RTA-Resummed Stochastic Diffusion Beyond White Noise: Markovization and W_n Cumulant Dynamics

Navid Abbasi

With **Mei Huang, Dirk Rischke, Pei Zhang**

首届喷注与重夸克物理研讨会 (The 1st Workshop on Jet and Heavy Quark Physics, JAQ 2026)

1. QCD CEP

- Landau-Ginzburg picture in thermal equilibrium

$$\Omega = \int d^3x \left[\frac{1}{2}(\nabla\sigma)^2 + \frac{m_\sigma^2}{2}\sigma^2 + \frac{\lambda_3}{3}\sigma^3 + \frac{\lambda_4}{4}\sigma^4 + \dots\right].$$

$$\xi = m_\sigma^{-1}.$$

$$\boxed{\begin{aligned} &\kappa_2 = \langle\sigma_0^2\rangle = \frac{T}{V}\,\xi^2\,; \qquad \kappa_3 = \langle\sigma_0^3\rangle = \frac{2\lambda_3 T}{V}\,\xi^6\,; \\ &\kappa_4 = \langle\sigma_0^4\rangle_c \equiv \langle\sigma_0^4\rangle - \langle\sigma_0^2\rangle^2 = \frac{6T}{V}[2(\lambda_3\xi)^2 - \lambda_4]\,\xi^8\,. \end{aligned}}$$

[Stephanov PRL (2009)]

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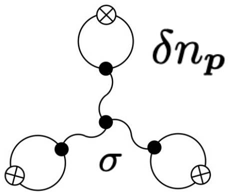
- Contribution to particle correlators

$$C_3 = \langle \delta n_{\boldsymbol{p}_1} \delta n_{\boldsymbol{p}_2} \delta n_{\boldsymbol{p}_3} \rangle_\sigma$$

$$\sim \kappa_3\, G_{\sigma\delta n}(\boldsymbol{p}_1) G_{\sigma\delta n}(\boldsymbol{p}_2) G_{\sigma\delta n}(\boldsymbol{p}_3)$$



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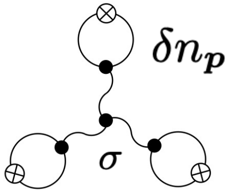


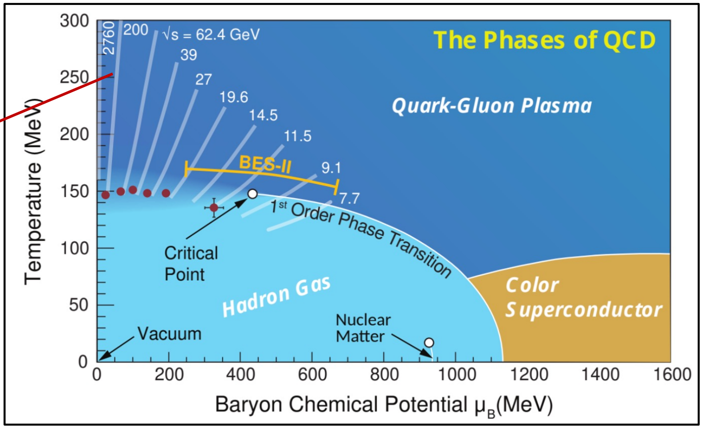
Fluctuations of the order parameter (baryon density fluctuations)

The fluctuations of the net proton number (cumulants)

2. CEP from experiment

QGP droplet

trajectory



[BEST, Nucl.Phys.A (2022)]

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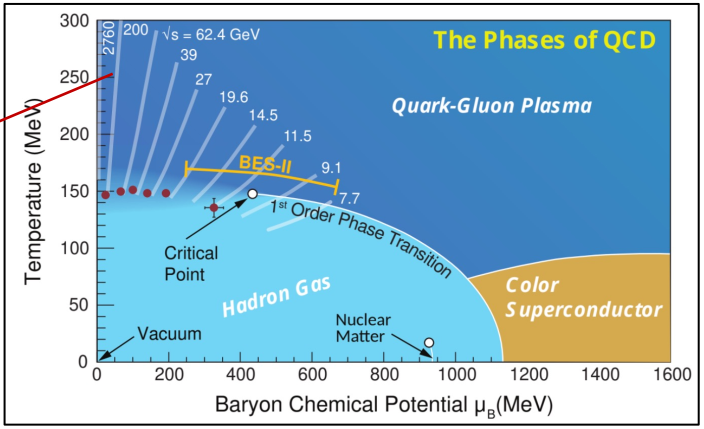

QGP droplet trajectory

- Experiment measures net-proton cumulants:

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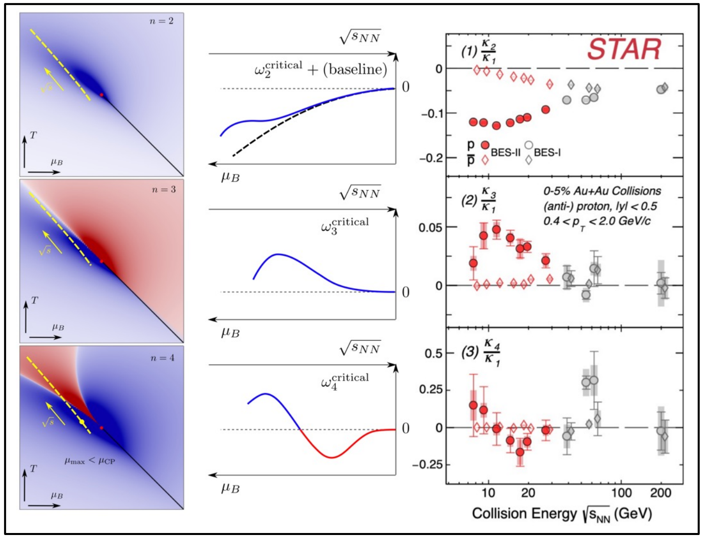

$$\omega_n \equiv \kappa_n / \kappa_1$$

[Stephanov 2024]

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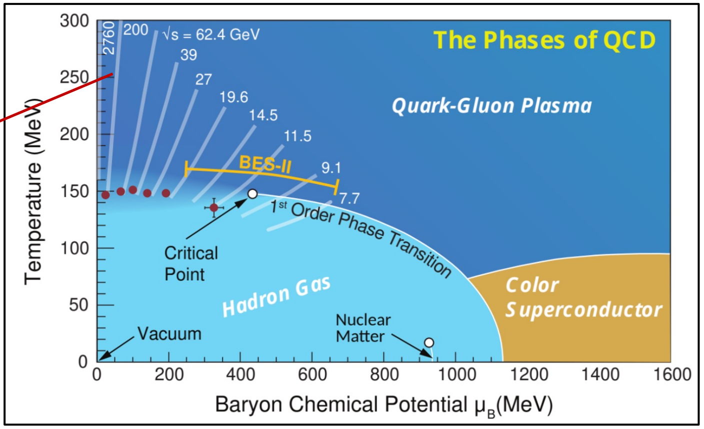

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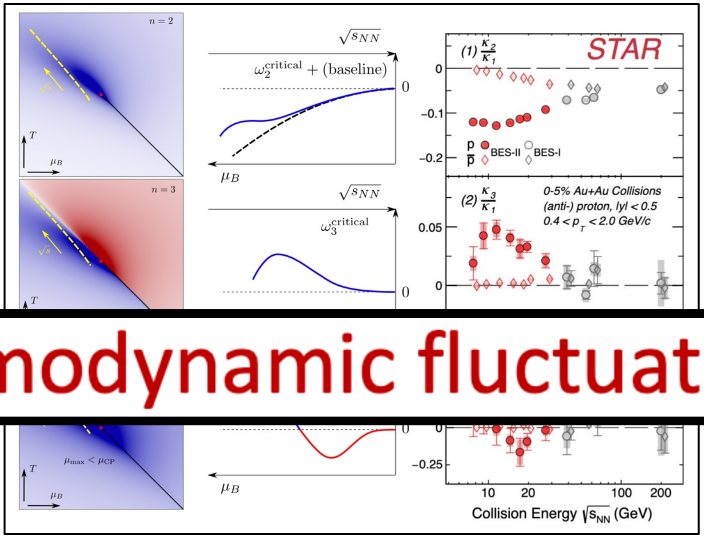
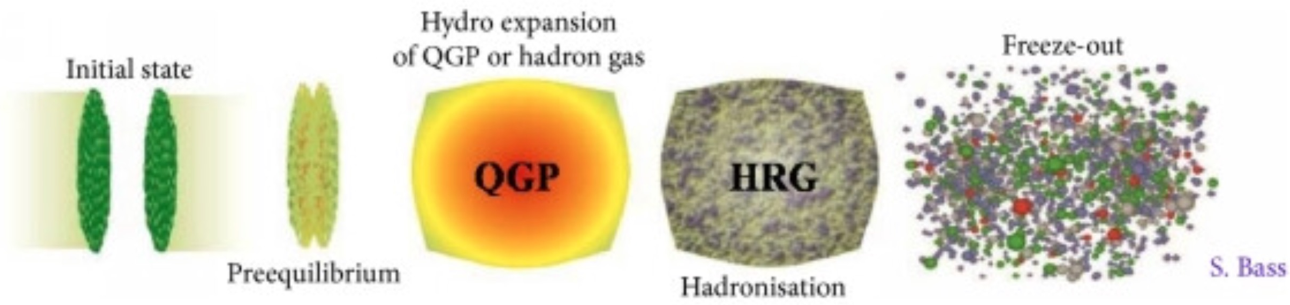


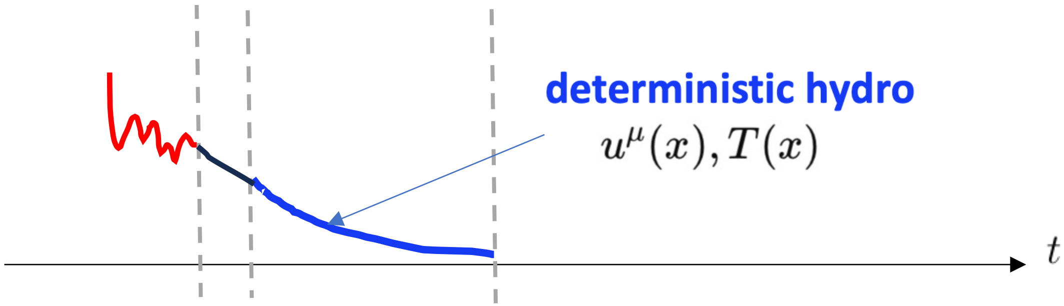
thermodynamic fluctuations!

[Stephanov 2024]

Theory needs **dynamical evolution + acceptance mapping**

3. QGP dynamics and Fluctuations





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- Sampling at freeze-out → additional event-by-event fluctuations [Bzdak, Koch, Shen PRC (2012)]

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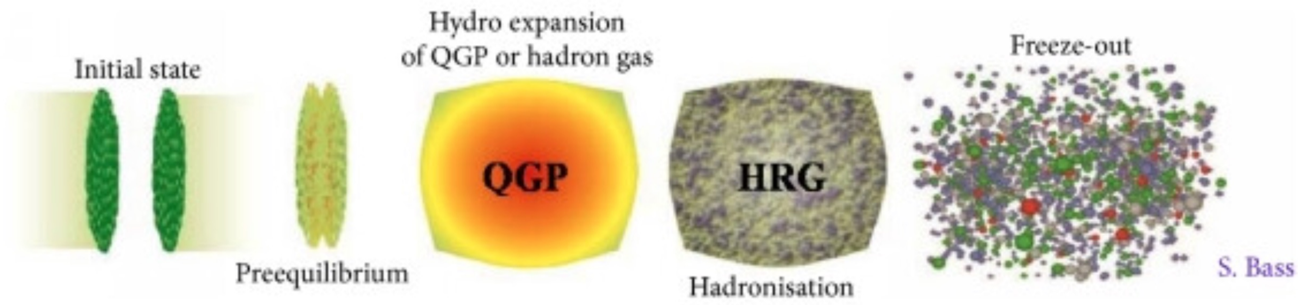

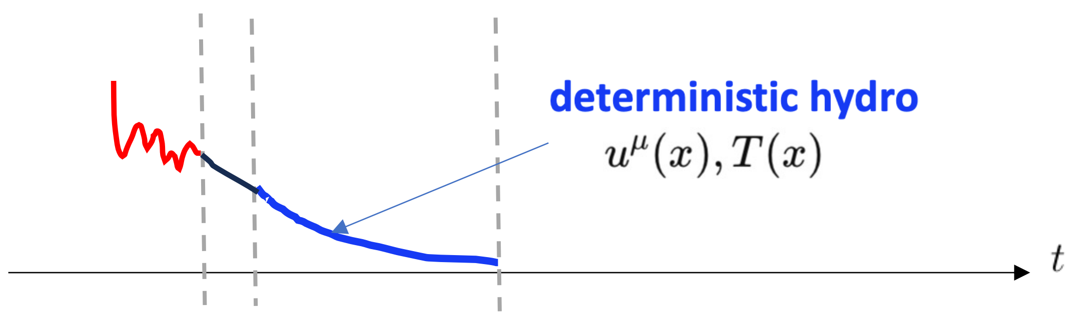

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- **The main assumption**: correlation functions are assumed to be in equilibrium.

(local equilibrium + white noise)

$$\delta e(t, \boldsymbol{x}) \equiv T^{00}(t, \boldsymbol{x}) - \langle T^{00} \rangle$$

$$\boxed{\langle \delta e(t, \boldsymbol{k}) \delta e(t, -\boldsymbol{k}') \rangle = \frac{(e+p)T}{c_s^2} (2\pi)^3 \delta^3(\boldsymbol{k} - \boldsymbol{k}')}$$

[Landau-Lifshitz Vol 9]

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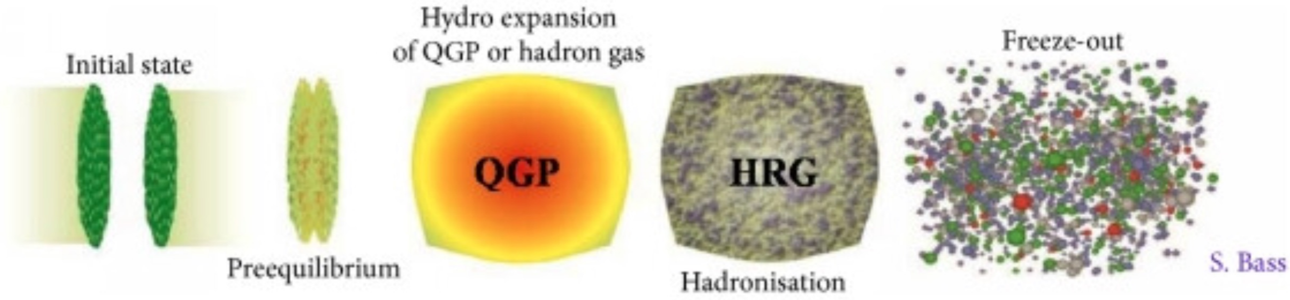

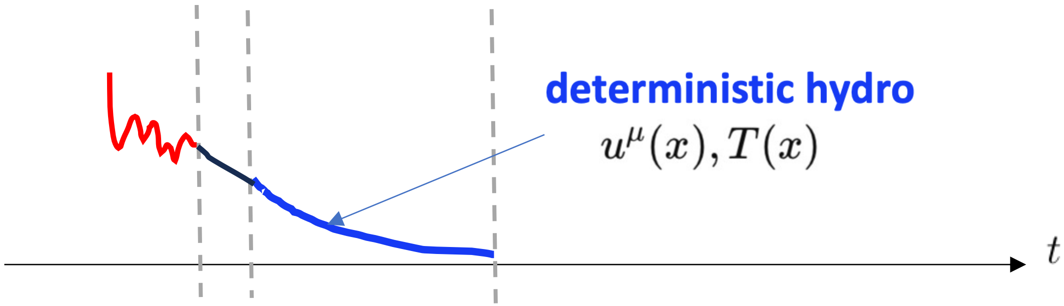

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[Landau-Lifshitz Vol 9]

what happens near CEP?

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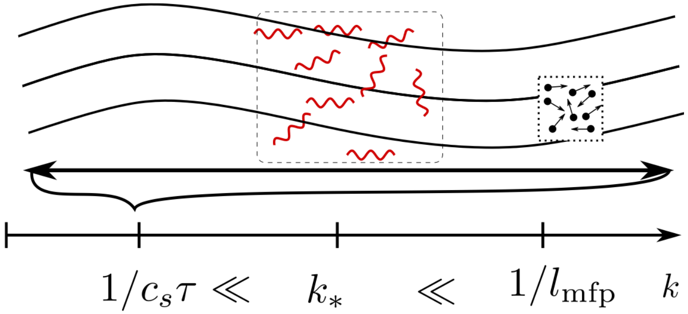
  - $\gamma_\eta k_*^2 \sim \theta\,, \quad k_* \sim \left(\frac{\theta}{\gamma_\eta}\right)^{1/2}$ separates equilibrated / non-equilibrated modes.
  - Wigner function for $q \sim k_*$ evolves locally:

$$\boxed{W_2(t, \boldsymbol{x}; \boldsymbol{q}) = \int_{\boldsymbol{q}} e^{-i\boldsymbol{d}\cdot\boldsymbol{y}} \langle \delta\phi(t, \boldsymbol{x} - \frac{\boldsymbol{y}}{2})\, \delta\phi(t, \boldsymbol{x} + \frac{\boldsymbol{y}}{2}) \rangle}$$



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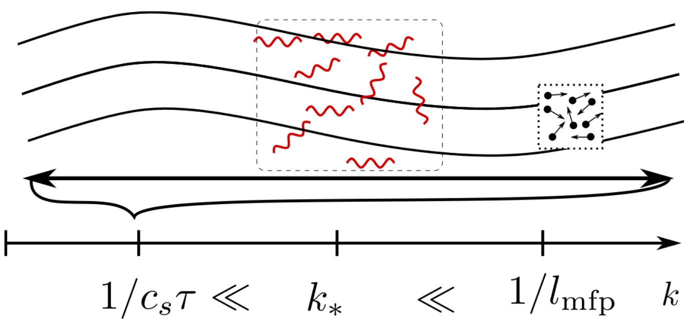

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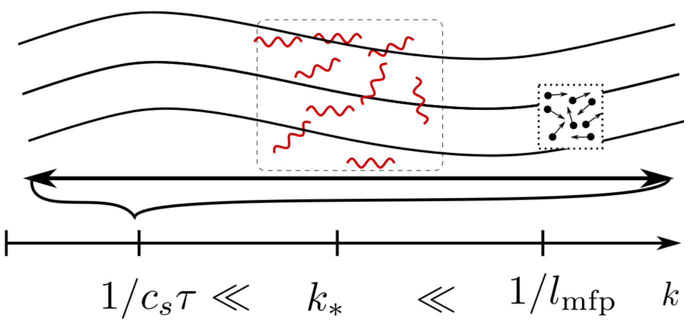

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- Evolution of W_2 in a Bjorken flow [Akumatsu, Mazeliauskas, Teaney PRC (2017)] [An, Basar, Stephanov, Yee PRC (2019)]
- Evolution of W_2 in relativistic hydro with a critical point [An, Basar, Stephanov, Yee PRC (2020)]

5. Deterministic W_n from stochastic dynamics

In the diffusion theory, response is **instantaneous**: [An, Basar, Stephanov, Yee PRL (2021)]

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This motivates going beyond white noise.

6. W_n and cumulants

Evolution equations should be constructed for

- The set of slow fields near CEP: $j\text{'s} = (m, p)$
- Coupled to an expanding background

$$u \cdot \partial G^{\mathrm{c}}_{mm}(x_1, x_2) = 2\Big[-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1} G^{\mathrm{c}}_{mm}(x_1, x_2) + L_{m,m}(x_1) G^{\mathrm{c}}_{mm}(x_1, x_2) + Q_{mm}(x_1, x_2) \Big]_{\overline{12}}, \tag{3.15a}$$

$$\begin{aligned} u \cdot \partial G^{\mathrm{c}}_{mmm}(x_1, x_2, x_3) = 3\Big[&-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1} G^{\mathrm{c}}_{mmm}(x_1, x_2, x_3) + L_{m,m}(x_1) G^{\mathrm{c}}_{mmm}(x_1, x_2, x_3) \\ &+ L_{m,p}(x_1) G^{\mathrm{c}}_{pmm}(x_1, x_2, x_3) + L_{m,mm}(x_1) G^{\mathrm{c}}_{mm}(x_1, x_2) G^{\mathrm{c}}_{mm}(x_1, x_3) \\ &+ 2Q_{mm,m}(x_1, x_2) G^{\mathrm{c}}_{mm}(x_1, x_3) \Big]_{\overline{123}}, \end{aligned} \tag{3.15b}$$

$$\begin{aligned} u \cdot \partial G^{\mathrm{c}}_{mmmm}(x_1, x_2, x_3, x_4) = 4\Big[&-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1} G^{\mathrm{c}}_{mmmm}(x_1, x_2, x_3, x_4) + L_{m,m}(x_1) G^{\mathrm{c}}_{mmmm}(x_1, x_2, x_3, x_4) \\ &+ L_{m,p}(x_1) G^{\mathrm{c}}_{pmmm}(x_1, x_2, x_3, x_4) + 3L_{m,mm}(x_1) G^{\mathrm{c}}_{mm}(x_1, x_2) G^{\mathrm{c}}_{mmm}(x_1, x_3, x_4) \\ &+ 3L_{m,mp}(x_1) G^{\mathrm{c}}_{mm}(x_1, x_2) G^{\mathrm{c}}_{pmm}(x_1, x_3, x_4) \\ &+ L_{m,mmm}(x_1) G^{\mathrm{c}}_{mm}(x_1, x_2) G^{\mathrm{c}}_{mm}(x_1, x_3) G^{\mathrm{c}}_{mm}(x_1, x_4) \\ &+ 3Q_{mm,p}(x_1, x_2) G^{\mathrm{c}}_{pmm}(x_1, x_3, x_4) + 3Q_{mm,m}(x_1, x_2) G^{\mathrm{c}}_{mmm}(x_1, x_3, x_4) \\ &+ 3Q_{mm,mm}(x_1, x_2) G^{\mathrm{c}}_{mm}(x_1, x_3) G^{\mathrm{c}}_{mm}(x_1, x_4) \Big]_{\overline{1234}}, \end{aligned} \tag{3.15c}$$

[An, Basar, Stephanov, Yee PRC (2023)]

the next step is to map W_n onto the particle multiplicity cumulant C_n

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$$u \cdot \partial G^{c}_{mm}(x_1, x_2) = 2\Big[-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1} G^{c}_{mm}(x_1, x_2) + L_{m,m}(x_1) G^{c}_{mm}(x_1, x_2) + Q_{mm}(x_1, x_2) \Big]_{\overline{12}}, \tag{3.15a}$$

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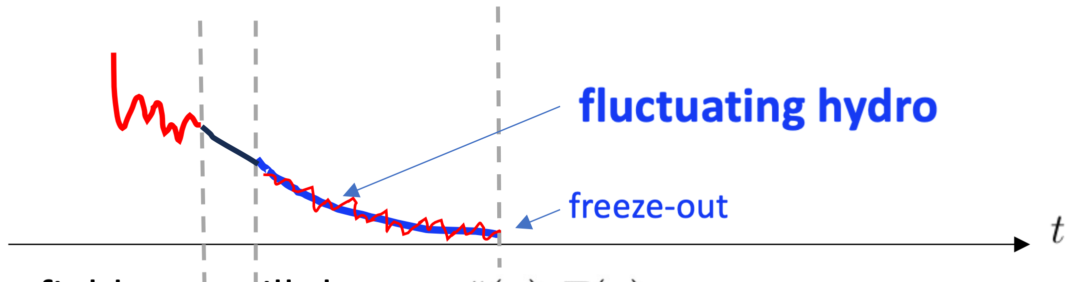
[An, Basar, Stephanov, Yee PRC (2023)]

the next step is to map W_n onto the particle multiplicity cumulant C_n



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\end{aligned}
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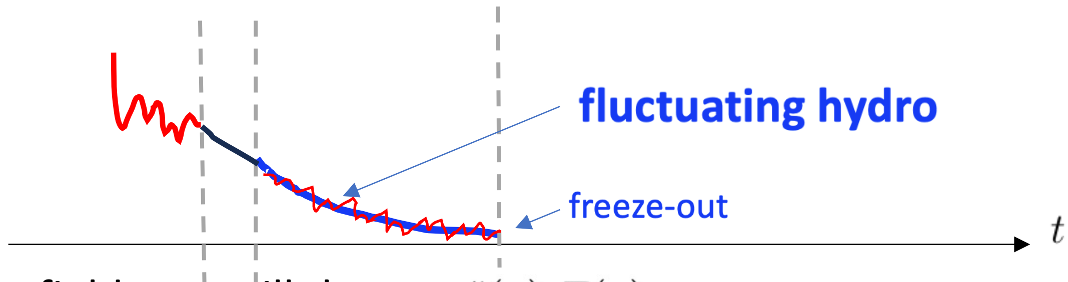

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$$\boxed{C_m(\Delta y) = \int \left[\prod_{i=1}^{m}\frac{dq_i}{2\pi}A(q_i)\right](2\pi)\delta\left(\sum_{i=1}^{m}q_i\right)W_m(q_1,\ldots,q_m;t_f).}$$

rapidity

$$A(q) = e^{iqy_0}\,\frac{2\sin(q\Delta y/2)}{q}$$

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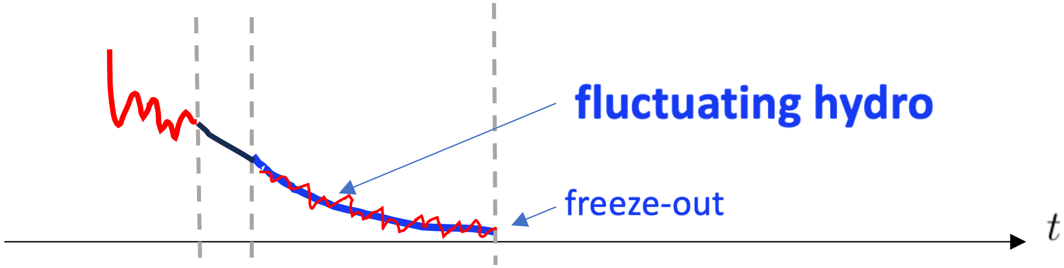

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$$A(q) = e^{iqy_0} \frac{2\sin(q\Delta y/2)}{q}$$

rapidity

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We aim to explore the effect of colored noise on the evolution of W_n,

and consequently on the cumulants, in an RTA setup

7. Noise in RTA diffusion

- Integrating RTA gives a **resummed theory of diffusion** [NA, Rischke JHEP (2025)]

$$p^\mu \partial_\mu f(x,\boldsymbol{p}) = \frac{p^\alpha u_\alpha}{\tau}\left[f(x,\boldsymbol{p}) - f^{(0)}(x,\boldsymbol{p})\right] \xrightarrow[\mathbf{D} \,=\, \partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla}]{} \boxed{\int_\Omega \frac{\mathbf{D}}{1+\tau\mathbf{D}} n(x) = \ 0}$$

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Derivative expansion: $\partial_t n + \nabla_i\left[-\frac{\tau}{3}\nabla_i n + \frac{\tau^3}{45}\nabla_i\nabla^2 n - \frac{2\tau^5}{945}\nabla_i\nabla^4 n + \dots\right] = 0$

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- FDT:

$$\langle \zeta(\omega,k)\,\zeta(\omega',k')\rangle = (2\pi)^{d+1}\delta(\omega+\omega')\delta(k+k')\,\mathcal{N}(\omega,k) \quad \frac{|1-L_p|^2}{\tau^2}\,\chi T\tau\left(\frac{L_p}{1-L_p}+\frac{L_p^*}{1-L_p^*}\right)$$

noise is colored $\boxed{L_p = \frac{1}{2i\tau k}\ln\left(\frac{\frac{i}{\tau}+\omega-k}{\frac{i}{\tau}+\omega+k}\right)}$

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we use the "**Markovian embedding technique**" [Gardiner 2004]

8. Markovian embedding

non-Markovian

$$\boxed{\left\langle \frac{\mathcal{D}}{1+\tau\mathcal{D}}\, n_k(t) \right\rangle_\mu = \zeta_k(t)}$$

$\langle f \rangle_\mu \equiv \frac{1}{2}\int_{-1}^{1} d\mu\, f(\mu)$ $\qquad$ $\mathcal{D} \equiv \partial_t + ik\mu$

Markovianization by introducing

a hidden variable:

$\mu = \cos\theta$

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$$\boxed{(1+\tau\mathcal{D})\, g_k(\mu,t) = n_k(t) + \tau\, \xi_k(\mu,t)}$$

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- $\zeta_k(t)$ is colored.
- $\xi_k(\mu,t)$ is chosen to be white.
- Integrating out $\mu = \cos\theta$ → the original non-Markovian dynamics:

$$\boxed{\langle \xi(\mu,t)\, \xi(\mu',t') \rangle = \frac{2\chi T}{\tau}\Big[\, 2\delta(\mu-\mu') - 1 \,\Big]\, \delta(t-t').}$$

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The goal is to approximately solve $g_k(\mu,t)$

9. Hierarchy of Markovian embeddings

- The key point is: $$\mu P_\ell = \frac{\ell+1}{2\ell+1}P_{\ell+1} + \frac{\ell}{2\ell+1}P_{\ell-1}$$

- The projected equation onto ℓ $$\tau \dot{g}_\ell + g_\ell + i\tau k\left(\frac{\ell}{2\ell+1}g_{\ell-1} + \frac{\ell+1}{2\ell+1}g_{\ell+1}\right) = \delta_{\ell 0}\, n + \tau\, \xi_\ell$$

- Hierarchy of the equations

g_0 g_1 g_2

$$\begin{aligned}
\ell = 0. &\qquad \dot{n} = -ikJ, \\
\ell = 1. &\qquad \tau \dot{J} + J + i\tau k\left(\tfrac{1}{3}n + \tfrac{2}{3}Q\right) = \eta_1, \\
\ell = 2. &\qquad \tau \dot{Q} + Q + i\tau k\left(\tfrac{2}{5}J\right) = \eta_2. \\
&\vdots
\end{aligned}$$

$$\boxed{\begin{aligned}
\langle \eta_1(t)\eta_1^*(t')\rangle &= \frac{2\chi T\tau}{3}\delta(t-t'), \\
\langle \eta_2(t)\eta_2^*(t')\rangle &= \frac{2\chi T\tau}{5}\delta(t-t'),
\end{aligned}}$$

Truncation at $\ell = \ell_{\max}$ gives a finite Markovian system.

- Evolution equation for W2 (truncated at $\ell_{\max} = 1$)

$$W_2 \equiv \langle n_k n_{-k}\rangle, \qquad P \equiv -i\langle J_k n_{-k}\rangle, \qquad V \equiv \langle J_k J_{-k}\rangle$$

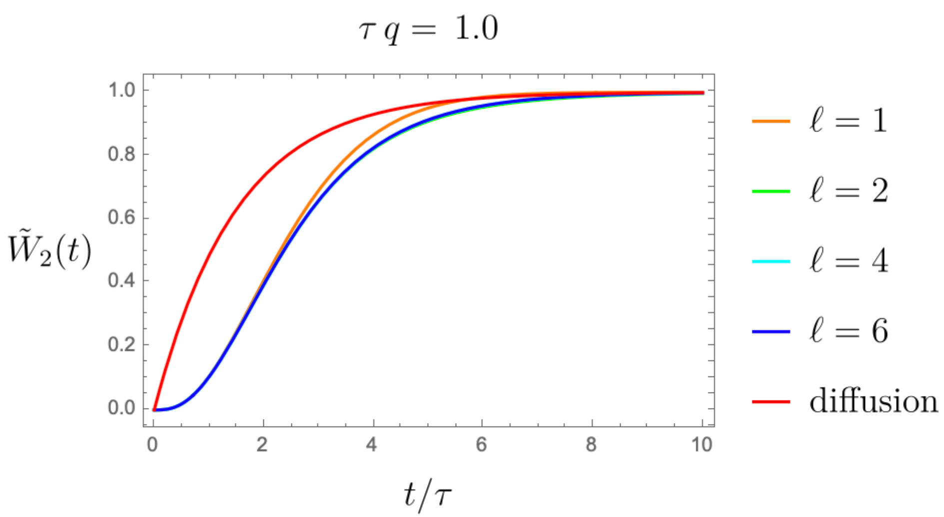
$$\boxed{\dot{W}_2 = 2kP}$$

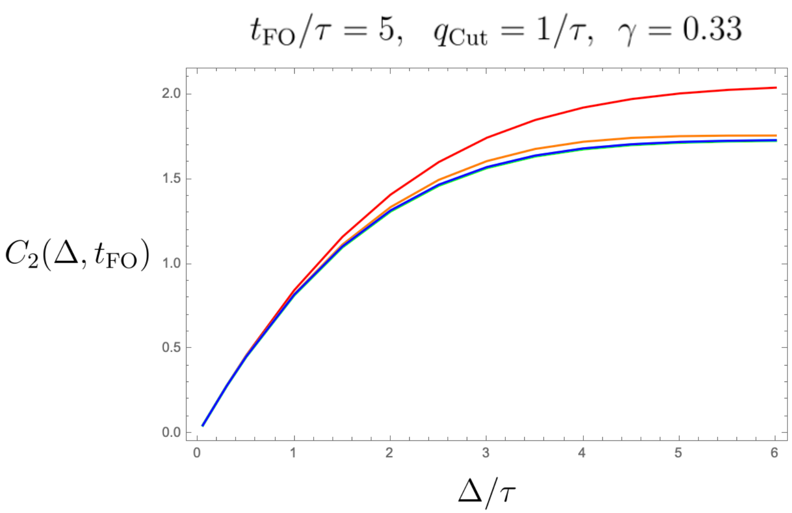
$$\boxed{\dot{P} = -\frac{1}{\tau}P - \frac{k}{3}W_2 + kV}$$

$$\boxed{\dot{V} = -\frac{2}{\tau}V - \frac{2k}{3}P + \frac{2\chi T}{3\tau}}$$

10. Numerical results: C_2

diffusion vs. RTA truncated at $\ell_{max} = 1, 2, 4, 6$





- Memory → lower equilibration of high q-modes
- W_2 rises more slowly → suppressed C_2
- Stable for $\ell_{max} \gtrsim 4$ → baseline benchmark for C_3 and C_4

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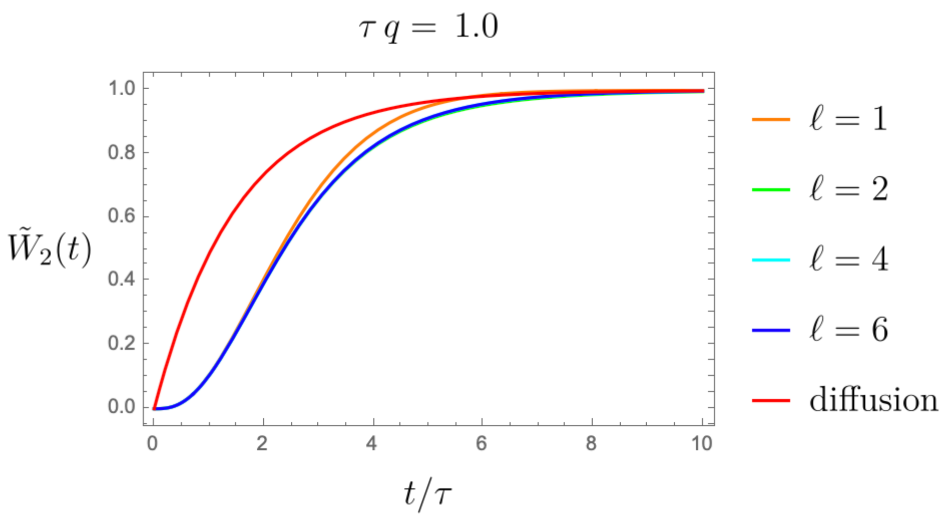

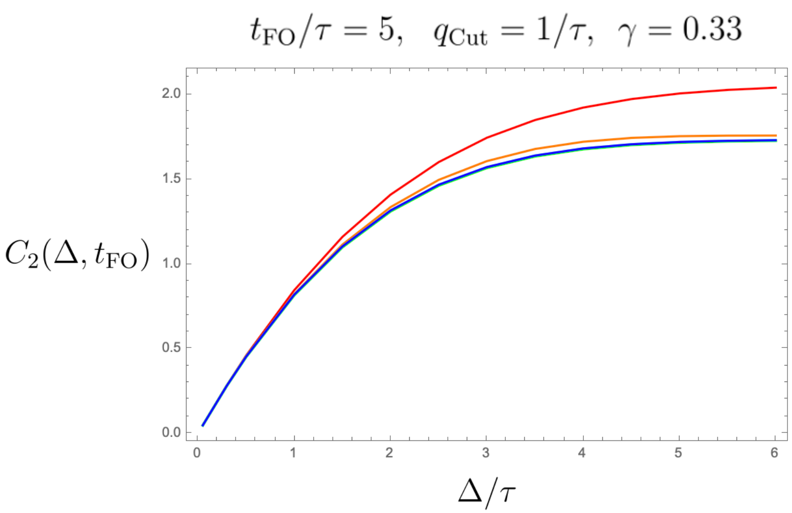

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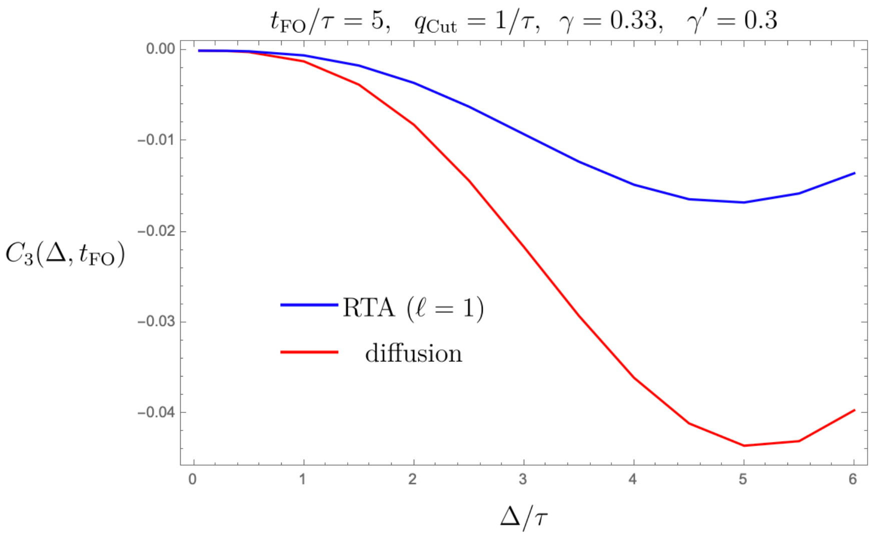
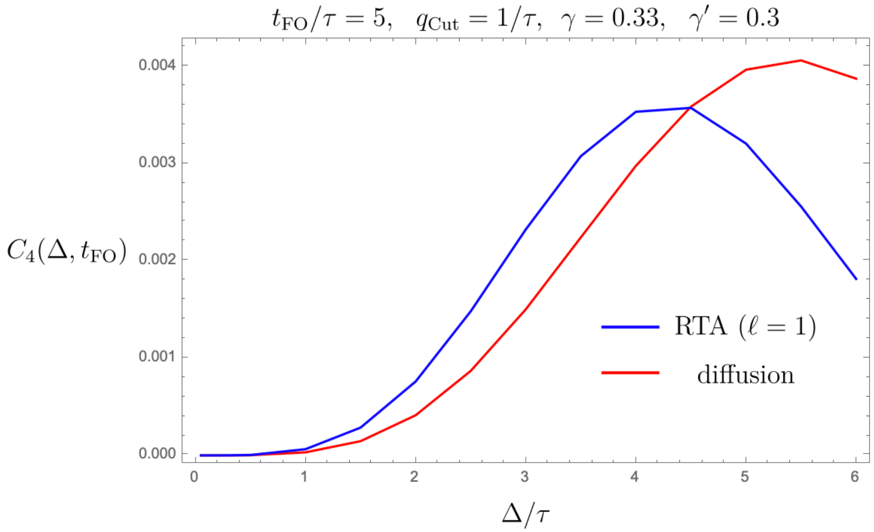
"C_2 suppression from memory"

11. Numerical results: C_3 and C_4

- Currently shown for $\ell_{max} = 1$





- C_3: the same trend as C_2
- C_4: ordering changes with Δ

  → C_4 is more sensitive to nonlinear source terms

colored-noise memory effects can change
the magnitude and also the shape of higher cumulants
→ Important for quantitative comparisons to data

12. Conclusion and outlook

1. Memory isn't a small detail!
   - it's required by FDT once relaxation is finite.
   - It mainly affects high-k,
   - so it shows up first at small rapidity windows.

2. We have done a **microscopic derivation** of the colored noise, tied to a transport model.

3. We developed a concrete, computable scheme for W_n hieracrchy.

3. The next step is to couple the equations to a **expanding background** and **critical EoS.**

4. Final goal: a fully fluctuating RTA + expanding background + critical EoS + slow mode

Thank you for your attention!