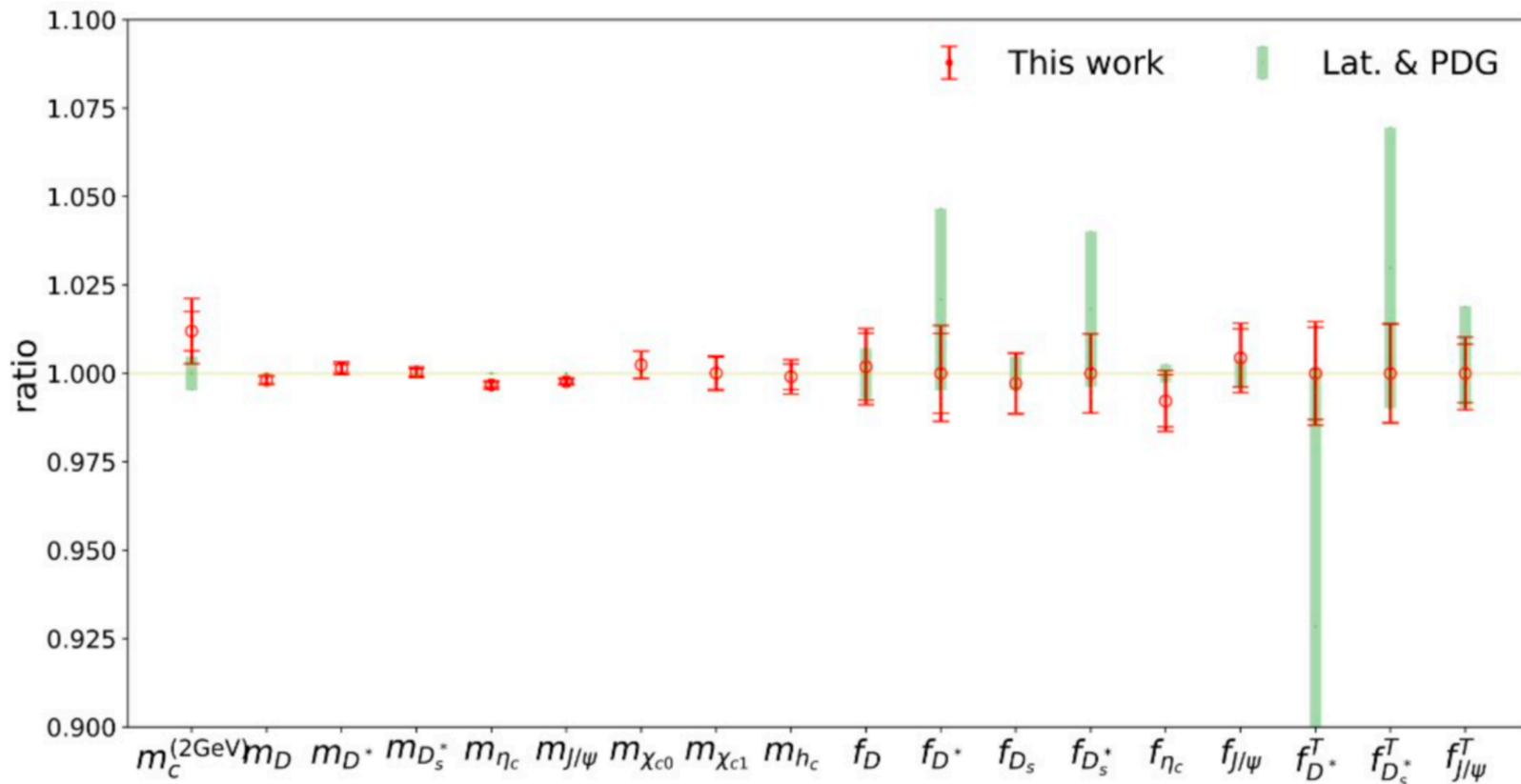


- P-wave charmonium masses also agree with PDG well, as  $m_{1P} - m_{1S} = 461(19)$  MeV.



B.-L. Hu, et. al., CLQCD, arXiv: 2411.18402

- Predict the ground state  $\frac{1}{2}^+$  and  $\frac{3}{2}^+$  masses at 1% level, while QED effects are missing;
- Agree with the previous results well with better precision in most of the cases, and more systematic.



Hai-Yang Du, B.L. Hu, et. al., CLQCD, PRD111(2025) 054504

$$f_{D^+} = 0.2113(33)_{\text{lat}} \text{ MeV}$$

$$\downarrow$$

$$f_{D^+} |V_{cd}| = 45.8(1.1)_{\text{exp}} \text{ MeV} \longrightarrow |V_{cd}| = 0.2168(33)_{\text{lat}}(52)_{\text{exp}}$$

$$f_{D_s^+} |V_{cs}| = 243.5(2.7)_{\text{exp}} \text{ MeV} \longrightarrow |V_{cs}| = 0.975(13)_{\text{lat}}(11)_{\text{exp}}$$

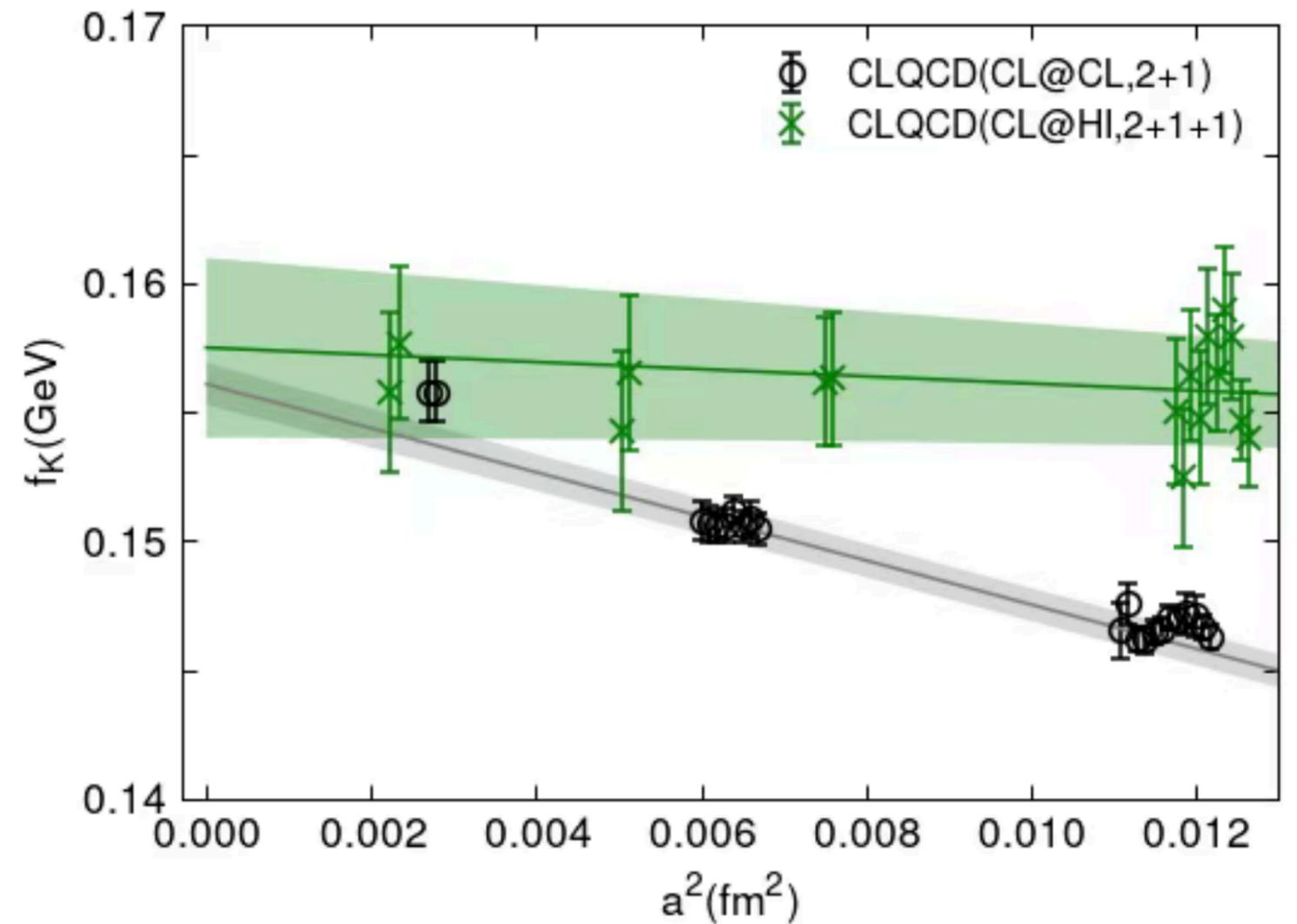
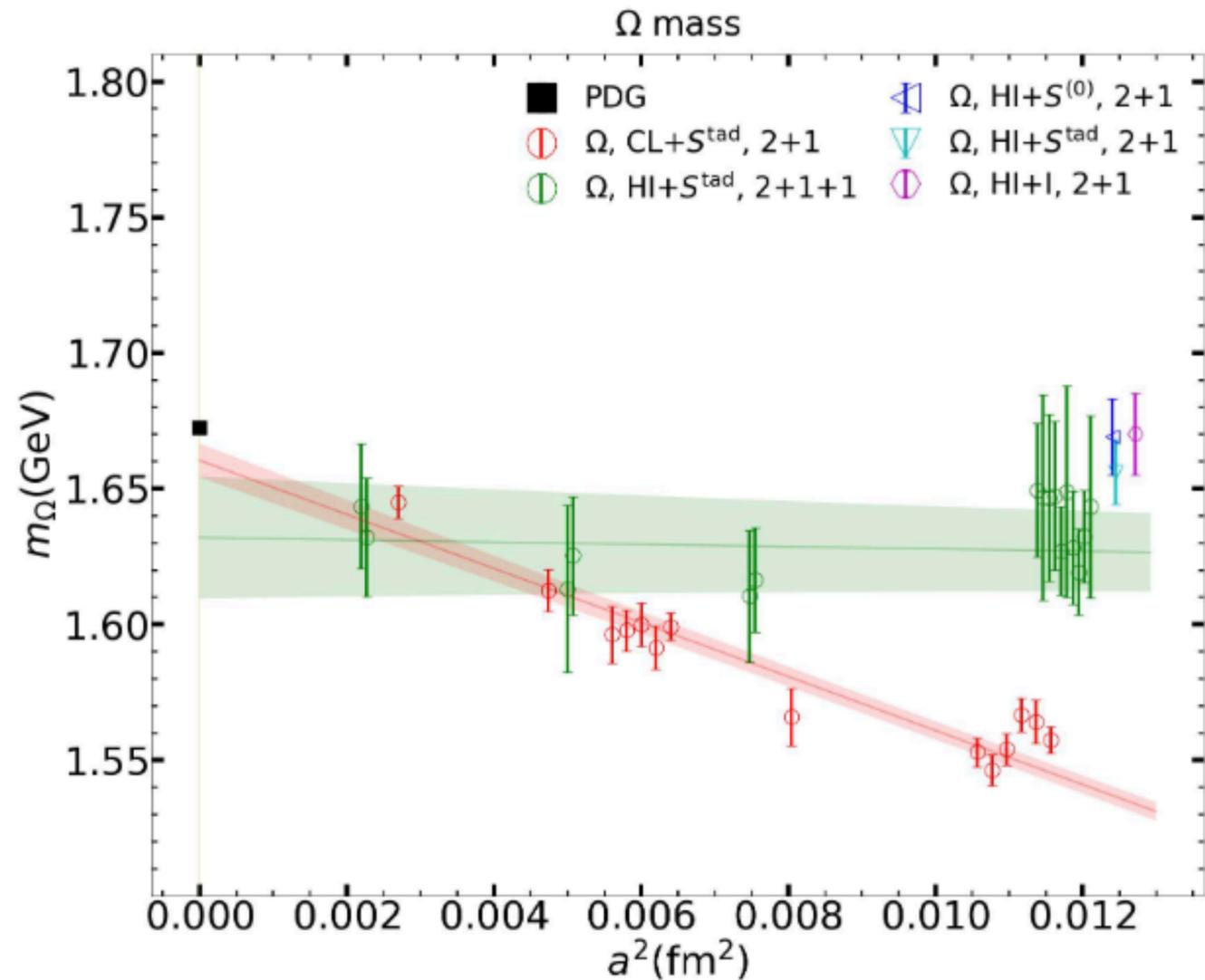
$$\uparrow$$

$$f_{D_s^+} = 0.2498(33)_{\text{lat}} \text{ MeV}$$

- Verified the unitarity of CKM matrix elements involving the charm quark:

$$|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 0.999(25)(22).$$

- Also provide the most precise  $f_{D^*}$  and  $f_{D_s^*}$  so far.



- Discretization errors are also smaller for  $m_\Omega$  and  $f_K$  with similar continuum limit which suggest negligible charm sea effects.

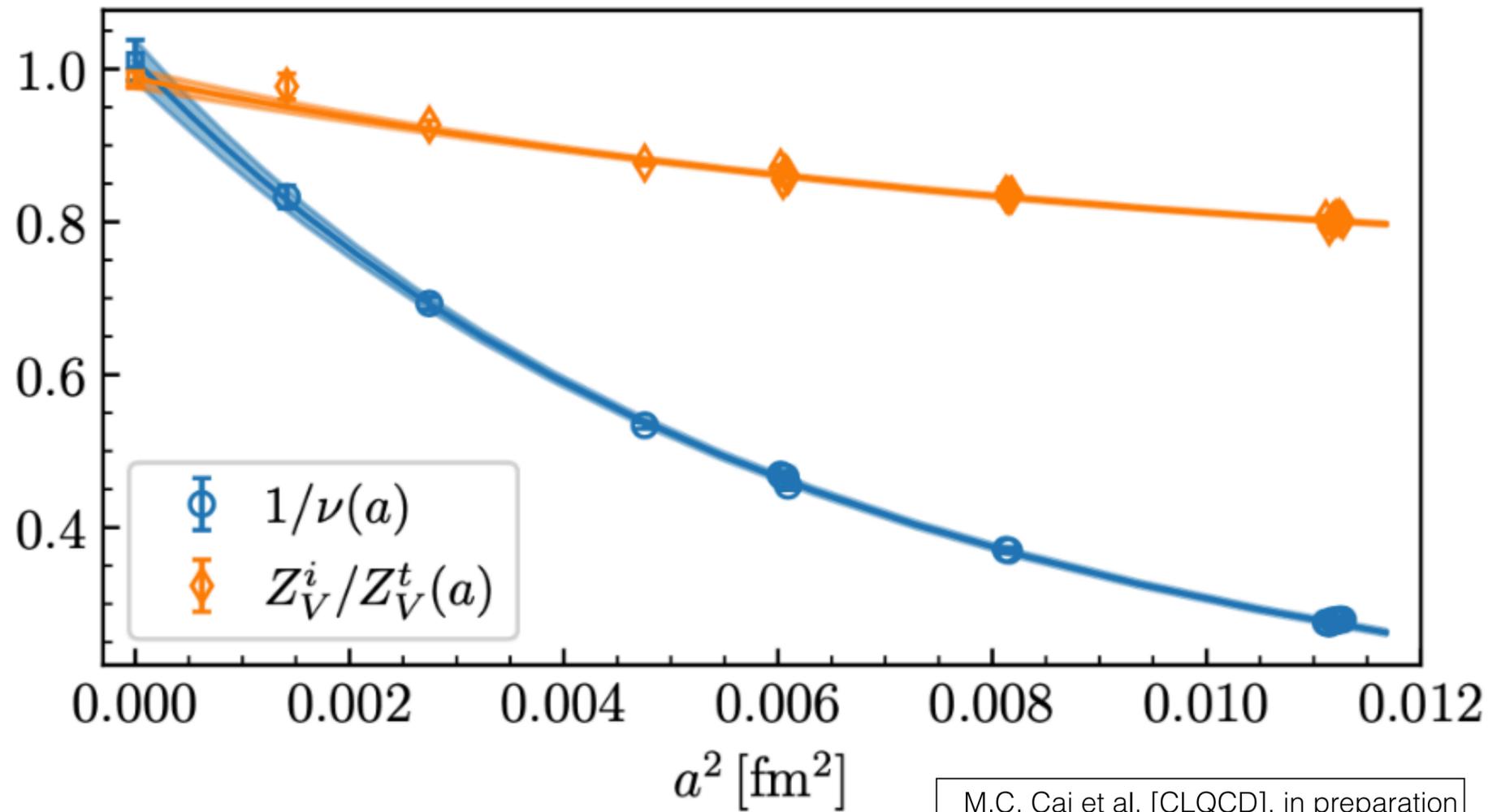


# Bottom quark

$$S_Q = a^4 \sum_x \bar{Q} \left[ m_Q + \gamma_4 \nabla_4 - \frac{a}{2} \nabla_4^2 + \nu \sum_{i=1}^3 \left( \gamma_i \nabla_i - \frac{a}{2} \nabla_i^2 \right) - c_E(\nu, u_0) \frac{a}{2} \sum_{i=1}^3 \sigma_{i4} F_{i4} - c_B(\nu, u_0) \frac{a}{2} \sum_{i>j=1}^3 \sigma_{ij} F_{ij} \right] Q,$$

$$c_E(\nu, u_0) = (1 + \nu)/(2u_0^3), \quad c_B(\nu, u_0) = \nu/(u_0^3)$$

L.M. Liu et al. PRD81(2010)094505



## Restoration of isotropy in the continuum limit

- Tune  $\nu$  to ensure  $E_H^2 = m_H^2 + p_H^2$ ;
- Practical calculation suggests
 
$$\nu = \frac{\sinh(c_\nu m_H a)}{c_\nu m_H a} (1.006(10) + 0.258 \text{ GeV}^{-2} (m_\pi^2 - m_{\pi, \text{phys}}^2))$$
 with  $c_\nu = 0.621(3)$ ;
- Comparable with the free quark case  $c_\nu = 0.5$ ;
- $\nu \rightarrow 1$  with either  $m_H \rightarrow 0$  or  $a \rightarrow 0$ ;
- The anisotropy of the matrix elements also approaches 1 in the continuum limit:

$$\frac{Z_{V_z}}{Z_{V_t}} \equiv \frac{\langle H | V_t | H \rangle p_z}{\langle H | V_z | H \rangle E} = 0.99(2) - 0.015(2) m_H^2 a^2.$$

# Bottom quark

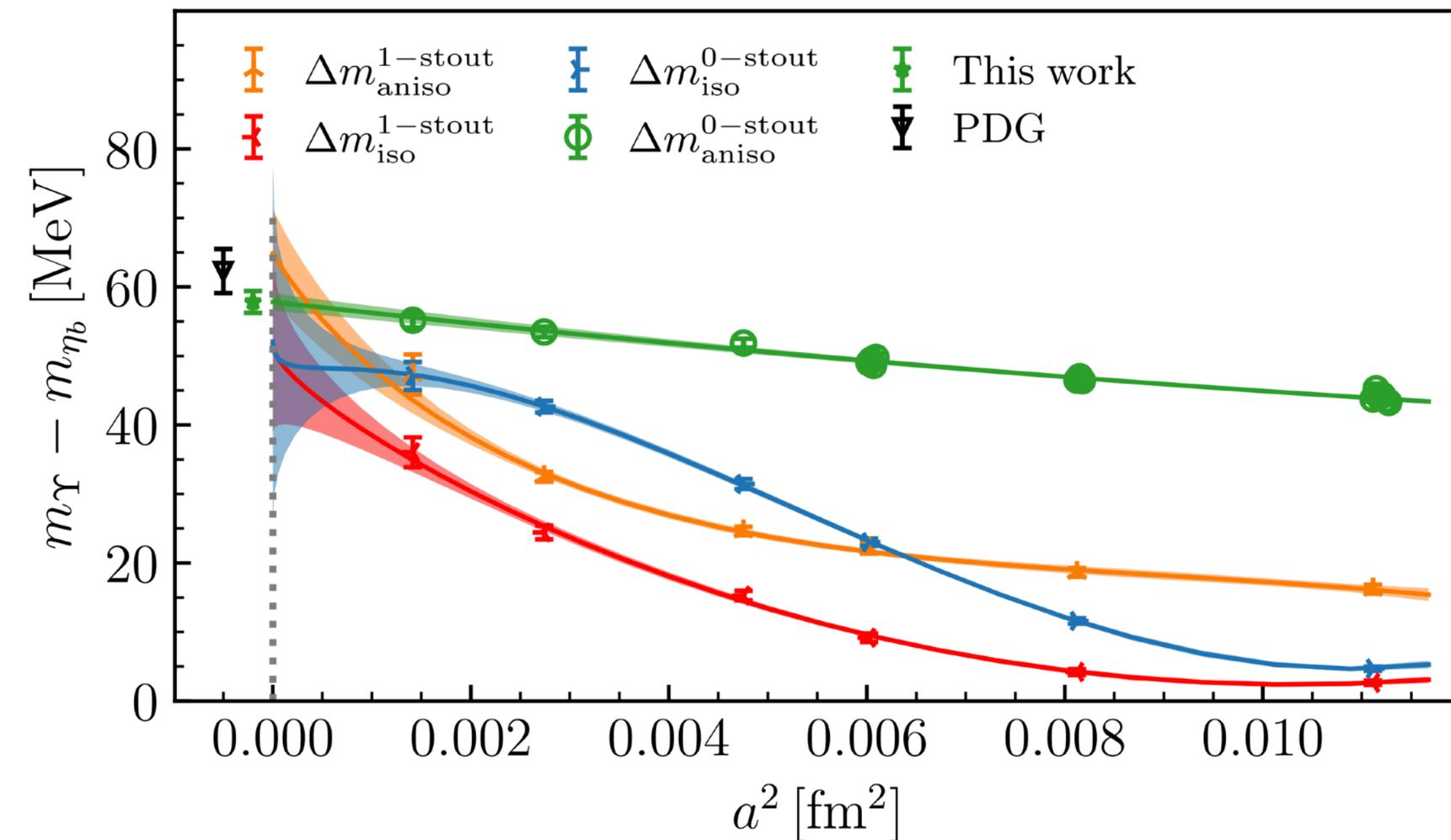
$$\Delta_{\text{HFS}}^b \equiv M_\Upsilon - M_{\eta_b}$$

$$S_Q = a^4 \sum_x \bar{Q} \left[ m_Q + \gamma_4 \nabla_4 - \frac{a}{2} \nabla_4^2 + \nu \sum_{i=1}^3 \left( \gamma_i \nabla_i - \frac{a}{2} \nabla_i^2 \right) - c_E(\nu, u_0) \frac{a}{2} \sum_{i=1}^3 \sigma_{i4} F_{i4} - c_B(\nu, u_0) \frac{a}{2} \sum_{i>j=1}^3 \sigma_{ij} F_{ij} \right] Q,$$

$$\Delta_{\text{HFS}}^b \equiv M_\Upsilon - M_{\eta_b}$$

$$c_E(\nu, u_0) = (1 + \nu)/(2u_0^3), \quad c_B(\nu, u_0) = \nu/(u_0^3)$$

L.M. Liu et al. PRD81(2010)094505



Isotropic action with smeared gauge field,  
as used by lighter flavors  
 $\Delta_{\text{HFS}}^b(a = 0.105 \text{ fm}, \nu = 1, u_0 \sim 0.95) \sim 3 \text{ MeV}$

Change gluon  
discretization

Introduce anisotropy

Isotropic action with  
**original** gauge field

**Anisotropic** action with  
smeared gauge field

$\Delta_{\text{HFS}}^b(a = 0.105 \text{ fm}, \nu = 1,$   
 $u_0 \sim 0.85) \sim 9 \text{ MeV}$

$\Delta_{\text{HFS}}^b(a = 0.105 \text{ fm}, \nu = 3.6,$   
 $u_0 \sim 0.95) \sim 15 \text{ MeV}$

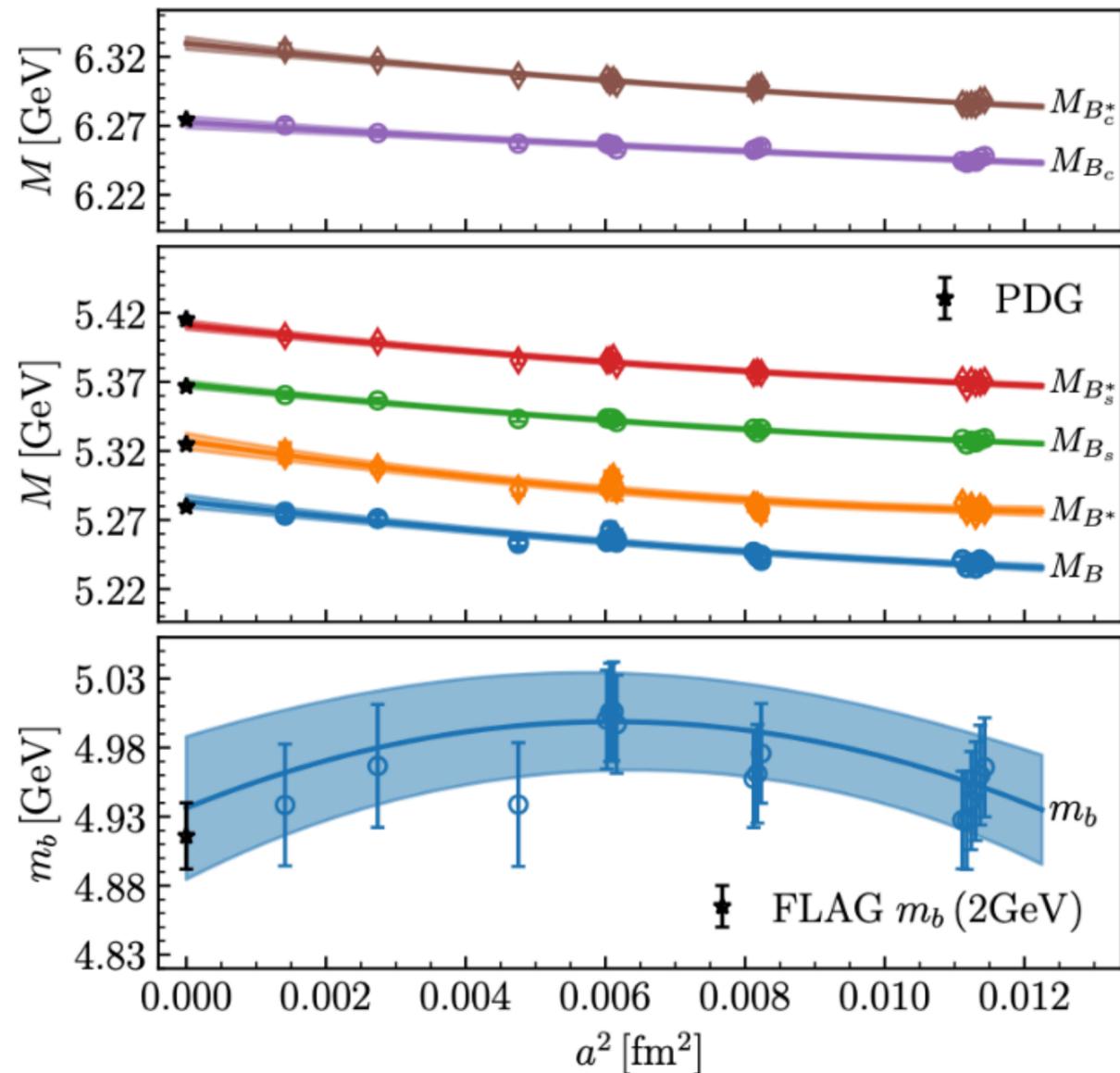
Introduce anisotropy

Change gluon  
discretization

**Anisotropic** action with  
**original** gauge field

$\Delta_{\text{HFS}}^b(a = 0.105 \text{ fm}, \nu = 3.6,$   
 $u_0 \sim 0.85) \sim 45 \text{ MeV}$

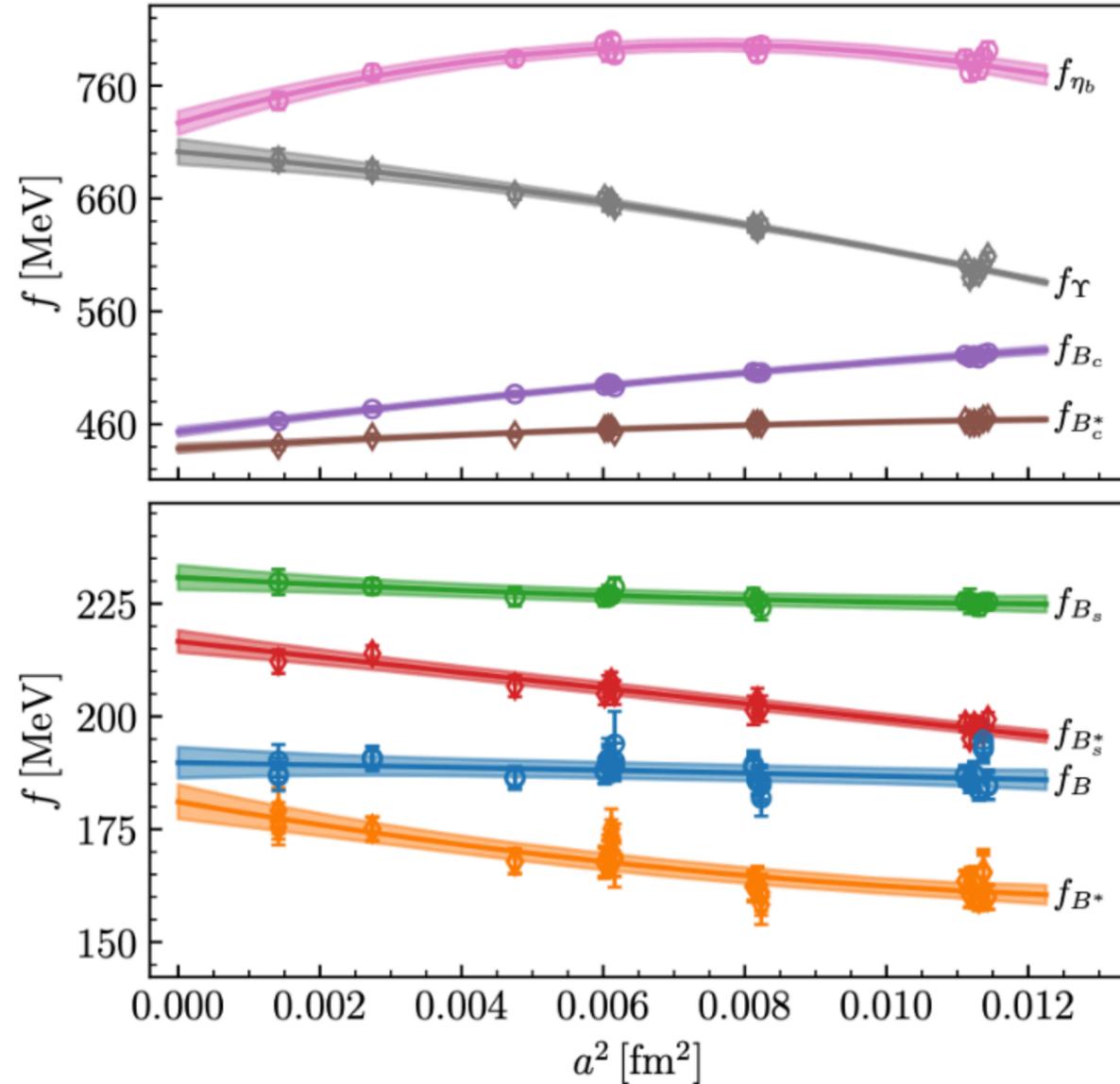
- Different discretized fermion action share the same continuum limit but very different discretization errors!



	$M_B$	$M_{B^*}$	$M_{B_s}$	$M_{B_s^*}$	$M_{B_c}$	$M_{B_c^*}$	$M_{\eta_b}$
$M$ [GeV]	5.2835(46)	5.3271(59)	5.3682(25)	5.4112(33)	6.2725(43)	6.3297(45)	9.4026(12)
PDG	5.2796(02)	5.3248(02)	5.3669(01)	5.4154(14)	6.2745(03)	6.3295(30)	9.3987(20)
$\chi^2/\text{dof}$	0.93	0.82	0.65	0.53	0.73	0.60	0.78
$Q$	0.60	0.81	0.84	0.93	0.76	0.88	0.71

- Discretization errors of both the bottom quark and hadron masses are at 1% level at the coarsest lattice spacing;
- Combining the ISB and QED corrections gives
 
$$m_{B^0}^{\text{phys}} = 5283.5(4.6) + 1.1(0.1)_{\text{ISB}} + 0.2(0.1)_{\text{QED}} = 5284.8(4.6) \text{ GeV}$$

$$m_{B^+}^{\text{phys}} = 5283.5(4.6) - 1.1(0.1)_{\text{ISB}} + 1.7(0.2)_{\text{QED}} = 5284.1(4.6) \text{ GeV}$$
 and  $m_{B_s}^{\text{phys}} = 5368.2(2.5) + 0.2(0.2)_{\text{QED}} = 5368.4(2.5) \text{ GeV}$ .
- Using the  $M_{B_s}^{\text{phys}}$  to determine the bottom quark mass can result in a 1.5(2.5) MeV shift on all the predictions (3(5) MeV for  $M_{\eta_b/\gamma}$ ).



	$f_B$	$f_{B^*}$	$f_{B_s}$	$f_{B_s^*}$	$f_{B_c}$	$f_{B_c^*}$	$f_{\eta_b}$	$f_{\Upsilon}$
$f$ [GeV]	0.1898(34)	0.1811(38)	0.2308(27)	0.2166(24)	0.4533(39)	0.4385(38)	0.727(10)	0.701(11)
$\chi^2/\text{dof}$	0.64	0.55	0.13	0.43	0.27	0.59	0.77	1.14
$Q$	0.97	1.00	1.00	0.97	1.00	0.89	0.73	0.31
FLAG $N_f = 3$	0.1920(43)		0.2284(37)					
FLAG $N_f = 4$	0.1900(13)		0.2303(13)					exp: 0.6897(46)(8)
HPQCD		0.1750(60)		0.2130(70)	0.4270(60)	0.4220(130)	0.7240(120)	0.6772(97)

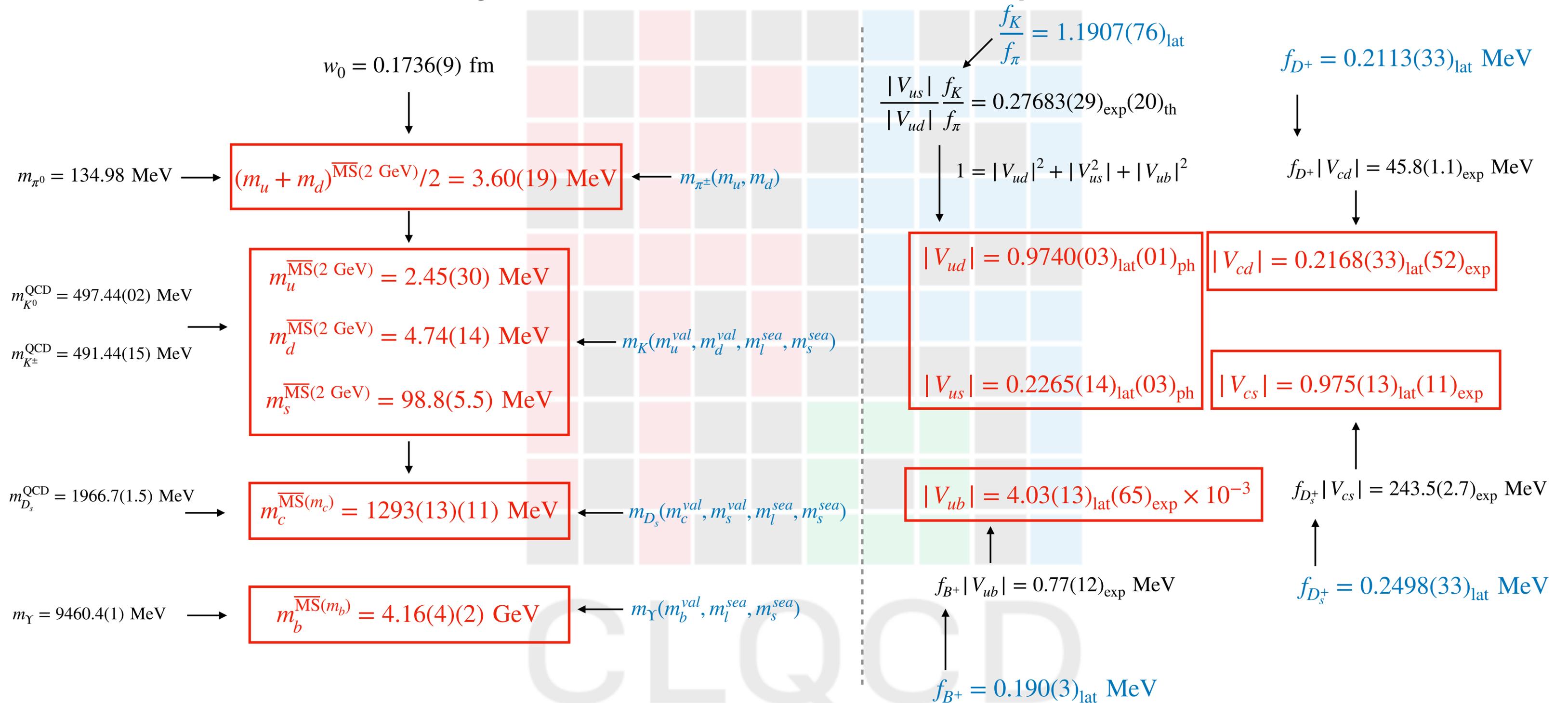
TABLE XV. Ratio of  $f_V/f_{PS}$ .

	$f_{B^*}/f_B$	$f_{B_s^*}/f_{B_s}$	$f_{B_c^*}/f_{B_c}$	$f_{\Upsilon}/f_{\eta_b}$
R	0.960(20)	0.943(12)	0.9636(89)	0.955(16)
HPQCD	0.941(26)	0.953(23)	0.988(27)	0.9454(99)
$\chi^2/\text{dof}$	0.49	0.59	0.17	0.21
$Q$	1.0	0.9	1.0	1.0

- Discretization errors of the decay constants of the bottomed hadron are highly suppressed compared to naive  $\mathcal{O}(m_b^2 a^2)$  power counting ( $m_b^2 a^2 \sim 7$  at  $a \sim 0.105$  fm) and also previous HPQCD result;
- Verify the relation  $f_{B^*}/f_B = f_{B_s^*}/f_{B_s} = f_{B_c^*}/f_{B_c} = f_{\Upsilon}/f_{\eta_b}$  with so far the highest precision.

# Bottom quark

## Summary of our standard model parameter determinations



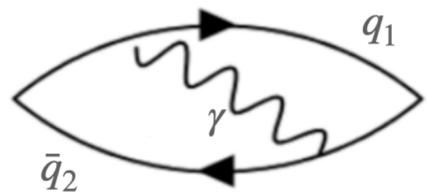


# QED corrections

## Subtraction scheme

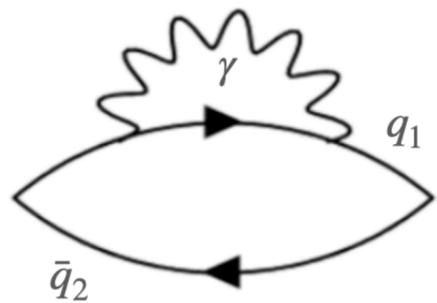
- The QCD+QED calculation can be done under the quenched QED approximation using  $\text{QED}_L$  for the valence fermion:

$$U_\mu^{\text{QCD+QED}} = U_\mu^{\text{QCD}} e^{-iee_q A_\mu}, \quad A_\mu(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} A_\mu(p), \quad P_{A_\mu(p)}|_{\vec{p} \neq 0} \propto e^{-\frac{1}{2N_V} \hat{p}^2 A_\mu^2(p)}$$



QED UV finite (at 1-loop level)

$$-e_{q_1} e_{q_2} \delta_{\text{QED}}^{\text{int}} m_\pi (m_q + \delta_{\text{QED}} m_q) \simeq -e_{q_1} e_{q_2} \delta_{\text{QED}}^{\text{int}} m_\pi (m_q)$$



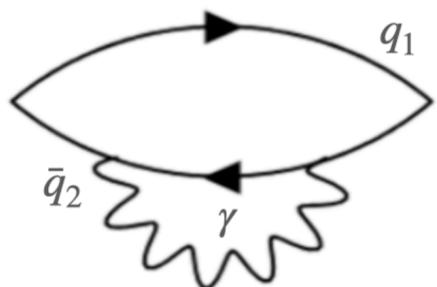
$$e_{q_1}^2 \delta_{\text{QED}}^{\text{self}} m_H$$

QED UV finite

$$\delta_{\text{QED}}^{\text{self}} m_\pi (m_q + \delta_{\text{QED}} m_q)$$

$$\simeq \delta_{\text{QED}}^{\text{self}} m_\pi (m_q) + \langle \bar{q}q \rangle_\pi \delta_{\text{QED}} m_q$$

QED UV divergent    QED UV divergent



$$e_{q_2}^2 \delta_{\text{QED}}^{\text{self}} m_H$$

- QED correction of the hadron masses suffer from additional UV divergence and require further renormalization in the quark mass;

- The PQU scheme requires

$$\delta_{\text{QED}} m_q = e_q^2 \frac{\delta_{\text{QED}}^{\text{int}} m_{\eta_q} - 2\delta_{\text{QED}}^{\text{self}} m_{\eta_q}}{2\langle \bar{q}q \rangle_{\eta_q}}$$

to cancel the UV divergence of  $\delta_{\text{QED}}^{\text{self}} m_{\eta_q}$ , and ensure

$$\delta_{\text{QED}} m_{\eta_q} = 0.$$

- And then the final correction to the hadron mass can be significantly smaller than 1% with proper quark mass renormalizations.

QED UV finite (at 1-loop level)

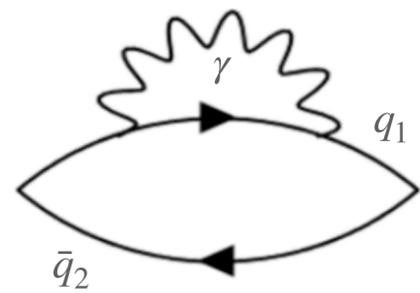
$$\delta_{\text{QED}}^{\text{int}} m_\pi(m_q + \delta_{\text{QED}} m_q) \simeq \delta_{\text{QED}}^{\text{int}} m_\pi(m_q)$$

QED UV finite

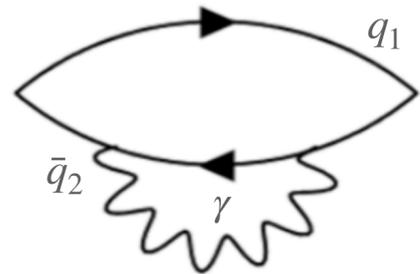
$$\delta_{\text{QED}}^{\text{self}} m_\pi(m_q + \delta_{\text{QED}} m_q) \simeq \delta_{\text{QED}}^{\text{self}} m_\pi(m_q) + \langle \bar{q}q \rangle_\pi \delta_{\text{QED}} m_q$$

QED UV divergent

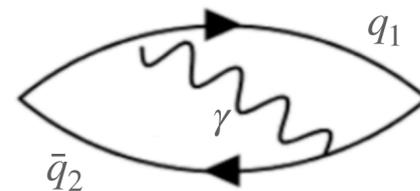
QED UV divergent



$$e_{q_1}^2 \delta_{\text{QED}}^{\text{self}} m_H$$



$$e_{q_2}^2 \delta_{\text{QED}}^{\text{self}} m_H$$



$$-e_{q_1} e_{q_2} \delta_{\text{QED}}^{\text{int}} m_H$$

$$\delta_{\text{QED}}^{\text{self}} m_\pi = 4.6 \text{ MeV}$$

$$\delta_{\text{QED}}^{\text{int}} m_\pi = 9.2 \text{ MeV}$$

$$\delta_{\text{QED}} m_{\pi^+} = (e_u^2 + e_d^2) \delta_{\text{QED}}^{\text{self}} m_\pi + e_u e_{\bar{d}} \delta_{\text{QED}}^{\text{int}} m_\pi$$

$$= \frac{5}{9} 4.6 \text{ MeV} + \frac{2}{9} 9.2 \text{ MeV} = 4.6 \text{ MeV}$$

$$\delta_{\text{QED}} m_{\pi^0} = (e_u^2 + e_d^2) \delta_{\text{QED}}^{\text{self}} m_\pi + \frac{1}{2} (e_u e_{\bar{u}} + e_d e_{\bar{d}}) \delta_{\text{QED}}^{\text{int}} m_\pi$$

$$= \frac{5}{9} 4.6 \text{ MeV} - \frac{1}{2} \frac{5}{9} 9.2 \text{ MeV} = 0$$

Self energy

+

Repulsive interaction

Self energy

+

Attractive interaction

# QED corrections

Self energy v.s.  
Interaction

$$\delta_{\text{QED}} m_H = \sum_q e_q^2 \delta_{\text{QED}}^{\text{self},q} m_H + \sum_q \sum_{q' \neq q} e_q e_{q'} \delta_{\text{QED}}^{\text{int},qq'} m_H$$

$$\begin{aligned} \delta_{\text{QED}} m_{B_0} &= e_b^2 \delta_{\text{QED}}^{\text{self},b} m_{B_0} + e_d^2 \delta_{\text{QED}}^{\text{self},d} m_{B_0} + e_b e_d \delta_{\text{QED}}^{\text{int},bd} m_{B_0} \\ &= \frac{1}{9} 5 \text{ MeV} + \frac{1}{9} 0.6 \text{ MeV} - \frac{1}{9} 4 \text{ MeV} \\ &= 0.2 \text{ MeV} \end{aligned}$$

Self energy

+

Attractive interaction

$$\begin{aligned} \delta_{\text{QED}} m_{B_+} &= e_b^2 \delta_{\text{QED}}^{\text{self},b} m_{B_+} + e_u^2 \delta_{\text{QED}}^{\text{self},u} m_{B_+} + e_b e_d \delta_{\text{QED}}^{\text{int},bd} m_{B_+} \\ &= \frac{1}{9} 5 \text{ MeV} + \frac{4}{9} 0.6 \text{ MeV} + \frac{2}{9} 4 \text{ MeV} \\ &= 1.6 \text{ MeV} \end{aligned}$$

Self energy

+

Repulsive interaction

$$\begin{aligned} \delta_{\text{QED}} m_{\Omega_{bbc}^*} &= 2e_b^2 \delta_{\text{QED}}^{\text{self},b} m_{\Omega_{bbc}^*} + e_c^2 \delta_{\text{QED}}^{\text{self},c} m_{\Omega_{bbc}^*} + e_b^2 \delta_{\text{QED}}^{\text{int},bb} m_{\Omega_{bbc}^*} + 2e_b e_c \delta_{\text{QED}}^{\text{int},bc} m_{\Omega_{bbc}^*} \\ &= 2 \frac{1}{9} 5 \text{ MeV} + \frac{4}{9} 3.5 \text{ MeV} + \frac{1}{9} 4 \text{ MeV} - 2 \frac{2}{9} 5 \text{ MeV} \\ &= 0.8 \text{ MeV} \end{aligned}$$

Self energy

+

Repulsive interaction

+

Attractive interaction

$$\begin{aligned} \delta_{\text{QED}} m_{\Omega_{ccc}} &= 3e_c^2 \delta_{\text{QED}}^{\text{self},c} m_{\Omega_{ccc}} + 3c_c^2 \delta_{\text{QED}}^{\text{int},cc} m_{\Omega_{ccc}} \\ &= 3 \frac{4}{9} 3.5 \text{ MeV} + 3 \frac{4}{9} 5 \text{ MeV} \\ &= 11 \text{ MeV} \end{aligned}$$

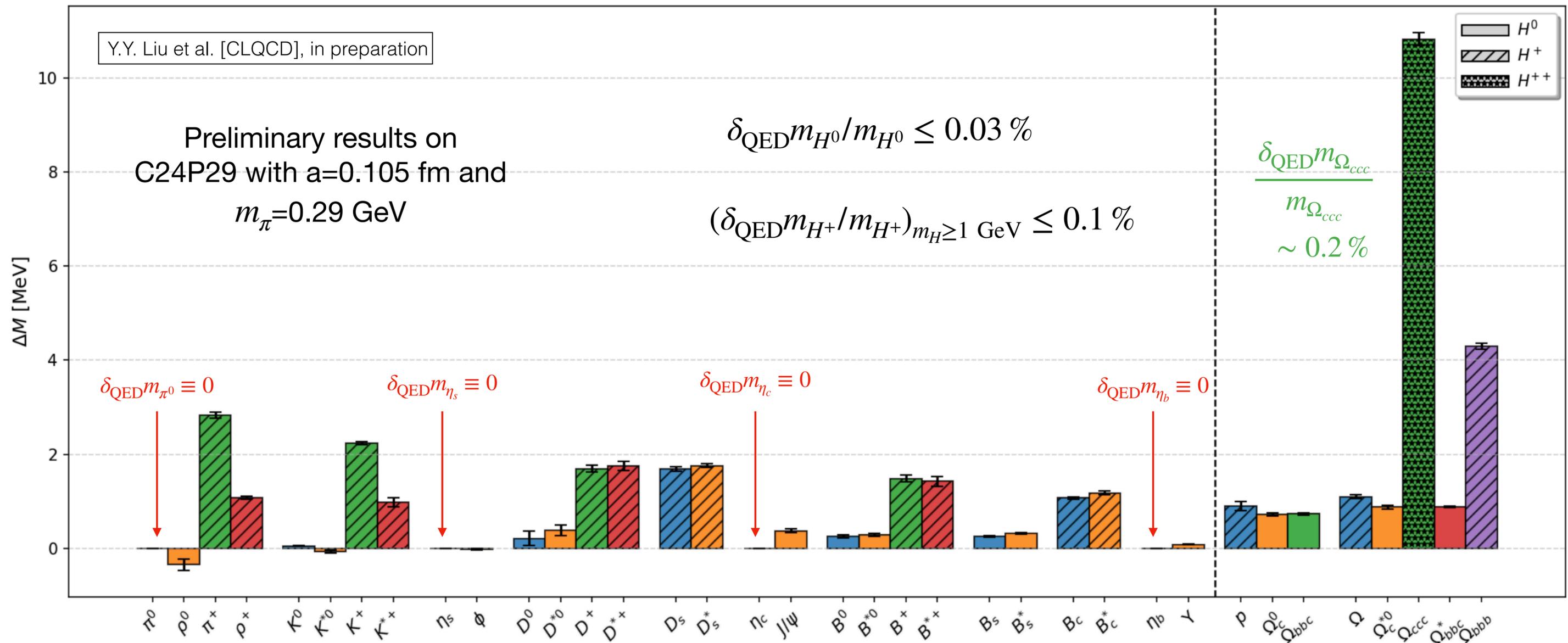
Self energy

+

Repulsive interaction

$$\delta_{\text{QED}} m_q = e_q^2 \frac{\delta_{\text{QED}}^{\text{int}, \bar{q}q} m_{\eta_q} - \delta_{\text{QED}}^{\text{self}, q} m_{\eta_q} - \delta_{\text{QED}}^{\text{self}, \bar{q}} m_{\eta_q}}{2 \langle \bar{q}q \rangle_{\eta_q}},$$

$$\delta_{\text{QED}} m_H = \sum_q e_q^2 (\delta_{\text{QED}}^{\text{self}, q} m_H + \delta_{\text{QED}} m_q \langle \bar{q}q \rangle_H) + \sum_q \sum_{q' \neq q} e_q e_{q'} \delta_{\text{QED}}^{\text{int}, qq'} m_H$$



# Summary

- Blending method provide new possibilities to improve the statistical precision for physical light quark mass;
- Charm and bottom physics can now be predicted with suppressed discretization error with kinds of the improvements;
- QED correction of the hadron masses can also be predicted precisely.

