Nonlinear evolution of the curvature perturbation and its applications

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HETH Forum

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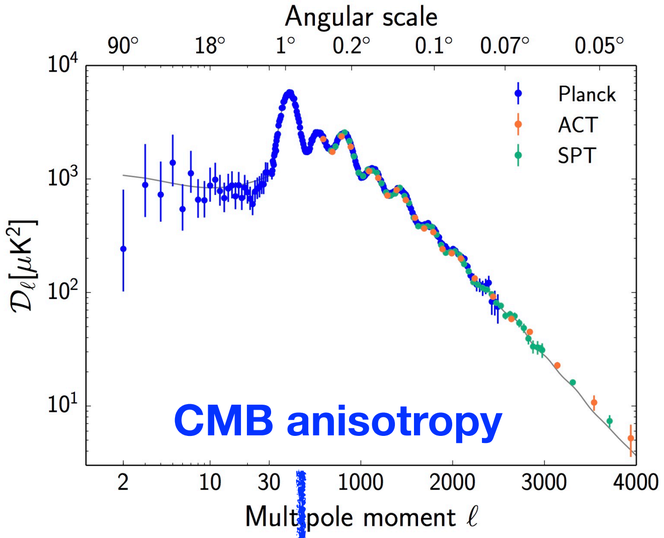
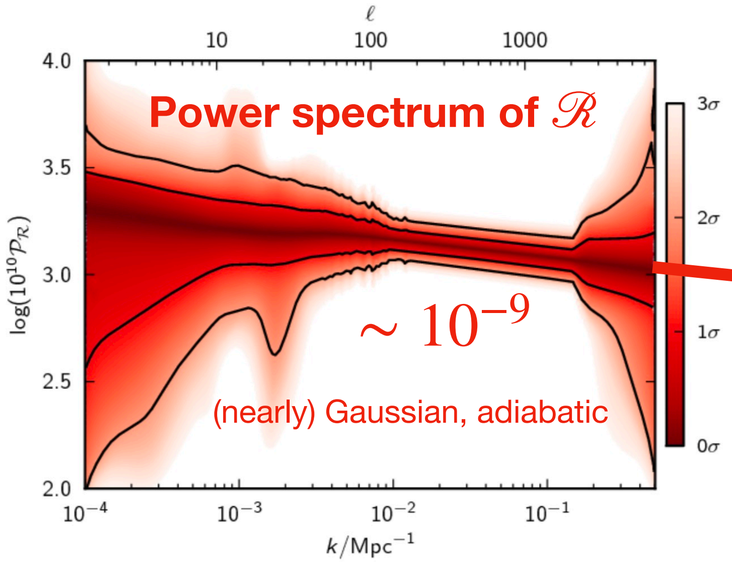
CONTENT

- Introduction: PBH and induced GW
- Nonlinear evolution of the curvature perturbation
- Application: gradient expansion and bubble relics
- Conclusion

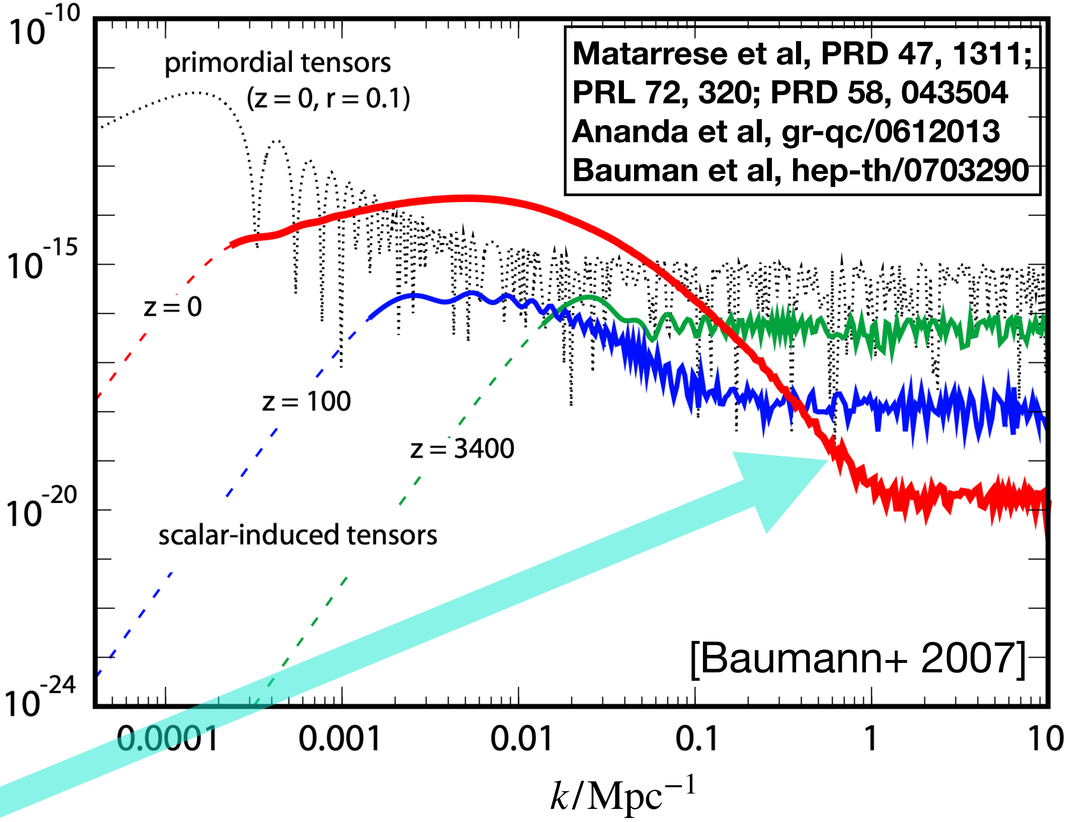


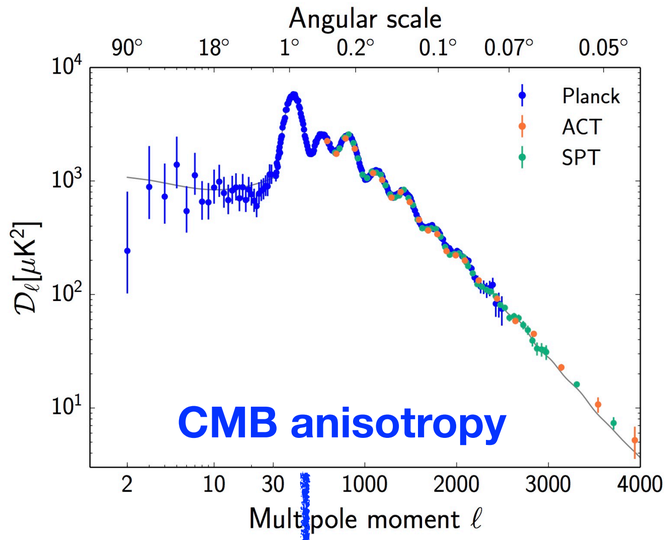
Reconstruction



$h\mathcal{R}\mathcal{R}$ coupling

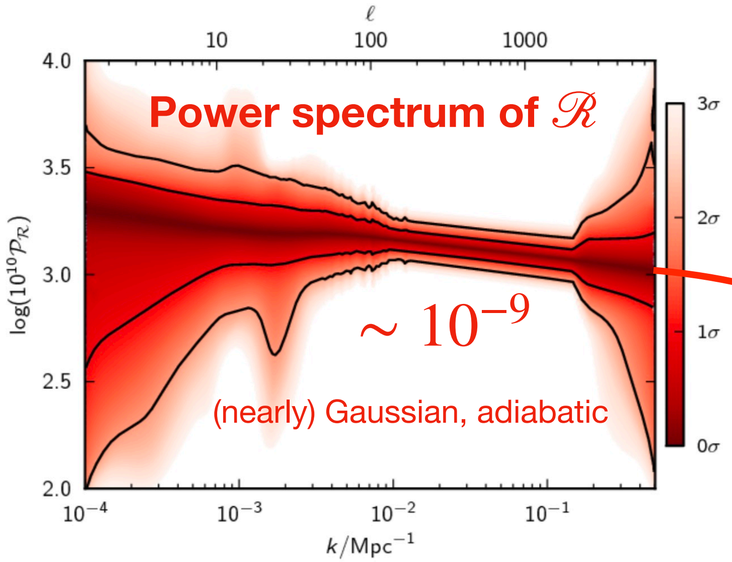
$$\Omega_{\rm GW} \sim 10^{-6}\mathcal{P}_\mathcal{R}^2 \sim 10^{-24}$$





CMB anisotropy

Reconstruction



$\sim 10^{-9}$

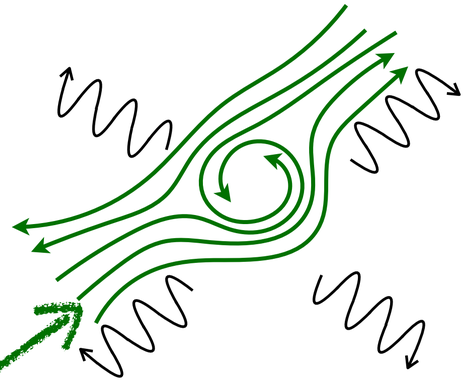
(nearly) Gaussian, adiabatic

Required by PBH formation

$\sim 10^{-2}$

Gaussian?
adiabatic?

nonlinear perturbation



Matarrese et al, PRD 47, 1311;
PRL 72, 320; PRD 58, 043504
Ananda et al, gr-qc/0612013
Bauman et al, hep-th/0703290

Scalar Perturbation Induced GW

crosscheck

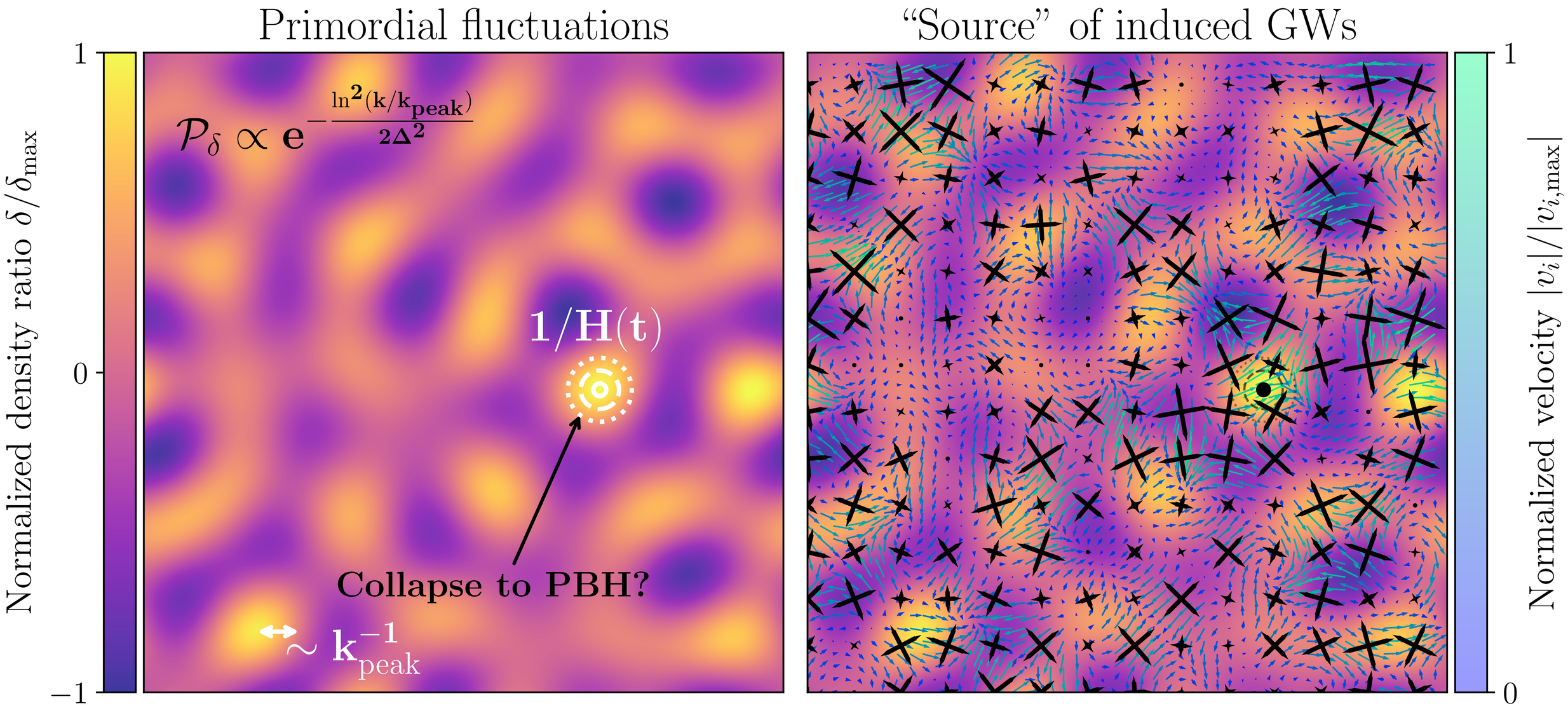
Saito, Yokoyama, 0812.4339
Cai, SP, Sasaki, 1810.11000

gravitational collapse

Primordial Black Hole

PBH

Zeldovich & Novikov 1966
Hawking 1971
Carr & Hawking 1974



Domenech, 2402.17388, PBH book, Ch. 17

PBH-IGW crosscheck

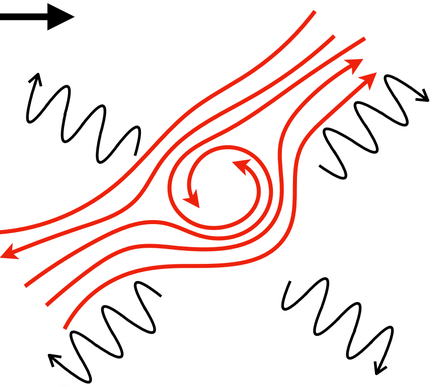
$$G_{\mu\nu} = T_{\mu\nu}$$
$$\nabla^2 \mathscr{R} \sim \delta_\ell$$
$$\Box h_{ij} \sim \nabla_i \mathscr{R} \nabla_j \mathscr{R}$$

$\mathbb{P}(\delta_\ell)$

$\delta_{\ell,\mathrm{cr}}$

$\beta \sim \mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_\delta}\right)$

$\Omega_{\mathrm{GW}} \sim 10^{-6}\mathscr{P}_{\mathscr{R}}^2$
$\sim 10^{-6}\sigma_\delta^4$



L_{Ph}

induced GWs

PBH

PBH

a/k_0

a/k_*

$1/H$

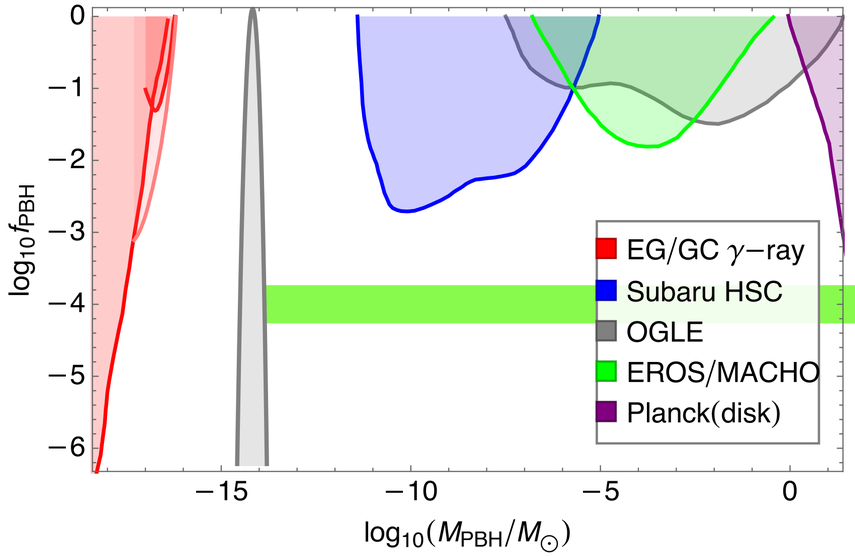
$f_{\mathrm{PBH}} \equiv \frac{\Omega_{\mathrm{PBH}}}{\Omega_{\mathrm{CDM}}}$
$= 4.11 \times 10^8 \beta(M) \left(\frac{M}{M_\odot}\right)^{-\frac{1}{2}}$

a

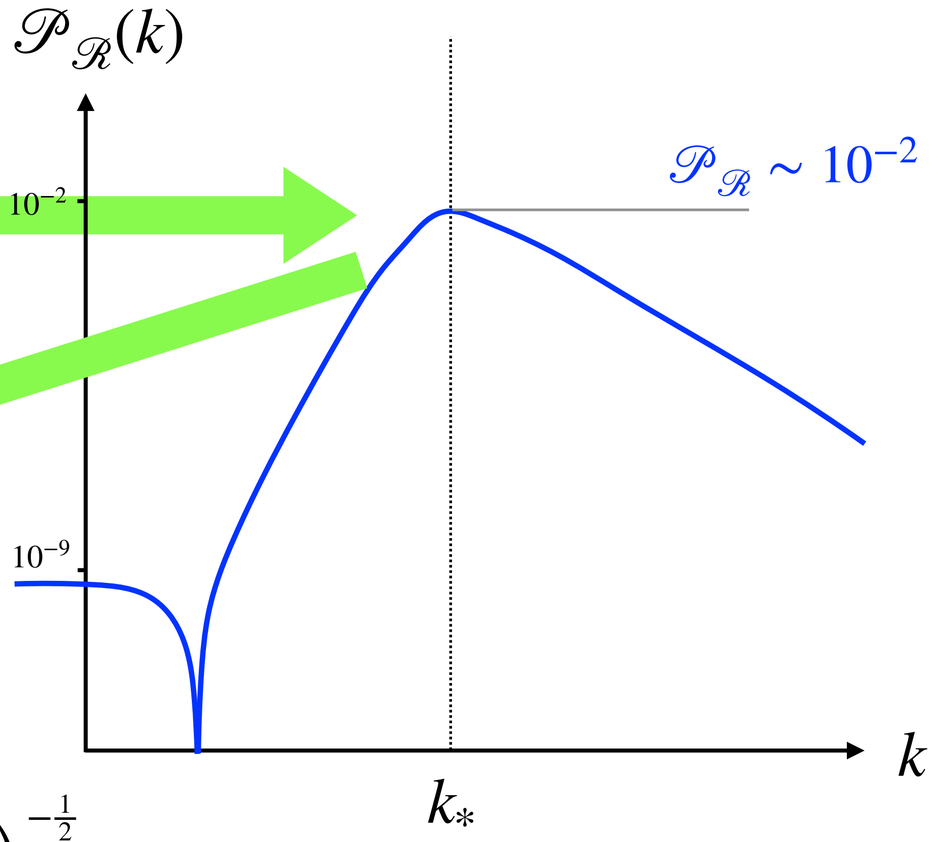
a_{ini} $a_* = k_*/H$ a_{end} $a_{\text{re-entry}}$ a_{eq} a_0

PBH-IGW crosscheck

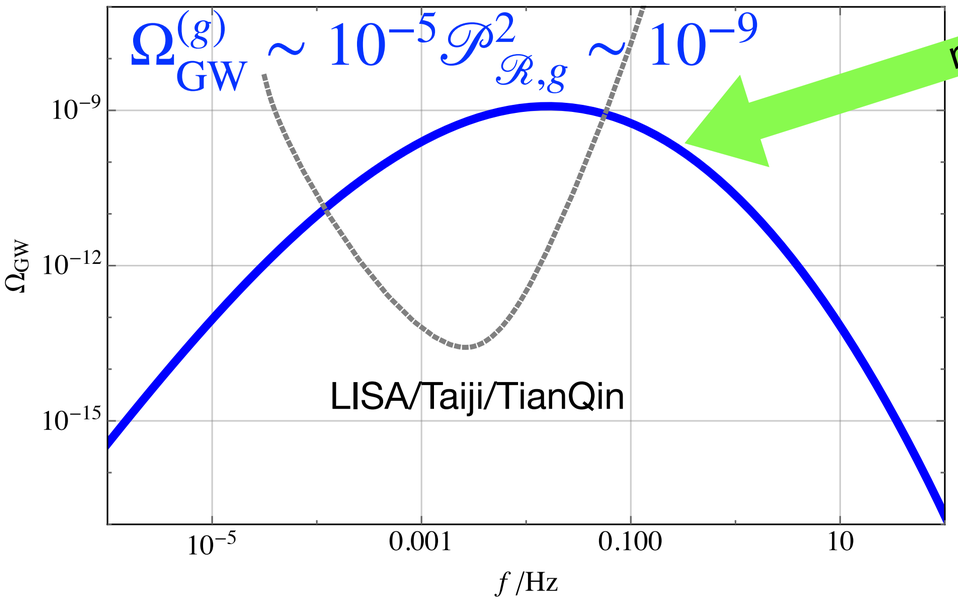
PBH=DM



gravitational collapse



nonlinear coupling $h\mathscr{R}\mathscr{R}$



$$f_{\rm IGW} \sim 3{\rm Hz}\left(\frac{M_{\rm PBH}}{10^{16}{\rm g}}\right)^{-\frac{1}{2}}$$

Saito & Yokoyama 0812.4339; 0912.5317
Bugaev & Klimai 0908.0664; 1012.4697
Escriva et al, 2211.05767

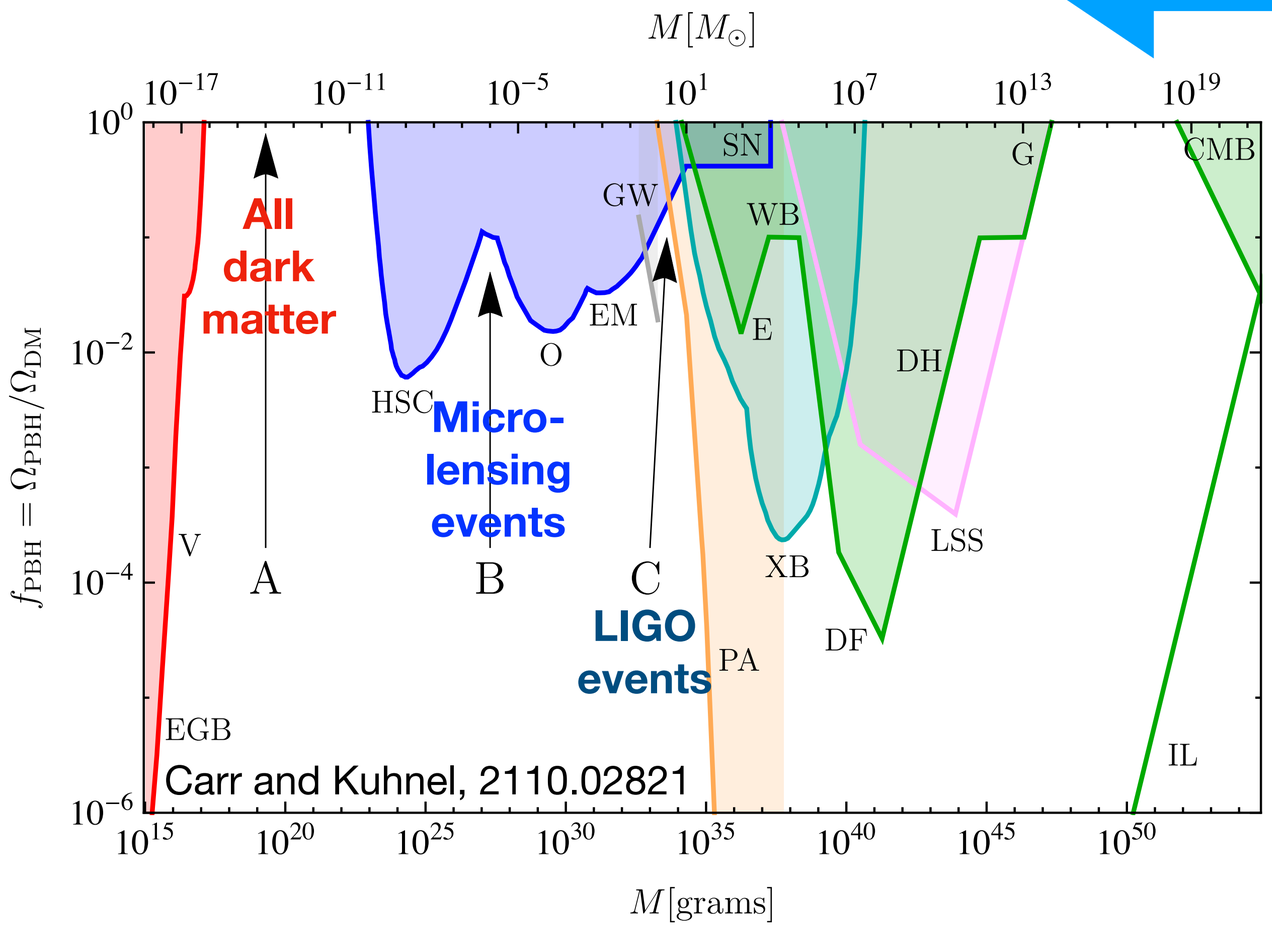
PBH-IGW crosscheck

$$f_{\rm IGW} \sim 3{\rm Hz}\left(\frac{M_{\rm PBH}}{10^{16}{\rm g}}\right)^{-\frac{1}{2}}$$

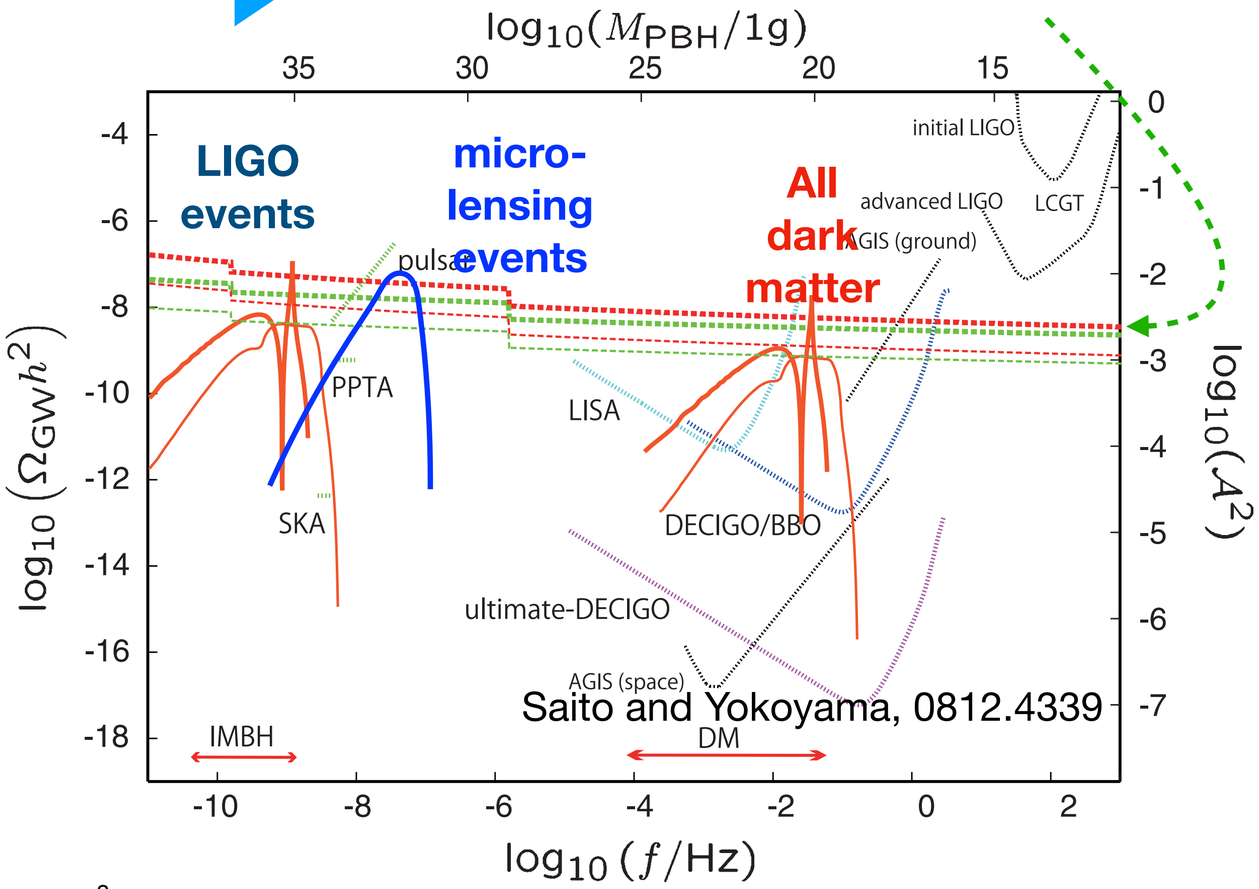
PBH constraints

crosscheck

Induced GW predictions

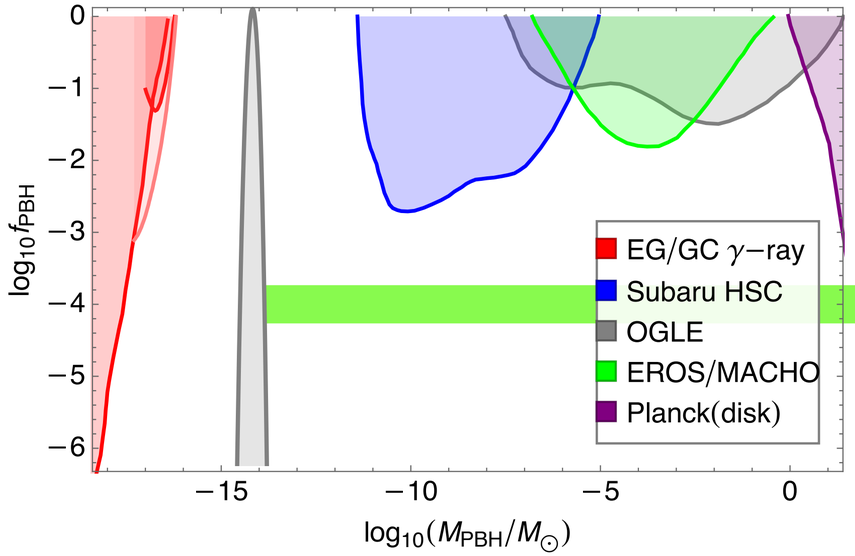


Constraint PBH≤DM

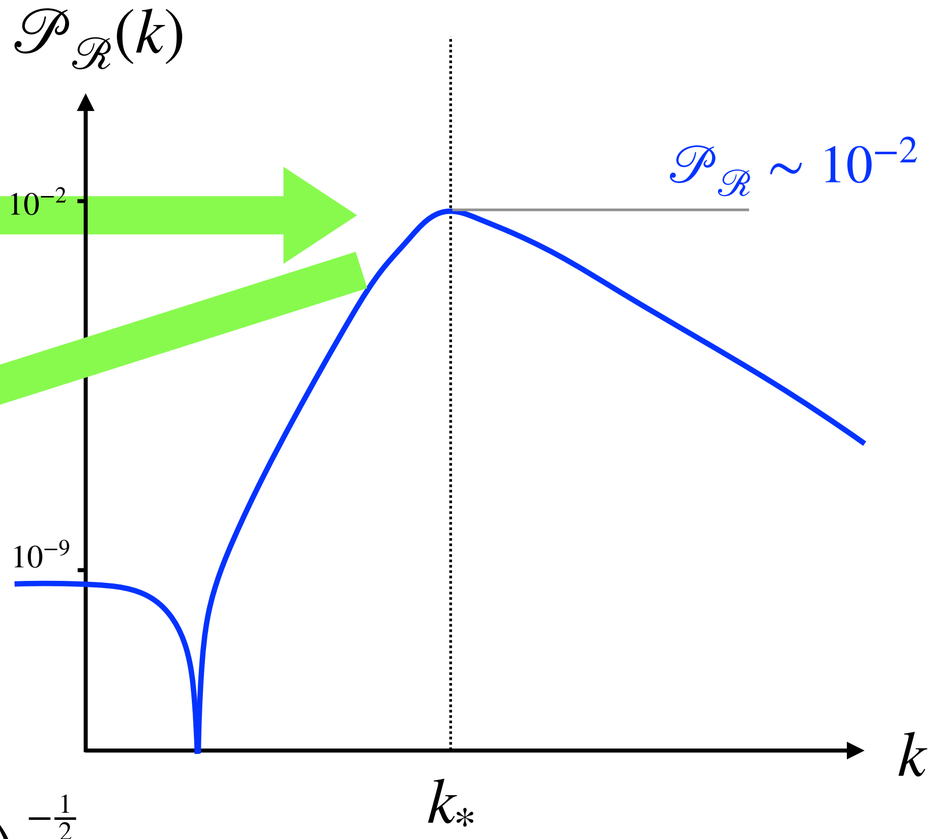


PBH-IGW crosscheck

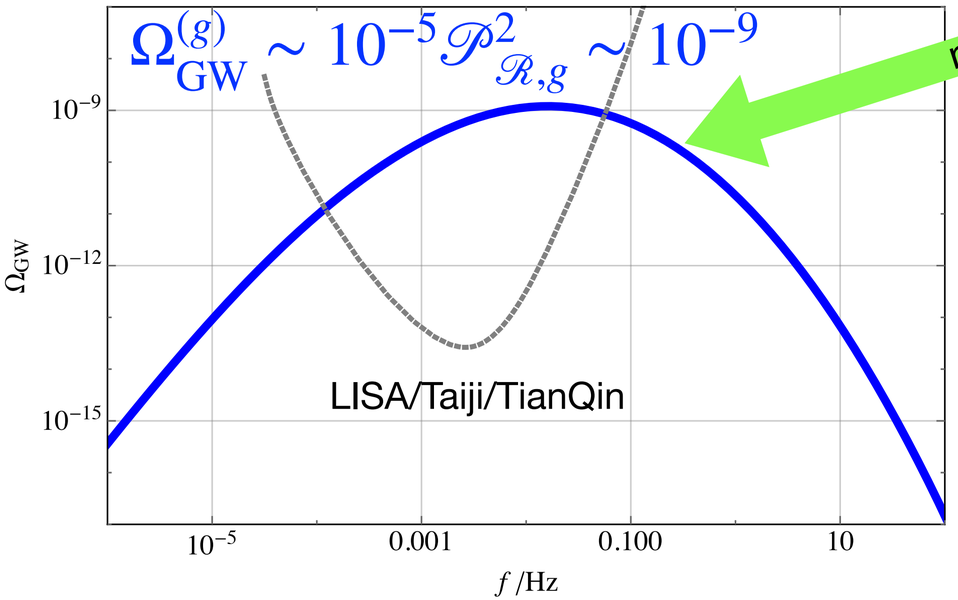
PBH=DM



gravitational collapse

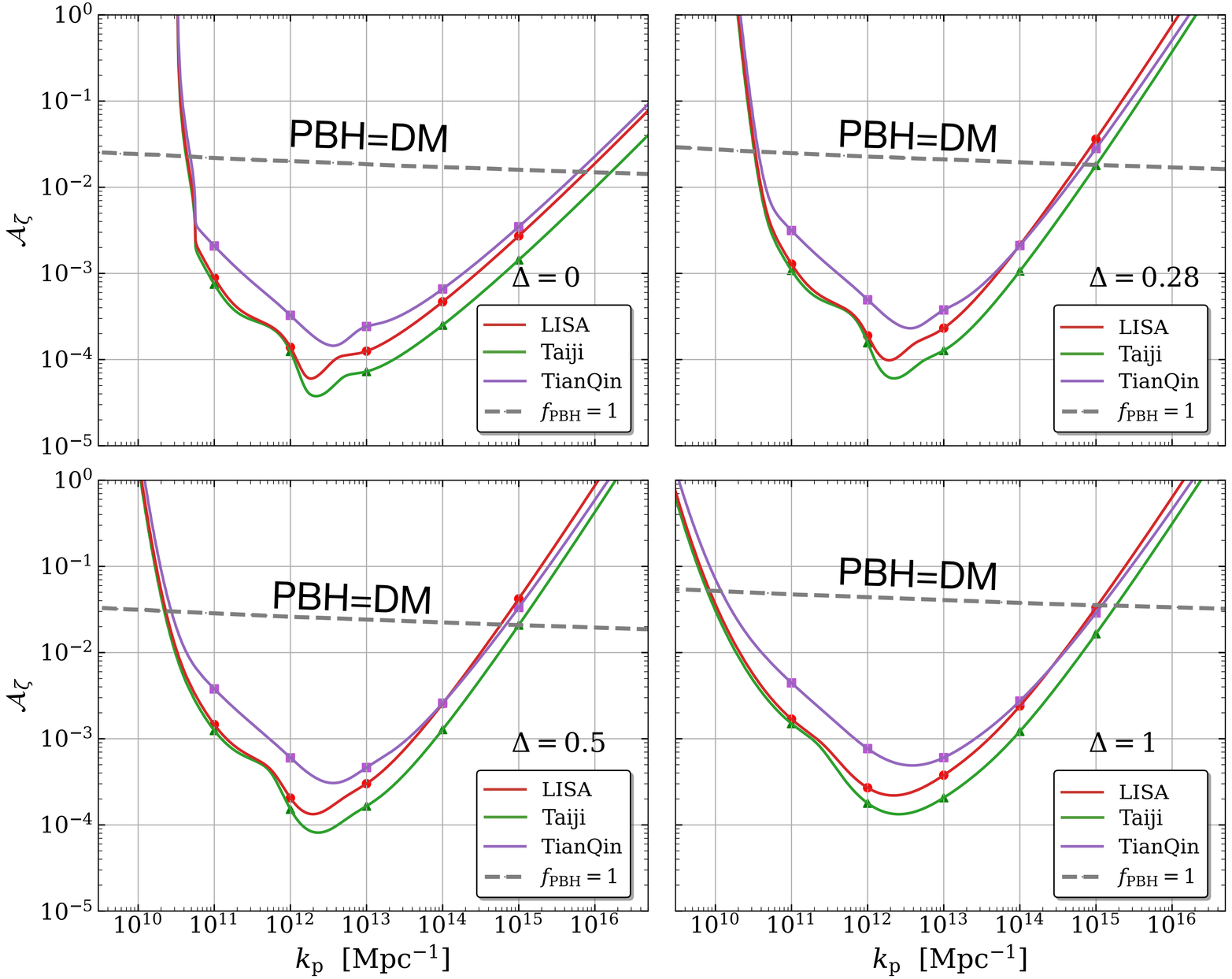


nonlinear coupling $h\mathscr{R}\mathscr{R}$



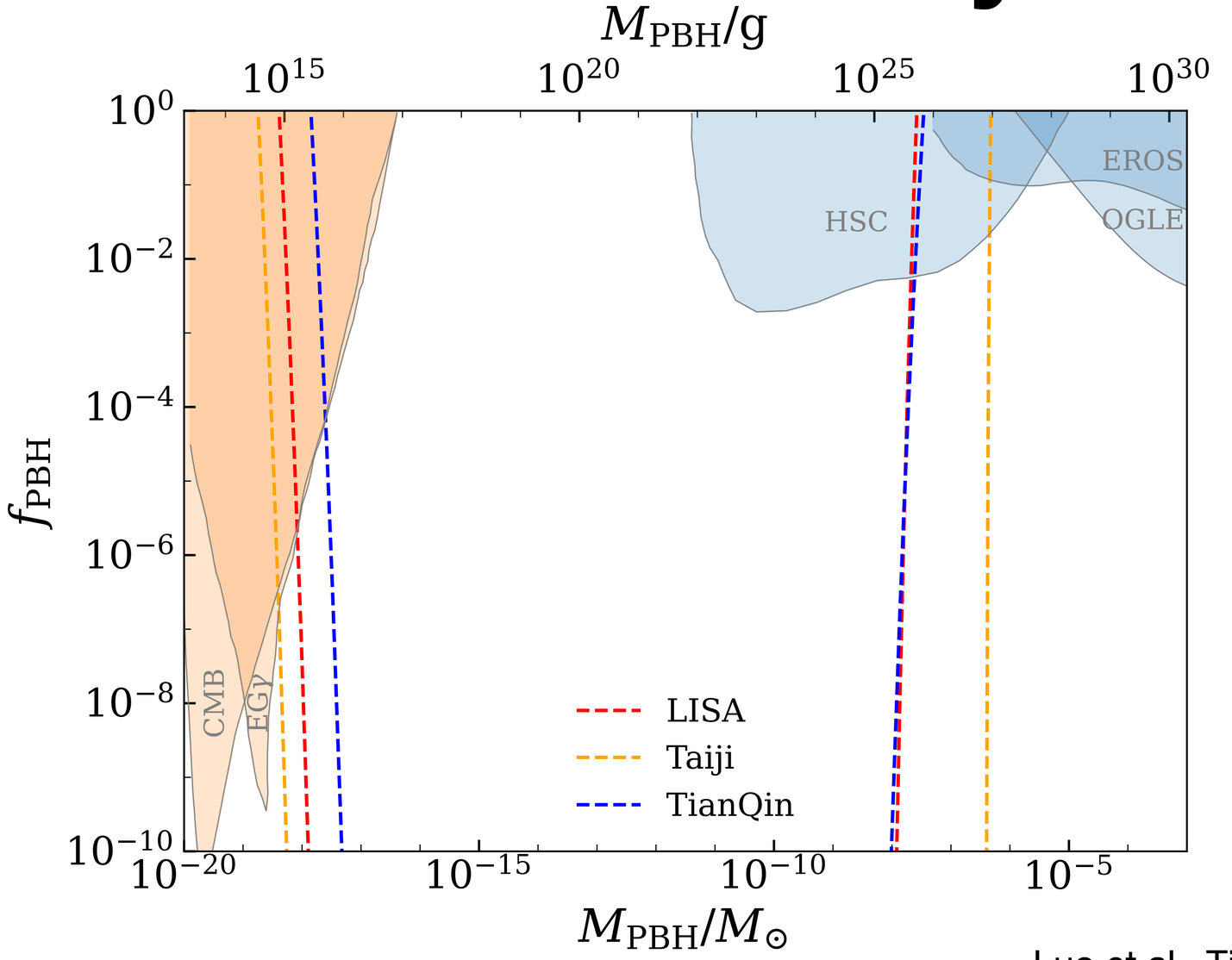
$$f_{\rm IGW} \sim 3{\rm Hz}\left(\frac{M_{\rm PBH}}{10^{16}{\rm g}}\right)^{-\frac{1}{2}}$$

Saito & Yokoyama 0812.4339; 0912.5317
Bugaev & Klimai 0908.0664; 1012.4697
Escriva et al, 2211.05767

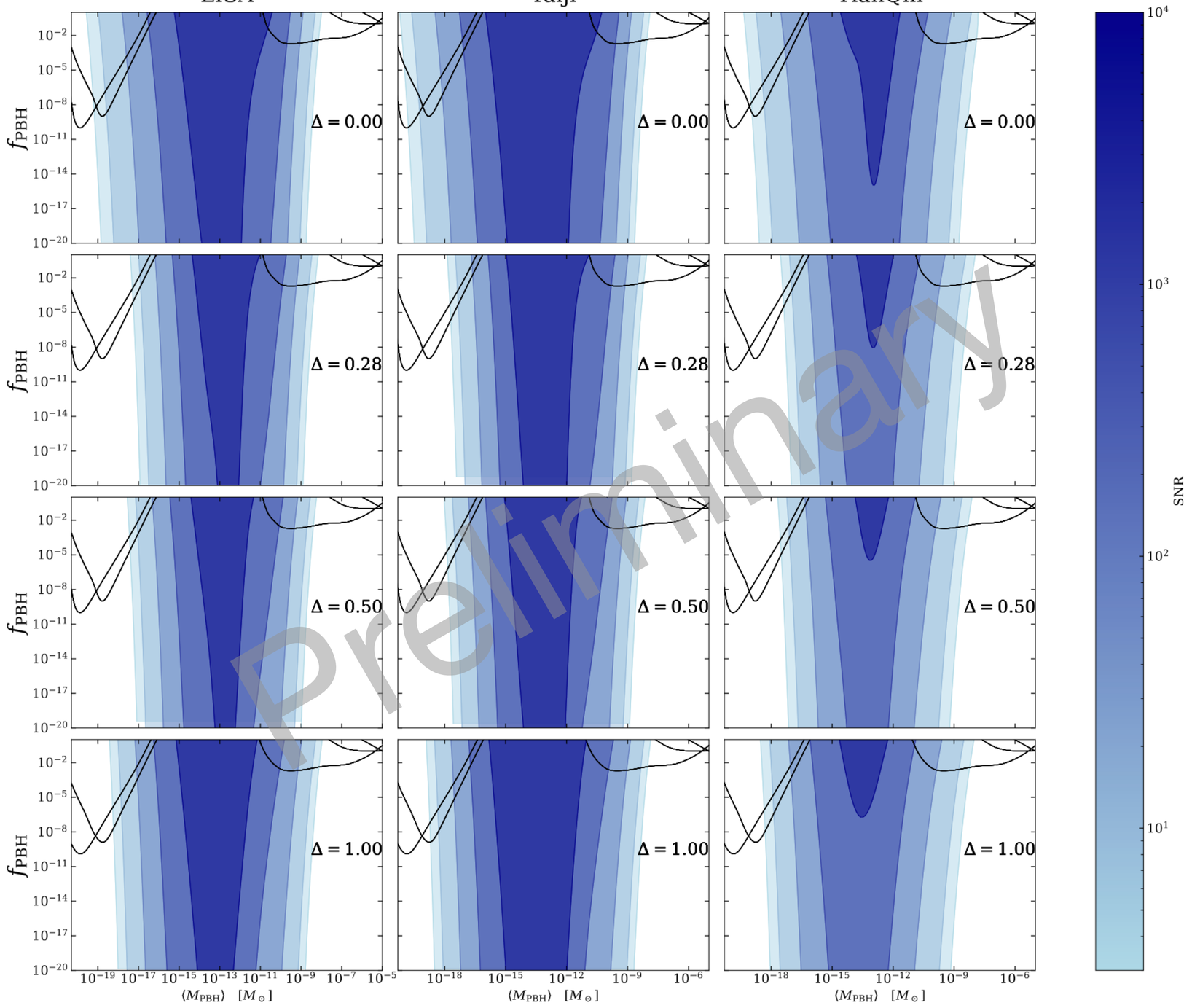


Luo et al., TianQin Whitebook, 2502.20138
Hong, Kuroyanagi, Pi, Wang, Zhang, to appear

Detectability



Luo et al., TianQin Whitebook, 2502.20138
Hong, Kuroyanagi, Pi, Wang, Zhang, to appear


LISA
Taiji
TianQin
$f_{\rm PBH}$
$\Delta = 0.00$
$\Delta = 0.28$
$\Delta = 0.50$
$\Delta = 1.00$
$\langle M_{\rm PBH} \rangle \quad [M_\odot]$
SNR
Preliminary

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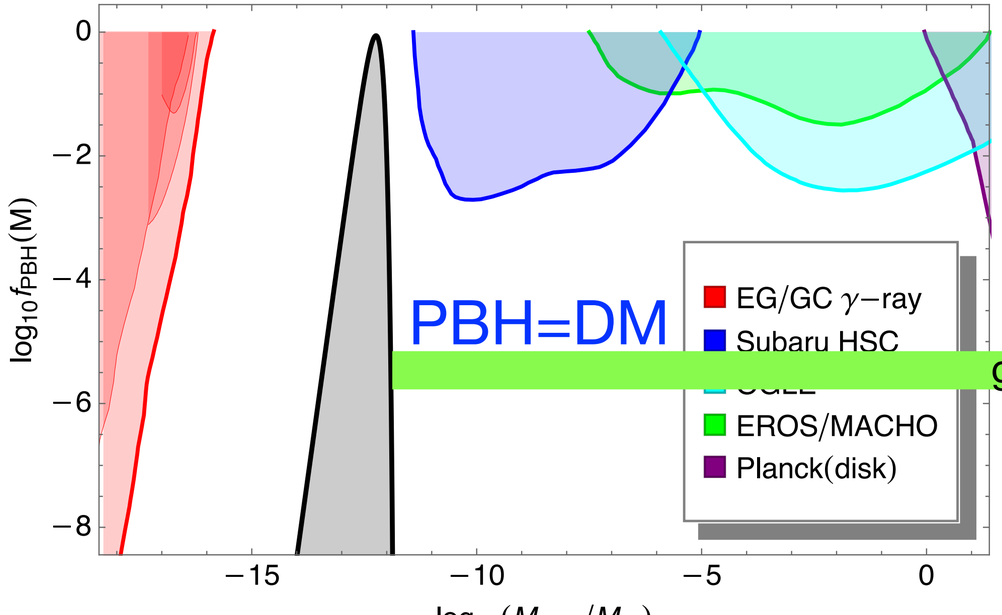
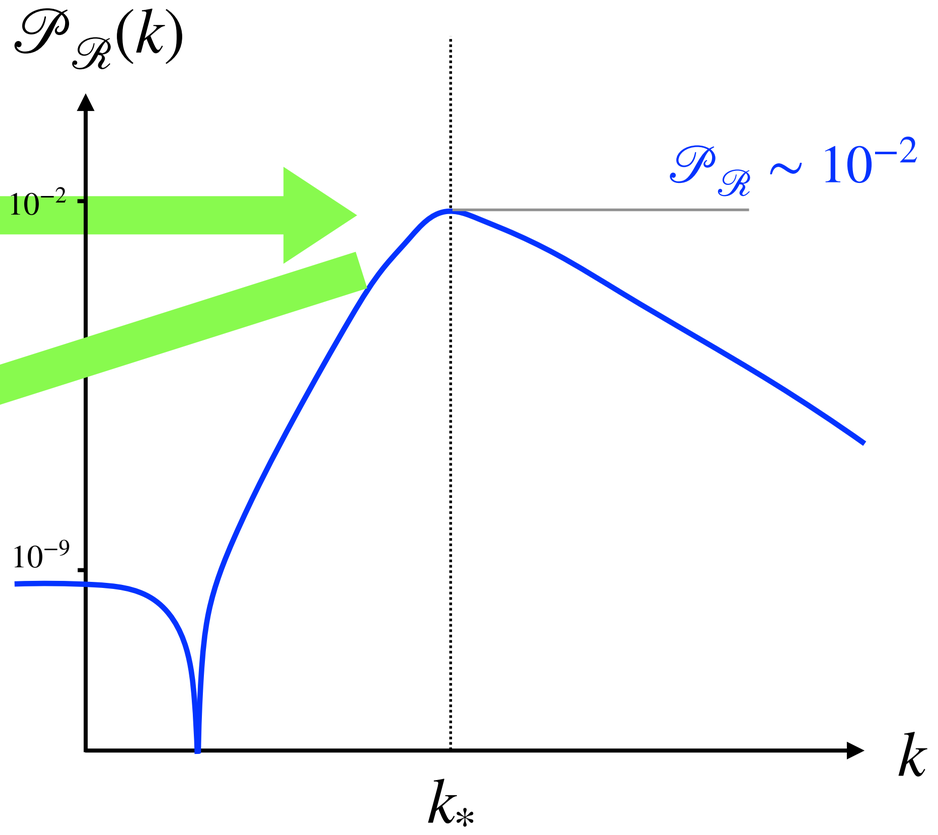
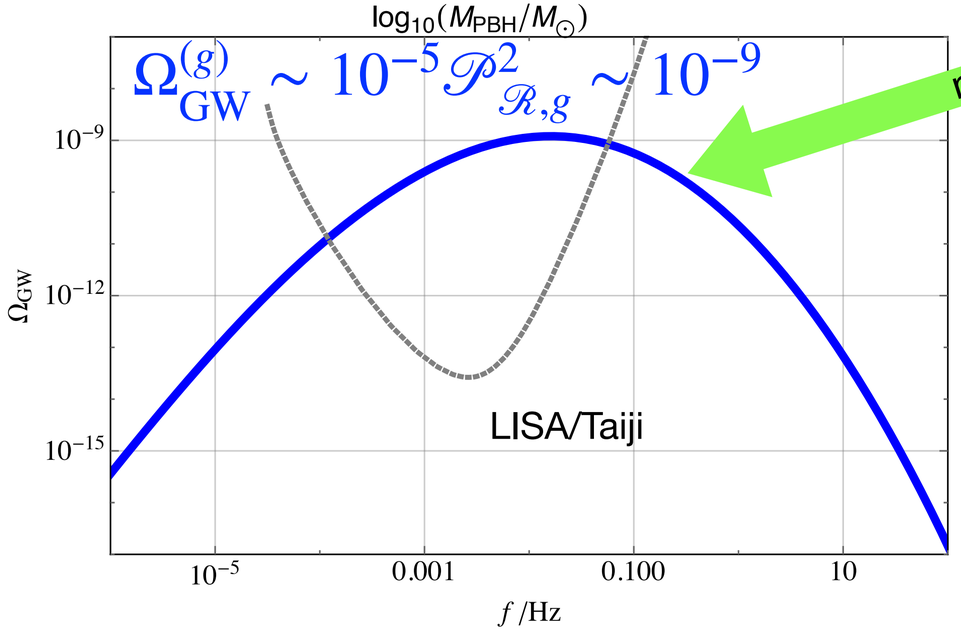
PBH-IGW crosscheck



gravitational collapse

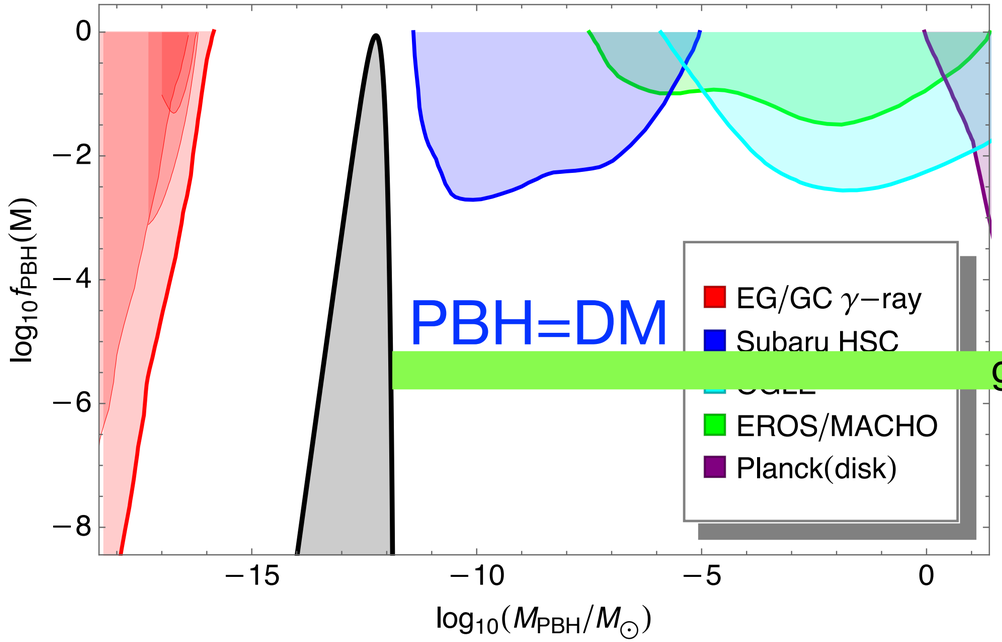


nonlinear coupling $h\mathscr{R}\mathscr{R}$



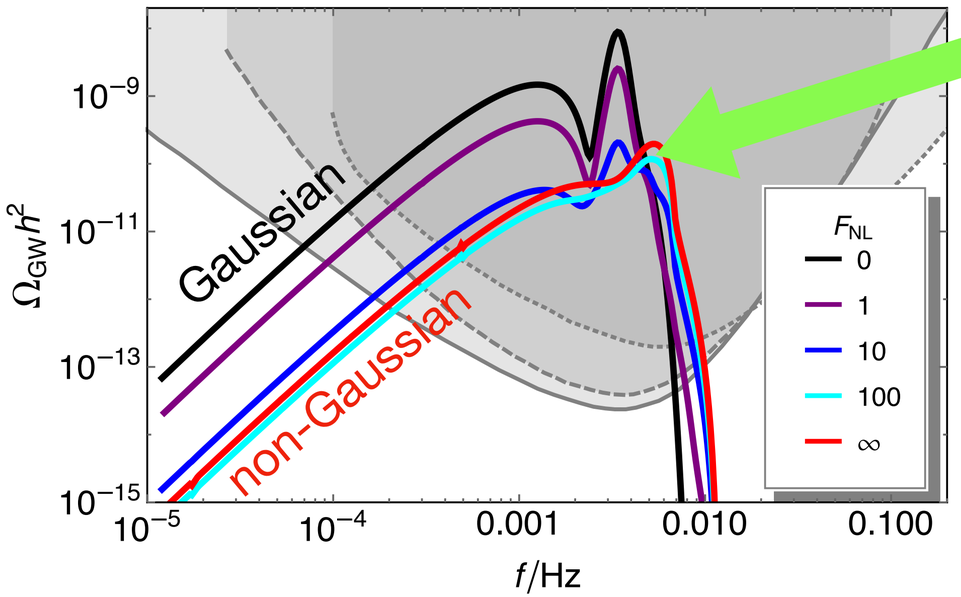
Saito & Yokoyama 0812.4339; 0912.5317
Bugaev & Klimai 0908.0664; 1012.4697
Escriva et al, 2211.05767

Including non-Gaussianity



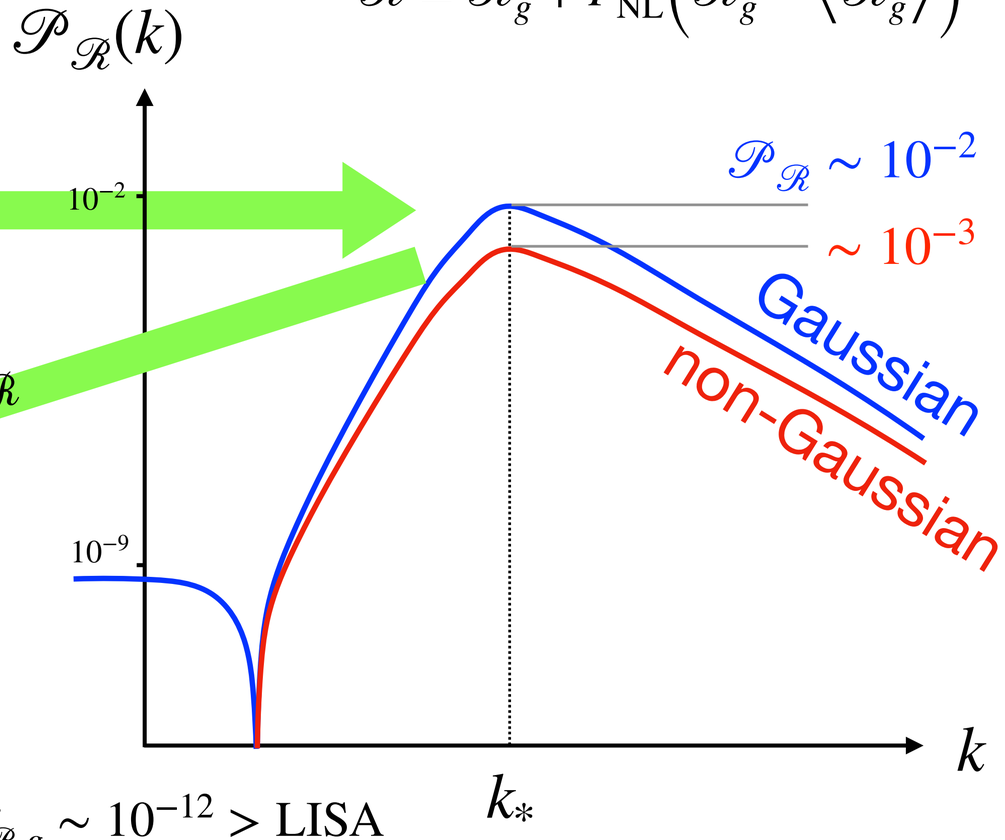
gravitational collapse

nonlinear coupling $h\mathscr{R}\mathscr{R}$ with $F_{\rm NL} > 0$



For perturbative non-Gaussianity:

$$\mathscr{R} = \mathscr{R}_g + F_{\rm NL}\left(\mathscr{R}_g^2 - \left\langle \mathscr{R}_g^2 \right\rangle\right)$$



$$F_{\rm NL}\mathscr{P}_{\mathscr{R}} \sim 10^{-2}$$

$$\Omega_{\rm GW}^{\rm (NG)} \sim 10^{-5} F_{\rm NL}^4 \mathscr{P}_{\mathscr{R},g}^4 \sim 10^{-12} > \text{LISA}$$

LISA/Taiji/TianQin can probe PBH-DM.

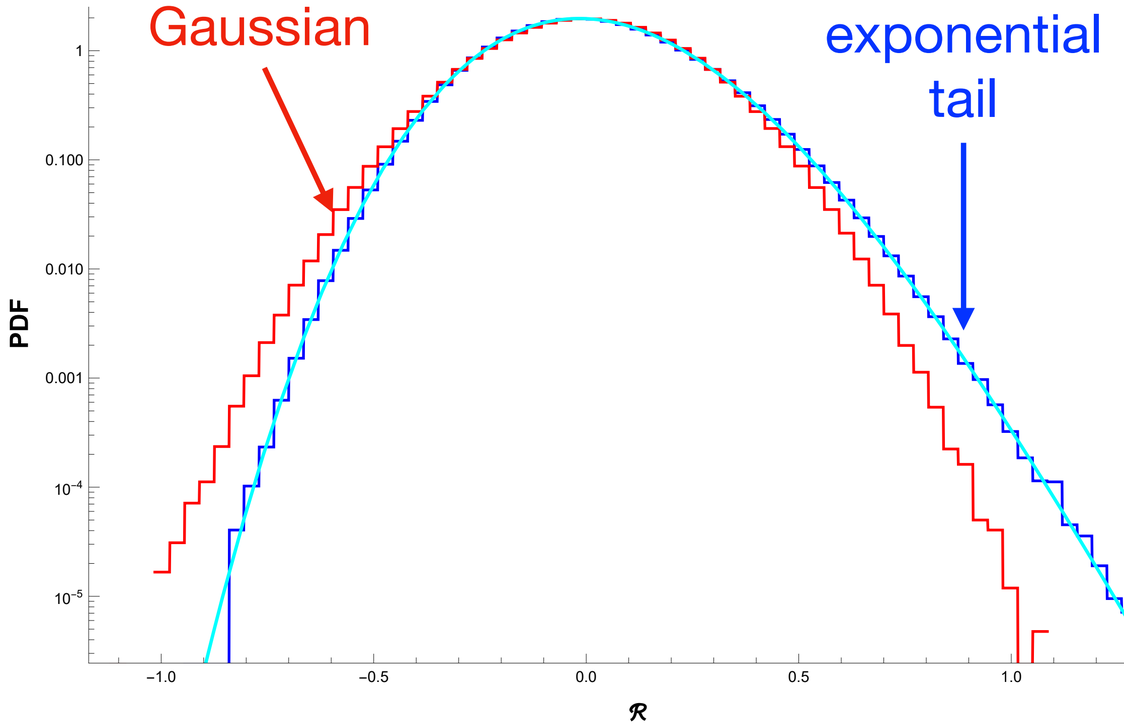
Cai, SP, Sasaki, 1810.11000

Why non-Gaussianity?

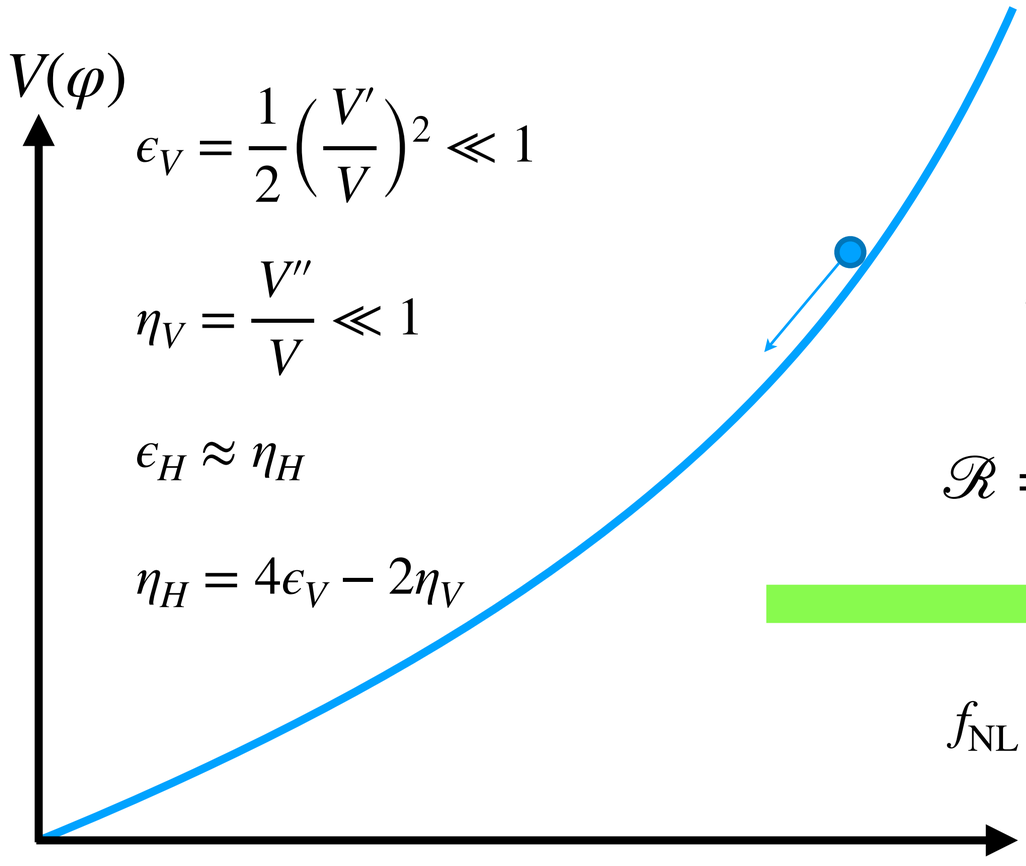
$$\left.\begin{array}{l} \mathscr{R} \xrightarrow{(1)} \mathscr{C}_\ell \\ \mathbb{P}(\mathscr{R}) \xrightarrow{(2)} \mathbb{P}(\mathscr{C}_\ell) \end{array}\right\} \xrightarrow[(4)\ \text{Window function}]{(3)\ \text{given}\ \mathscr{C}_{\rm cr}} \beta = \int_{\mathscr{C}_{\ell,{\rm cr}}}^{4/3} \mathbb{P}(\mathscr{C}_\ell)\frac{M(\mathscr{C}_\ell)}{M_H}{\rm d}\mathscr{C}_\ell$$

Non-Gaussianity must be taken into account：

- (1) Use compaction function $\mathscr{C}$ which nonlinearly depends on $\mathscr{R}$. (Harada et al 1503.03934; De Luca et al 1904.00970; Gow et al 2211.08348)
- (2) Primordial non-Gaussianity of $\mathscr{R}$ is usually non-perturbative. (This talk, and many others)
- (3) $\mathscr{C}_{\rm cr}$ depends on profile. (Musco 1809.02127; Escrivà et al 1907.13311)



Gaussian Curvature Perturbation



$$\epsilon_V = \frac{1}{2}\left(\frac{V'}{V}\right)^2 \ll 1$$

$$\eta_V = \frac{V''}{V} \ll 1$$

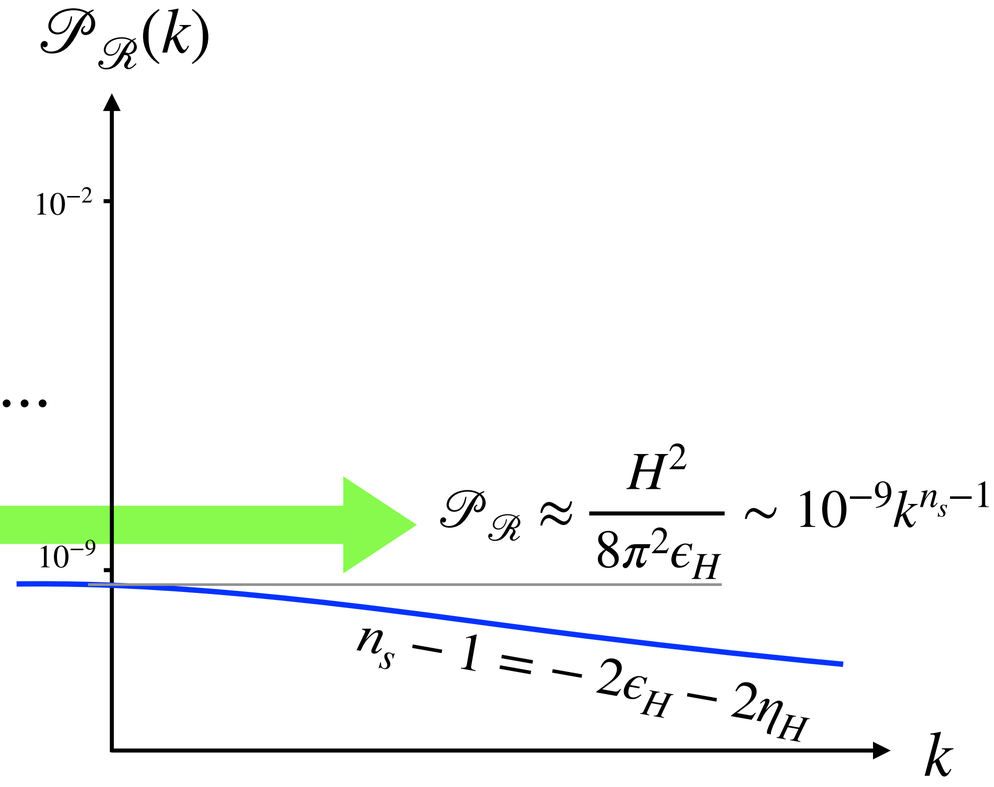
$$\epsilon_H \approx \eta_H$$

$$\eta_H = 4\epsilon_V - 2\eta_V$$

$$\mathscr{R}_g = -\frac{H}{\dot{\varphi}}\delta\varphi$$

$$\mathscr{R} = \mathscr{R}_g + \frac{3}{5} f_{\rm NL} \mathscr{R}_g^2 + \cdots$$

$$f_{\rm NL} = -\frac{5}{12}(n_s - 1)$$

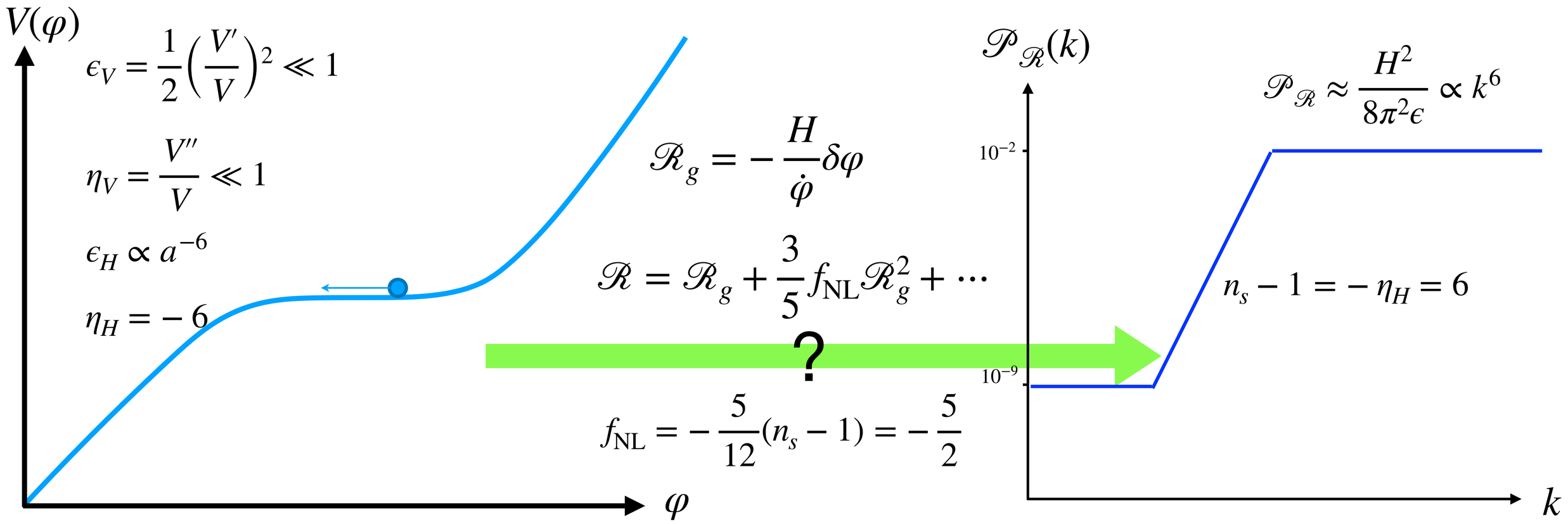


$$\mathscr{P}_{\mathscr{R}} \approx \frac{H^2}{8\pi^2\epsilon_H} \sim 10^{-9} k^{n_s - 1}$$

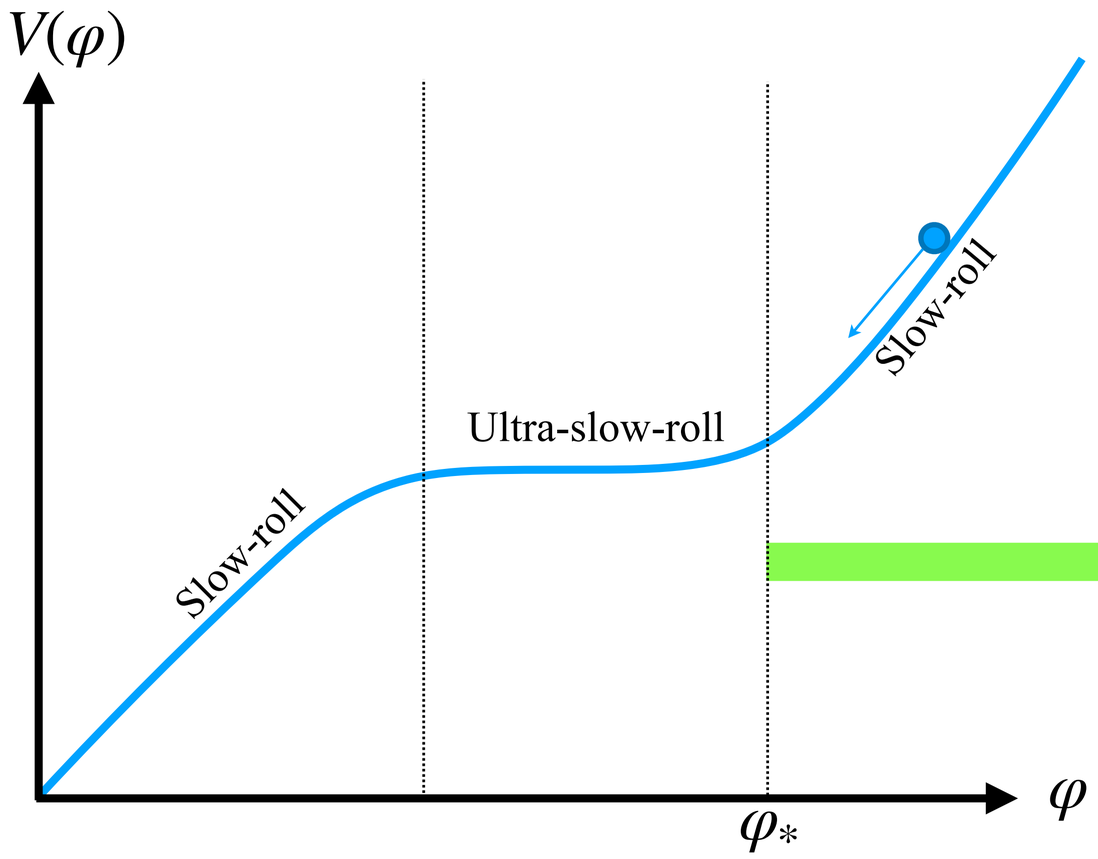
$$n_s - 1 = -2\epsilon_H - 2\eta_H$$

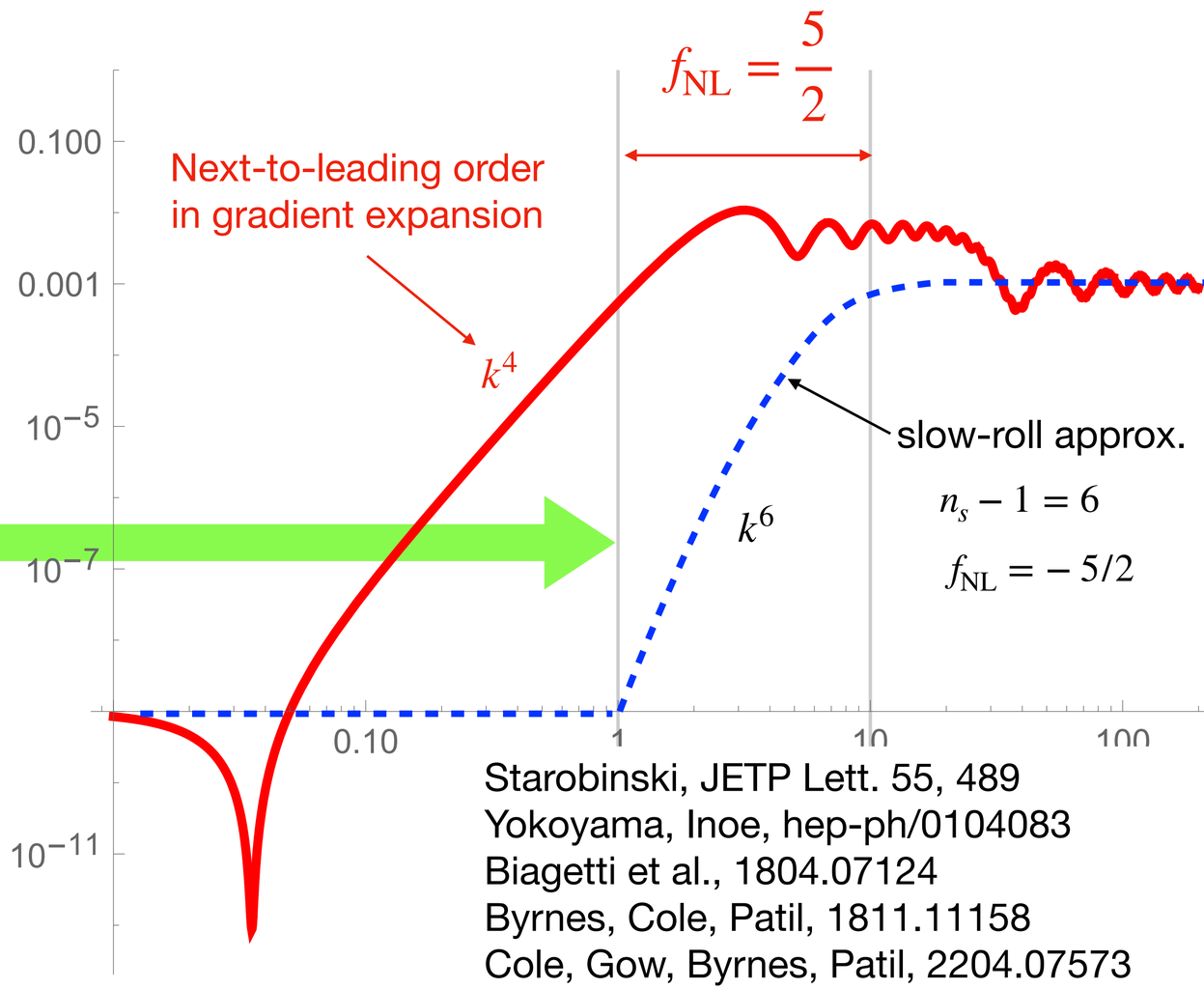
Stewart and Sasaki, astro-ph/9507001
Lyth and Roquigez, astro-ph/0504045
Maldadena, astro-ph/0210603

Gaussian Curvature Perturbation

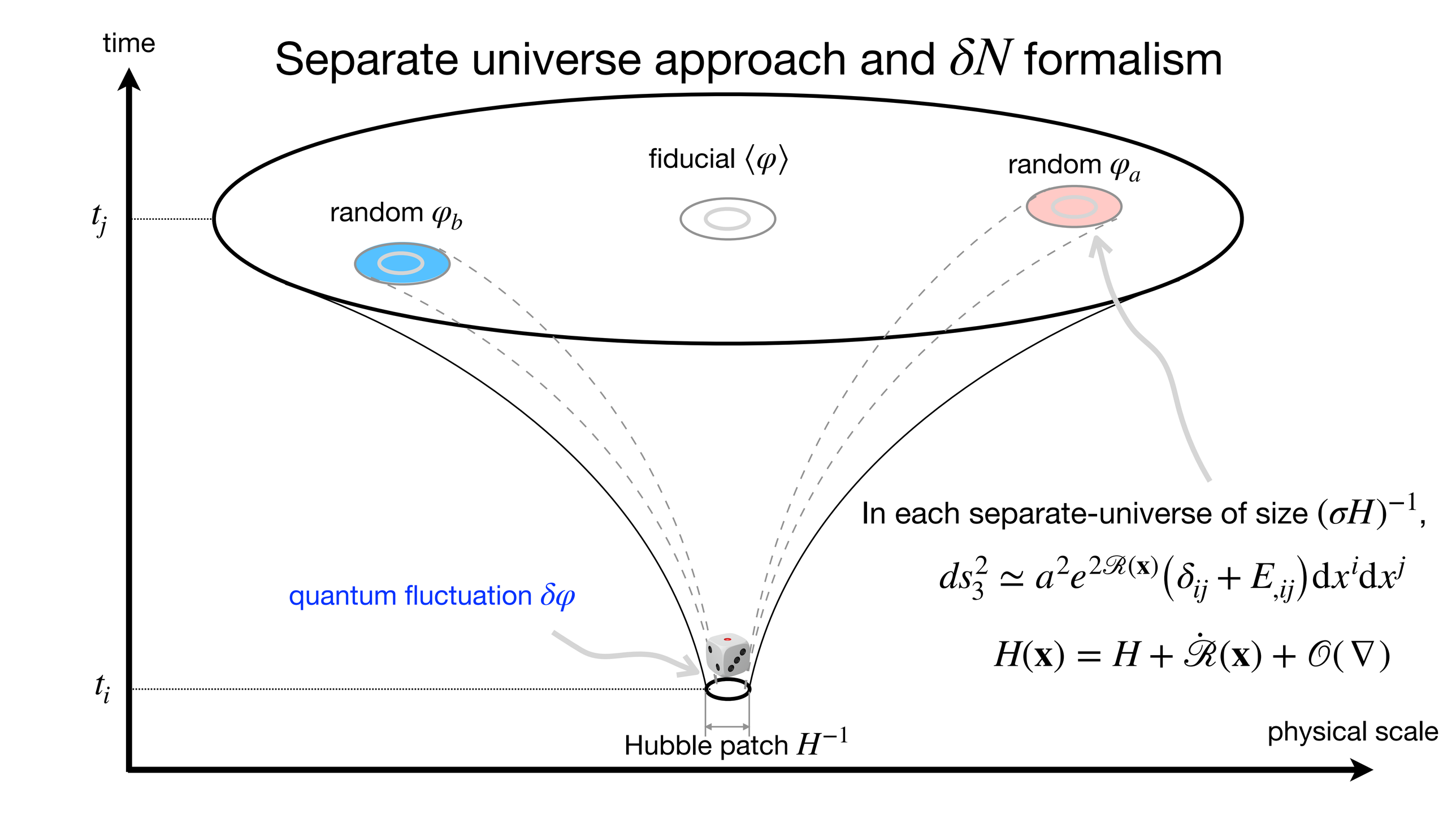


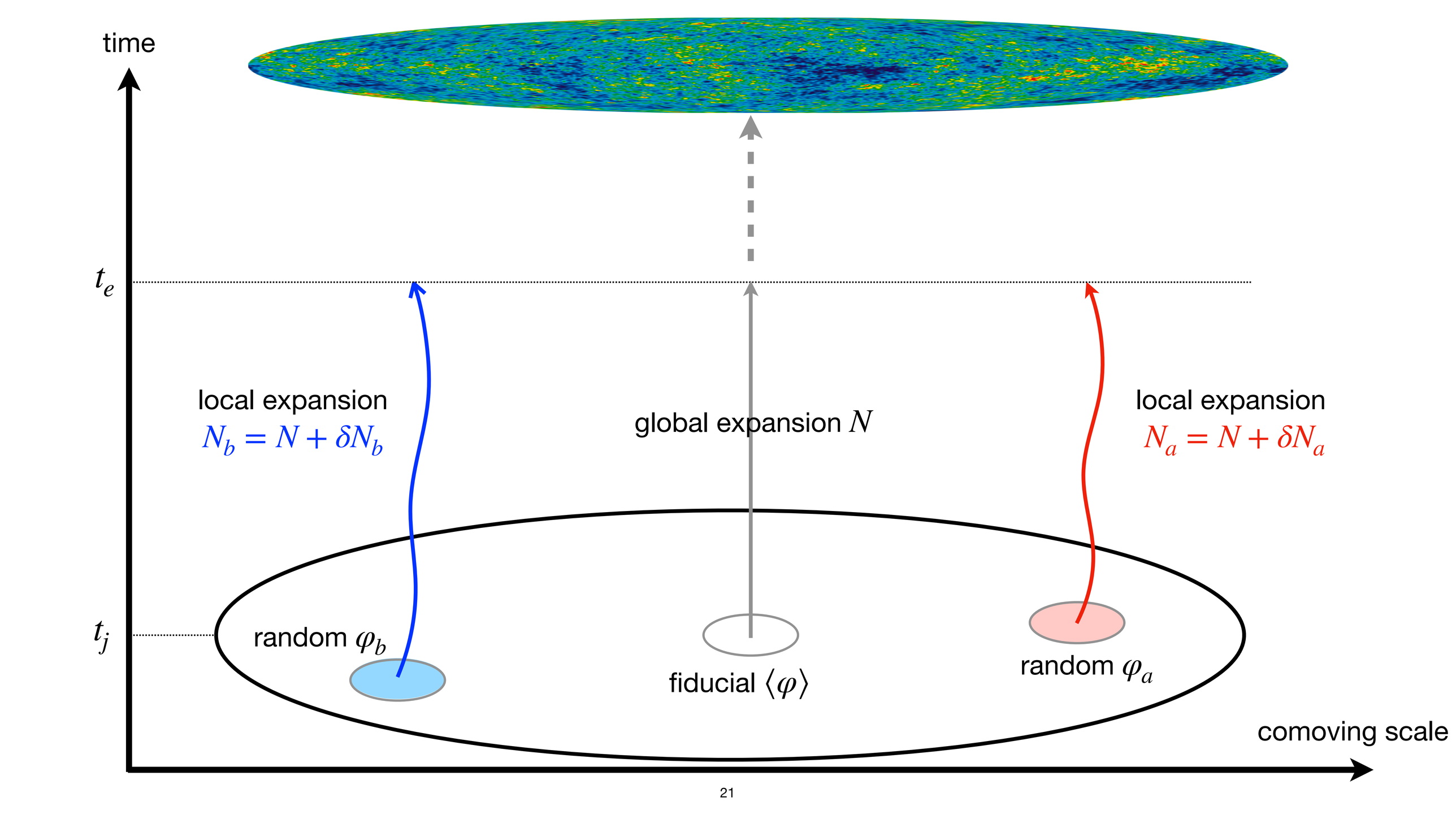
Ultra-slow-roll inflation





Starobinski, JETP Lett. 55, 489
Yokoyama, Inoe, hep-ph/0104083
Biagetti et al., 1804.07124
Byrnes, Cole, Patil, 1811.11158
Cole, Gow, Byrnes, Patil, 2204.07573
SP, Wang, 2209.14183
Domenech, Vargas, Vargas, 2309.05750

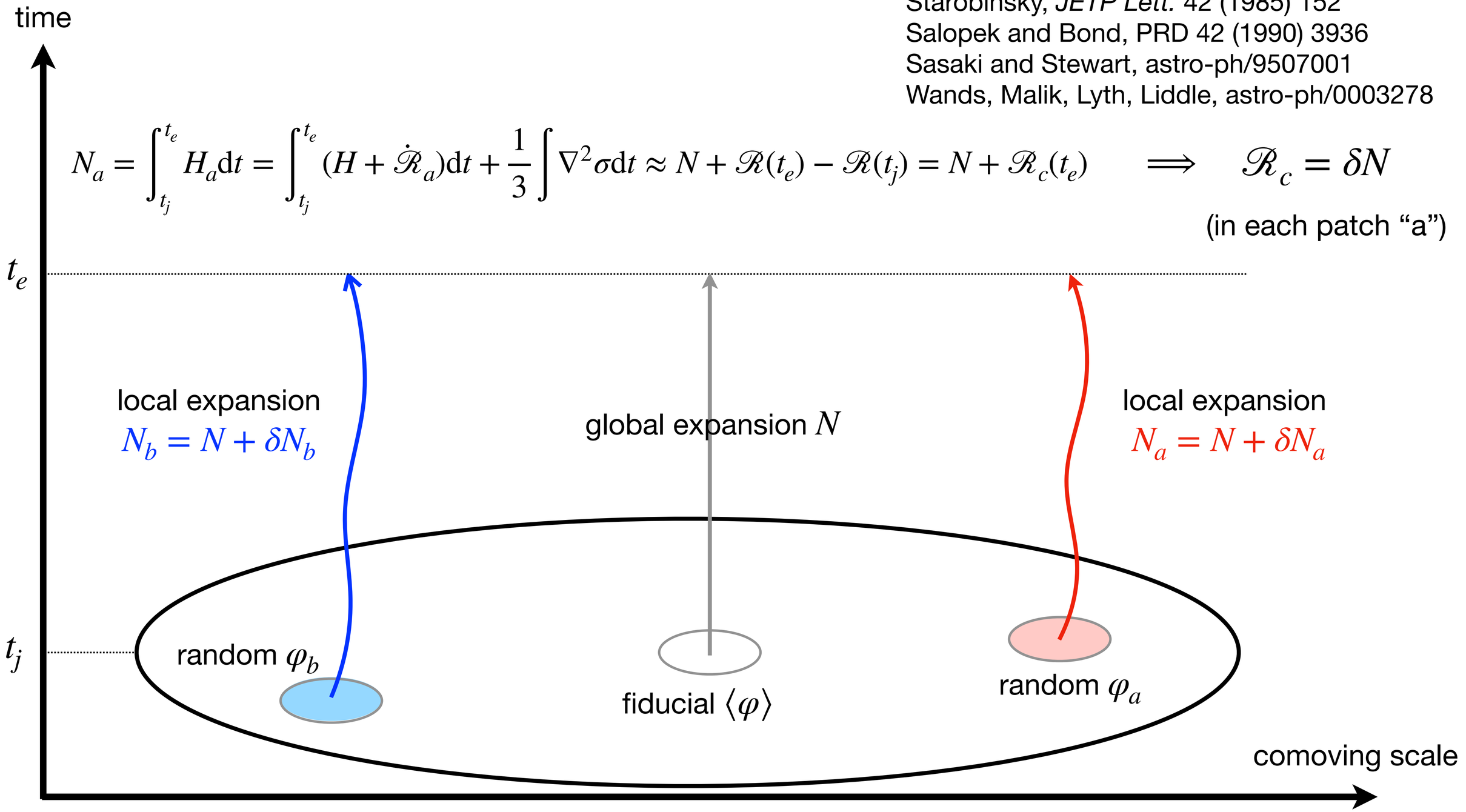

Separate universe approach and δN formalism
time
t_j
t_i
fiducial $\langle\varphi\rangle$
random φ_a
random φ_b
quantum fluctuation $\delta\varphi$
Hubble patch H^{-1}
physical scale
In each separate-universe of size $(\sigma H)^{-1}$,
$ds_3^2 \simeq a^2 e^{2\mathcal{R}(\mathbf{x})}\left(\delta_{ij} + E_{,ij}\right)\mathrm{d}x^i \mathrm{d}x^j$
$H(\mathbf{x}) = H + \dot{\mathcal{R}}(\mathbf{x}) + \mathcal{O}(\nabla)$


time
t_e
t_j
local expansion
$N_b = N + \delta N_b$
global expansion N
local expansion
$N_a = N + \delta N_a$
random φ_b
fiducial $\langle\varphi\rangle$
random φ_a
comoving scale

Starobinsky, *JETP Lett.* 42 (1985) 152
Salopek and Bond, PRD 42 (1990) 3936
Sasaki and Stewart, astro-ph/9507001
Wands, Malik, Lyth, Liddle, astro-ph/0003278

$$N_a = \int_{t_j}^{t_e} H_a \mathrm{d}t = \int_{t_j}^{t_e} (H + \dot{\mathscr{R}}_a)\mathrm{d}t + \frac{1}{3}\int \nabla^2 \sigma \mathrm{d}t \approx N + \mathscr{R}(t_e) - \mathscr{R}(t_j) = N + \mathscr{R}_c(t_e) \quad \Longrightarrow \quad \mathscr{R}_c = \delta N$$

(in each patch "a")



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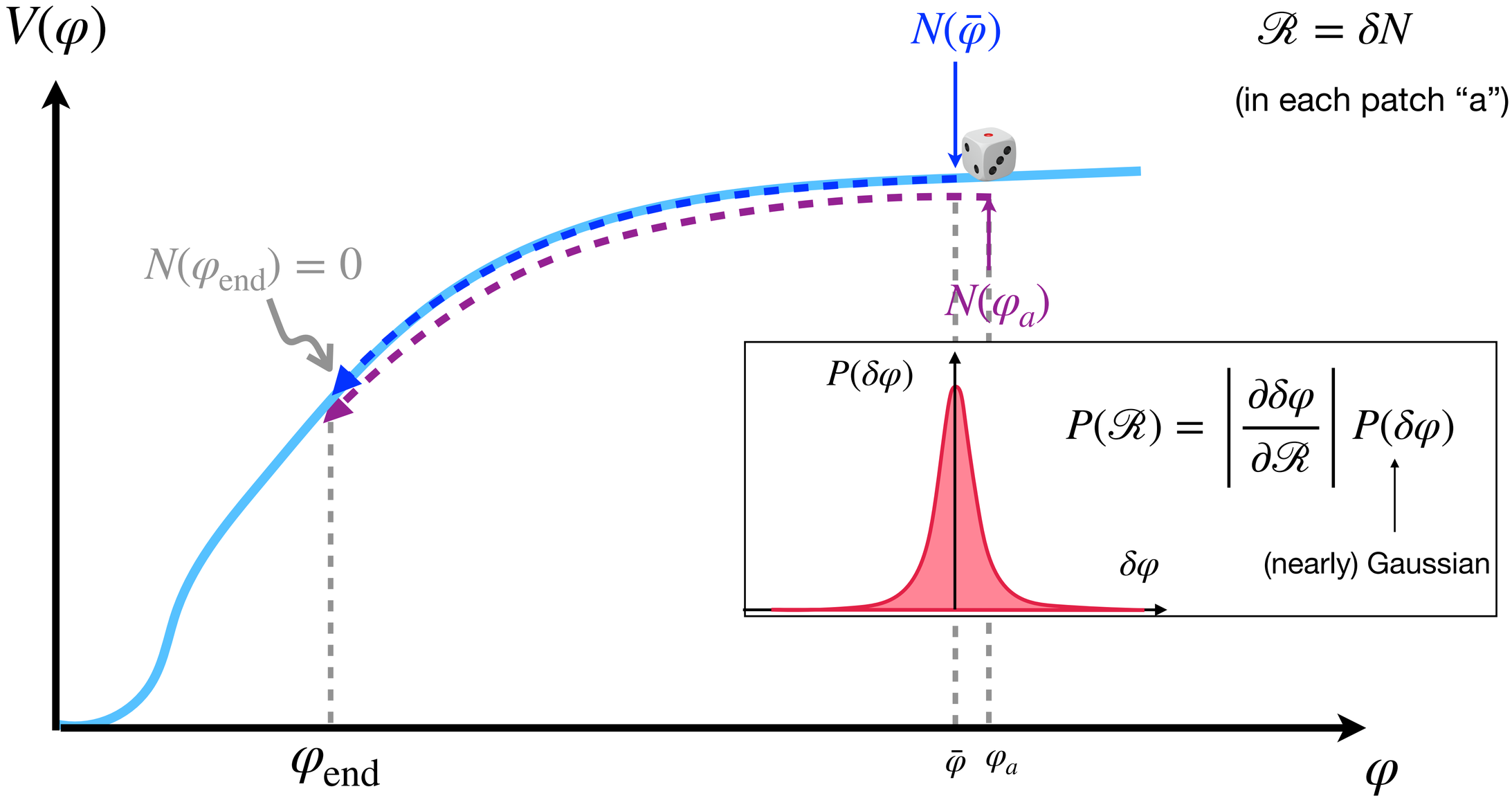
Salopek and Bond, PRD 42 (1990) 3936
Sasaki and Stewart, astro-ph/9507001
Wands, Malik, Lyth, Liddle, astro-ph/0003278

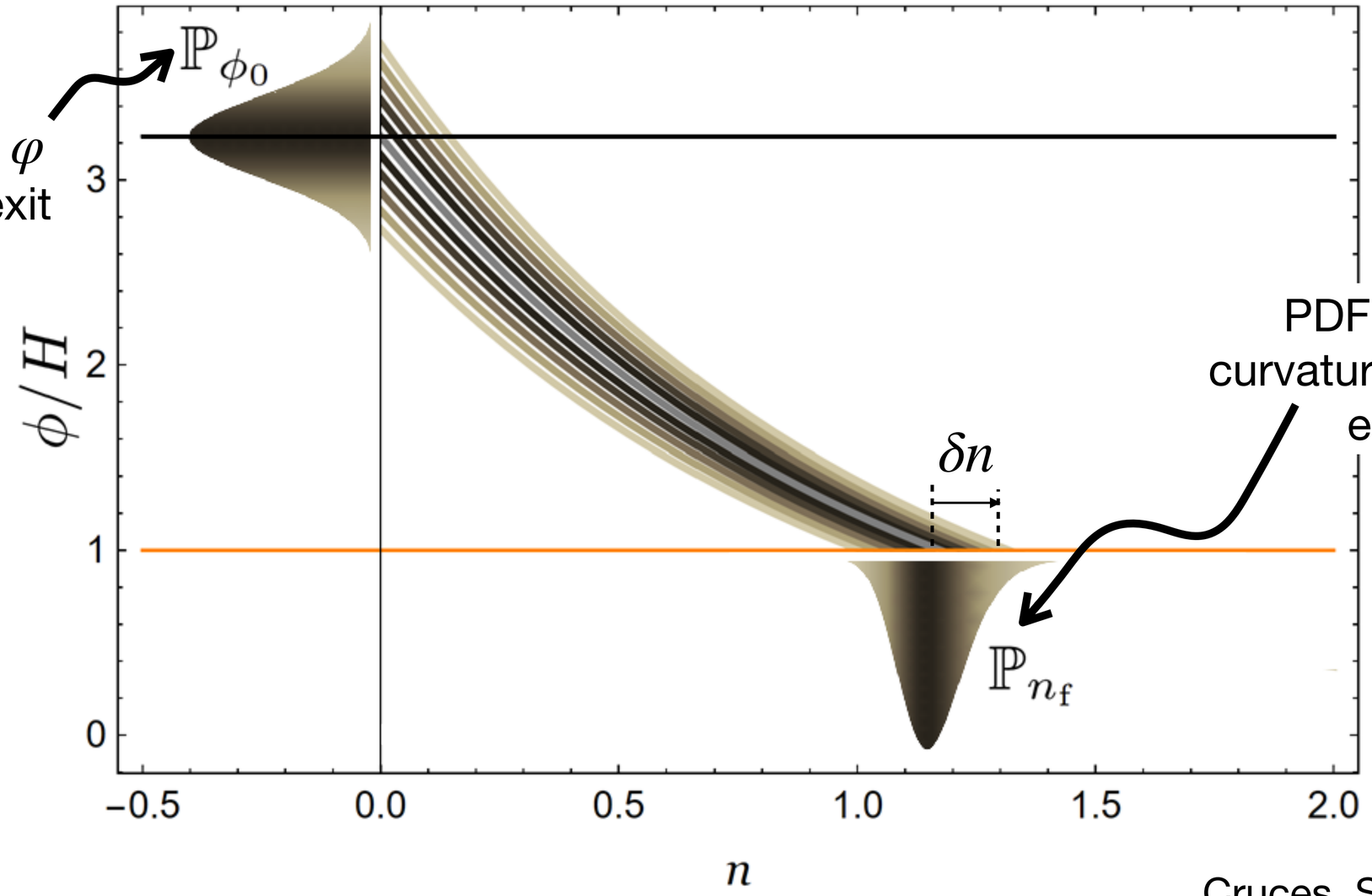
$\mathcal{R} = \delta N$

(in each patch "a")



δN formalism

PDF of inflaton φ at the horizon exit

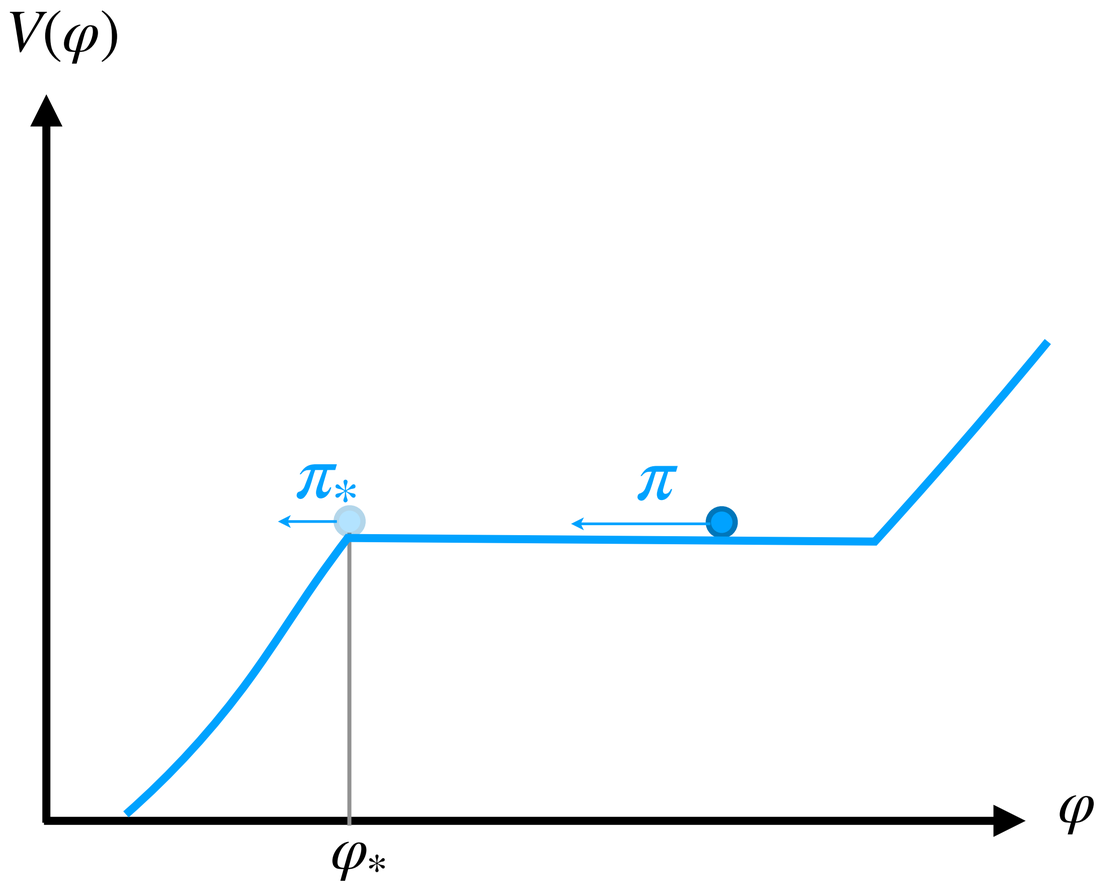


PDF of the comoving curvature perturbation at the end of inflation

Cruces, SP, Sasaki, 2505.24590

Ultra-slow-roll inflation

$$\mathrm{d}N = H\mathrm{d}t$$



$$\varphi'' + 3H\varphi' = 0 \quad \text{(superhorizon, not background)}$$

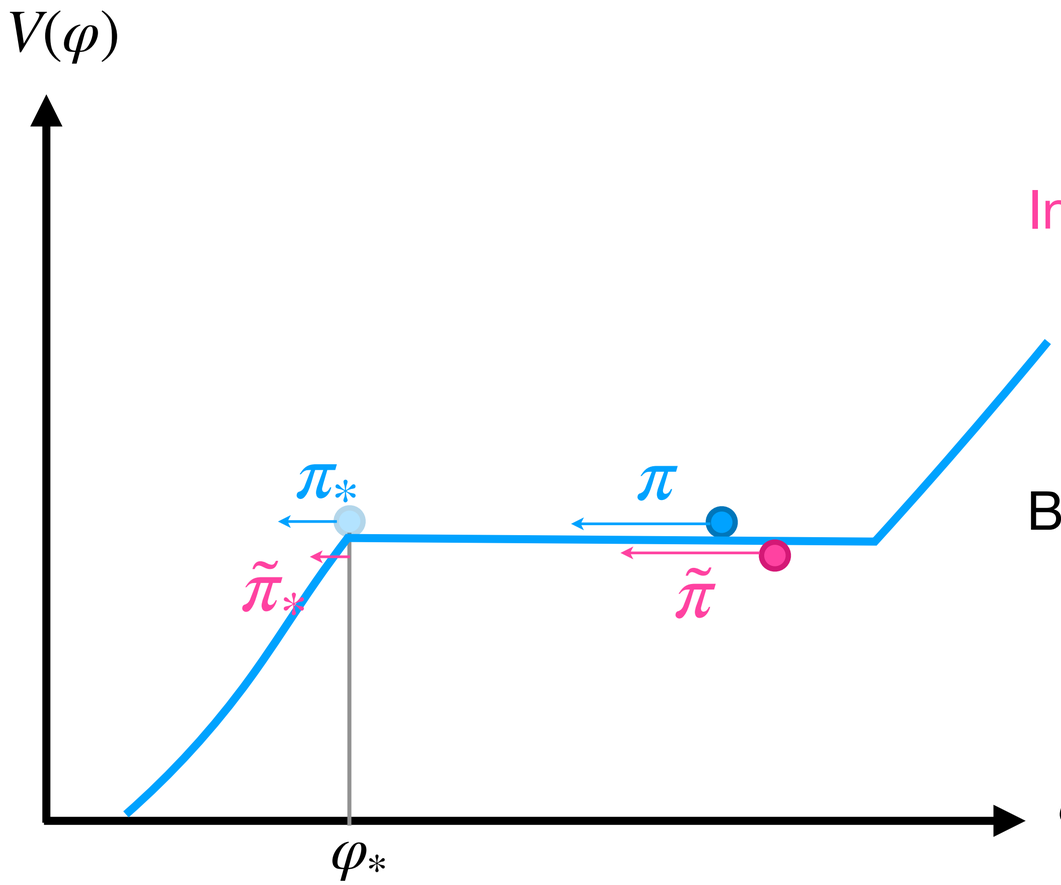
$$\pi' + 3\pi = 0 \qquad \left(\pi \equiv \frac{\mathrm{d}\varphi}{\mathrm{d}N}\right)$$

$$\pi = \pi_* e^{-3(N-N_*)} \quad (\pi = 0 \text{ is the attractor solution})$$

$$N - N_* = -\frac{1}{3}\ln\frac{\pi(N)}{\pi_*}$$

SP and Sasaki, 2211.13932
SP, 2404.06151

Ultra-slow-roll inflation



In the “fiducial” patch

$$\bar{N} - N_* = -\frac{1}{3}\ln\frac{\pi(\bar{N})}{\pi_*}$$

In a perturbed patch

$$N - N_* = -\frac{1}{3}\ln\frac{\pi(N)}{\pi_*}$$

By δN formalism, the curvature perturbation is

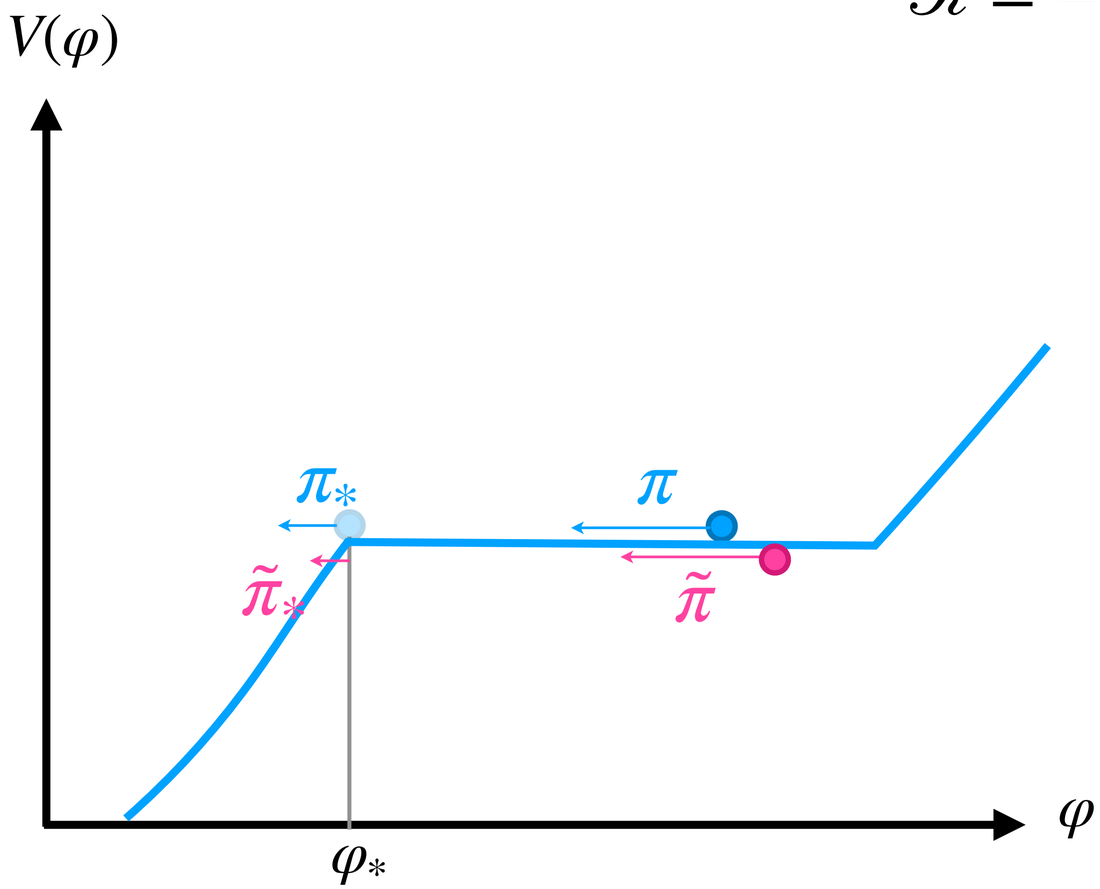
$$\mathscr{R} = \delta N = N - \bar{N} = -\frac{1}{3}\ln\frac{\pi(N)}{\pi(\bar{N})}$$

$$= -\frac{1}{3}\ln\left(1 + \frac{\delta\pi(N)}{\pi(\bar{N})}\right)$$

Ultra-slow-roll inflation

$$\mathscr{R} = -\frac{1}{3}\ln\left(1+\frac{\delta\pi(N)}{\pi(\bar{N})}\right) = -\frac{1}{3}\ln\left(1+\frac{\delta\pi}{\bar{\pi}}\right)$$

$$\left(f_{\rm NL}=\frac{5}{2},\quad g_{\rm NL}=-\frac{25}{3},\ \cdots\right)$$



Namjoo, Firouzjahi, Sasaki, 1210.3692
Chen, Firouzjahi, Komatsu, Namjoo, Sasaki, 1308.5341
Cai, Chen, Namjoo, Sasaki, Wang, Wang, 1712.09998
Biagetti, Franciolini, Kehagias, Riotto, 1804.07124
Passaglia, Hu, Motohashi, 1812.08243
SP and Sasaki, 2211.13932
SP, 2404.06151, PBH book ch.7
Jackson et al 2410.13683, Cruces et al 2410.17987
Ballestores et al 2412.14106, Caravano et al 2506.11795
Artigas, SP, Tanaka, 2408.09964
Escriva, Garriga, SP, 2512.04986

piecewise quadratic potential

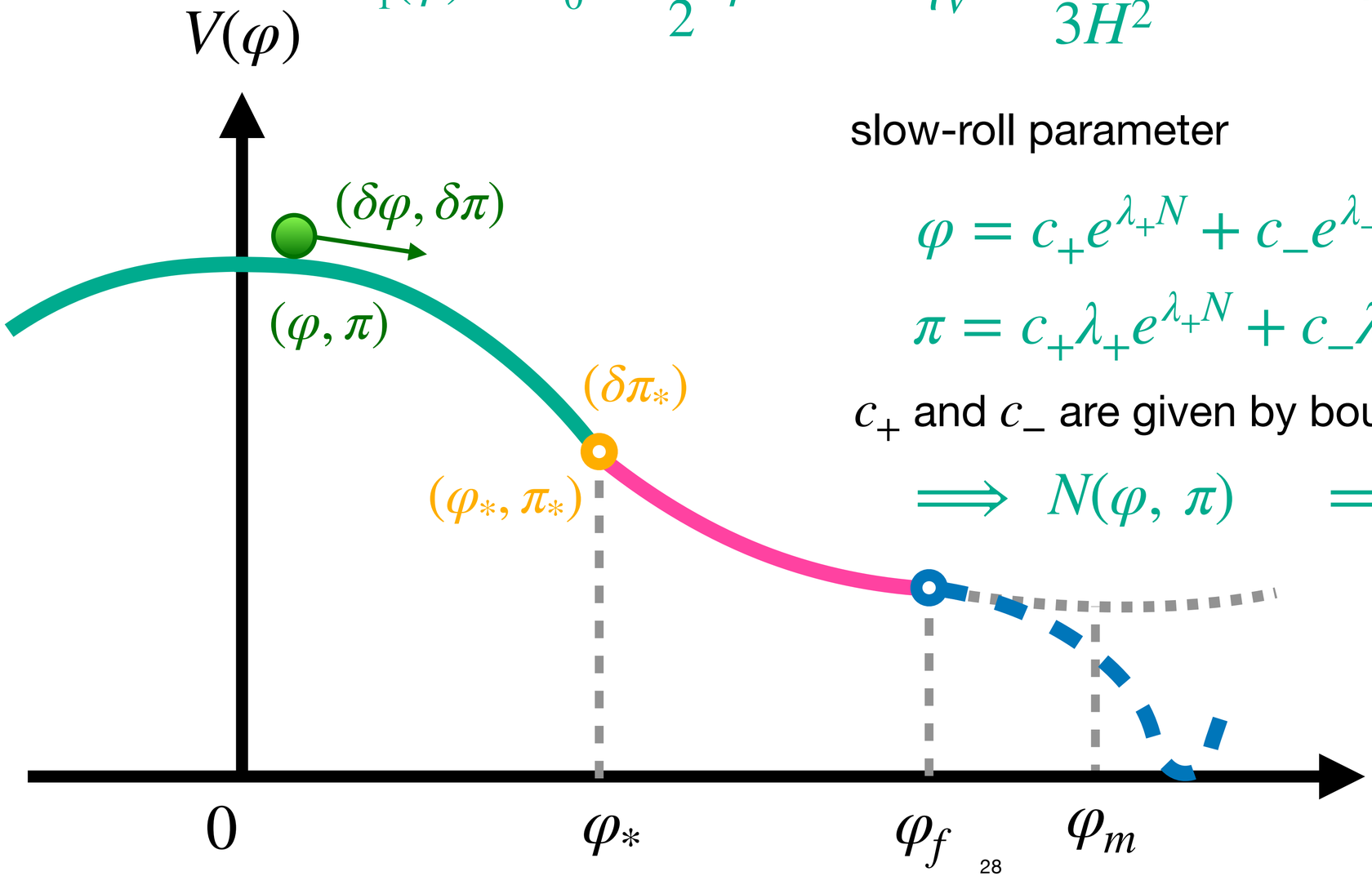
$$V_1(\varphi) = V_0 + \frac{m_1^2}{2}\varphi^2$$

$$\eta_V = \frac{m_1^2}{3H^2}$$

slow-roll parameter

$$\lambda_\pm = \frac{3 \pm \sqrt{9 - 12\eta_V}}{2}$$

Characteristic roots
(Liyapnov exponent)



$$\varphi = c_+ e^{\lambda_+ N} + c_- e^{\lambda_- N}$$

$$\pi = c_+ \lambda_+ e^{\lambda_+ N} + c_- \lambda_- e^{\lambda_- N}$$

c_+ and c_- are given by boundary conditions (φ_*, π_*)

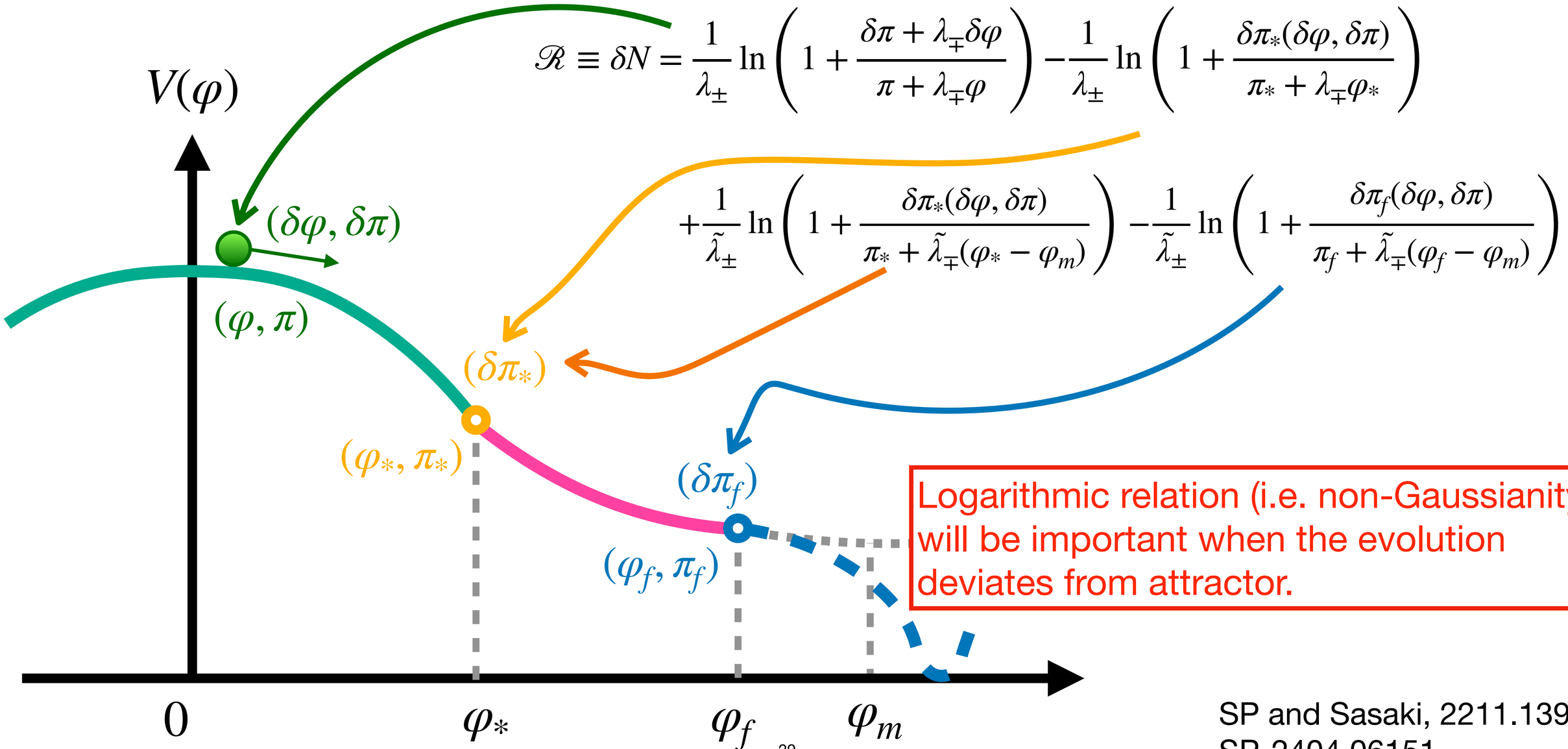
$$\Longrightarrow \; N(\varphi,\, \pi) \quad \Longrightarrow \; \mathscr{R} = \delta N = \delta N(\delta\varphi, \delta\pi)$$

SP and Sasaki, 2211.13932
SP, 2404.06151

Logarithmic Duality

$$\mathcal{R} \equiv \delta N = \frac{1}{\lambda_\pm} \ln\left(1 + \frac{\delta\pi + \lambda_\mp \delta\varphi}{\pi + \lambda_\mp \varphi}\right) - \frac{1}{\lambda_\pm} \ln\left(1 + \frac{\delta\pi_*(\delta\varphi, \delta\pi)}{\pi_* + \lambda_\mp \varphi_*}\right)$$

$$+ \frac{1}{\tilde{\lambda}_\pm} \ln\left(1 + \frac{\delta\pi_*(\delta\varphi, \delta\pi)}{\pi_* + \tilde{\lambda}_\mp(\varphi_* - \varphi_m)}\right) - \frac{1}{\tilde{\lambda}_\pm} \ln\left(1 + \frac{\delta\pi_f(\delta\varphi, \delta\pi)}{\pi_f + \tilde{\lambda}_\mp(\varphi_f - \varphi_m)}\right)$$



Logarithmic relation (i.e. non-Gaussianity) will be important when the evolution deviates from attractor.

SP and Sasaki, 2211.13932
SP, 2404.06151

Logarithmic Duality

SP and Sasaki, 2211.13932
SP, 2404.06151

$$\mathscr{R}(\delta\varphi, \delta\pi)$$

$$\mathscr{R} = \frac{1}{\lambda_\pm} \ln\left(1 + \frac{\delta\pi + \lambda_\mp \delta\varphi}{\pi + \lambda_\mp \varphi}\right) - \frac{1}{\lambda_\pm} \ln\left(1 + \frac{\delta\pi_*}{\pi_* + \lambda_\mp \varphi_*}\right) + \cdots$$

c.f. consistency relation

$$f_{\rm NL} = \frac{5}{2}\epsilon_V - \frac{5}{6}\eta_V$$

$$\left(f_{NL} = -\frac{5}{6}\lambda_- \approx -\frac{5}{6}\eta\right)$$

$\lambda_- \ll 1$
$\lambda_+ \approx 3$

$$\mathscr{R} = -H\frac{\delta\varphi}{\dot\varphi} + \frac{3}{5} f_{\rm NL}\left(-H\frac{\delta\varphi}{\dot\varphi}\right)^2$$

Slow-roll inflation
Stewart and Sasaki, 1995
Lyth and Roquigez, 2005

$\lambda_- = -1/\mu$
$\lambda_+ = 3 - 1/\mu$

$$\mathscr{R} = -\mu \ln\left(1 - \frac{\mathscr{R}_g}{\mu}\right)$$

Constant-roll
Atal, Garriga, Marcos-Caballero, 1905.13202
Atal, Cid, Escrivà, Garriga, 1908.11357
Escrivà, Atal, Garriga, 2306.09990
Inui, Motohashi, SP, et al, 2409.13500

$\lambda_- = 0$
$\lambda_+ = 3$

$$\mathscr{R} = -\frac{1}{3}\ln\left(1 + \frac{\delta\pi_*}{\pi_*}\right)$$

Ultra-slow-roll
Namjoo, Firouzjahi, Sasaki, 1210.3692
Cai, Chen, et al 1712.09998
Biagetti et al 1804.07124
Passaglia et al 1812.08243

$\lambda_- = -3/2$
$\lambda_- = 9/2$

$$\mathscr{R} = \frac{2}{3}\ln(1+\delta)$$

Curvaton scenario,
SP and Sasaki, 2112.12680
Ferrante et al, 2211.01728
Hooper et al. 2308.00756

$$\mathscr{R} = -\frac{1}{\lambda}\ln\left(f(\mathscr{R}_{\rm G})\right)$$

Extensions,
Celoria et al, 2103.09244
Kawaguchi et al, 2305.18140

Probability Distribution Function

SP and Sasaki, 2211.13932
SP, 2404.06151

For the simplest single-logarithm case:

$$\mathscr{R} \equiv \delta N = \frac{1}{\lambda_-} \ln\left(1 + \frac{\delta\pi + \lambda_+ \delta\varphi}{\pi + \lambda_+ \varphi}\right)$$

$$\Downarrow$$

$$P(\mathscr{R})d\mathscr{R} = P(\delta\varphi)d\delta\varphi$$

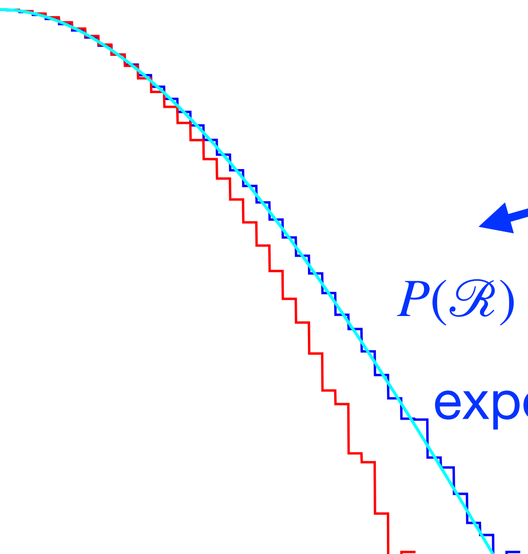
Gaussian PDF with variance $\sigma^2_{\delta\varphi}$

$$P(\mathscr{R}) = \frac{e^{\lambda_- \mathscr{R}}}{\sqrt{2\pi}\sigma_{\delta\varphi}} \left|\lambda_-\right| \varphi \exp\left[-\frac{\varphi^2}{2\sigma^2_{\delta\varphi}}\left(e^{\lambda_- \mathscr{R}} - 1\right)^2\right]$$

$\lambda_- < 0$

$P(\mathscr{R}) \sim e^{\lambda_- \mathscr{R}}$

exponential tail

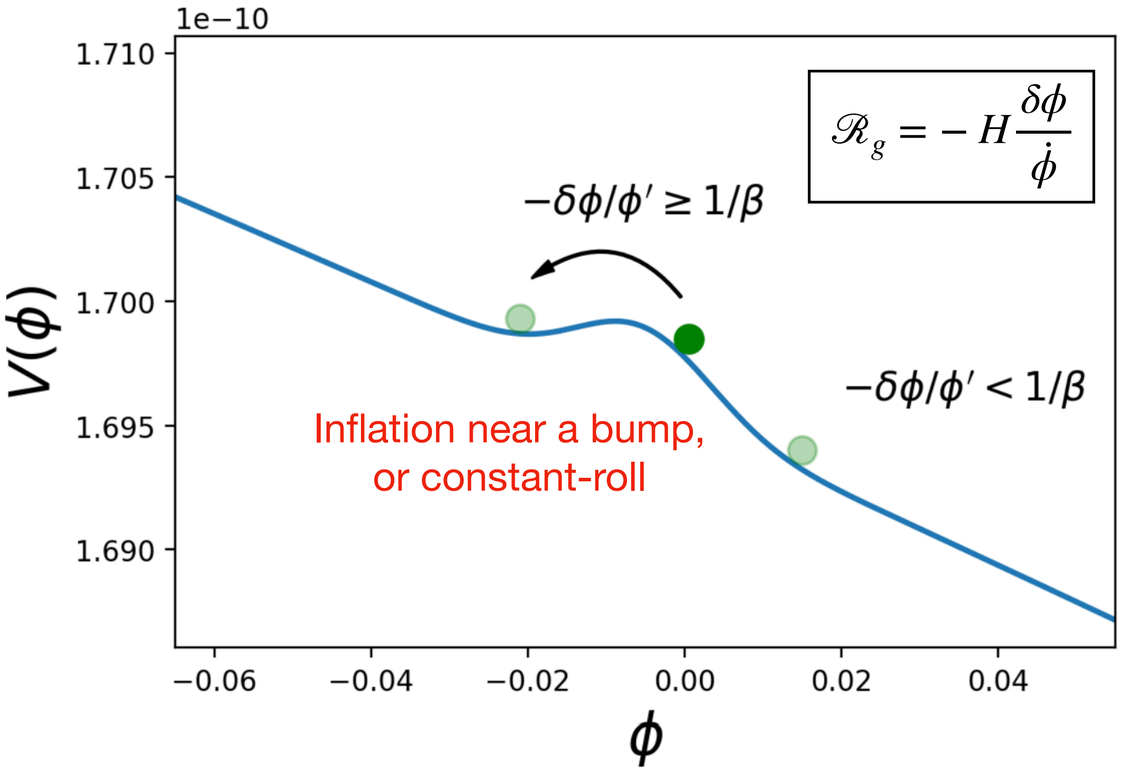


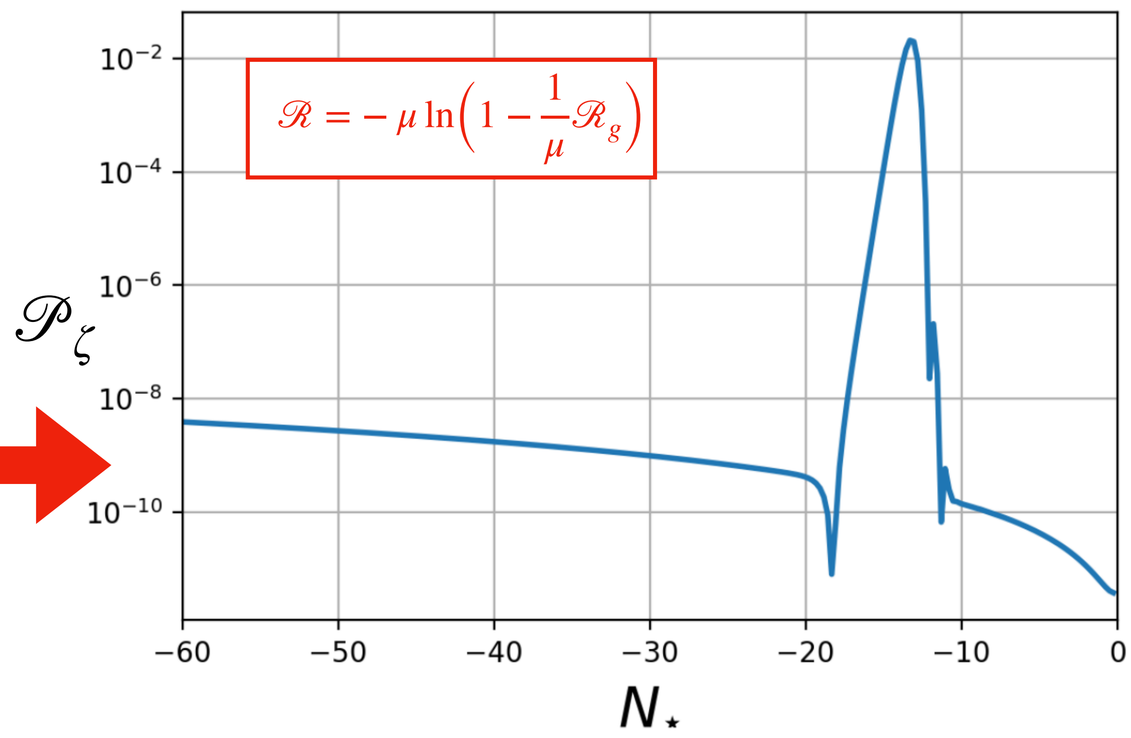
$\lambda_- > 0$

$P(\mathscr{R}) \sim \exp\left(-c^2 e^{2\lambda_- \mathscr{R}}\right)$

Gumbel-distribution-like tail

Application to Constant-roll

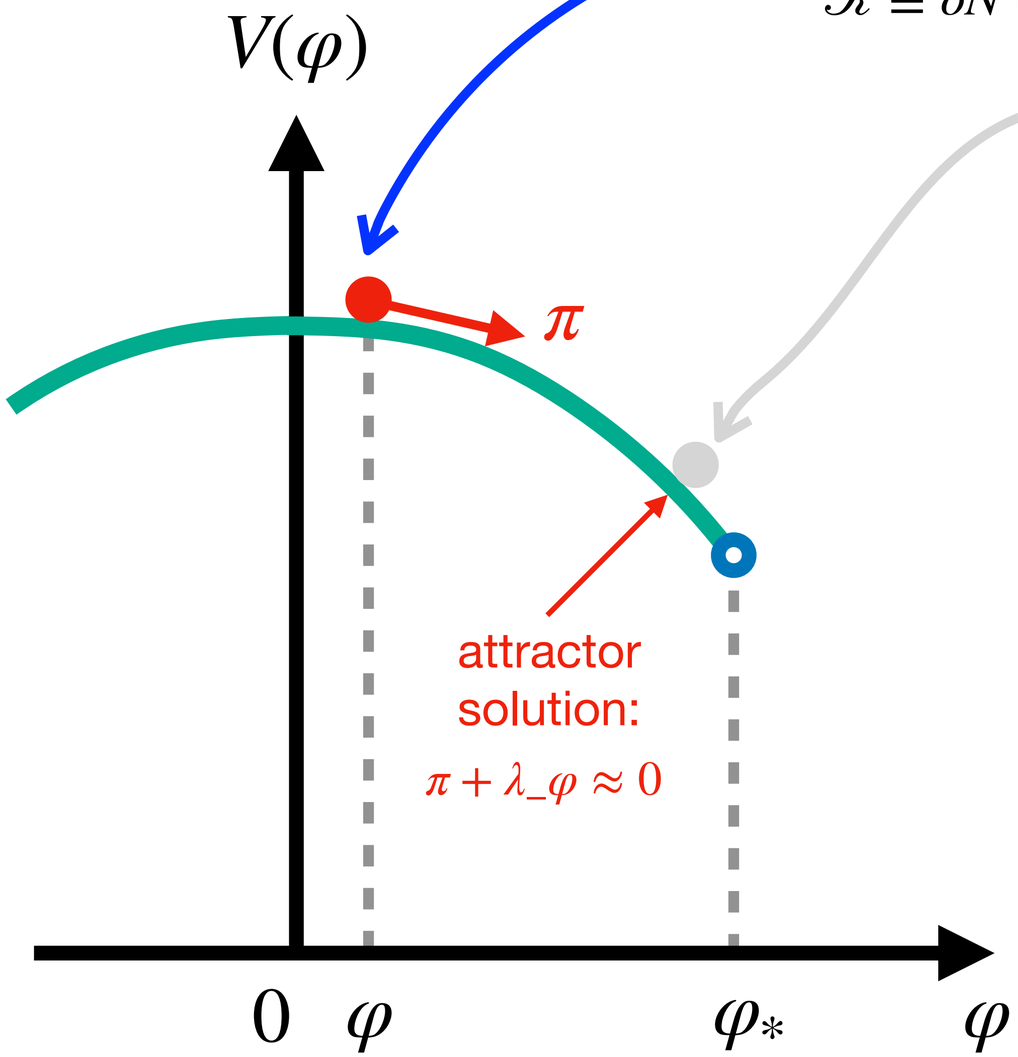


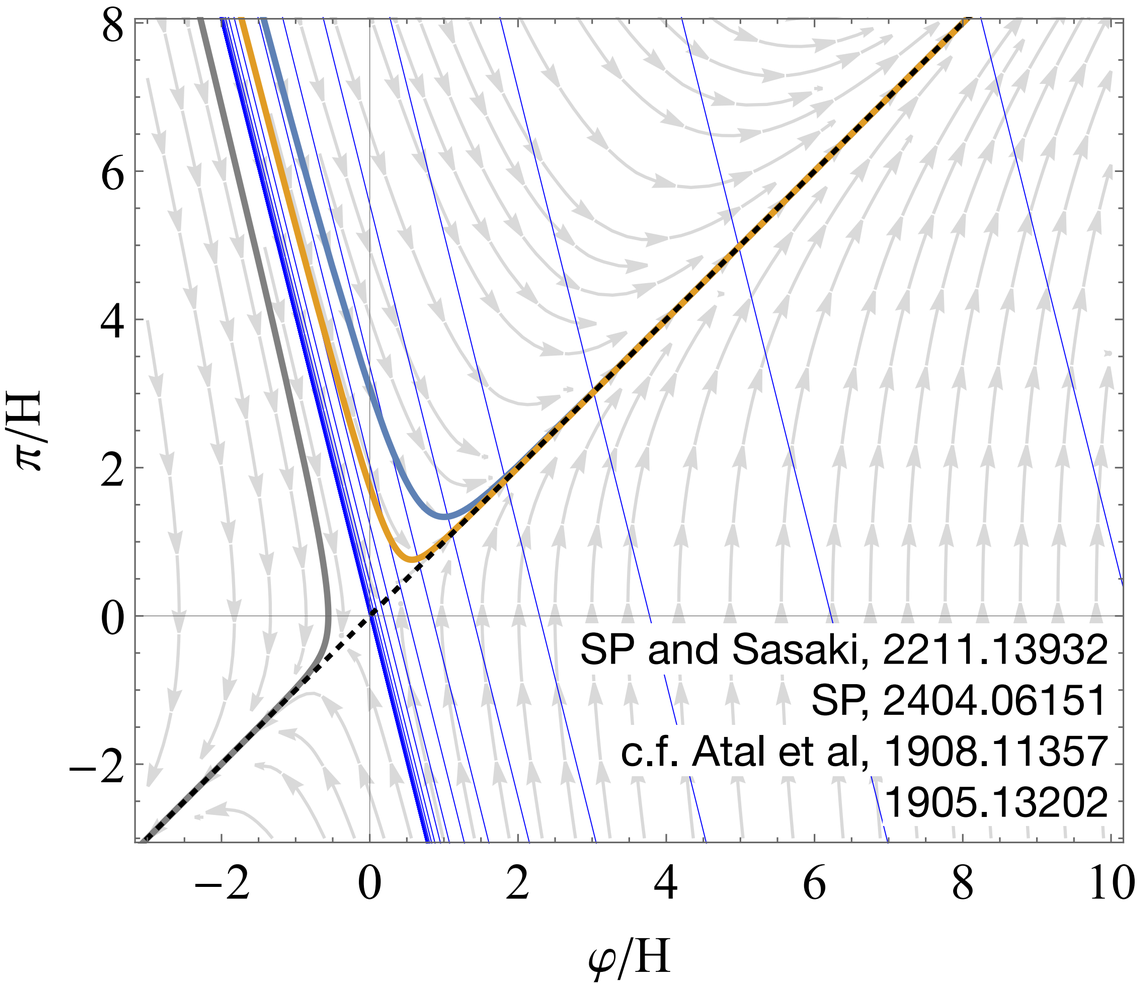


Atal, Garriga, Marcos-Caballero, 1905.13202
Atal, Cid, Escrivà, Garriga, 1908.11357
Escrivà, Atal, Garriga, 2306.09990

Application to Constant-roll

$$\mathcal{R} \equiv \delta N = \frac{1}{\lambda_-} \ln\left(1 + \frac{\delta\pi + \lambda_+ \delta\varphi}{\pi + \lambda_+ \varphi}\right) - \frac{1}{\lambda_-} \ln\left(1 + \frac{\delta\pi_*}{\pi_* + \lambda_+ \varphi_*}\right)$$



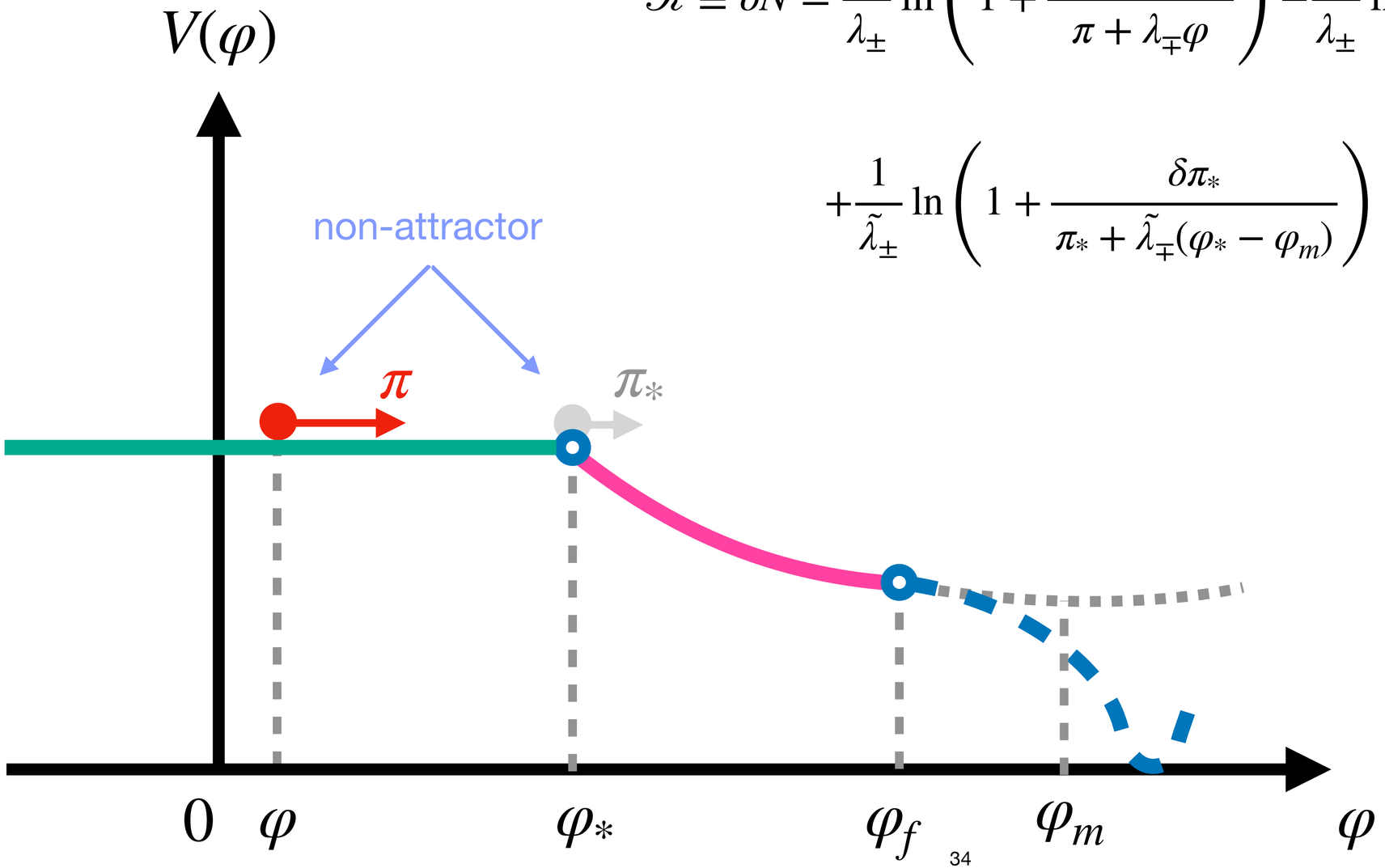


USR-to-SR transition

$$(\lambda_- = 0, \quad \lambda_+ = 3)$$
$$(\tilde{\lambda}_- = \tilde{\eta}, \quad \tilde{\lambda}_+ = 3 - \tilde{\eta})$$

$$\mathscr{R} \equiv \delta N = \frac{1}{\lambda_\pm} \ln\left(1 + \frac{\delta\pi + \lambda_\mp \delta\varphi}{\pi + \lambda_\mp \varphi}\right) - \frac{1}{\lambda_\pm} \ln\left(1 + \frac{\delta\pi_*}{\pi_* + \lambda_\pm \varphi_*}\right)$$

$$+ \frac{1}{\tilde{\lambda}_\pm} \ln\left(1 + \frac{\delta\pi_*}{\pi_* + \tilde{\lambda}_\mp(\varphi_* - \varphi_m)}\right) - \frac{1}{\tilde{\lambda}_\pm} \ln\left(1 + \frac{\delta\pi_f}{\pi_f + \tilde{\lambda}_\pm(\varphi_f - \varphi_m)}\right)$$

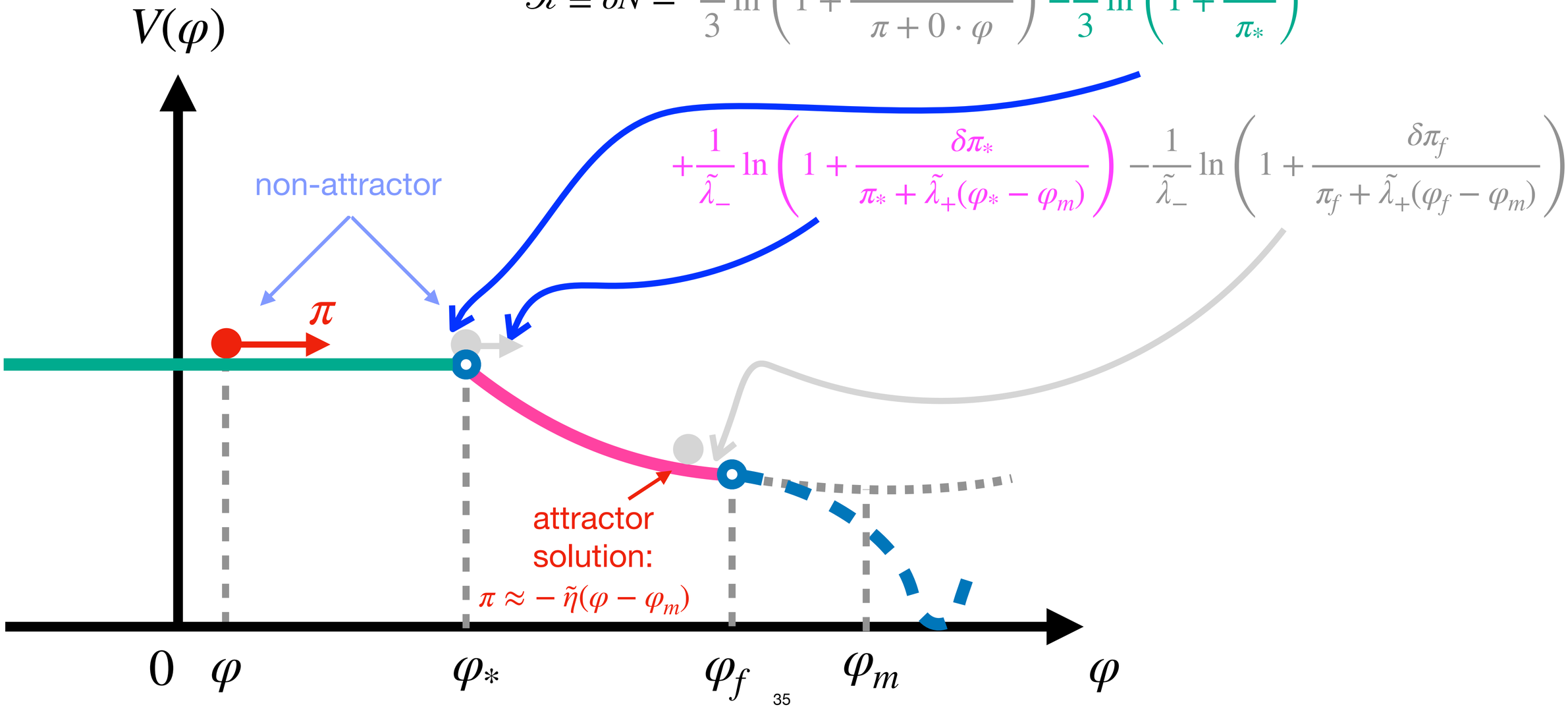


USR-to-SR transition

$$(\lambda_- = 0, \quad \lambda_+ = 3)$$
$$(\tilde{\lambda}_- = \tilde{\eta}, \quad \tilde{\lambda}_+ = 3 - \tilde{\eta})$$

$$\mathcal{R} \equiv \delta N = \frac{1}{3}\ln\left(1 + \frac{\delta\pi + 0\cdot\delta\varphi}{\pi + 0\cdot\varphi}\right) - \frac{1}{3}\ln\left(1 + \frac{\delta\pi_*}{\pi_*}\right)$$

$$+\frac{1}{\tilde{\lambda}_-}\ln\left(1 + \frac{\delta\pi_*}{\pi_* + \tilde{\lambda}_+(\varphi_* - \varphi_m)}\right) - \frac{1}{\tilde{\lambda}_-}\ln\left(1 + \frac{\delta\pi_f}{\pi_f + \tilde{\lambda}_+(\varphi_f - \varphi_m)}\right)$$



USR-to-SR transition

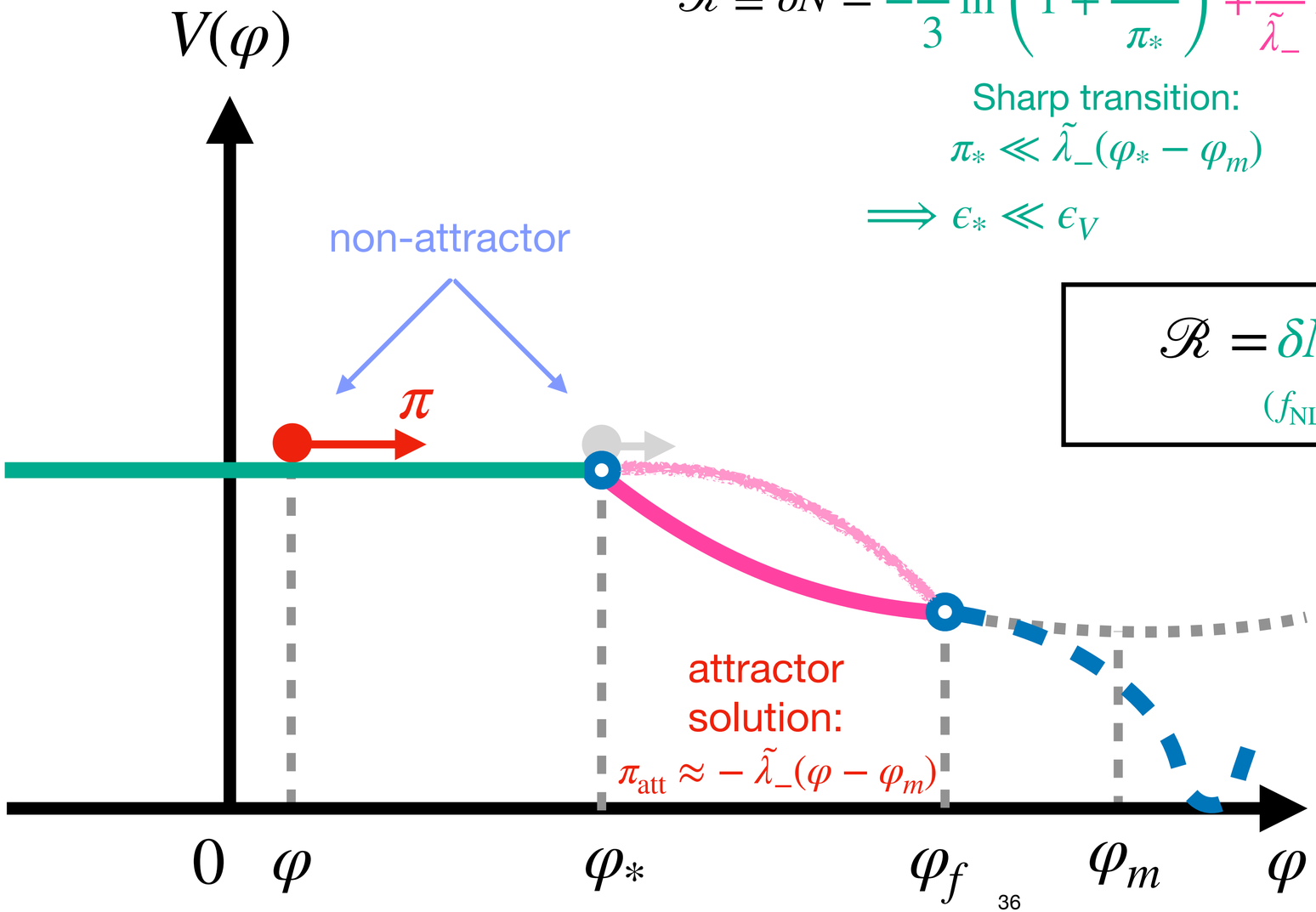
$$(\lambda_- = 0, \quad \lambda_+ = 3)$$
$$(\tilde{\lambda}_- = \tilde{\eta}, \quad \tilde{\lambda}_+ = 3 - \tilde{\eta})$$

$$\mathscr{R} \equiv \delta N = -\frac{1}{3}\ln\left(1+\frac{\delta\pi_*}{\pi_*}\right) + \frac{1}{\tilde{\lambda}_-}\ln\left(1+\frac{\delta\pi_*}{\pi_* + \tilde{\lambda}_+(\varphi_* - \varphi_m)}\right)$$

Sharp transition:
$\pi_* \ll \tilde{\lambda}_-(\varphi_* - \varphi_m)$
$\Longrightarrow \epsilon_* \ll \epsilon_V$

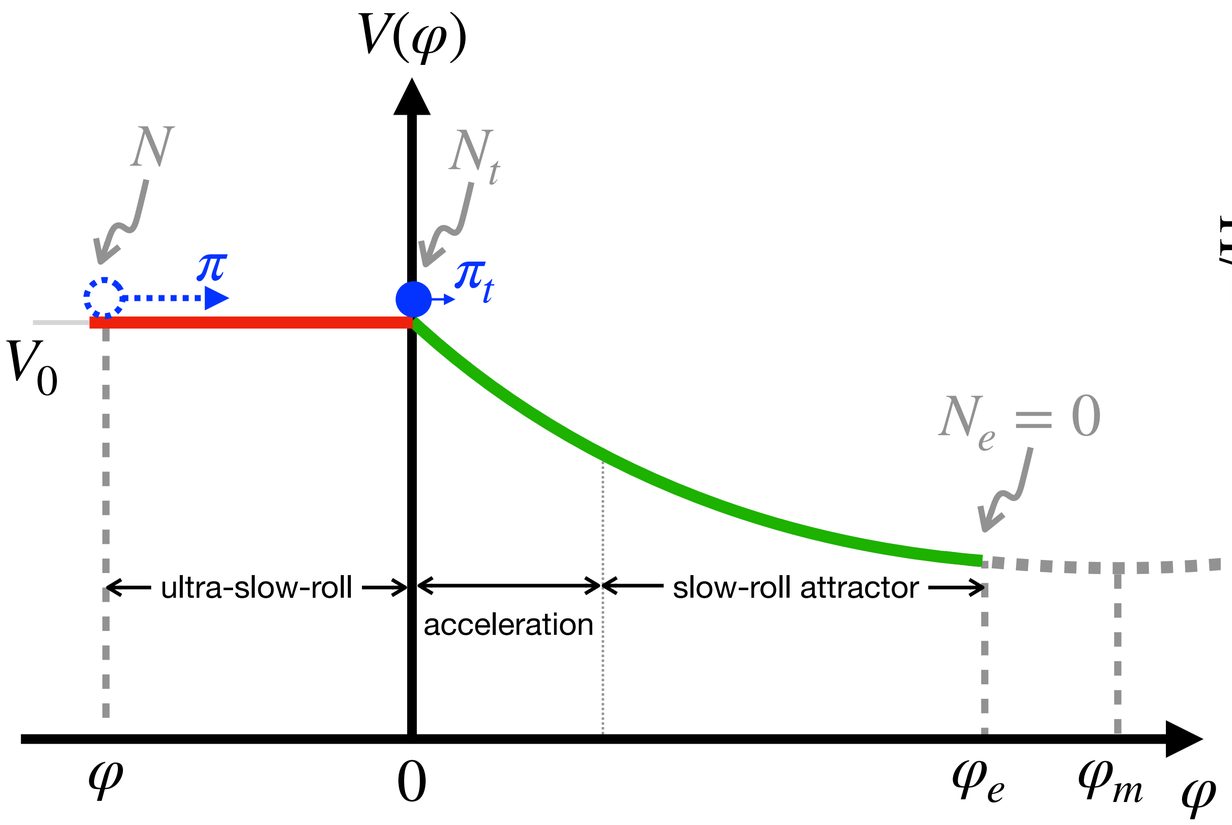
Smooth transition:
$\pi_* \gg \tilde{\lambda}_-(\varphi_* - \varphi_m)$
$\Longrightarrow \epsilon_* \gg \epsilon_V$

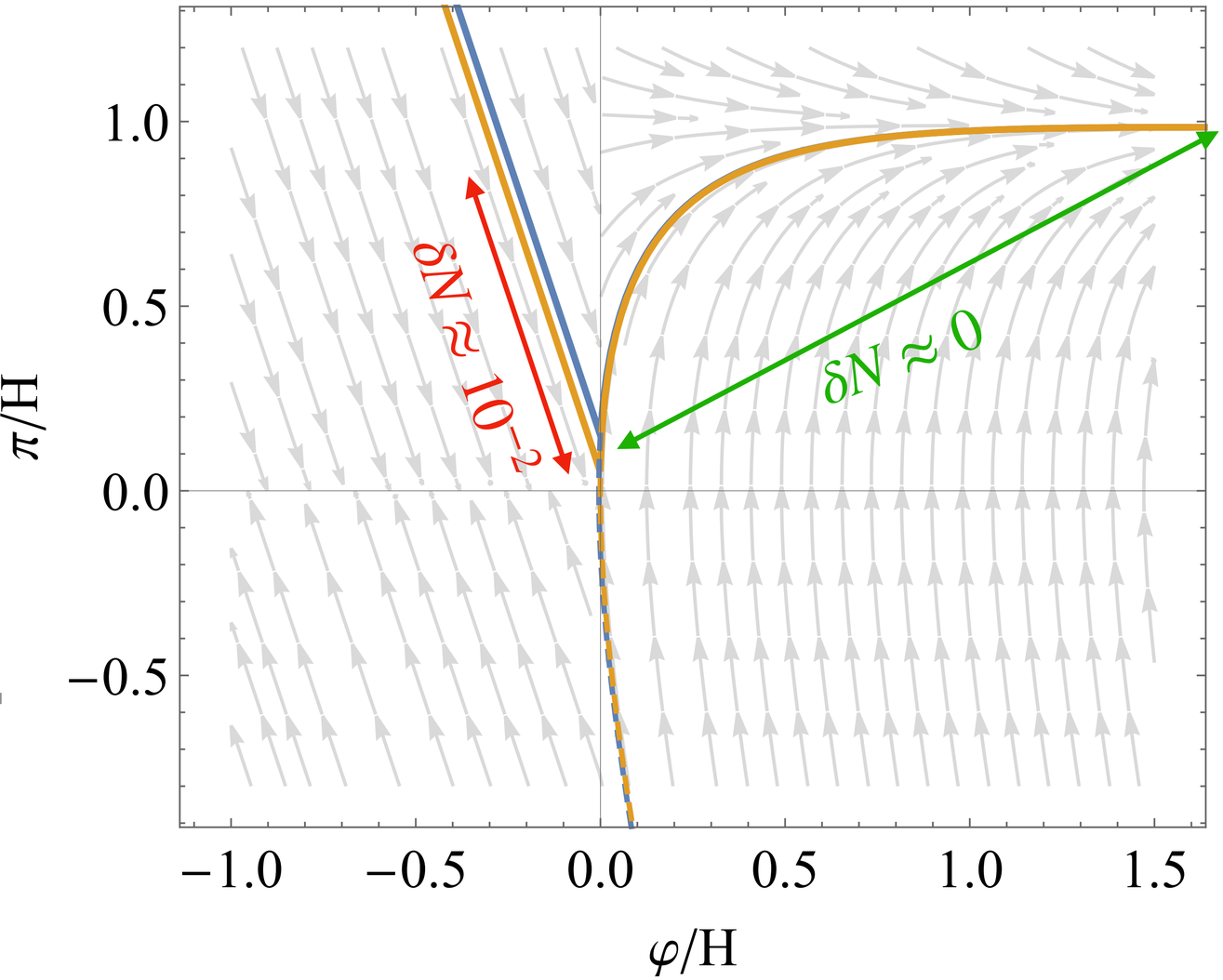
$$\mathscr{R} = \delta N_{\rm USR} + \delta N_{\rm SR}$$
$$(f_{\rm NL} = 5/2) \quad (f_{\rm NL} = -5\tilde{\eta}/6)$$



- Sharp (smooth) transition means acceleration (deceleration) in the following slow-roll stage.
- When $\epsilon_* \sim \epsilon_V$, two logarithms are comparable, thus no simple exponential tail.

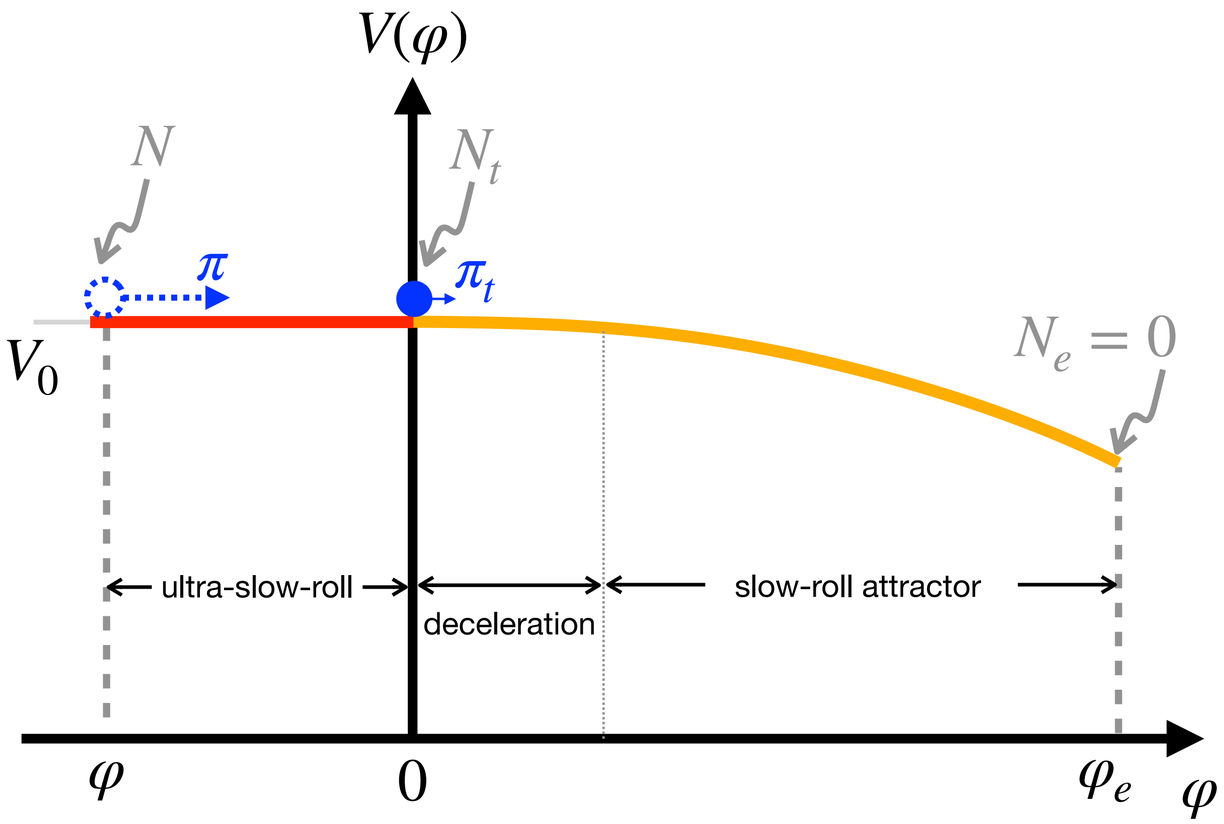
Sharp transition

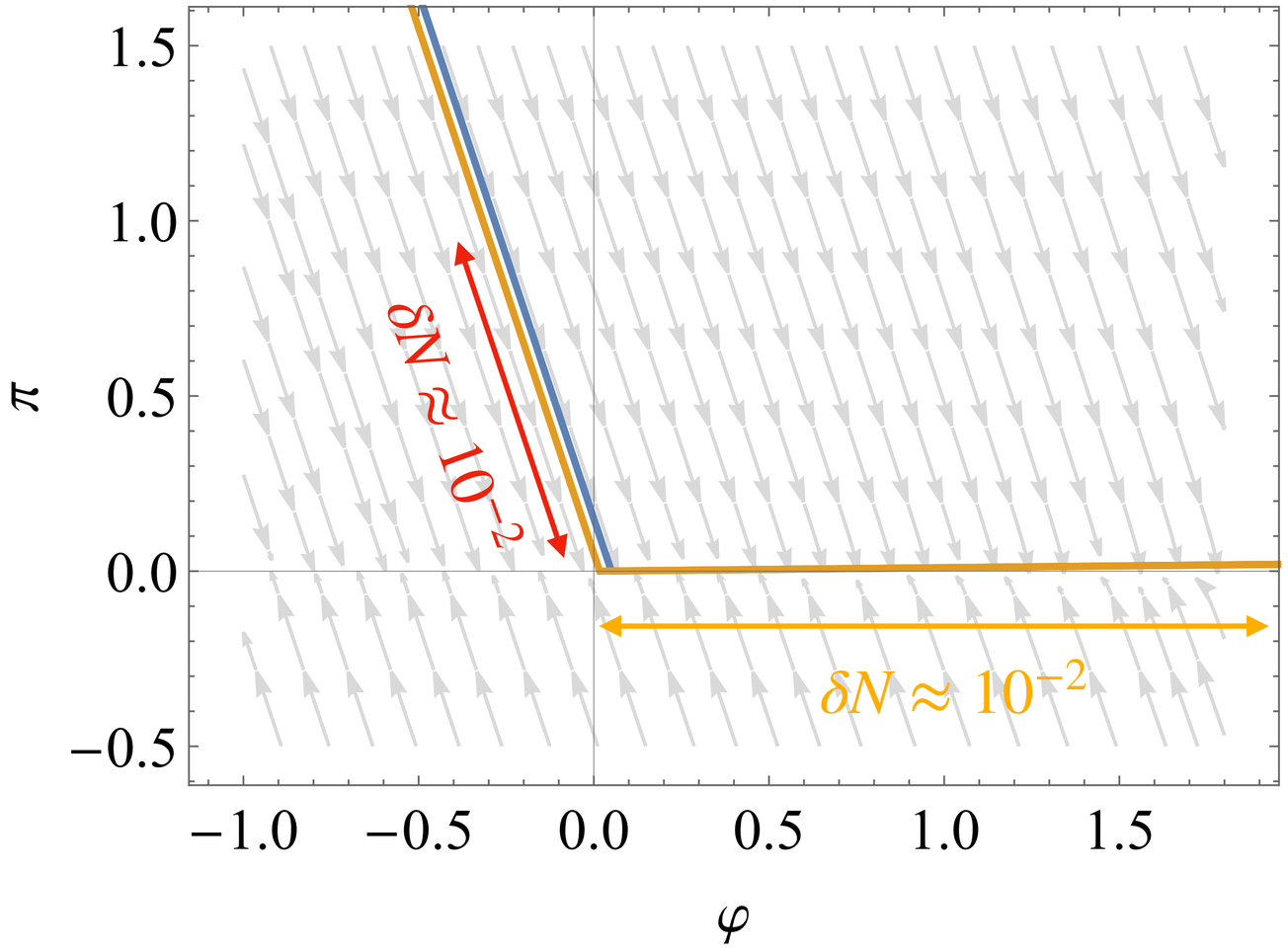




SP and Sasaki, 2211.13932
SP, 2404.06151

Smooth transition



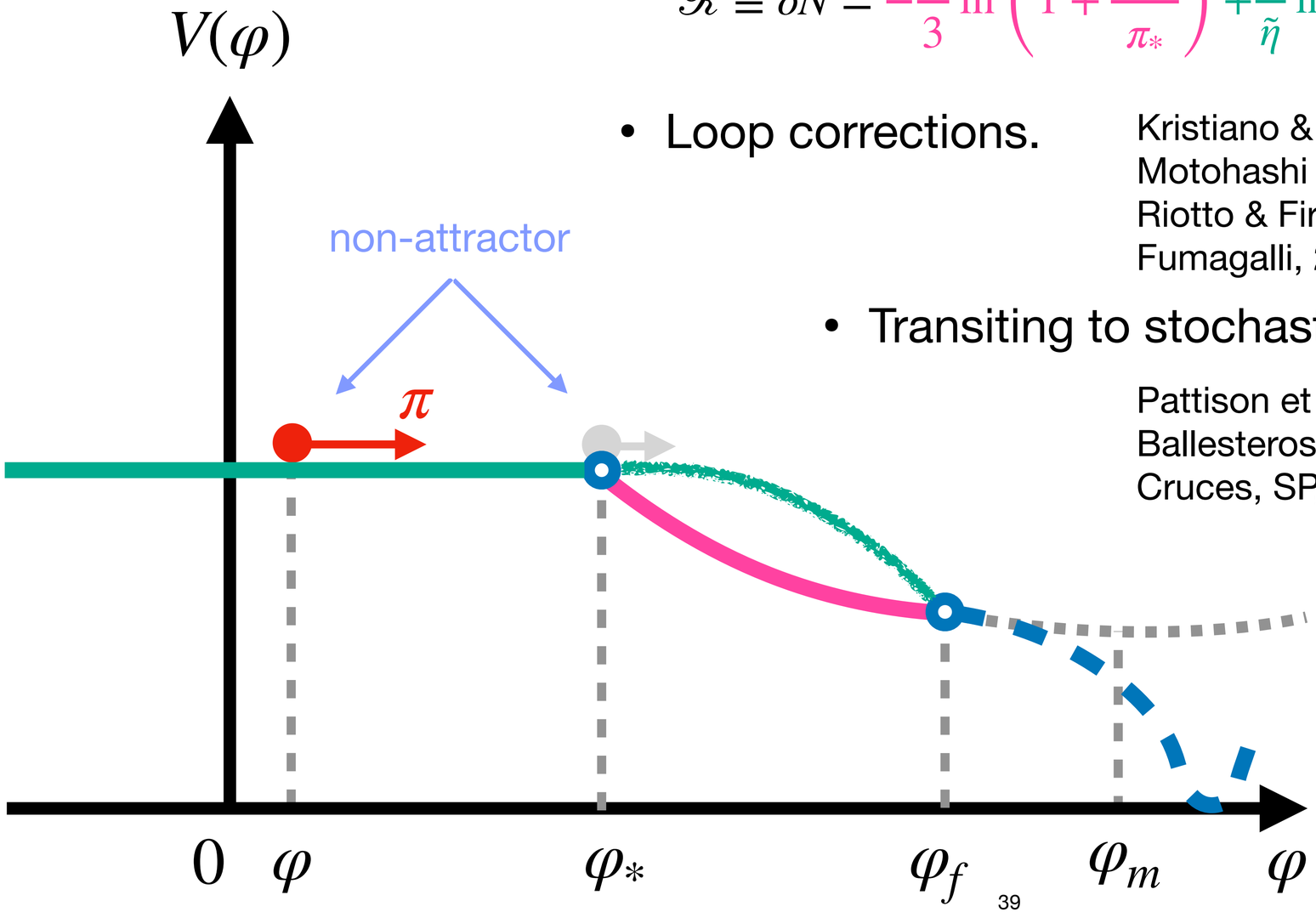


SP and Sasaki, 2211.13932
SP, 2404.06151

USR-to-SR transition

$$(\lambda_- = 0, \quad \lambda_+ = 3)$$
$$(\tilde{\lambda}_- = \tilde{\eta}, \quad \tilde{\lambda}_+ = 3 - \tilde{\eta})$$

$$\mathcal{R} \equiv \delta N = -\frac{1}{3}\ln\left(1 + \frac{\delta\pi_*}{\pi_*}\right) + \frac{1}{\tilde{\eta}}\ln\left(1 + \frac{\delta\pi_*}{\pi_* + (3-\tilde{\eta})(\varphi_* - \varphi_m)}\right)$$



- Loop corrections.

Kristiano & Yokoyama, 2211.03395, 2405.12145
Motohashi & Tada, 2303.16035
Riotto & Firouzjahi, 2304.07801
Fumagalli, 2305.19263

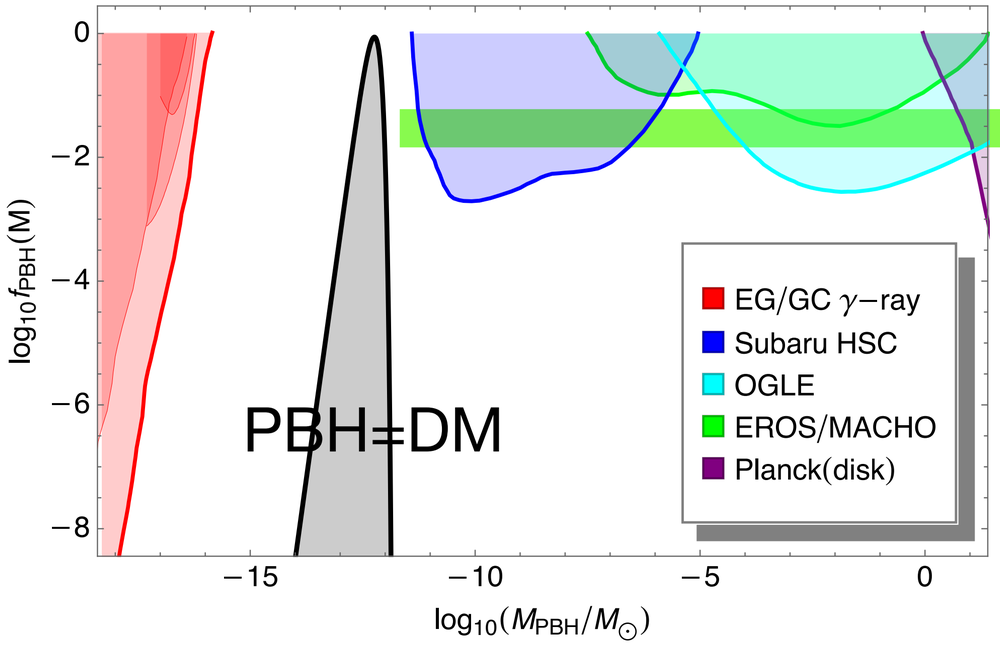
- Transiting to stochastic approach

Pattison et al., 2101.05741
Ballesteros et al 2406.02417
Cruces, SP, Sasaki, 2505.24590

- Sharp transition will make the separate universe approach (thus δN formalism) invalid transiently.

Domenech et al., 2309.05750
Jackson et al., 2311.03281
Artigas, SP, Tanaka, 2408.09964
Briaud et al., 2509.05124

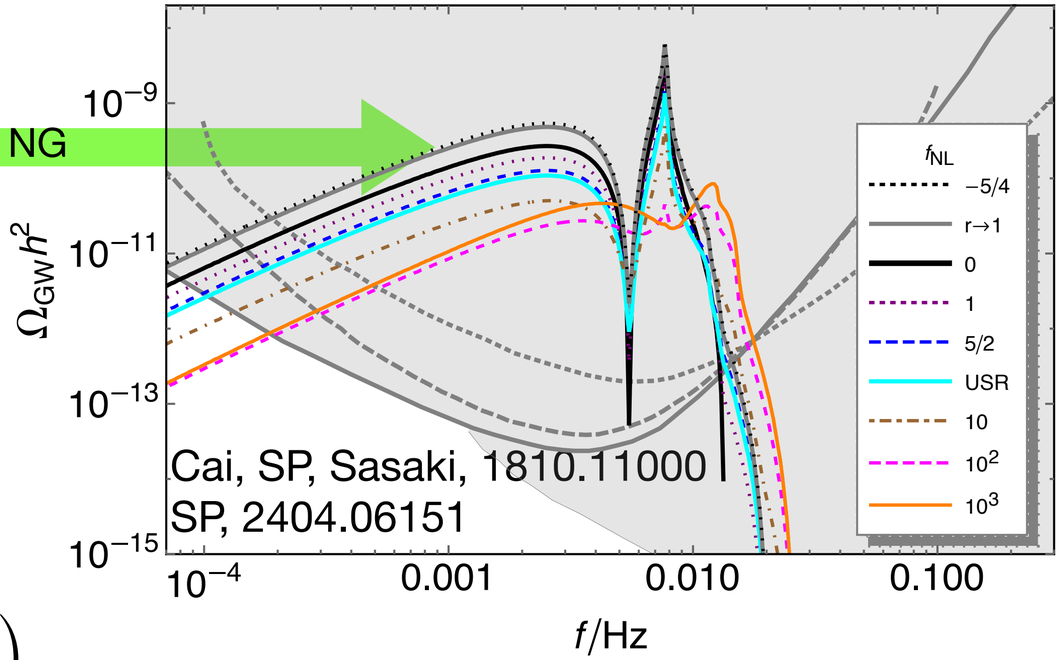
Application: PBH as DM



Press-Schechter-type w/. NG

$$\mathscr{R} = \mathscr{R}_g + \frac{3}{5} f_{\rm NL} \mathscr{R}_g^2$$

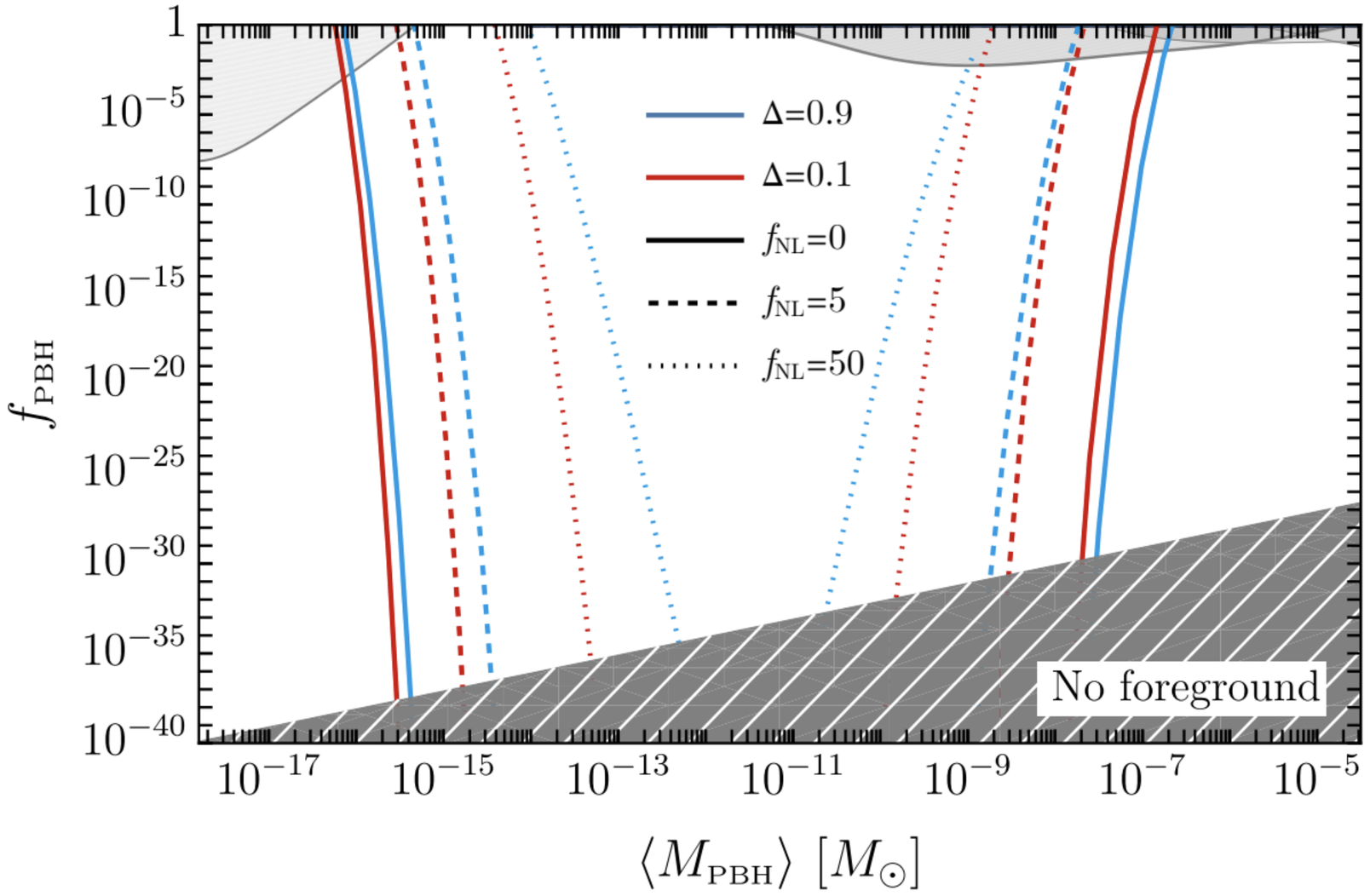
$$\mathscr{R} = -\frac{1}{3} \ln\left(1 - 3\mathscr{R}_g\right)$$



- Primordial NG must be taken into account when calculating PBH abundance
- When fixing PBH abundance, NG impact on SGWB is mild
- LISA/Taiji/TianQin can probe the induced GW when PBH=DM

- Quadratic: Cai, SP, Sasaki 1810.11000; Unal 1811.09151
- Higher orders: $f_{\rm NL}$: Adshead+ 2105.01659; Abe+ 2209.13891; Garcia-Saenz+ 2207.14267. $g_{\rm NL}$: Yuan+, 2308.07155; Li+, 2309.07792. $i_{\rm NL}$: Perna+, 2403.06962
- When fixing PBH abundance, NG impact on SGWB is mild

Detectability

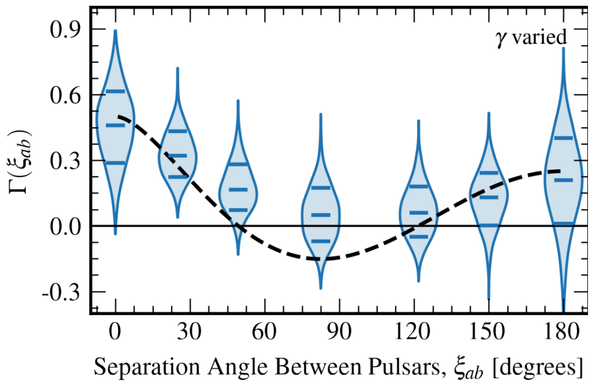


c.f. Iovino et al. 2512.13648

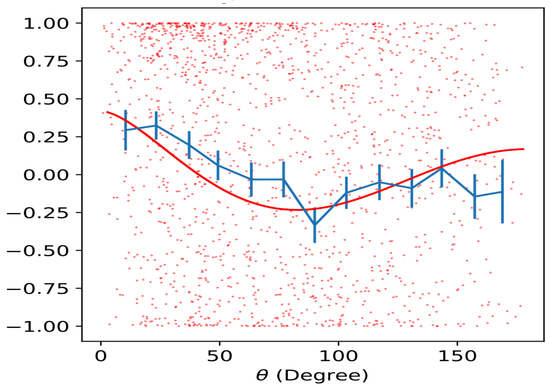
Application: nHz SGWB

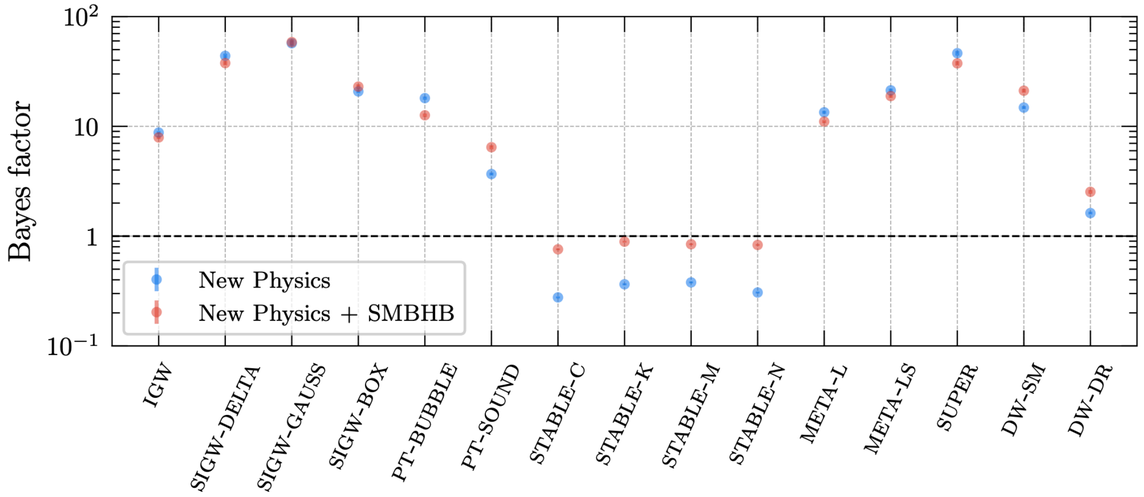
$$\Omega_{\rm GW}(f_{\rm yr}) = 3.6^{+2.4}_{-1.6} \times 10^{-9}$$

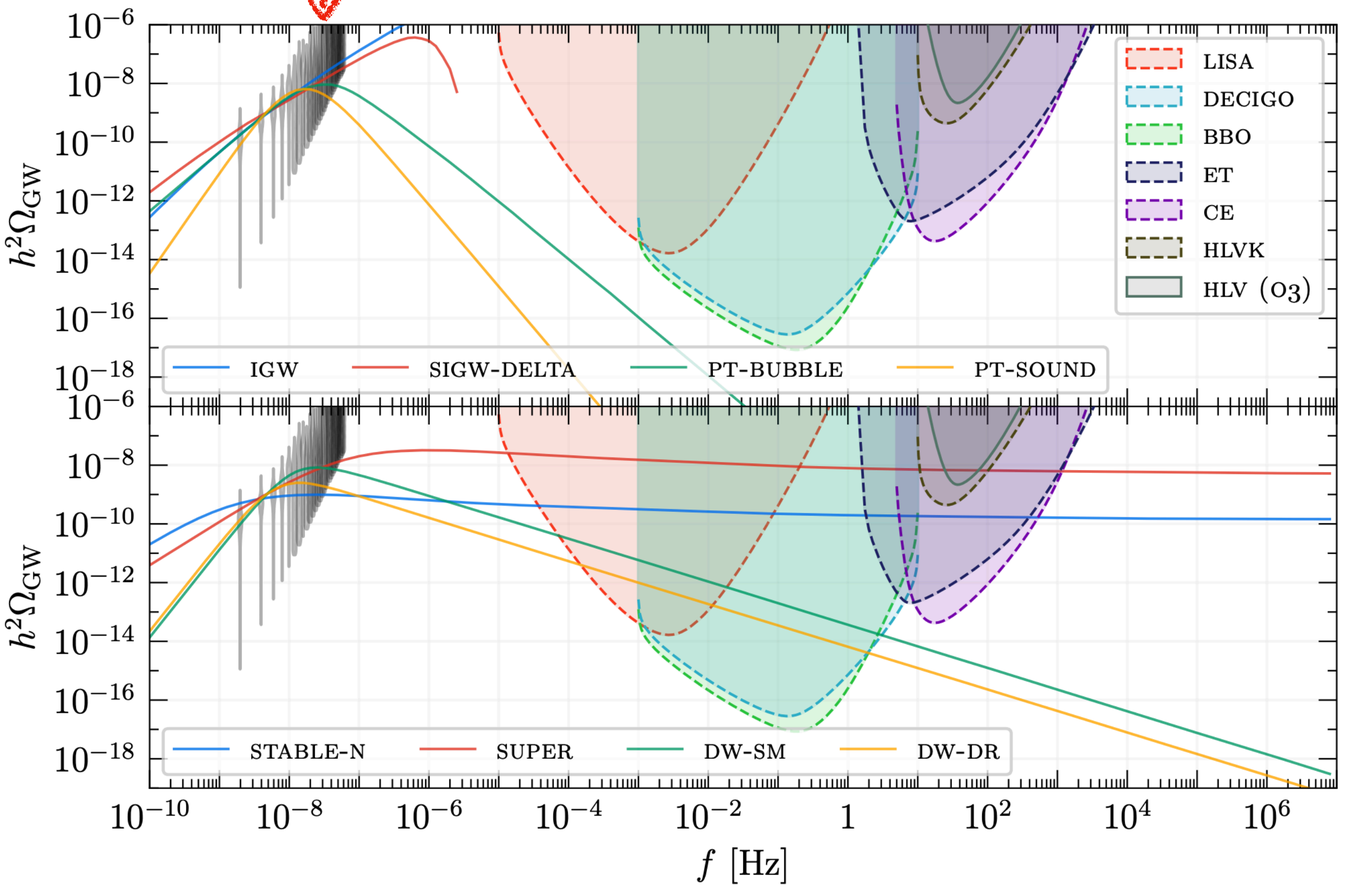
NANOGrav, 2306.16213



CPTA, 2306.16216



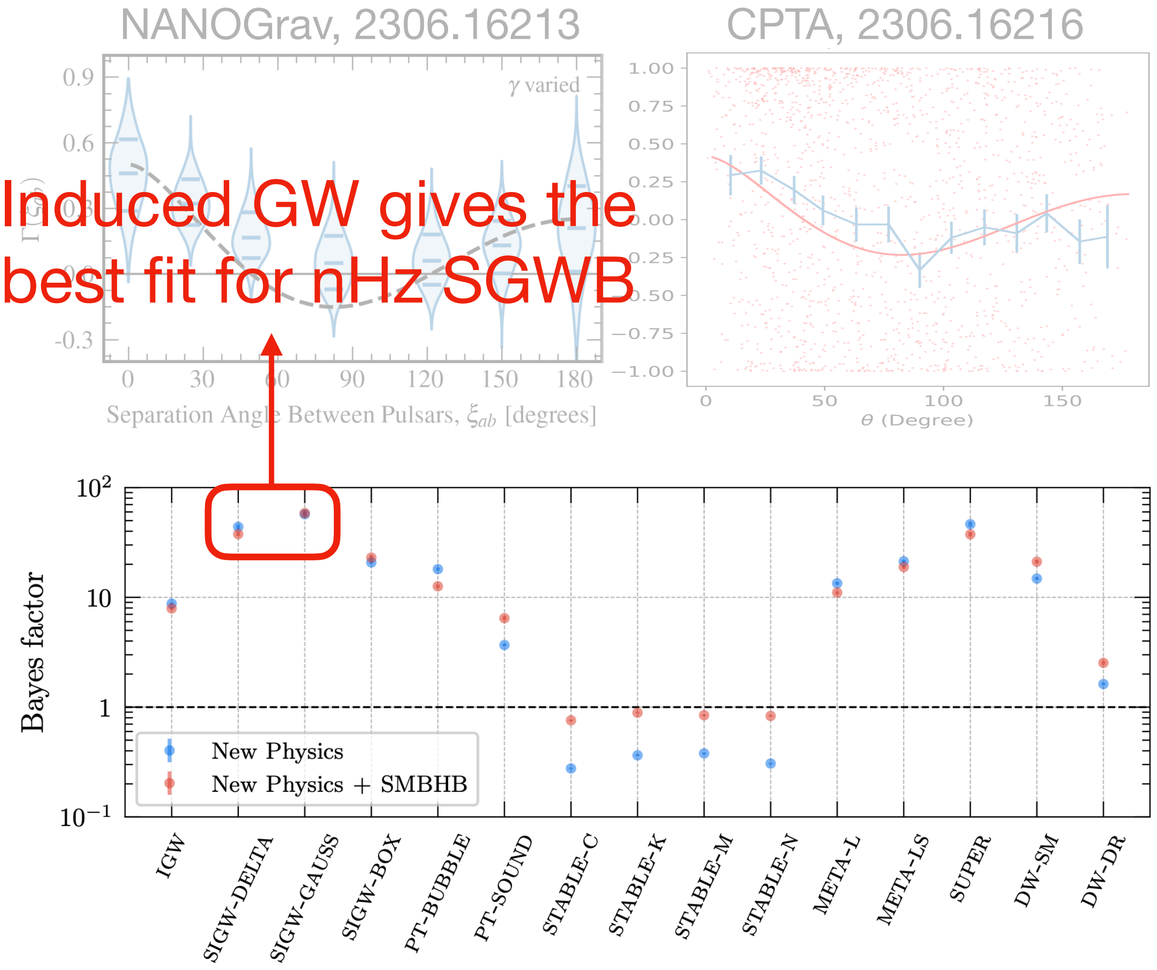




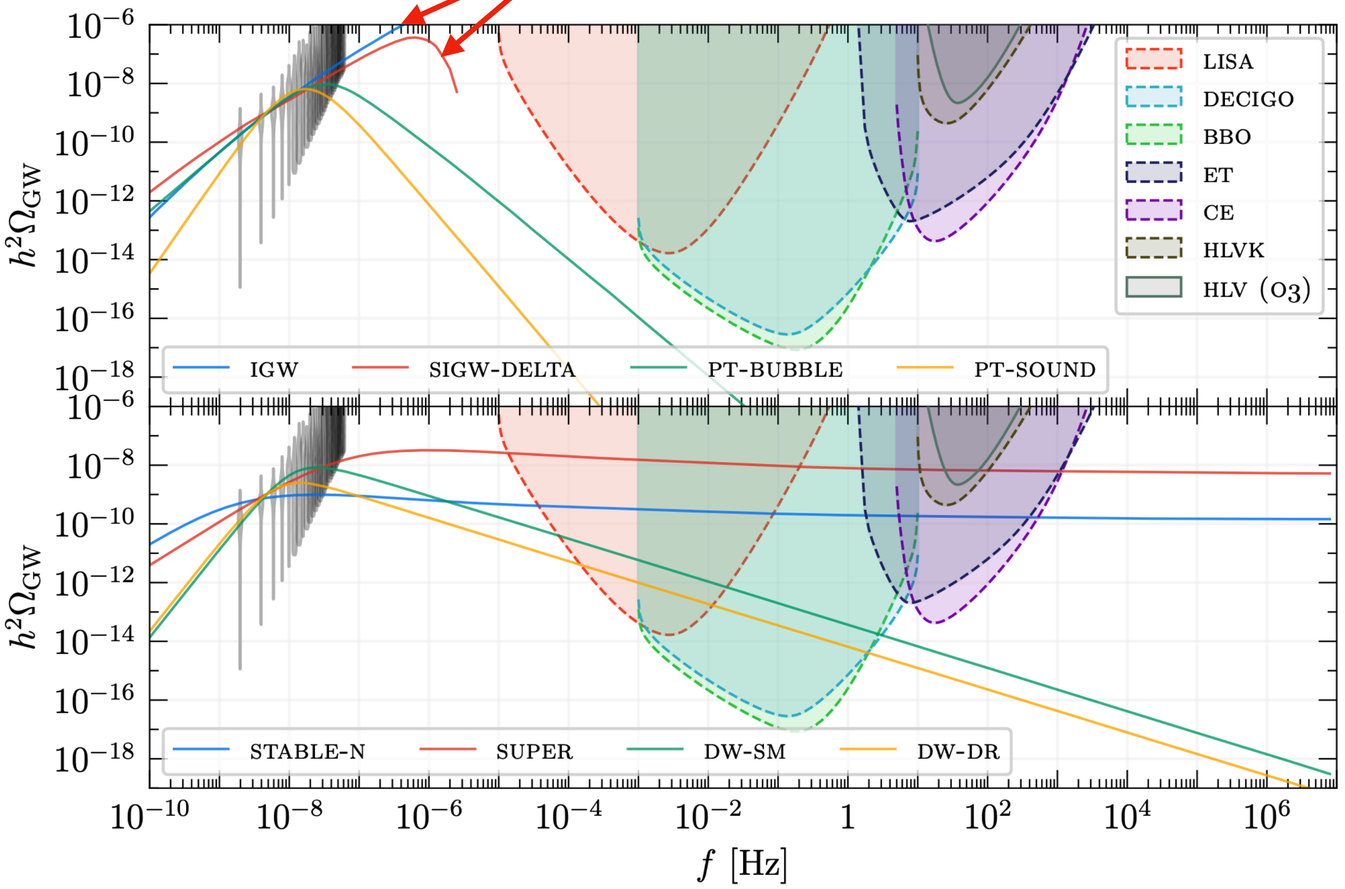
NANOGrav, 2306.16219

Application: nHz SGWB

Induced GW gives the best fit for nHz SGWB

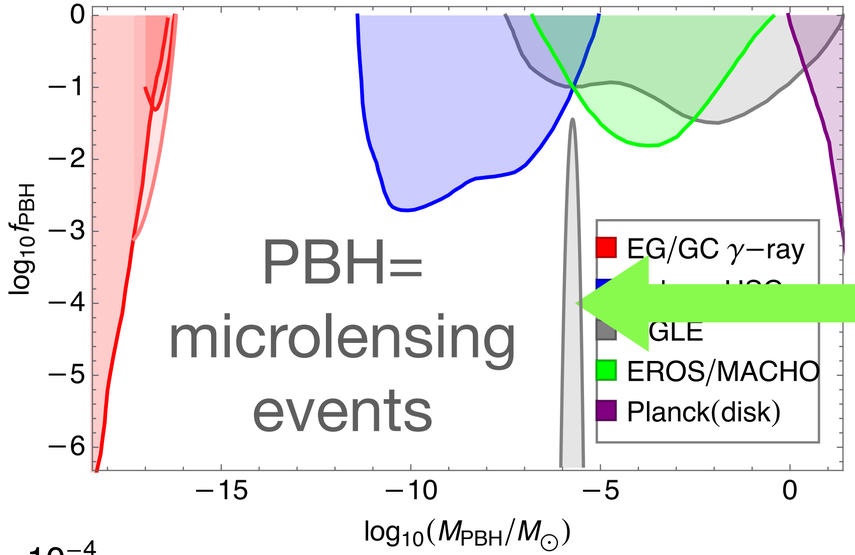


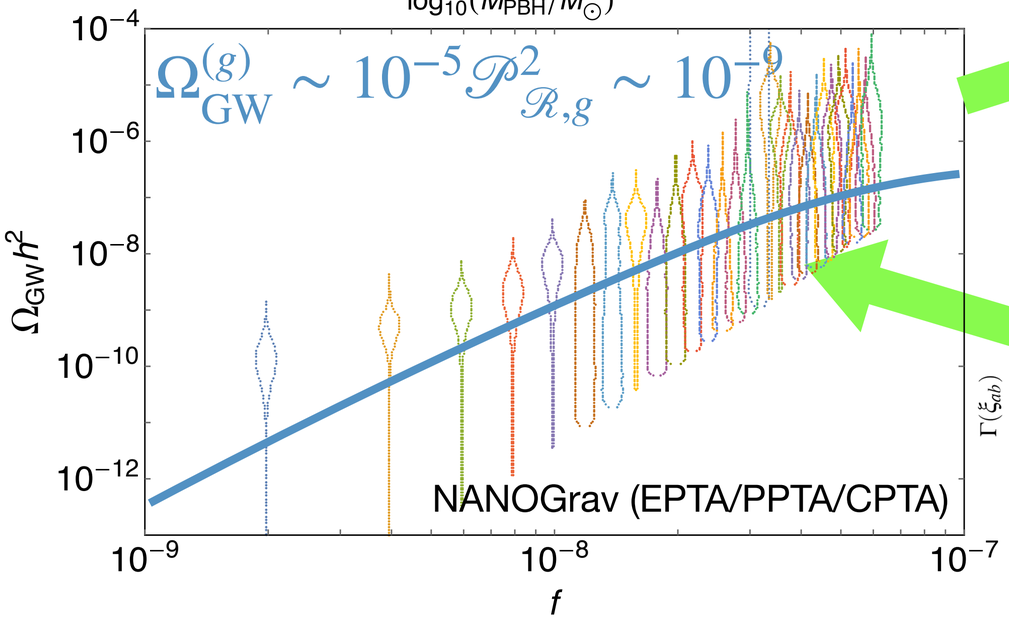
Induced GW

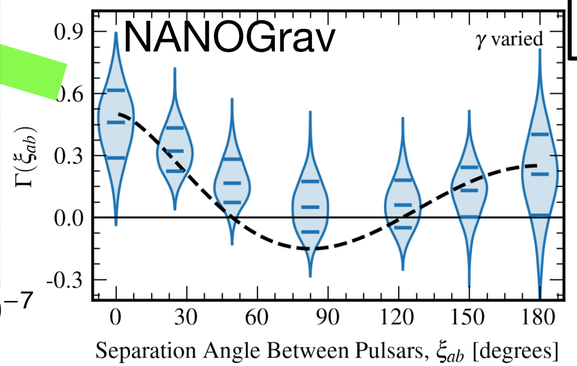


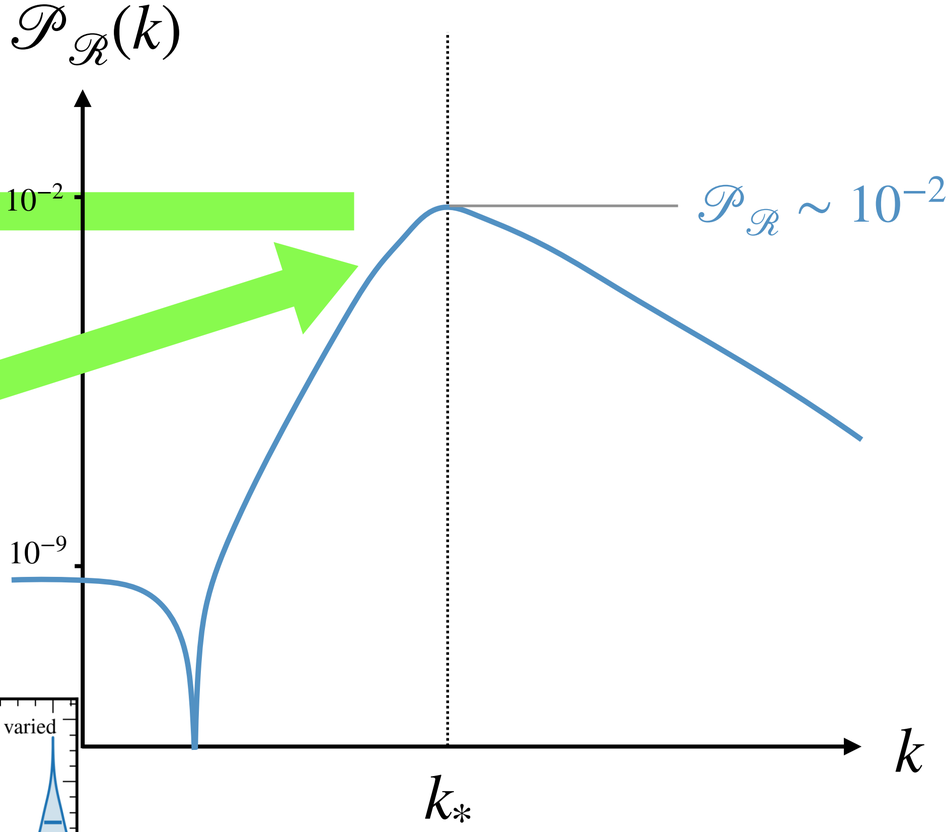
NANOGrav, 2306.16219

Crosscheck by PBH and IGW



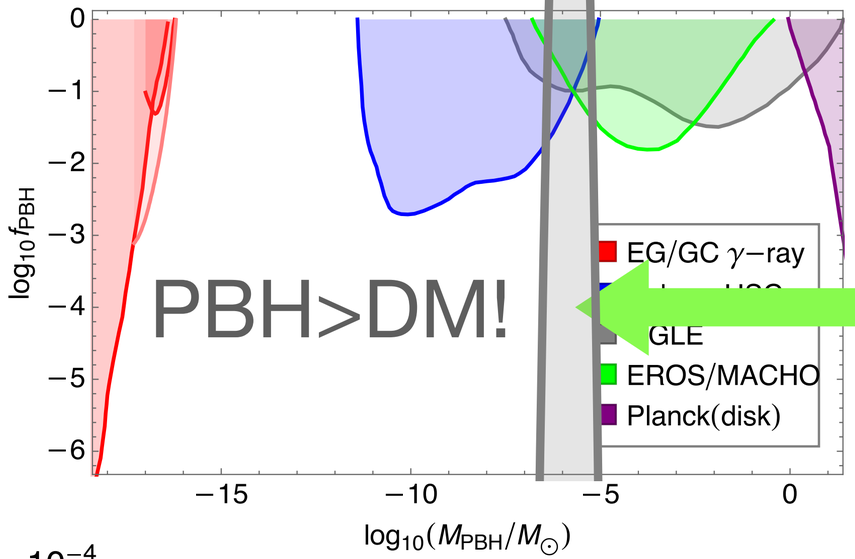


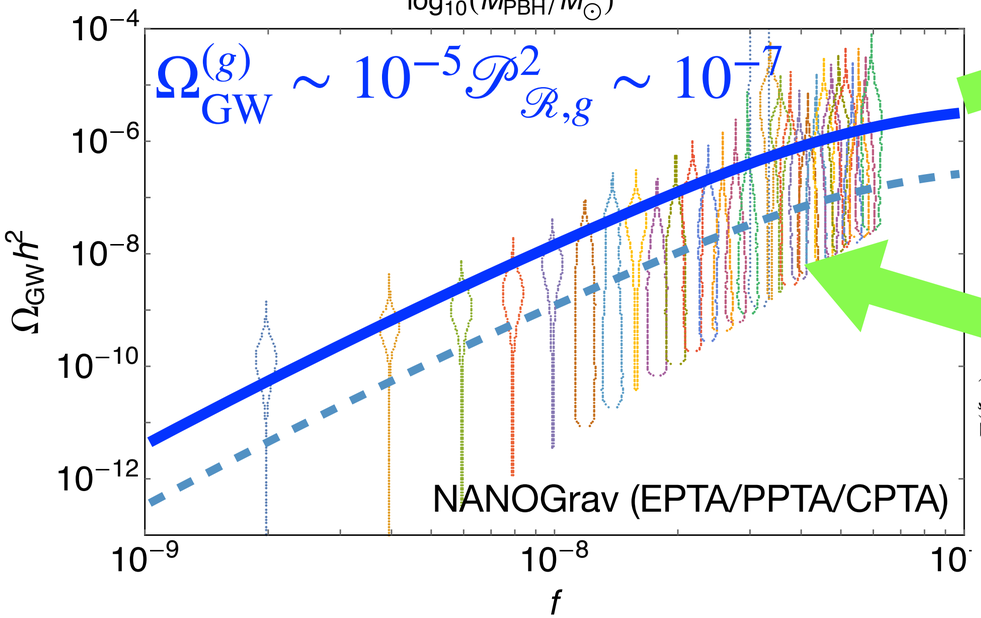


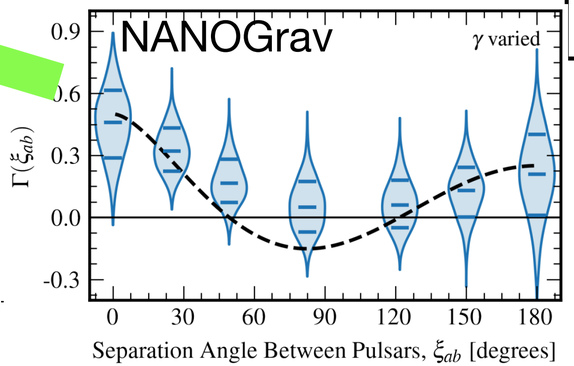


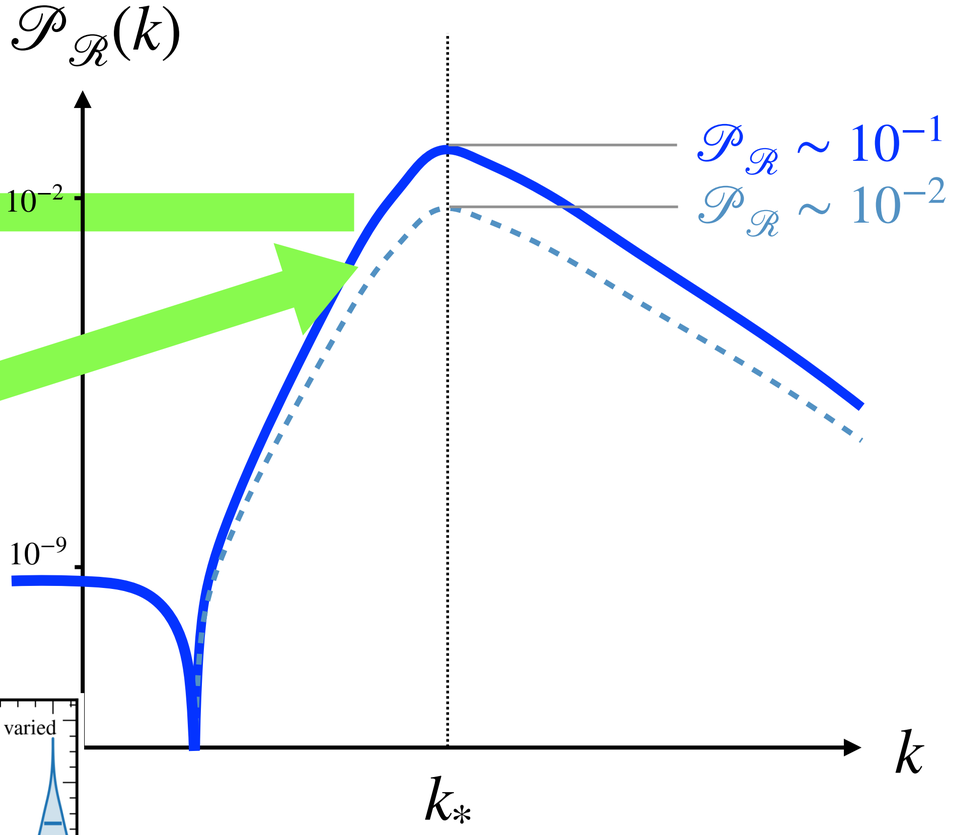
Kohri, Terada, 2009.11853
SP and Domenech, 2010.03976
NANOGrav, 2306.16219
Inomata, Kohri, Terada, 2306.17834

Crosscheck by PBH and IGW



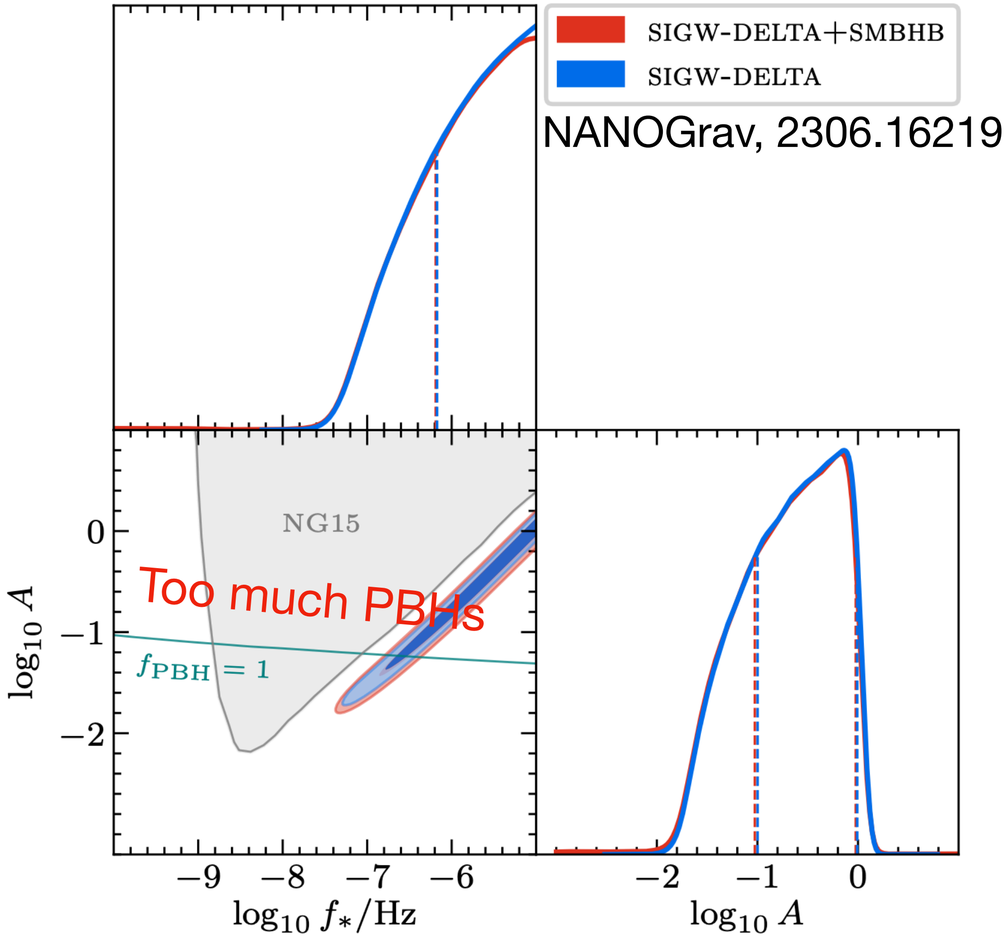




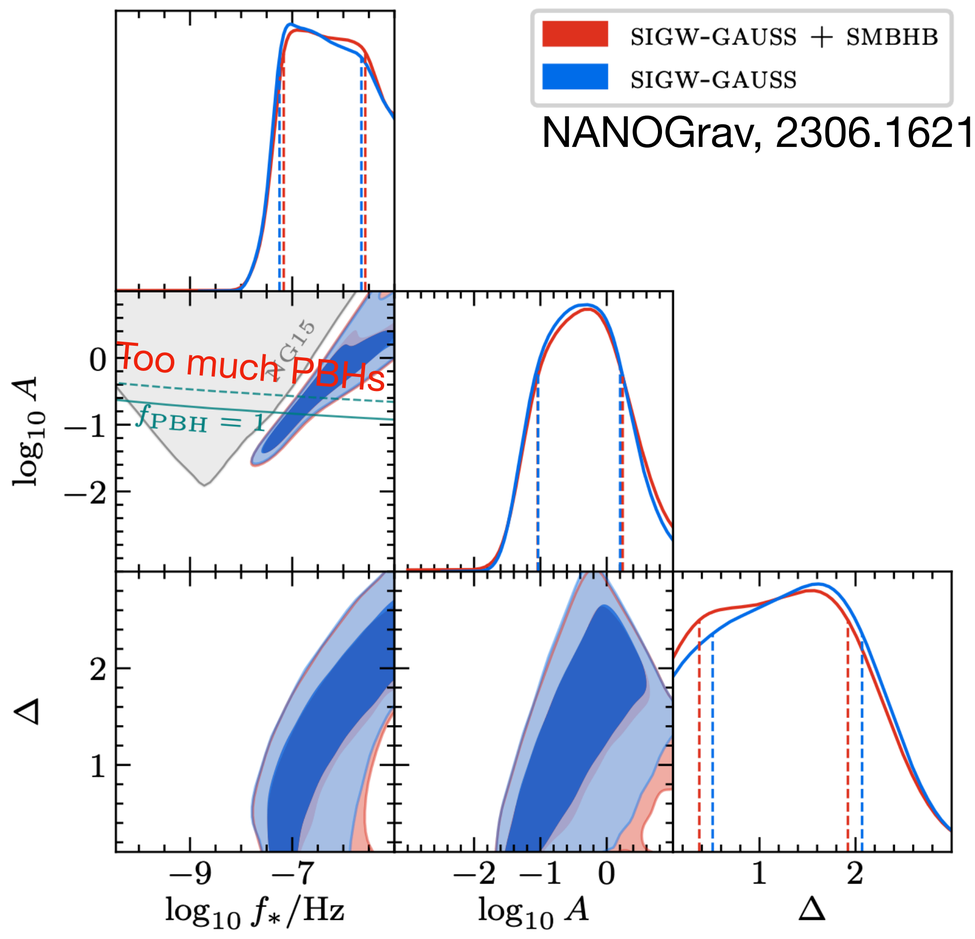


NANOGrav, 2306.16219
Inomata, Kohri, Terada, 2306.17834

IGW as nHz SGWB



NANOGrav, 2306.16219



NANOGrav, 2306.16219

$$\mathscr{P}_{\mathscr{R}} = A\delta(\ln k - \ln k_*)$$

monochromatic

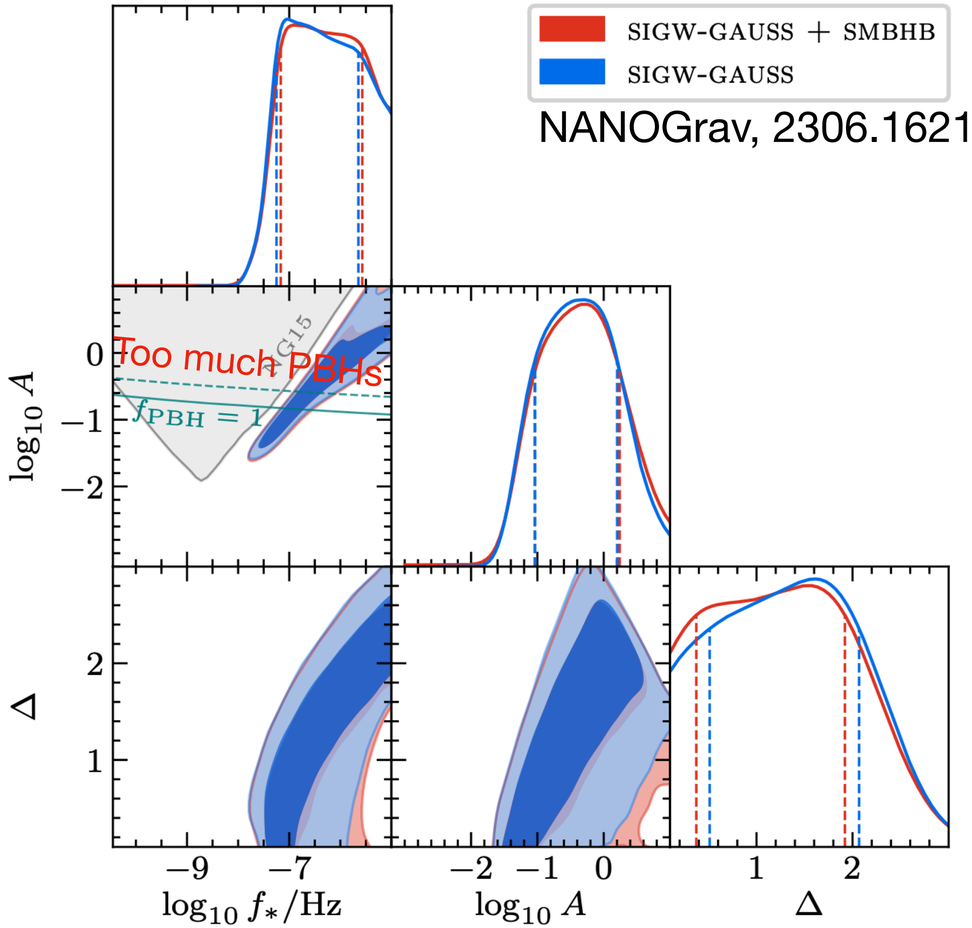
$$\mathscr{P}_{\mathscr{R}} = \frac{A}{\sqrt{2\pi}\Delta}\exp\left(-\frac{(\ln k - \ln k_*)^2}{2\Delta^2}\right)$$

lognormal [SP and Sasaki 2005.12306]

IGW as nHz SGWB

How to solve the PBH overproduction

- (1) Use more conservative method to calculate. (Inomata et al 2306.17834; Iovino et al 2406.20089)
- (2) Suppress PBH abundance by increase the threshold, usually by changing the equation-of-state. (Domenech and SP, 2010.03976; Domenech, SP, et al, 2402.18965)
- (3) Suppress PBH abundance by negative non-Gaussianity, where curvaton scenario is the only known model. (SP and Sasaki 2112.12680; Franciolini et al. 2306.17149)



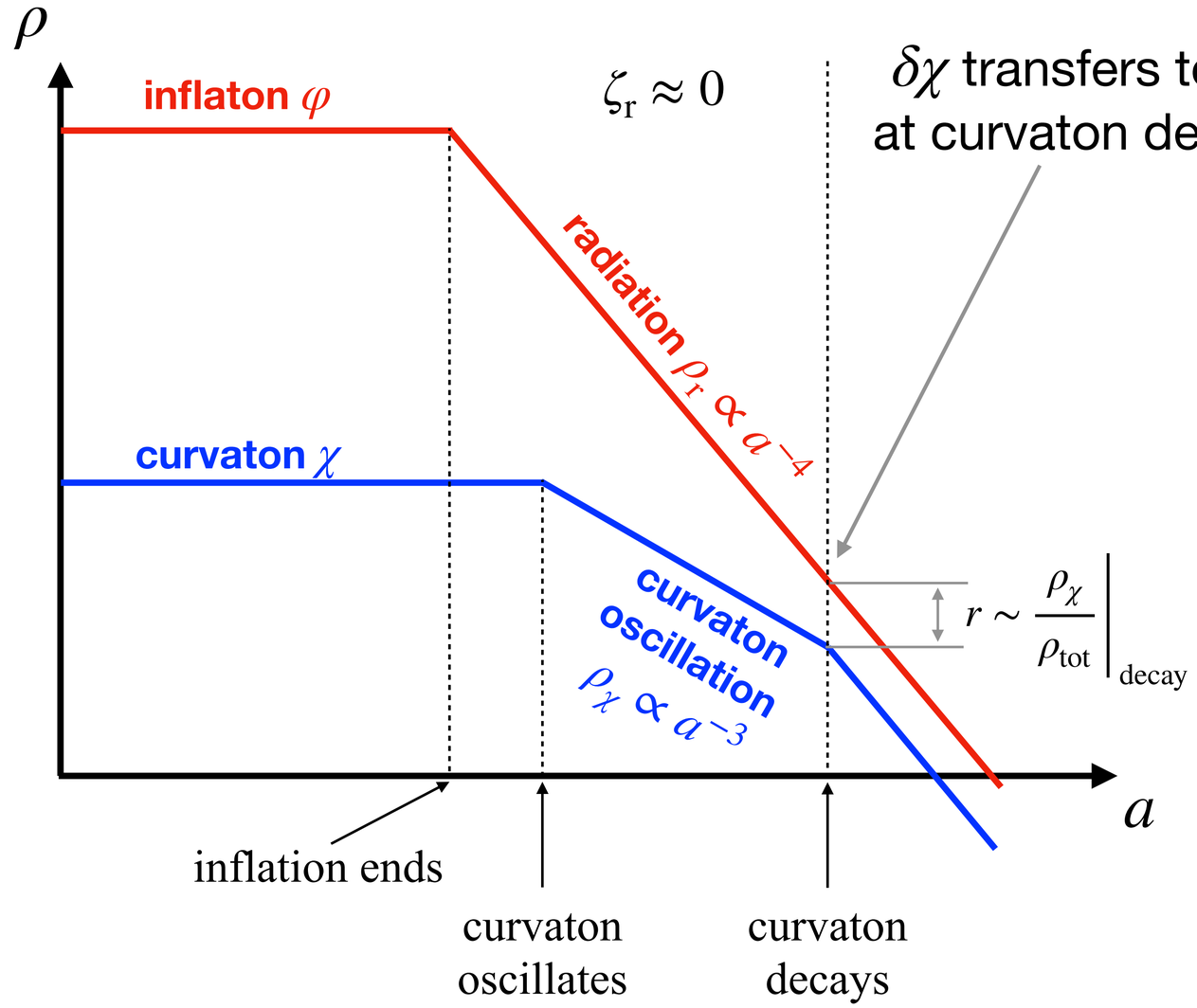
NANOGrav, 2306.16219

$$\mathscr{P}_{\mathscr{R}} = \frac{A}{\sqrt{2\pi}\Delta} \exp\left(-\frac{(\ln k - \ln k_*)^2}{2\Delta^2}\right)$$

lognormal [SP and Sasaki 2005.12306]

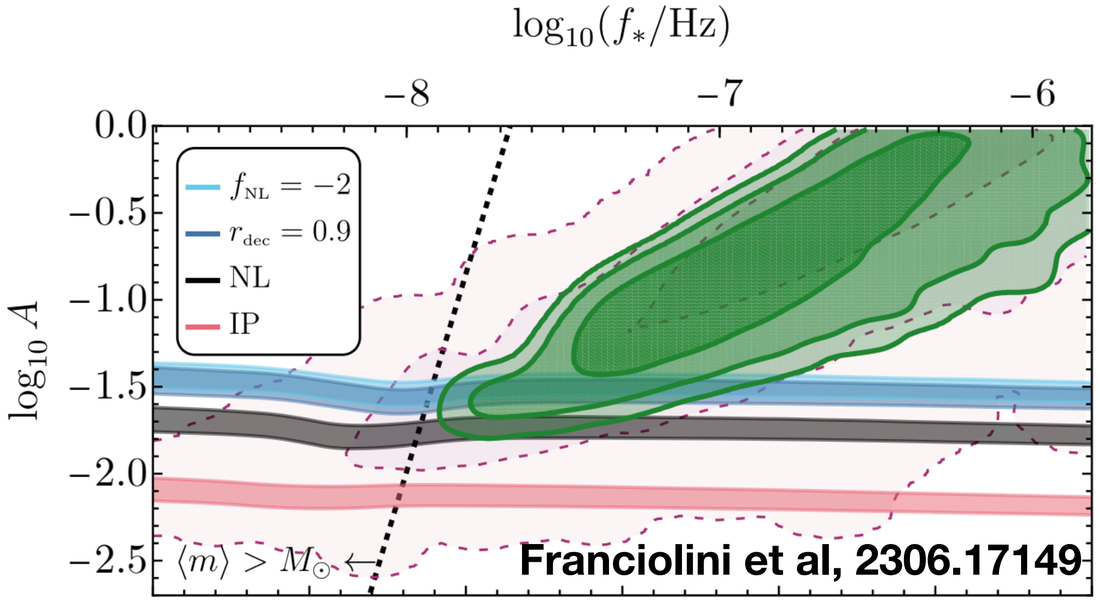
Curvaton and PBH

SP and Sasaki, 2112.12680
SP and Sasaki, 2211.13932
Ferrante et al, 2211.01728



$$\zeta = \zeta(\delta\chi/\chi) \longrightarrow \begin{cases} \frac{r}{3}\left[2\frac{\delta\chi}{\chi} + \left(\frac{\delta\chi}{\chi}\right)^2\right] & \text{when } r \ll 1 \\ \frac{2}{3}\ln\left|1+\frac{\delta\chi}{\chi}\right| & \text{when } r \sim 1 \end{cases}$$

- $\zeta(\delta\chi)$ degenerates to a logarithmic relation ($f_{\rm NL} = -5/4$) when the curvaton dominates.

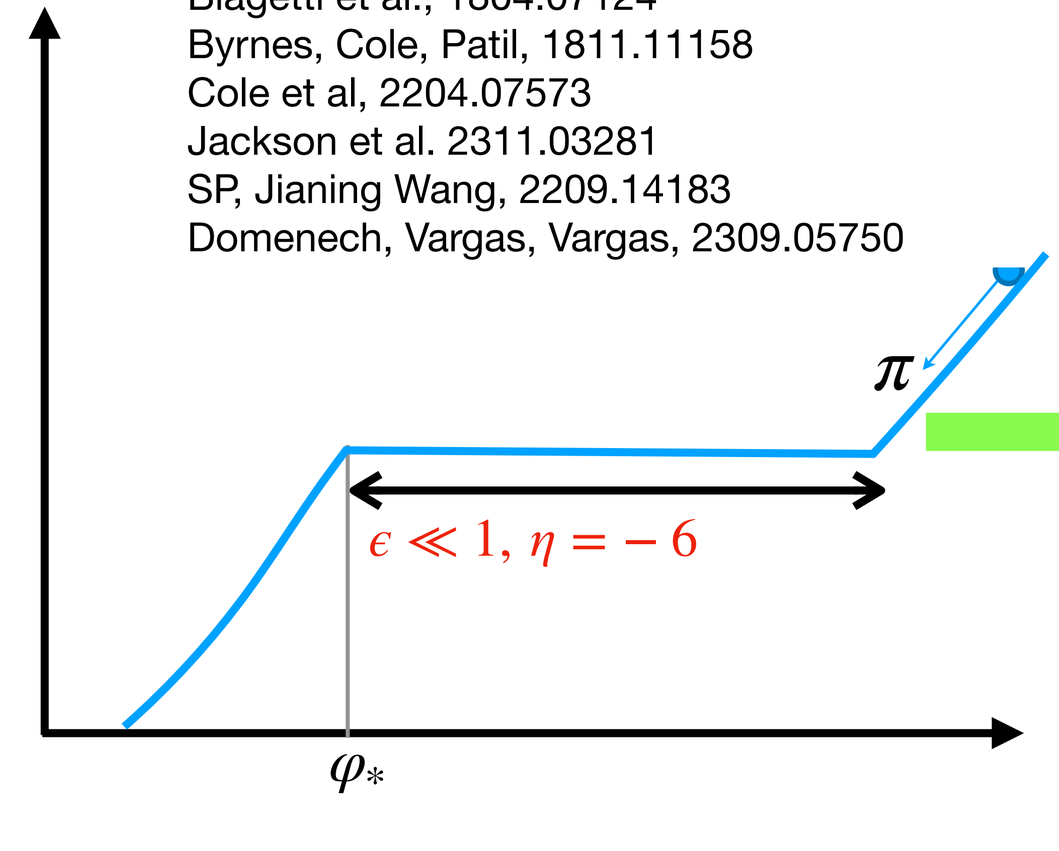


CONTENT

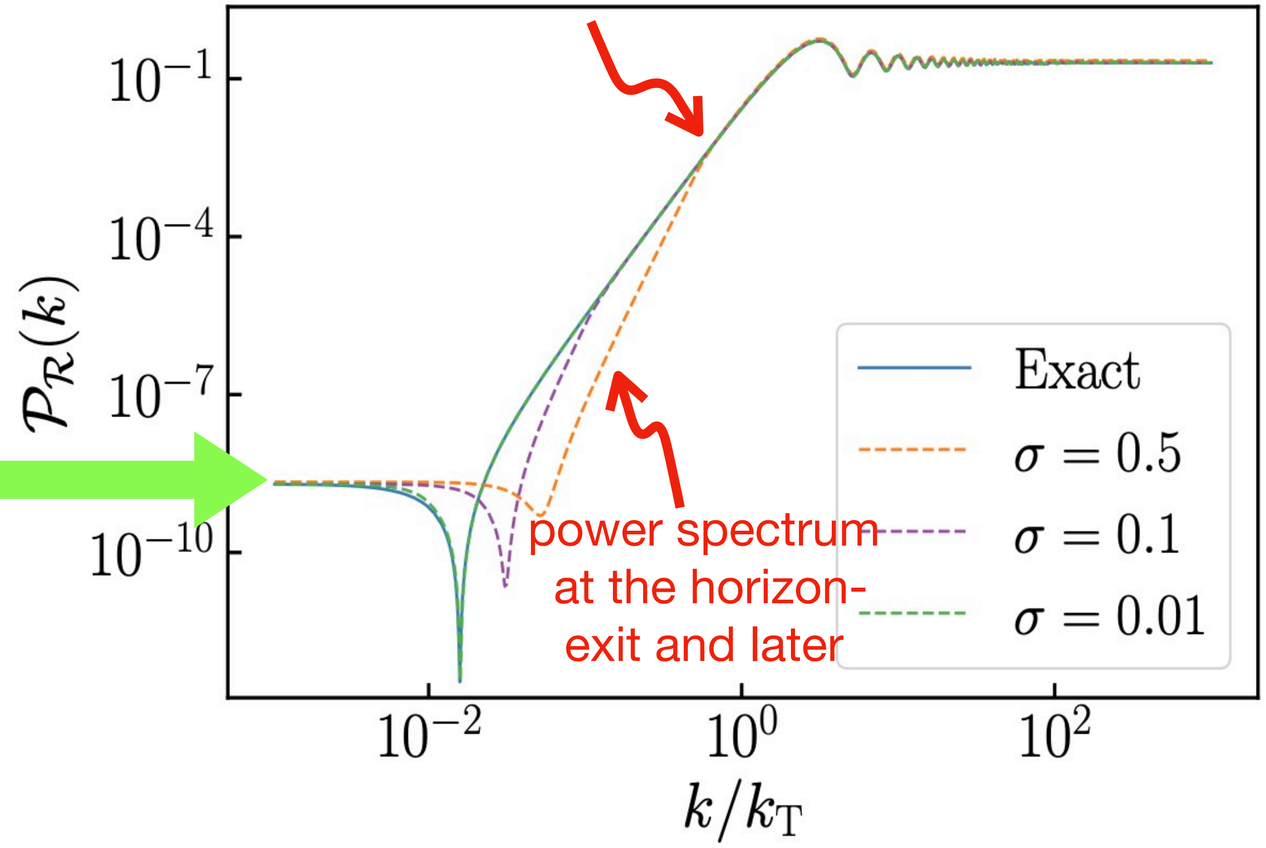
- Introduction: PBH and induced GW
- Nonlinear evolution of the curvature perturbation
- Application: gradient expansion and bubble relics
- Conclusion

Topic 1: Gradient Expansion

Starobinski, JETP Lett. 55, 489
Yokoyama, Inoe, hep-ph/0104083
Biagetti et al., 1804.07124
Byrnes, Cole, Patil, 1811.11158
Cole et al, 2204.07573
Jackson et al. 2311.03281
SP, Jianing Wang, 2209.14183
Domenech, Vargas, Vargas, 2309.05750



power spectrum
at the end of inflation



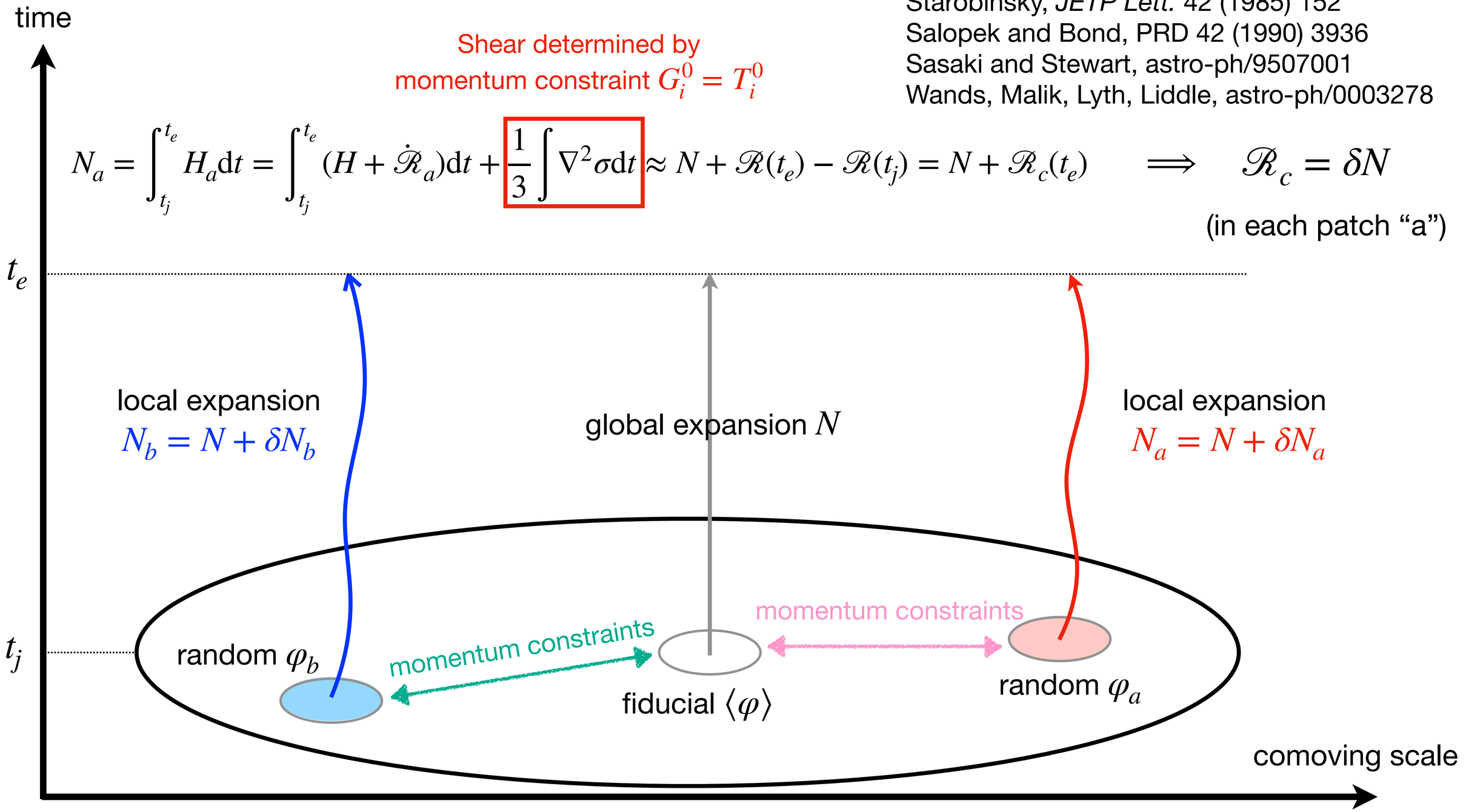
power spectrum
at the horizon-
exit and later

Starobinsky, *JETP Lett.* 42 (1985) 152
Salopek and Bond, PRD 42 (1990) 3936
Sasaki and Stewart, astro-ph/9507001
Wands, Malik, Lyth, Liddle, astro-ph/0003278

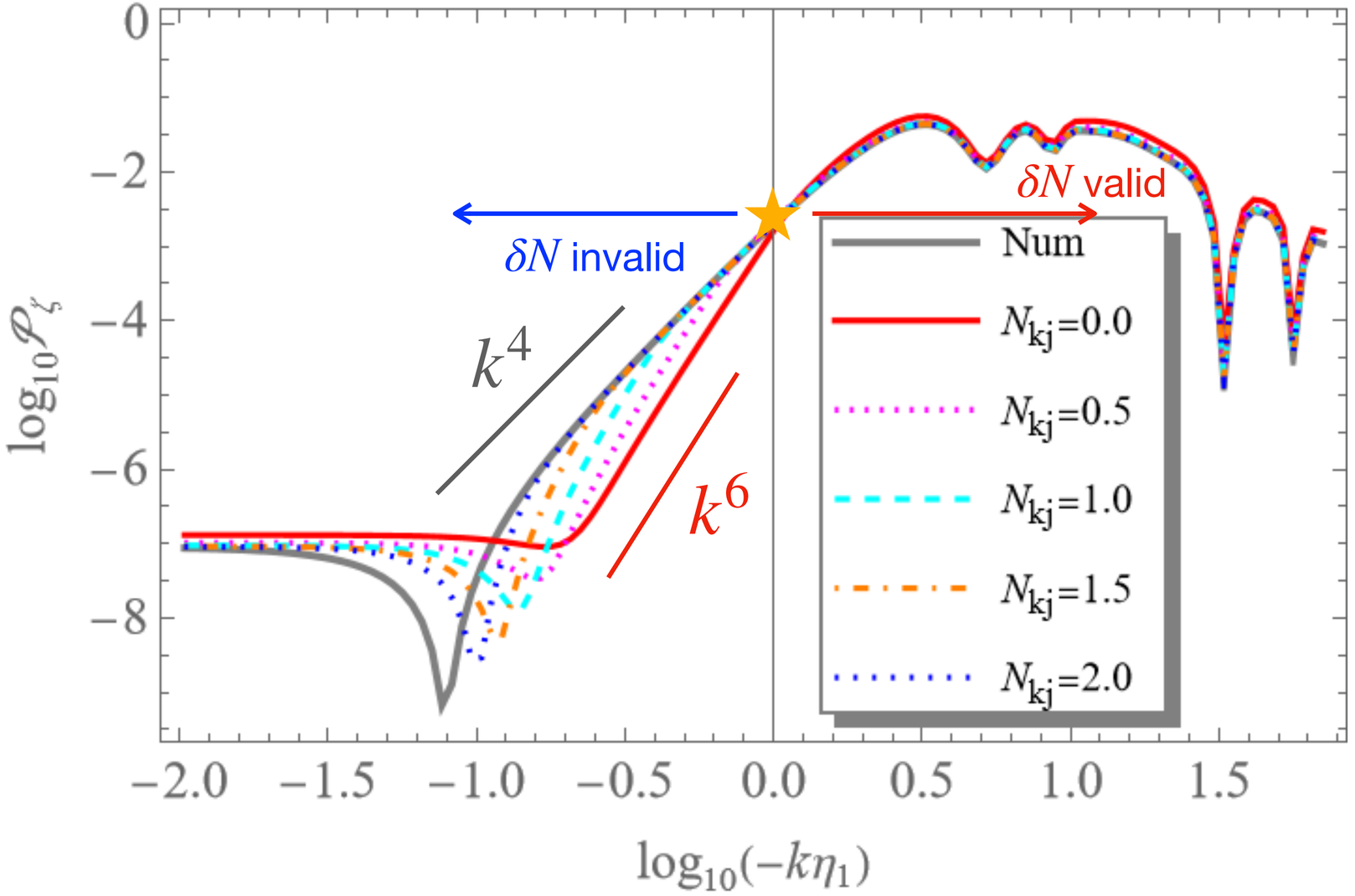
Shear determined by
momentum constraint $G_i^0 = T_i^0$

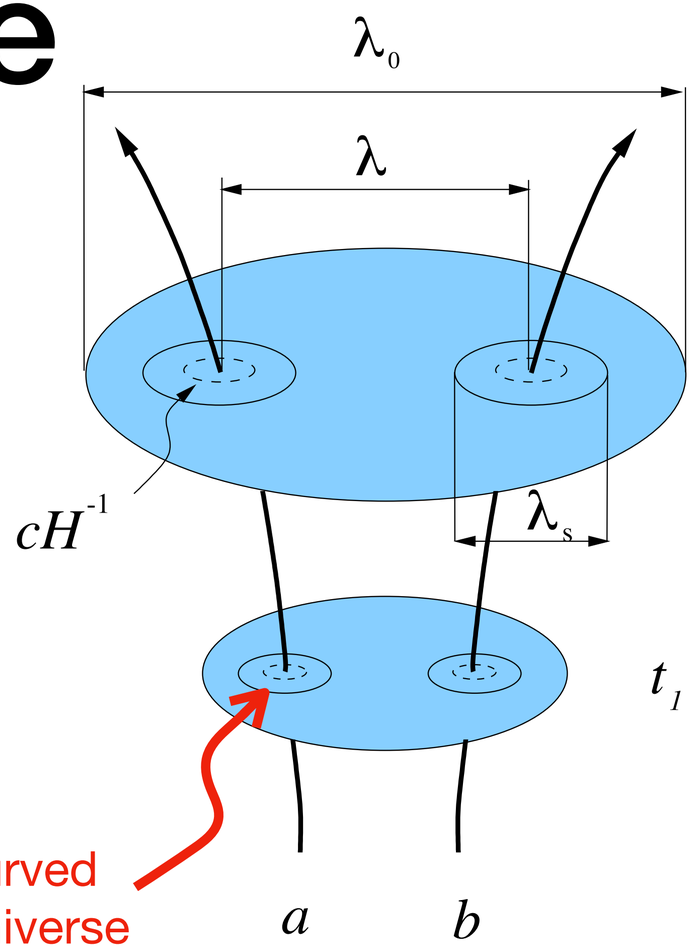
$$N_a = \int_{t_j}^{t_e} H_a \mathrm{d}t = \int_{t_j}^{t_e} (H + \dot{\mathscr{R}}_a)\mathrm{d}t + \boxed{\frac{1}{3}\int \nabla^2 \sigma \mathrm{d}t} \approx N + \mathscr{R}(t_e) - \mathscr{R}(t_j) = N + \mathscr{R}_c(t_e) \quad \Longrightarrow \quad \mathscr{R}_c = \delta N$$

(in each patch "a")

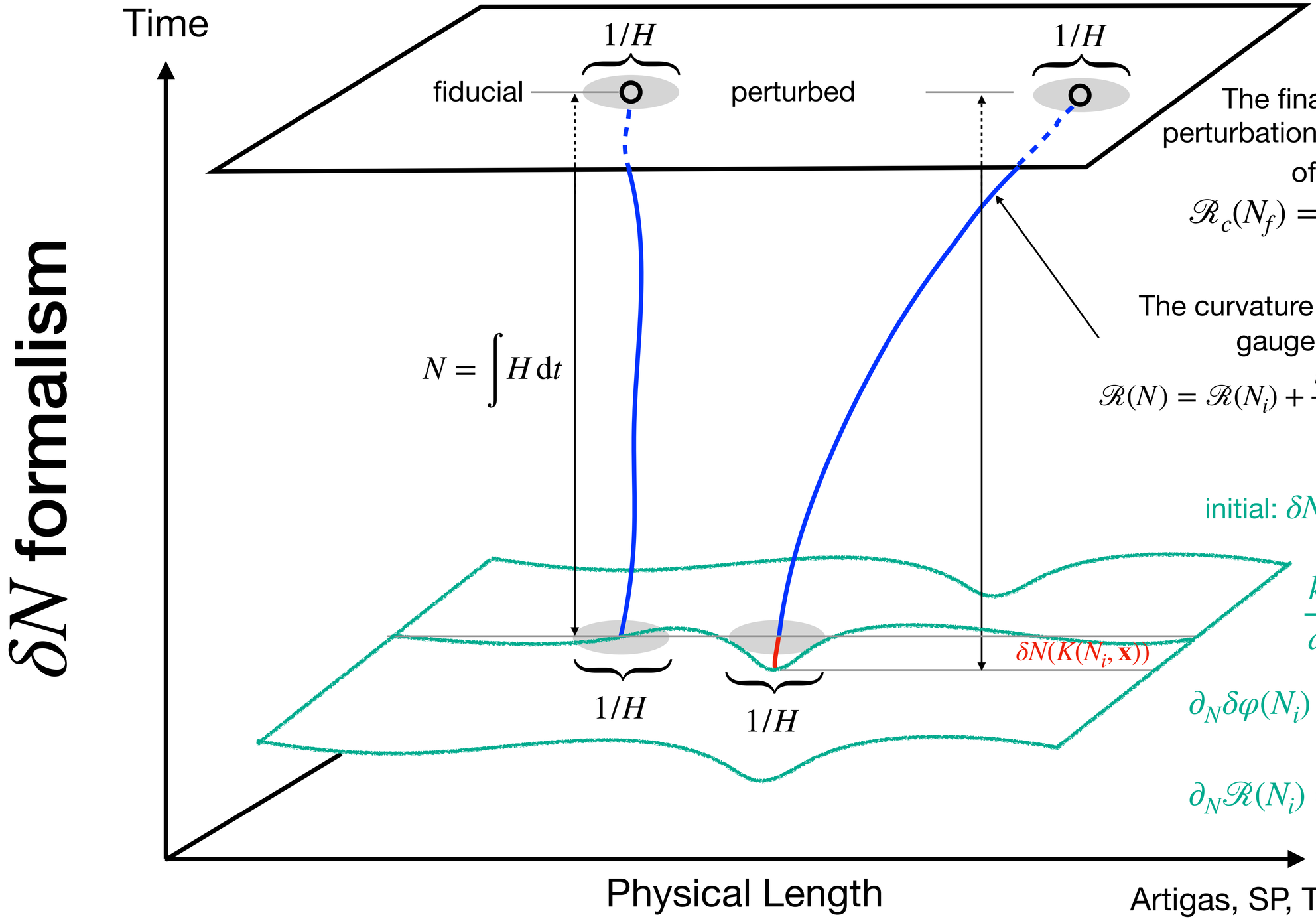


Separate Universe





$$\mathscr{R}(N_e) = \delta N(\delta\varphi(N_i))$$



The final comoving curvature perturbation is a gauge transformation of $\mathscr{R}$ in δN gauge

$$\mathscr{R}_c(N_f) = \mathscr{R}(N_f) + \delta N(K(N_i, \mathbf{x}))$$

The curvature perturbation in δN gauge evolves as

$$\mathscr{R}(N) = \mathscr{R}(N_i) + \frac{k^2 \mathscr{R}(N_i)}{6H^2}\left(e^{-2N_i} - e^{-2N}\right)$$

initial: δN gauge, comoving slice

$$\delta\varphi(N_i) = 0,$$

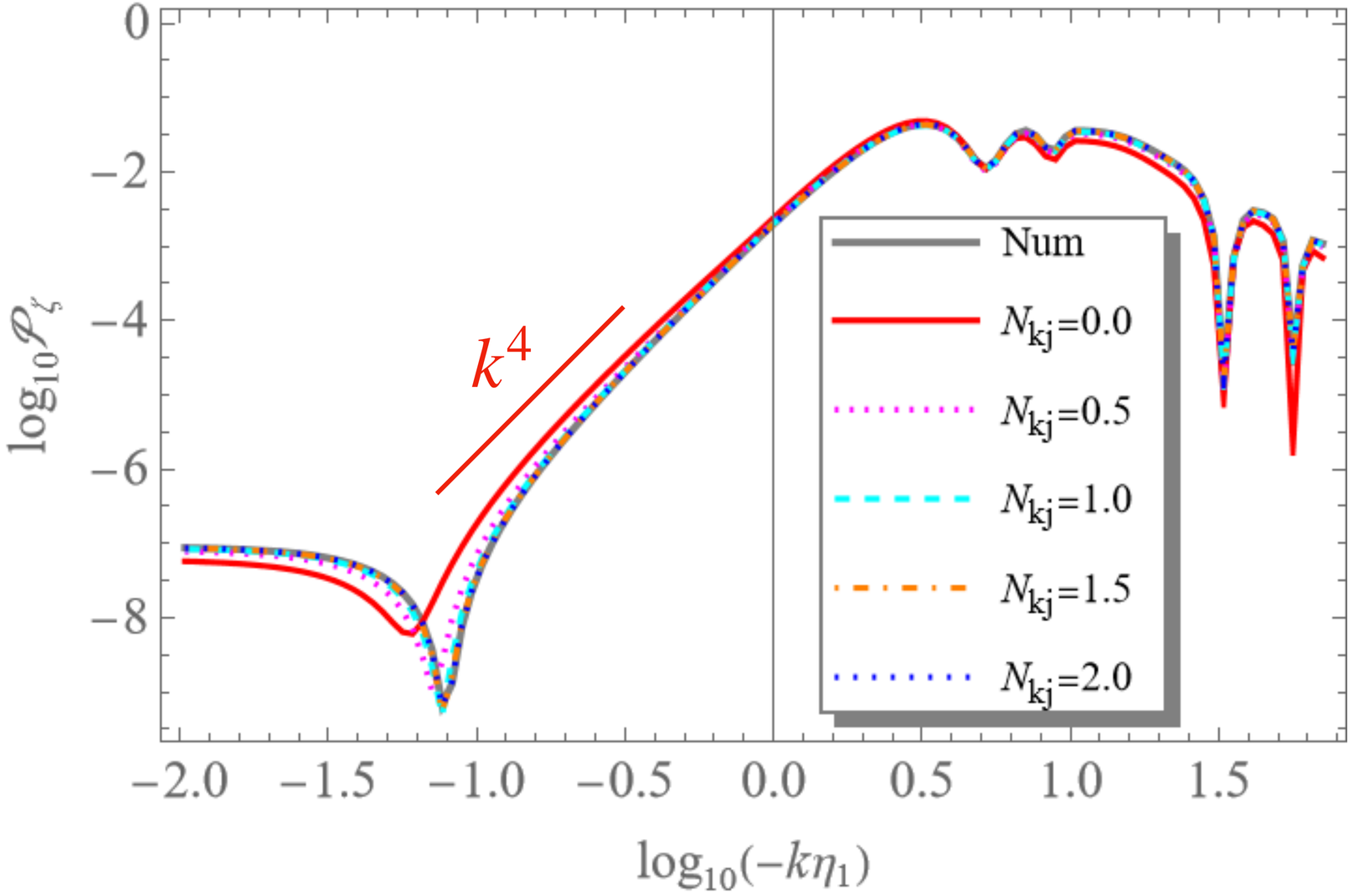
$$\frac{k^2}{a_i^2}\mathscr{R}(N_i) = \frac{3}{2}K$$

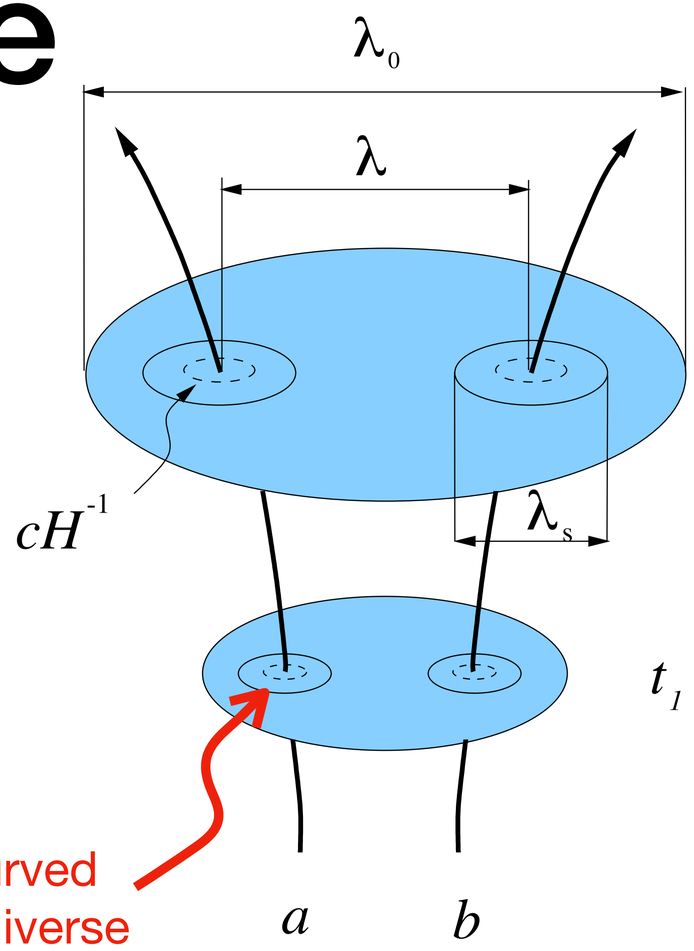
$$\partial_N \delta\varphi(N_i) = \frac{\varphi_N}{3}\frac{k^2}{a_i^2 H_0^2}\mathscr{R}(N_i) + \cdots$$

$$\partial_N \mathscr{R}(N_i) = \frac{k^2}{a_i^2 H_0^2}\mathscr{R}(N_i) + \mathcal{O}(\varphi_N)$$

Artigas, SP, Tanaka, 2408.09964

Separate Universe





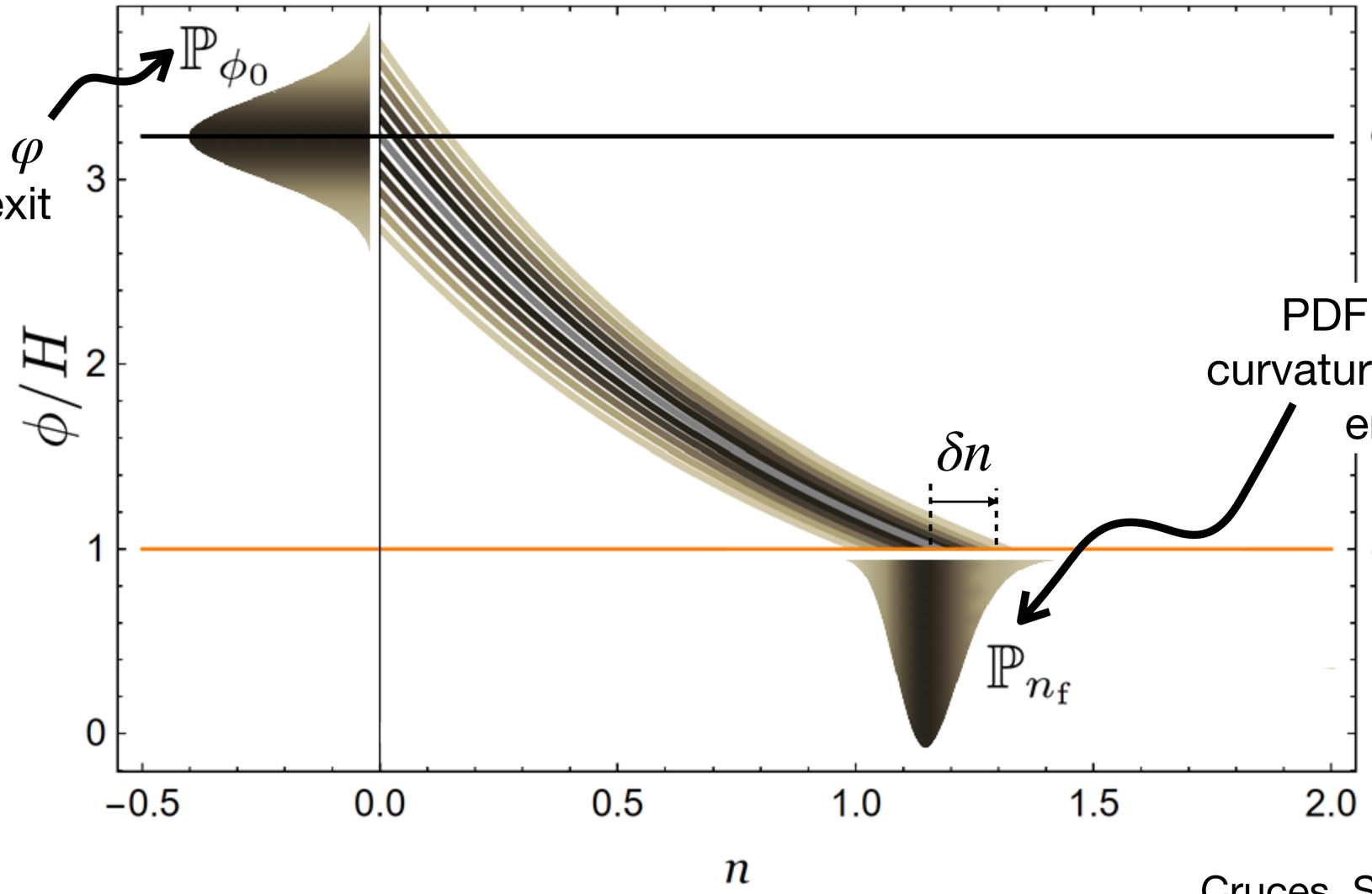
$$\mathscr{R}(N_e) = \mathscr{R}(N_i) + \delta N(k^2 \mathscr{R}(N_i))$$

(Note that $k^2\mathscr{R} \sim K$)

Artigas, SP, Tanaka, 2408.09964

Topic 2: Bubble relics

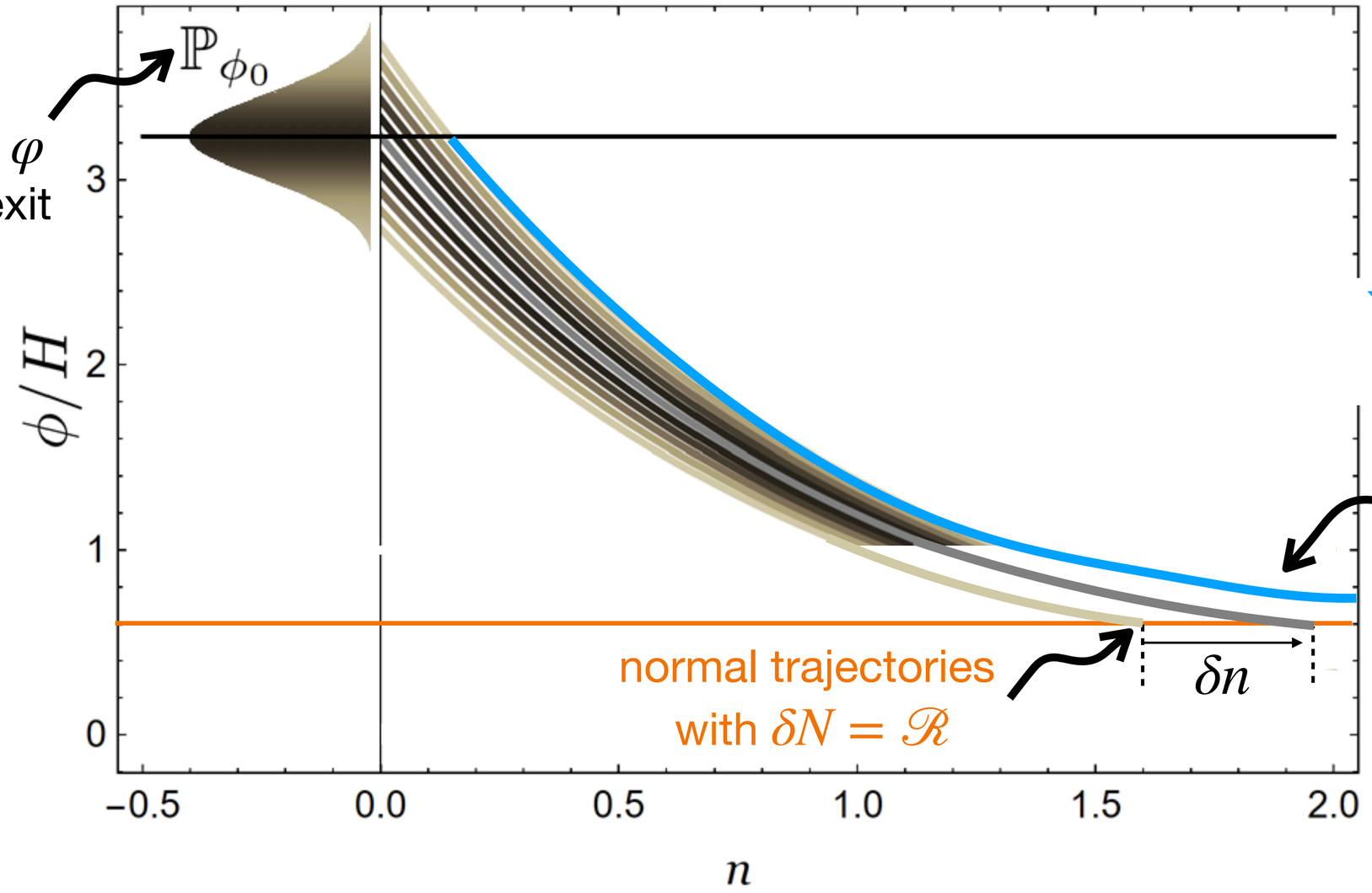
PDF of inflaton φ at the horizon exit



PDF of the comoving curvature perturbation at the end of inflation

Cruces, SP, Sasaki, 2505.24590

Topic 2: Bubble relics



PDF of inflaton φ
at the horizon exit

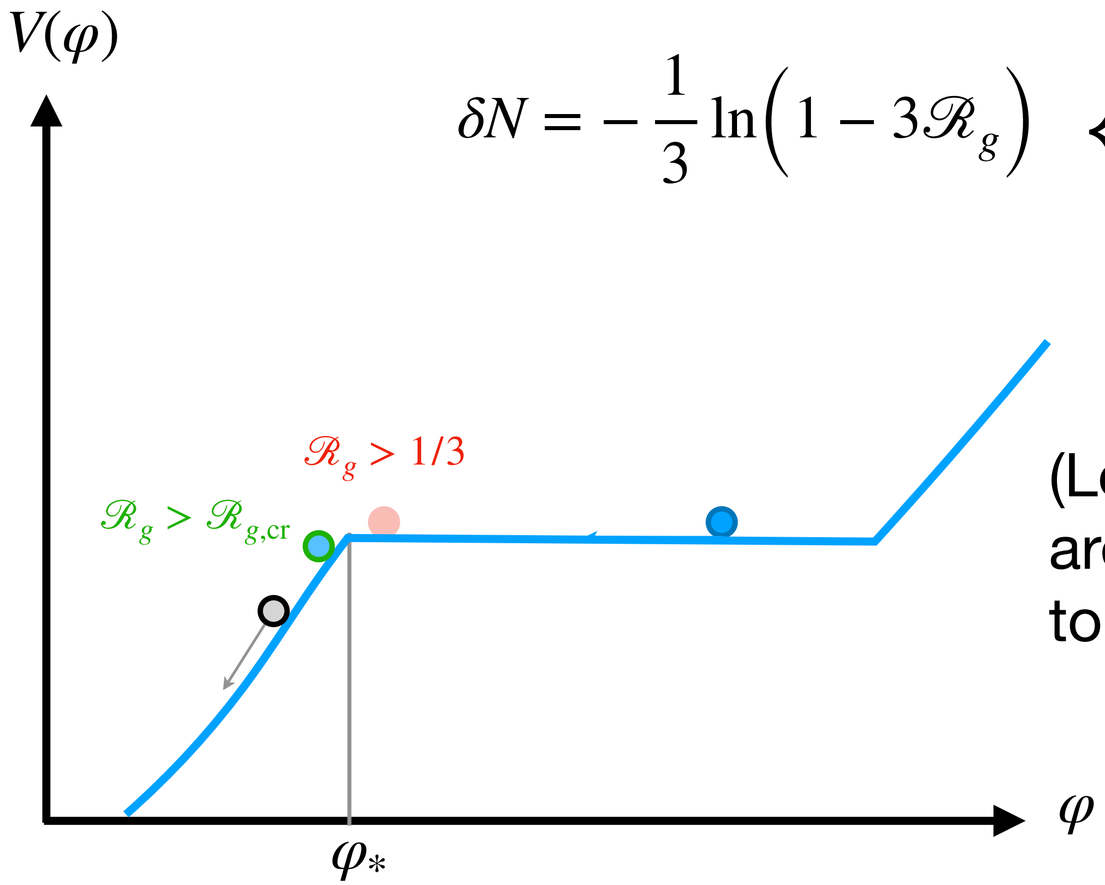
``stuck" trajectories
with $\delta N \to \infty$

Eternal inflation

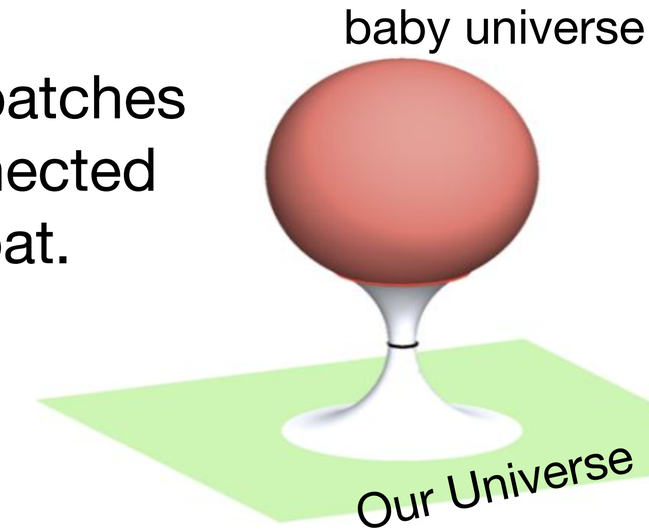
normal trajectories
with $\delta N = \mathscr{R}$

Ultra-slow-roll inflation

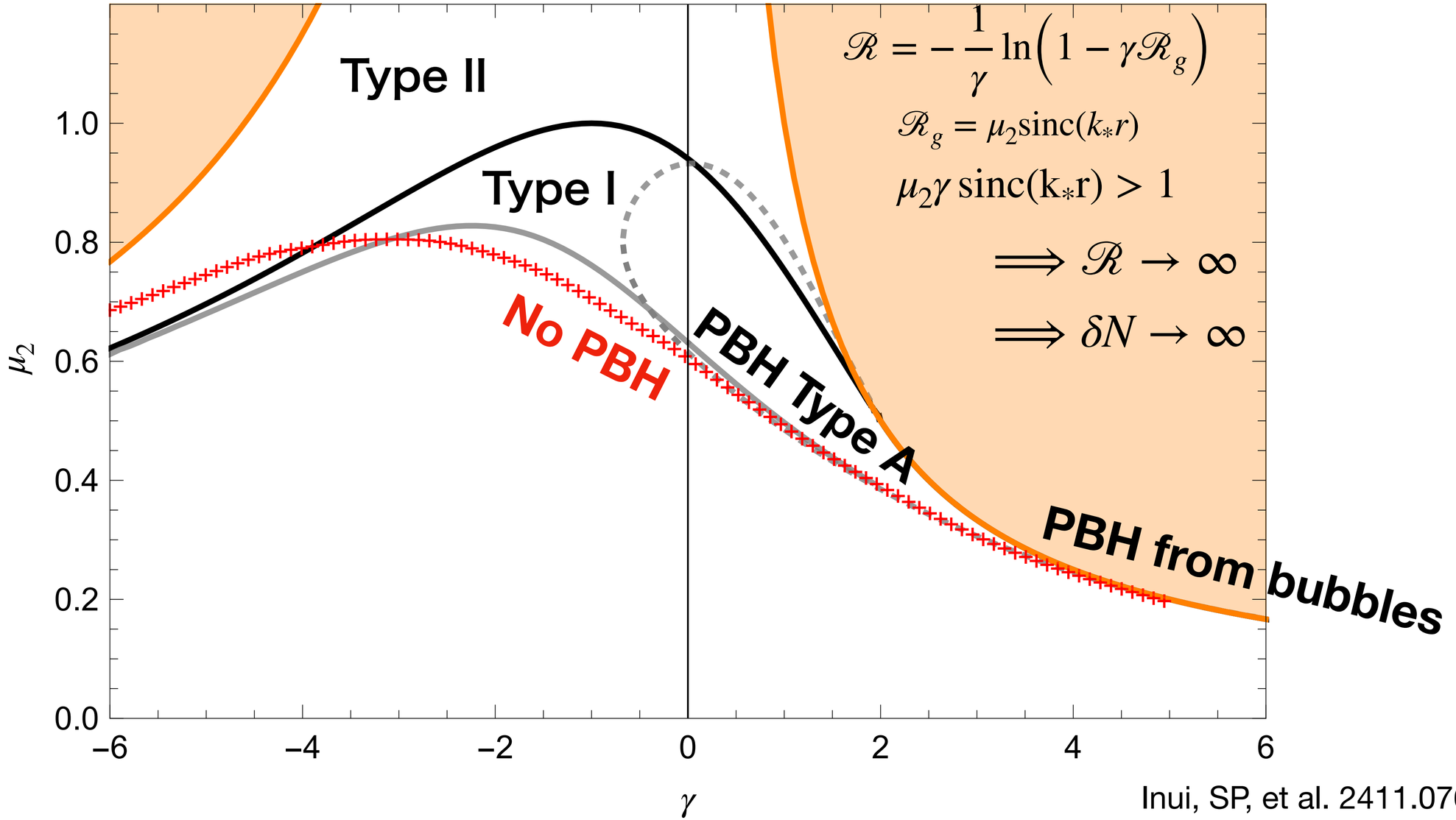
$$\delta N = -\frac{1}{3}\ln\left(1-3\mathscr{R}_g\right) \begin{cases} = \mathscr{R} & \text{with } \mathscr{R}_g \ll 1 \quad \text{vanilla universe} \\ = \text{PBH} & \text{with } \mathscr{R}_g > \mathscr{R}_{g,\text{th}} \quad \text{local closed universe} \\ \to \infty & \text{with } \mathscr{R}_g > 1/3 \quad \text{baby universe} \end{cases}$$



(Local) eternal inflating patches are baby universes connected to our universe by a throat.

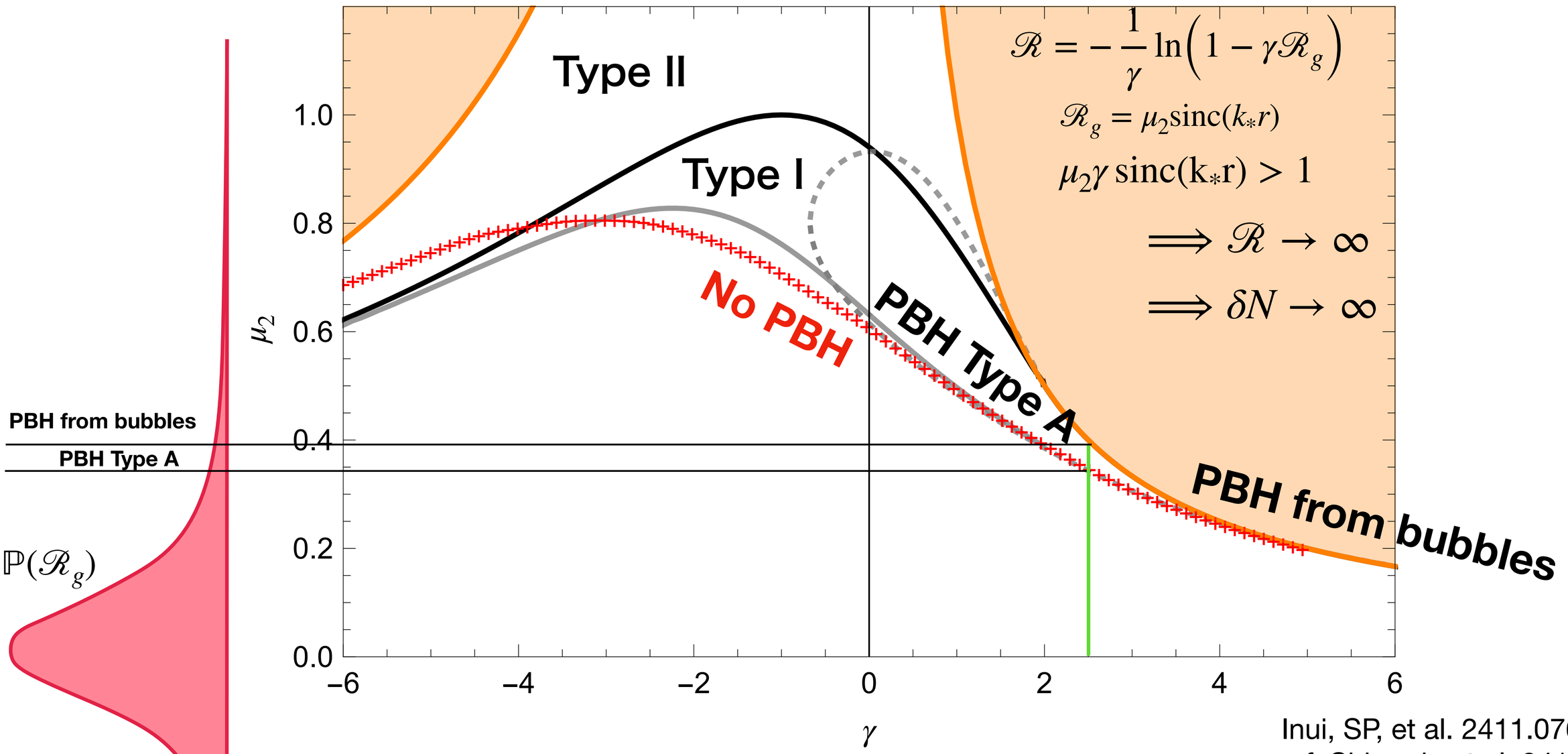


PBH in logarithmic $\mathscr{R}$



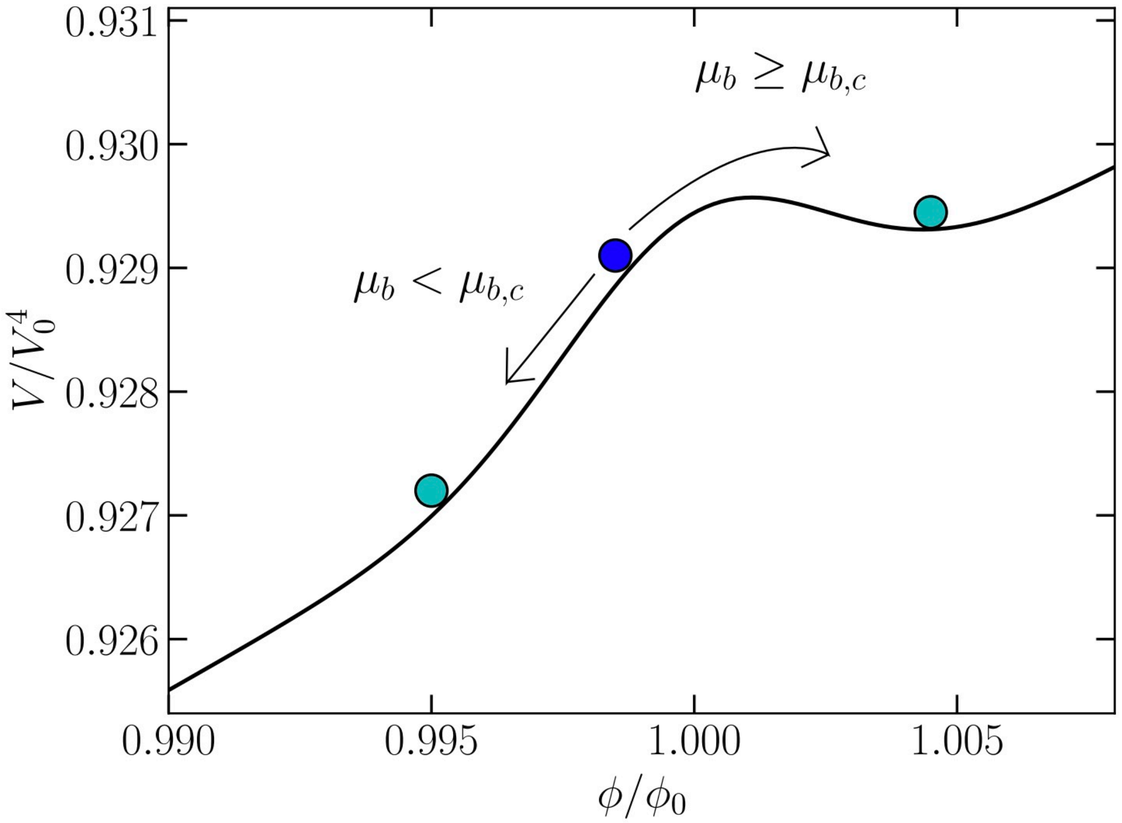
Inui, SP, et al. 2411.07647
c.f. Shimada et al. 2411.07648

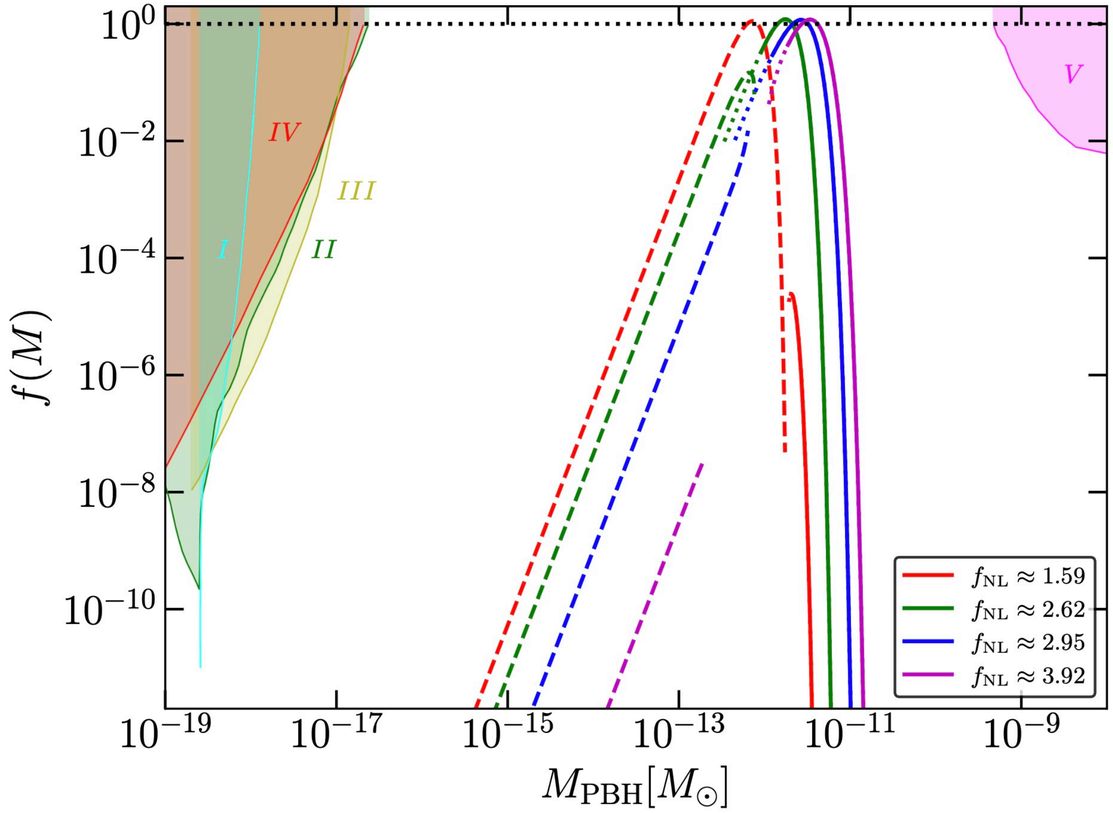
PBH in logarithmic $\mathscr{R}$



Inui, SP, et al. 2411.07647
c.f. Shimada et al. 2411.07648

Constant-roll inflation and bubble channel



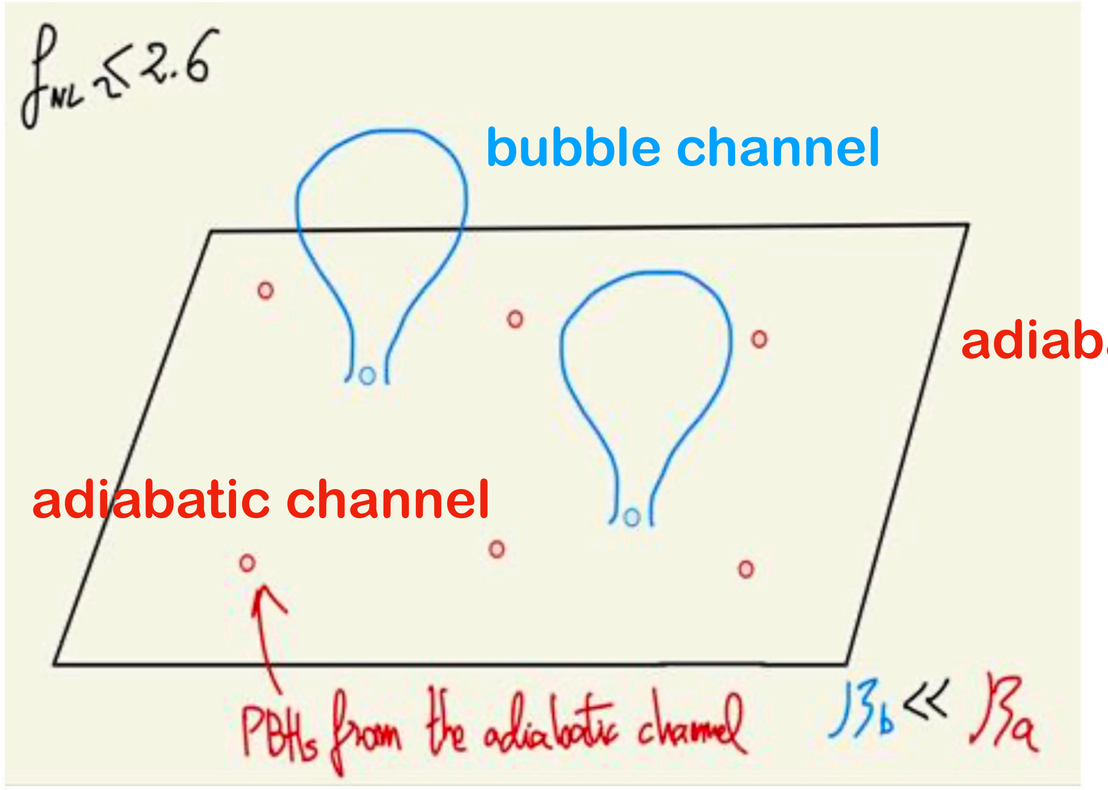


Escriva, Atal, Garriga, 2306.09990

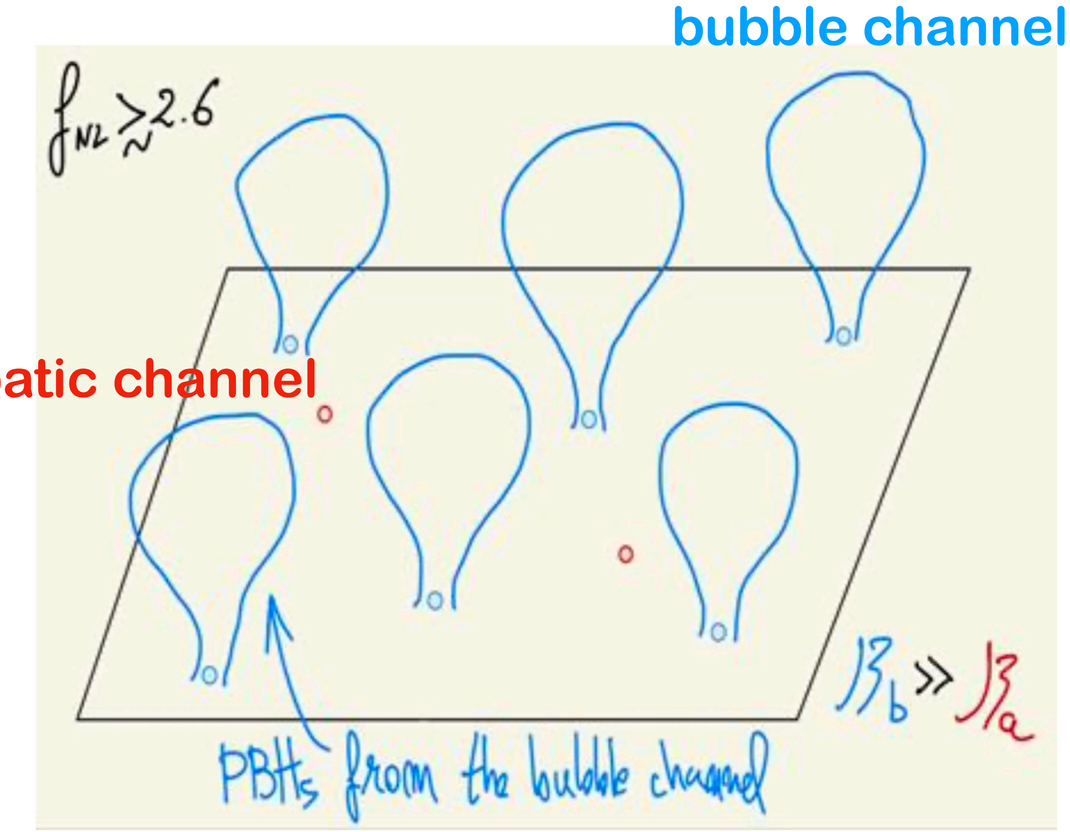
From Jaume Garriga's slide at Paris IHT

Qualitative picture

Fluctuations type I-> dominant contribution

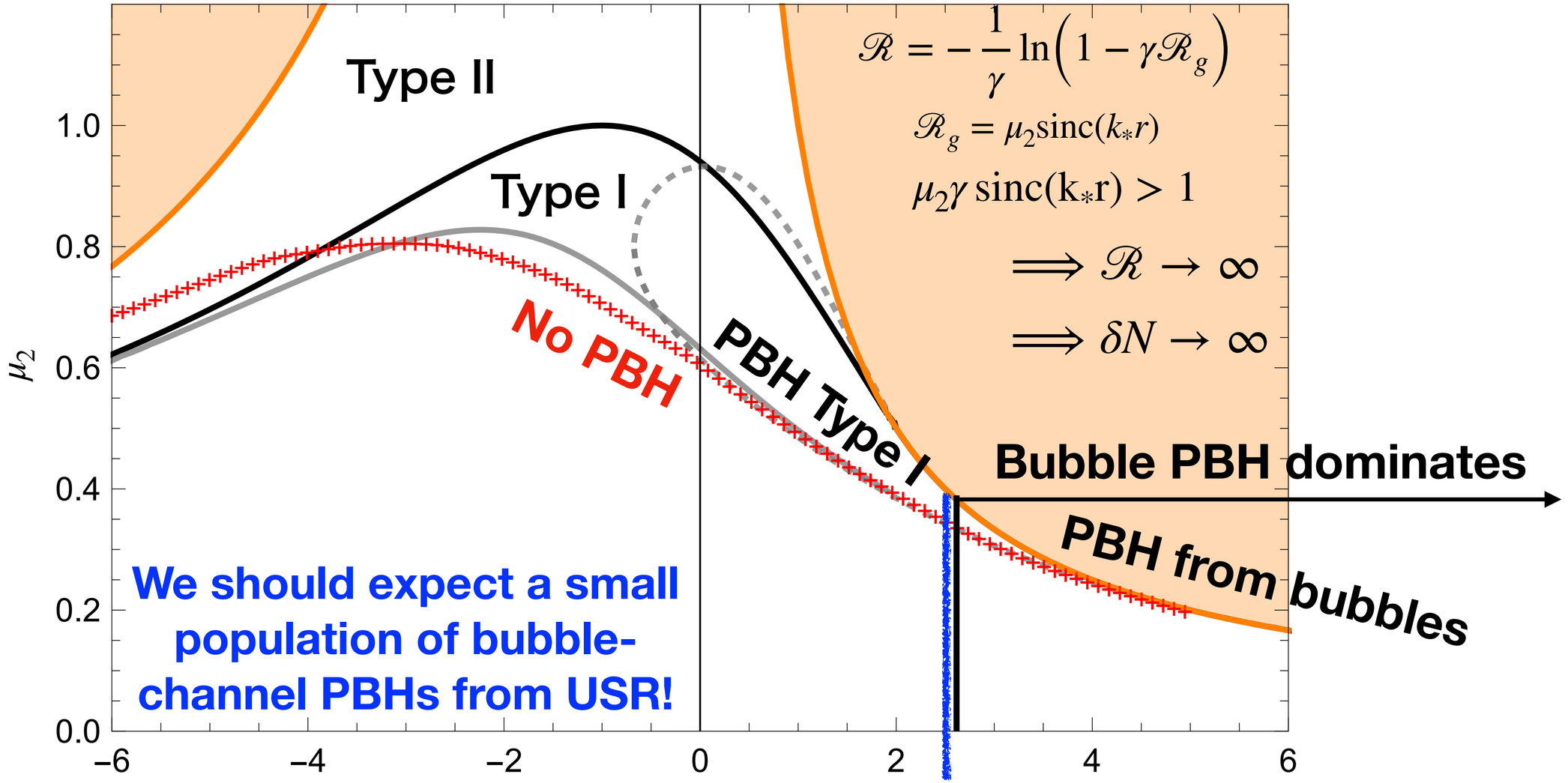


Fluctuations type II-> dominant contribution



Escriva, Atal, Garriga, 2306.09990

PBH in logarithmic $\mathscr{R}$



USR $f_{\rm NL} = 5/2$

Inui, SP, et al. 2411.07647
c.f. Shimada et al. 2411.07648

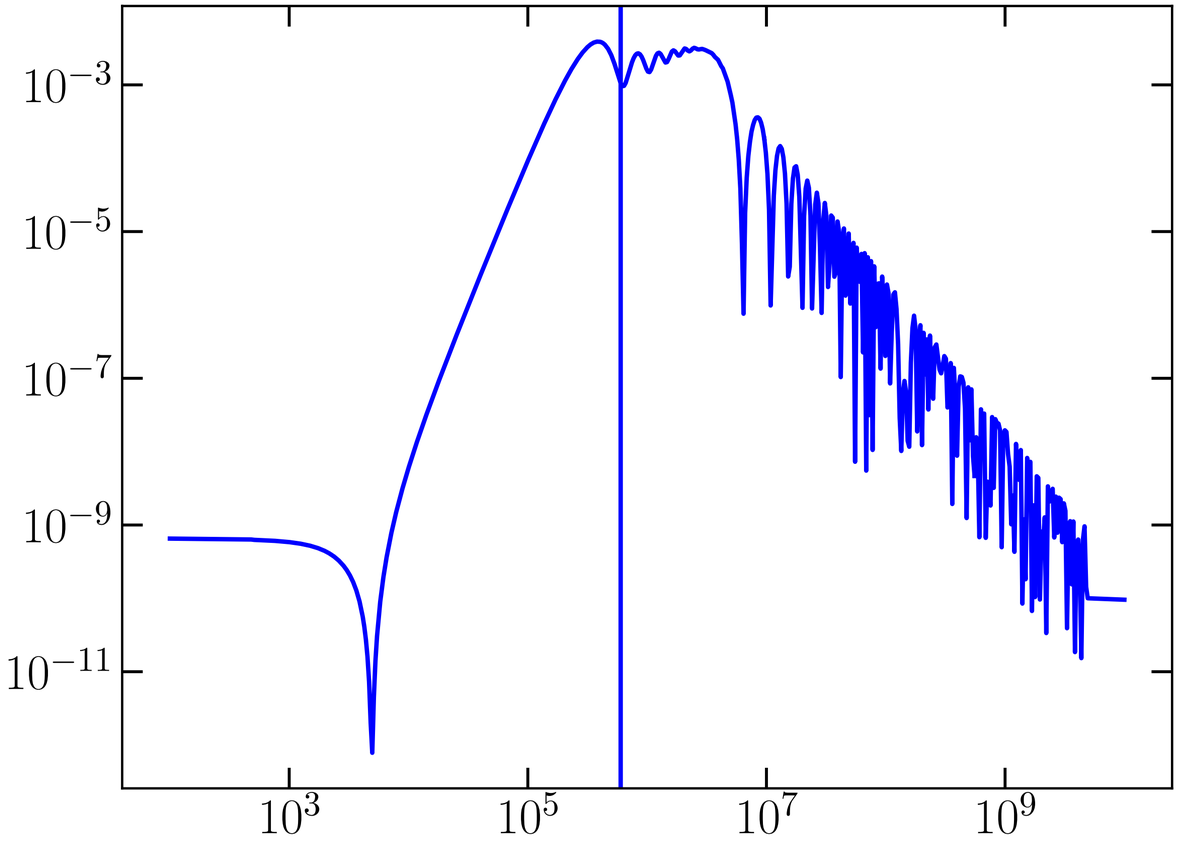
Astrophysics > Cosmology and Nongalactic Astrophysics

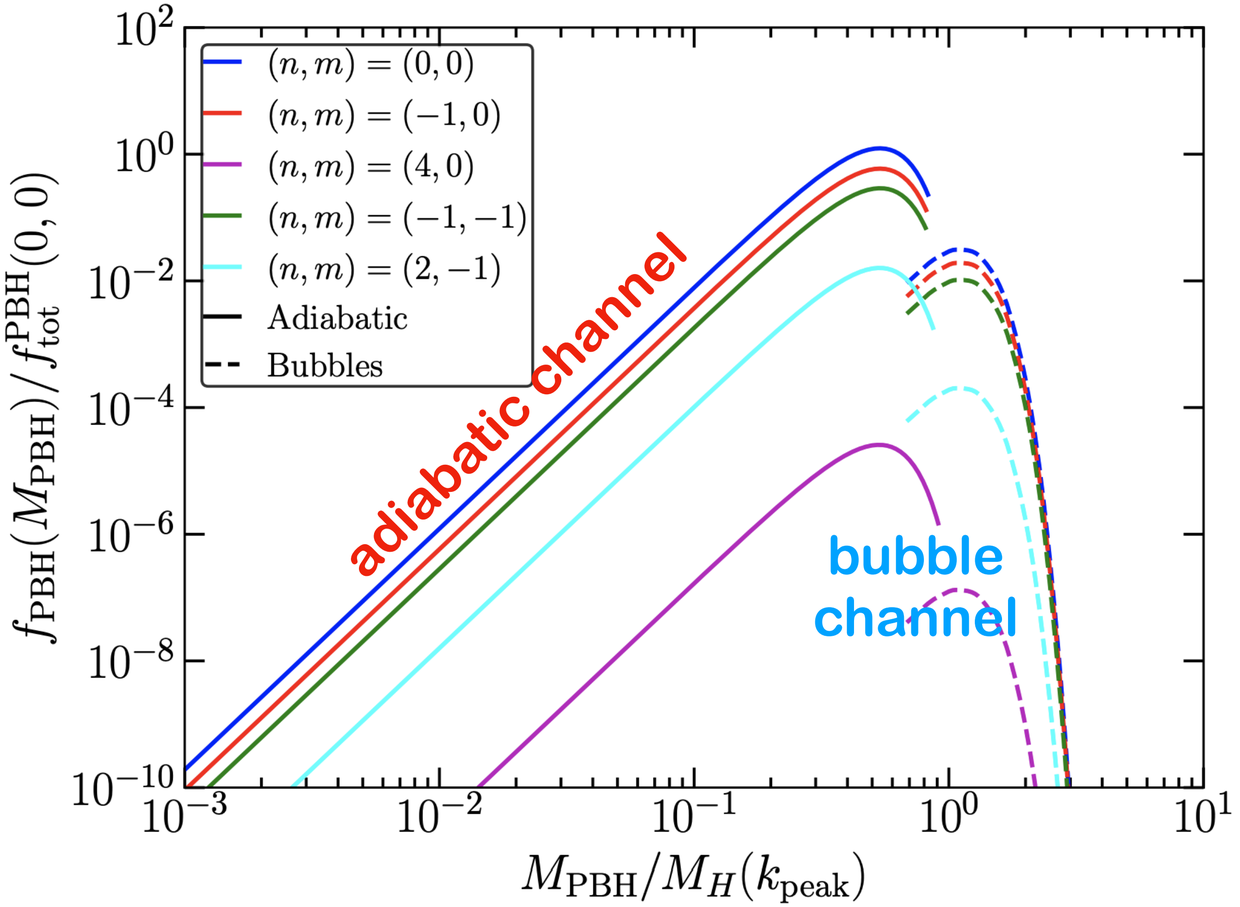
[Submitted on 4 Dec 2025]

Escriva, Garriga, SP, 2512.04986

Inflationary relics from an Ultra-Slow-Roll plateau

Albert Escrivà, Jaume Garriga, Shi Pi





Astrophysics > Cosmology and Nongalactic Astrophysics

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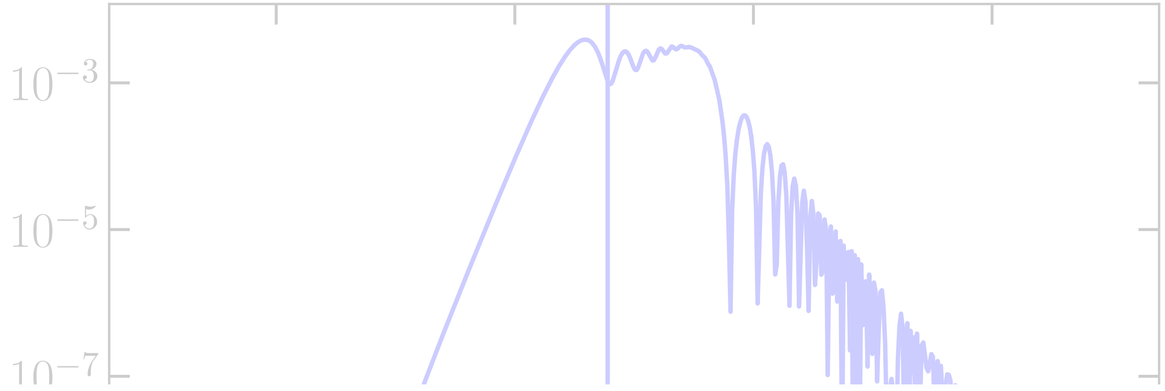
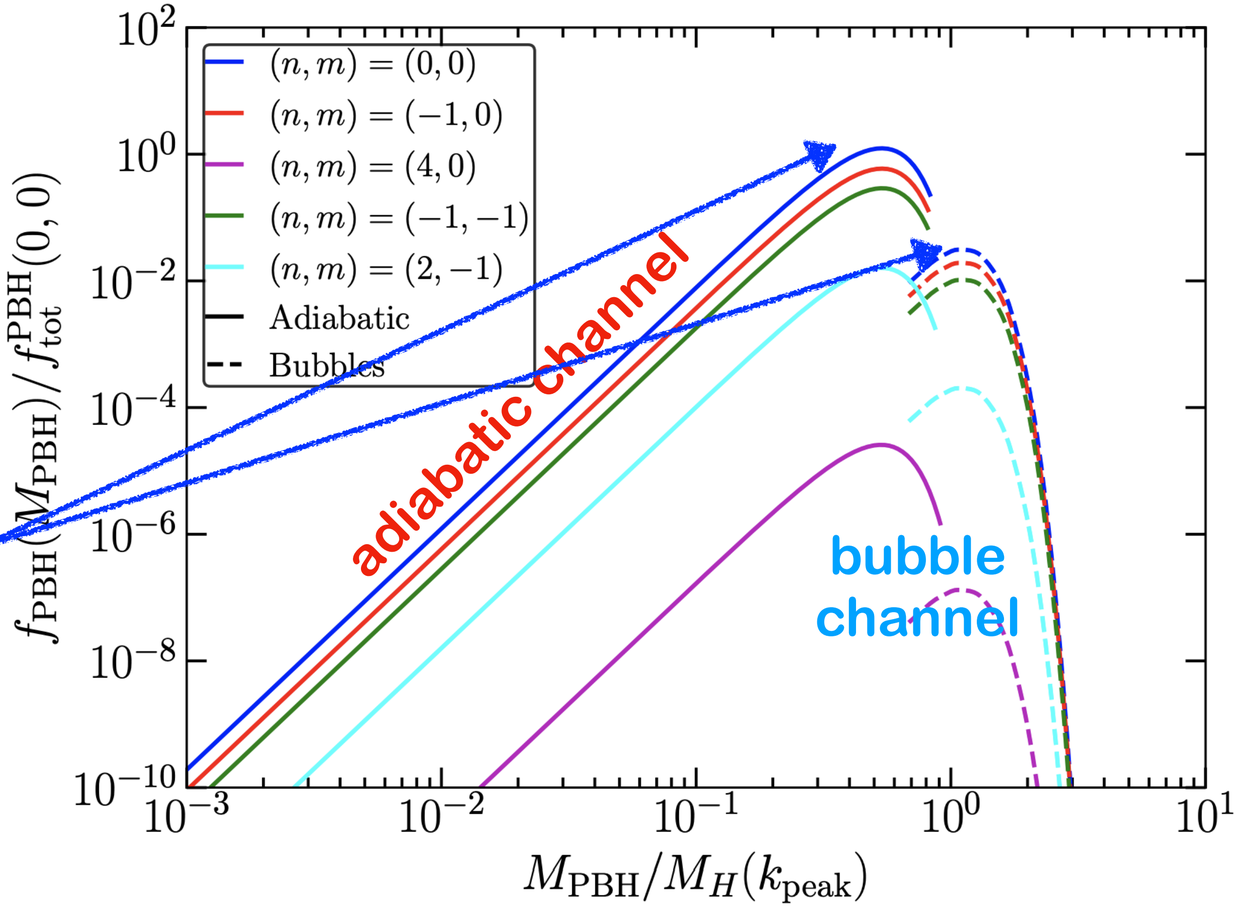


n	m	$f^{\rm tot}_{\rm PBH}(n,m)/f^{\rm tot}_{\rm PBH}(0,0)$	$f^{\rm tot}_{\rm a}(n,m)/f^{\rm tot}_{\rm b}(n,m)$
0.0	0.0	1.0	51.68
−1.0	0.0	4.78×10^{-1}	39.71
−1.0	−1.0	2.35×10^{-1}	36.05
2.0	−1.0	1.29×10^{-2}	104.06
4.0	0.0	2.08×10^{-5}	260.07



Conclusion

- On super-horizon scales, the nonlinear evolution of the curvature perturbation can be well described by δN formalism, which gives a sum of logarithmic functions of $\mathscr{R}(\delta\varphi, \dot{\delta\varphi})$.

- The extended δN considers spatial curvature $\mathscr{K}$, which takes into account the NLO term in gradient expansion.

- Ultra-slow-roll inflation inevitably generates Type-B PBHs formed by the bubble channel, which account for a few percent of all PBHs.