The equations of motion of a spinning particle in a background Yang-Mills field

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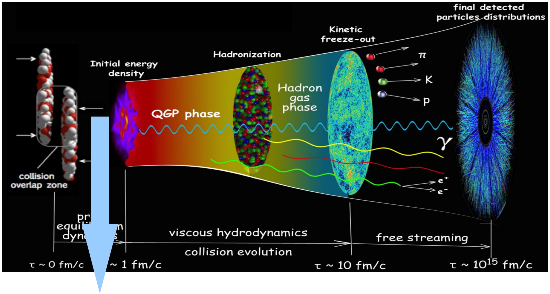
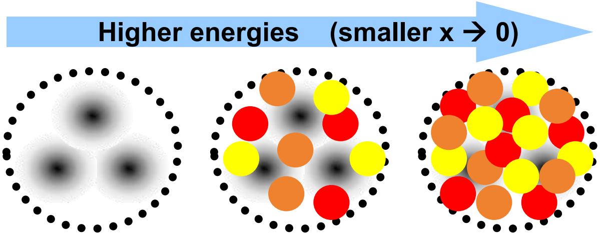
based on arXiv: 2511.20083 and work in progress

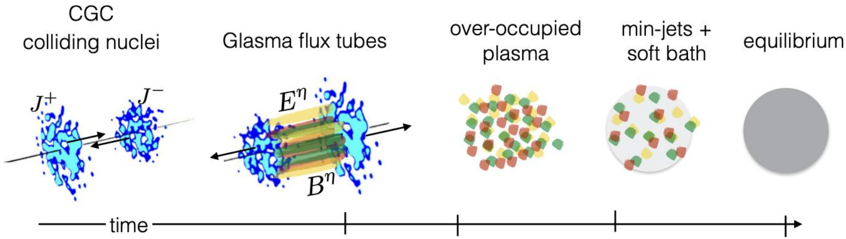
Outline

Strong Chromo-EM field in RHICs



Color Glass Condensate (CGC)

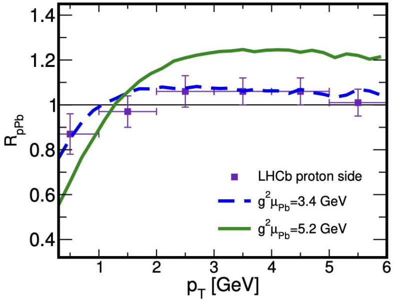




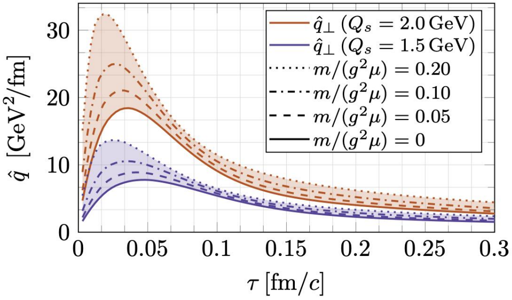
Field strength: $gB^a \sim 100m_\pi^2$
Lifetime: $\tau \lesssim 1$ fm/c

Phenomena

- Gluon saturation; Collective flow
- H.F. and Jet momentum diffusion
- Spin polarization & alignment



M. Ruggieri et al., PRD 98, 094024 (2018)



A. Ipp et al., PLB 810, 135810 (2020)

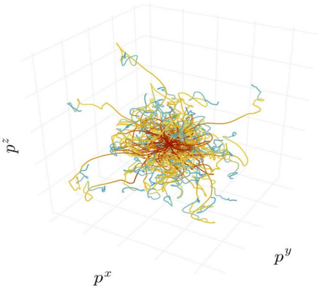
Framework: Classical colored particle dynamics (Wong Equations)

S.K.Wong, Il Nuovo Cimento 65A, 689 (1970)

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{p^\mu}{m}$$

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = gQ^a F^{\mu\nu,a}\frac{p_\nu}{m}$$

$$\frac{\mathrm{d}Q^a}{\mathrm{d}\tau} = -g f^{abc} A^b_\mu Q^c \frac{p^\mu}{m}$$

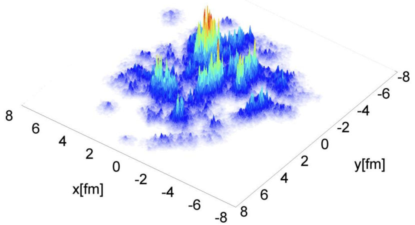


M. Ruggieri et al., PRD 98, 094024 (2018)
Y. Sun et al., PLB 798, 134933 (2019)
A. Ipp et al., PRD 102, 074001 (2020)
K. Boguslavski et al., JHEP 09 (2020) 077
C. Andres et al., PLB 803, 135318 (2020)
K. Boguslavski et al., NPA 1005, 121970 (2021)
J.-H. Liu et al., PRD 103, 034029 (2021)
P. Khowal et al., EPJP 137, 307 (2022)
M. Ruggieri et al., PRD 106, 034032 (2022)
M. E. Carrington et al., PLB 834, 137464 (2022)
C. Andres et al., JHEP 03 (2023) 189
D. Avramescu et al., PRL 134, 172301 (2025)

Spin Degrees of freedom are missing for quarks and gluons

Why spin matters

- Stern-Gerlach force in highly non-uniform Glasma field $\vec{\nabla}(\vec{\mu}\cdot\vec{B})$



- Spin polarization & alignment (heavy flavor hadrons)

Non-Relativistic Treatment

Issue: Not applicable for RHICs

$$\frac{dx^i}{dt} = \frac{P^i}{m}$$
$$\frac{dP^i}{dt} = -\frac{g}{m} Q_a \left(m F_a^{i0} - P^j F_a^{ij} \right) - \frac{g}{m} Q_a S^j (\mathcal{D}^i B_a^j)$$
$$\frac{dQ_a}{dt} = \frac{g}{m} f_{abc} Q_c \left(P^i A_b^i + S^i B_b^i \right)$$
$$\frac{dS^i}{dt} = -\frac{g}{m} Q_a F_a^{ij} S^j$$

N. Linden, A. J. Macfarlane, and J. W. van Holten, CJP 46, 209 (1996)
Pooja, S. K. Das, V. Greco, and M. Ruggieri, EPJP 138, 313 (2023)

Grassmann Variables

Issue: Numerically impractical

$$\dot{x}^\mu = \epsilon P^\mu$$
$$\dot{P}^\mu = \epsilon g F^{a\,\mu\nu} Q^a P_\nu - \frac{i\epsilon g}{2} \psi^\alpha (D^\mu F_{\alpha\beta})^a Q^a \psi^\beta$$
$$\dot{\psi}^\mu = \epsilon g F^{a\,\mu\nu} Q^a \psi_\nu$$
$$\dot{\lambda}_a^\dagger = -igv^\mu t_{ab}^c A_\mu^c \lambda_b^\dagger - \frac{\epsilon g}{2} \psi^\mu F_{\mu\nu}^b t_{ac}^b \lambda_c^\dagger \psi^\nu$$
$$\dot{\lambda}_a = igv^\mu t_{ab}^c A_\mu^c \lambda_b + \frac{\epsilon g}{2} \psi^\mu F_{\mu\nu}^b t_{ac}^b \lambda_c \psi^\nu$$
$$Q^a \equiv \lambda_c^\dagger t_{cd}^a \lambda_d \qquad S_{\mu\nu} = -i\psi_\mu \psi_\nu$$

A. P. Balachandran, P. Salomonson, B.-S. Skagerstam, and J.-O. Winnberg, PRD 15, 2308 (1977)
N. Mueller and R. Venugopalan, PRD 99, 056003 (2019)

EOMs of Spinning and Colored Particles

Dirac-Equation Based

Issue: Constraints not preserved

$$\mathcal{H}\psi \equiv -\frac{1}{2m}\left\{\left[\pi^\mu + \frac{g}{c}\hat{A}^\mu\right]^2 + \frac{g\hbar}{2c}\sigma^{\mu\nu}\hat{F}_{\mu\nu} - m^2c^2\right\}\psi = 0$$

$$\frac{dx^\mu}{d\tau} = \frac{p^\mu}{m}$$

$$\frac{dp^\mu}{d\tau} = g\,Q^a F^{a\,\mu\nu}\,u_\nu + \frac{g}{2mc}\,Q_a\,(D^\mu F_{\lambda\nu})^a S^{\lambda\nu}$$

$$\frac{dQ^a}{d\tau} = -g f_{abc}\left[u^\mu A^b_\mu + \frac{1}{2mc}S^{\mu\nu}F^b_{\mu\nu}\right]Q^c$$

$$\frac{dS^{\mu\nu}}{d\tau} = \frac{g}{mc}Q^a\left(F^\mu_{a\,\lambda}\,S^{\lambda\nu} - F^\nu_{a\,\lambda}\,S^{\lambda\mu}\right)$$

U. W. Heinz, PLB 144, 228

$\Rightarrow$

$$u_\mu = \frac{p_\mu}{mc}$$

$$u^\mu u_\mu = 1$$

$$u^\mu D_\mu\left(F_{\lambda\nu}S^{\lambda\nu}\right) = 0$$

$$P^\mu S_{\mu\nu} = 0$$

$$S^{\mu\nu}S_{\mu\nu} = 8s^2$$

Inconsistency

Wigner-Function Based

Issue: Constraints not preserved

$$\delta(p^2-m^2)p^\mu\left(D_\mu\hat{f}_V + \frac{1}{2}\{F_{\nu\mu},\partial_p^\nu\hat{f}_V\}_c\right) = 0$$

$$\begin{aligned}0 = \delta(p^2-m^2)\Big(&p\cdot\tilde{\Delta}\,\hat{a}^\mu + \frac{1}{2}\{F^{\nu\mu}\,\hat{a}_\nu\}_c - \frac{i\hbar}{2}[\tilde{F}^{\mu\nu}p_\nu\,\hat{f}_\nu]_c \\ &- \frac{\hbar}{8}\epsilon^{\mu\nu\rho\sigma}p_\sigma\left[F_{\alpha\nu}F_{\beta\rho}\,\partial_p^\alpha\partial_p^\beta\hat{f}_\nu\right]_c \\ &- \frac{\hbar}{4}\epsilon^{\mu\nu\rho\sigma}p_\rho\left\{(D_\sigma F_{\beta\nu})\,\partial_p^\beta\hat{f}_\nu\right\}_c \\ &+ \frac{\hbar}{2}\delta'(p^2-m^2)\Big(p_\nu\{\tilde{F}^{\mu\nu}\,p\cdot\tilde{\Delta}\,\hat{f}_\nu\}_c \\ &+ \epsilon^{\mu\rho\alpha\beta}p_\rho\left\{F_\alpha{}^\nu\,\{F_{\beta\nu}\,\hat{f}_\nu\}_c\right\}_c + \frac{p^\rho p^\nu}{2}\left[[F_{\sigma\rho}\,\tilde{F}_{\mu\nu}]\,\partial_p^\sigma\hat{f}_\nu\right]_c\Big)\end{aligned}$$

X.-L. Luo and J.-H. Gao, JHEP 11 (2021) 115

D.-L. Yang, JHEP 06, 140 (2022)

EOM Requirements

- Lorentz covariance
- Constraint preservation

$$P^2 - \frac{1}{2}\mu g\, q^a\, F^a_{\mu\nu}\, S^{\mu\nu} - m^2 = 0$$

Mass-Shell Condition

$$P^\mu S_{\mu\nu} = 0$$

$$S^{\mu\nu} S_{\mu\nu} = 8s^2$$

Tulczyjew-Dixon Spin Supplementary Condition

$$\dot{Q}_n = 0 \quad Q_n = d^{abc\ldots} q^a q^b q^c \ldots .$$

Classical Casimirs Conservation

- Arbitrary chromo-magnetic moment (Landé g-factor ≠ 2)

Constrained Hamiltonian

- Systematic treatment on singular Lagrangians
- Constrained orbit, Gauge theories, general relativity
- Classical dynamics: Hamiltonian formulism easier than Lagrangian with constraints
- Quantum Mechanics: QCD (Faddeev–Popov), general relativity

Examples: charged particle in e.m. field

Lagrangian Formalism

$$S = -m\int d\tau - e\int A_\mu dx^\mu$$

$$\frac{d}{dt}(\gamma m\boldsymbol{v}) = e\boldsymbol{E} + e(\boldsymbol{v}\times\boldsymbol{B})$$

Hamiltonian Formalism

$$H = \sqrt{(p-eA)^2 + m^2} + e\Phi$$

$$\frac{d\boldsymbol{x}}{dt} = \frac{\boldsymbol{P}}{\sqrt{\boldsymbol{P}^2 + m^2}}$$

$$\frac{d\boldsymbol{P}}{dt} = e\boldsymbol{E} + e\left(\frac{d\boldsymbol{x}}{dt}\times\boldsymbol{B}\right)$$

m=0 limit? Lorentz Covariance?

$$L = \frac{1}{2g}(\dot{x}^\mu\dot{x}_\mu) + \frac{1}{2}gm^2 + eA_\mu\dot{x}^\mu$$

$$H_P = \frac{g}{2}\left[(p-eA)^2 - m^2\right] + \lambda p_g$$

Constraints

$$S^{\mu\nu} = 2(\omega^\mu \pi^\nu - \omega^\nu \pi^\mu)$$

$$T_1 : -\frac{1}{2}(P^2 - 2\mu g q^a F^a_{\mu\nu}\omega^\mu \pi^\nu - m^2),$$
$$T_2 : -\frac{1}{2}(\pi^2 - a_2), \quad T_3 : -\frac{1}{2}(\omega^2 - a_3),$$
$$T_4 : -P\omega, \quad T_5 : -P\pi, \quad T_6 : \omega\pi,$$

$$H_P = \frac{1}{2}g_1\left(P^2 - 2\mu g q^a F^a_{\mu\nu}\,\omega^\mu\pi^\nu - m^2\right) + \frac{1}{2}g_2(\pi^2 - a_2)$$
$$+\frac{1}{2}g_3(\omega^2 - a_3) + g_5(P\pi) + \lambda_4(P\omega) + \lambda_6(\omega\pi)$$
$$+\lambda_{g_i}\pi_{g_i}$$

$$[\hat{q}^a, \hat{q}^b] = i\hbar f^{abc}\hat{q}^c$$
$$[\hat{x}^\mu, \hat{p}^\nu] = i\hbar g^{\mu\nu}$$
$$[\hat{S}^{\mu\nu}, \hat{S}^{\rho\sigma}] = 2i\hbar\left(g^{\nu\rho}\hat{S}^{\mu\sigma} - g^{\mu\rho}\hat{S}^{\nu\sigma} + g^{\mu\sigma}\hat{S}^{\nu\rho} - g^{\nu\sigma}\hat{S}^{\mu\rho}\right)$$
$$[\hat{\omega}^\mu, \hat{\pi}^\nu] = i\hbar g^{\mu\nu}$$

$$\{q^a, q^b\}_{\mathrm{PB}} = f^{abc}q^c$$
$$\{x^\mu, p^\nu\}_{\mathrm{PB}} = g^{\mu\nu}$$
$$\{S^{\mu\nu},\, S^{\rho\sigma}\}_{\mathrm{PB}} = 2(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} + g^{\mu\sigma}S^{\nu\rho} - g^{\nu\sigma}S^{\mu\rho})$$
$$\{\omega^\mu, \pi^\nu\}_{\mathrm{PB}} = g^{\mu\nu}$$

J. Zhou, Y. S. Zhao and Y. Sun, arXiv: 2511.20083

Dirac-Bergmann Algorithm and EOMS

$$\dot{F} = \{F, H_P\}_{\mathrm{PB}} \quad \dot{\phi}_a = \{\phi_a, H_P\}_{\mathrm{PB}} \approx 0$$

Dirac-Bergmann Algorithm

- Generate new constraints or impose restrictions on the Lagrange multipliers

$$\dot{x}^\mu = \lambda_1 P^\mu + \frac{g}{\Delta}\lambda_1 S^{\mu\nu}\left\{\frac{\mu}{4}\,\mathrm{Tr}[qD_\nu F_{\rho\sigma}]\,S^{\rho\sigma} - (\mu-1)\,\mathrm{Tr}[qF_{\nu\rho}]\,P^\rho\right\}$$

$$\dot{P}^\mu = 2gg^{\mu\nu}\,\mathrm{Tr}[qF_{\nu\rho}]\,\dot{x}^\rho + \frac{1}{2}\mu\lambda_1 g\,\mathrm{Tr}[qD^\mu F_{\rho\sigma}]\,S^{\rho\sigma}$$

$$\dot{S}^{\mu\nu} = 2\mu\lambda_1 g\,\mathrm{Tr}[qF_{\rho\sigma}]\,g^{\mu[\rho}S^{\sigma]\nu} + \frac{2g}{\Delta}\lambda_1 P^{[\mu}S^{\nu]\alpha}\left\{\frac{\mu}{4}\,\mathrm{Tr}[qD_\alpha F_{\rho\sigma}]\,S^{\rho\sigma} - (\mu-1)\,\mathrm{Tr}[qF_{\alpha\rho}]\,P^\rho\right\}$$

$$\dot{q} = ig[A_\mu q]\,\dot{x}^\mu + i\frac{g}{4}\mu\lambda_1[F_{\mu\nu}q]\,S^{\mu\nu}$$

$$D_\mu F_{\rho\sigma} = \partial_\mu F_{\rho\sigma} - ig[A_\mu, F_{\rho\sigma}] \qquad \Delta = m^2 + q^a(2\mu+1)F^a_{\mu\nu}\omega^\mu\pi^\nu = P^2 + \tfrac{1}{4}q^a F^a_{\mu\nu}S^{\mu\nu}$$

EOMs of Spin Vector

$$S^\mu = \frac{1}{2\sqrt{P^2}}\epsilon^{\mu\nu\alpha\beta}P_\nu S_{\alpha\beta}$$ **not** $$S^\mu = \frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}P_\nu S_{\alpha\beta}$$

$$P^\mu S_\mu = 0,$$
$$S^\mu S_\mu = -4s^2,$$

$$\dot{x}^\mu = \lambda_1 P^\mu + \frac{g}{\Delta\sqrt{P^2}}\lambda_1\epsilon^{\mu\nu\alpha\beta}P_\alpha S_\beta\left\{\frac{\mu}{4}\mathrm{Tr}\left[qD_\nu F_{\rho\sigma}\right]\epsilon^{\rho\sigma\alpha'\beta'}P_{\alpha'}S_{\beta'} - (\mu-1)\mathrm{Tr}\left[qF_{\nu\rho}\right]P^\rho\right\}$$

$$\dot{P}^\mu = 2gg^{\mu\nu}\mathrm{Tr}\left[qF_{\nu\rho}\right]\dot{x}^\rho + \frac{g}{2\sqrt{P^2}}\mu\lambda_1\epsilon^{\rho\sigma\alpha\beta}P_\alpha S_\beta\mathrm{Tr}\left[qD^\mu F_{\rho\sigma}\right]$$

$$\dot{S}^\mu = 2\mu\lambda_1 g\mathrm{Tr}\left[qF^{\nu\mu}\right]S_\nu + \frac{1}{P^2}P^\mu\left\{2\mu\lambda_1 g\mathrm{Tr}\left[qF^{\rho\sigma}\right]P_\rho S_\sigma - S^\nu\dot{P}_\nu\right\}$$

$$\dot{q} = ig[A_\mu, q]\dot{x}^\mu + i\frac{g}{4\sqrt{P^2}}\mu\lambda_1[F_{\mu\nu}, q]\epsilon^{\mu\nu\alpha\beta}P_\alpha S_\beta$$

Implications

- Additional spin-dependent velocity
- Uniform fields → reduces to TBMT-type equations but with modifications
- Richer spin dynamics

Applications for Chiral Particles

Chiral kinetic equations

$$S^{0i} = 0$$
$$S^{ij} = h\,\epsilon_{ijk}\,\hat{p}^k$$

$$N^\mu S_{\mu\nu} = 0$$
$$P^\mu S_{\mu\nu} = 0$$
$$S^{\mu\nu} S_{\mu\nu} = 8s^2$$
$$P^2 - \frac{\mu}{2}\, q\, e\, F^{\mu\nu}\, S_{\mu\nu} = 0$$

$\mu = 0$

$$\sqrt{G}\,\dot{\boldsymbol{x}} = \hat{\boldsymbol{p}} + \boldsymbol{E} \times \boldsymbol{b} + \boldsymbol{B}(\hat{\boldsymbol{p}} \cdot \boldsymbol{b})\,;$$
$$\sqrt{G}\dot{\boldsymbol{p}} = \boldsymbol{E} + \hat{\boldsymbol{p}} \times \boldsymbol{B} + \boldsymbol{b}(\boldsymbol{E} \cdot \boldsymbol{B})\,.$$
$$G = (1+\boldsymbol{b}\cdot\boldsymbol{B})^2$$

M. A. Stephanov and Y. Yin, PRL 109, 162001 (2012)

$$\left\{\partial_t + \frac{1}{\sqrt{G}}(\tilde{\mathbf{v}} + \hbar\epsilon Q(\tilde{\mathbf{v}}\cdot\mathbf{b}_h)\mathbf{B} + \hbar\epsilon Q\tilde{\mathbf{E}}\times\mathbf{b}_h)\cdot\nabla_{\mathbf{x}} + \frac{\epsilon Q}{\sqrt{G}}(\tilde{\mathbf{E}} + \tilde{\mathbf{v}}\times\mathbf{B} + \hbar\epsilon Q(\tilde{\mathbf{E}}\cdot\mathbf{B})\mathbf{b}_h)\cdot\nabla_{\mathbf{p}}\right\} f_h^\epsilon(x,\mathbf{p}) = 0$$

$\mu = 1$

$$\sqrt{G} = (1 + \hbar\epsilon Q\mathbf{b}_h\cdot\mathbf{B}), \qquad \tilde{\mathbf{E}} = \mathbf{E} - \frac{1}{\epsilon Q}\nabla_{\mathbf{x}} E_{\mathbf{p}},$$
$$E_{\mathbf{p}} = |\mathbf{p}|(1 - \hbar\epsilon Q\mathbf{B}\cdot\mathbf{b}_h), \qquad \tilde{\mathbf{v}} = \frac{\partial E_{\mathbf{p}}}{\partial\mathbf{p}} = \hat{\mathbf{p}}(1 + 2\hbar\epsilon Q\mathbf{B}\cdot\mathbf{b}_h) - \hbar\epsilon Q b_h\mathbf{B}$$

A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, PRD 98, 036010 (2018)

Derived EOMs contain quantum corrections as Wigner function approach

$$\dot{\xi}^a = \{H, \xi^a\} = -\{\xi^a, \xi^b\}\partial_b H$$

$$\{p_i, p_j\} = -\frac{\epsilon_{ijk} B_k}{1 + \mathbf{B}\cdot\mathbf{\Omega}},$$

$$\{x_i, x_j\} = \frac{\epsilon_{ijk} \Omega_k}{1 + \mathbf{B}\cdot\mathbf{\Omega}},$$

$$\{p_i, x_j\} = \frac{\delta_{ij} + \Omega_i B_j}{1 + \mathbf{B}\cdot\mathbf{\Omega}},$$

$$\{\xi^a, \xi^b\} = (\omega^{-1})^{ab} \equiv \omega^{ab}$$

$$\mathrm{d}\Gamma = \sqrt{\omega}\,\mathrm{d}\xi = (1 + \mathbf{B}\cdot\mathbf{\Omega})\frac{\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{x}}{(2\pi)^3}$$

D.T. Son and N. Yamamoto, PRL 109, 181602 (2012)

Dirac Bracket

$$\{F, G\}_{\mathrm{DB}} \equiv \{F, G\}_{\mathrm{PB}} - \{F, \xi_\alpha\}_{\mathrm{PB}}\,\Delta^{-1}_{\alpha\beta}\,\{\xi_\beta, G\}_{\mathrm{PB}}$$

$$\Delta_{\alpha\beta} = \{\xi_\alpha, \xi_\beta\}_{\mathrm{PB}}$$

$$\{p_i, p_j\}_{\mathrm{DB}} = -\frac{\epsilon_{ijk} B_k}{1 + \mathbf{B}\cdot\mathbf{\Omega}}$$

$$\{x_i, x_j\}_{\mathrm{DB}} = \frac{\epsilon_{ijk} \Omega_k}{1 + \mathbf{B}\cdot\mathbf{\Omega}}$$

$$\{x_i, p_j\}_{\mathrm{DB}} = \frac{\delta_{ij} + \Omega_i B_j}{1 + \mathbf{B}\cdot\mathbf{\Omega}}$$

Established covariant EOMs of Spinning & colored particles in Yang–Mills fields

- Lorentz covariance
- Constraint preservation
- Arbitrary chromo-magnetic moment

Outlook

- Phenomenology: Heavy-flavor momentum diffusion; Spin polarization & alignment
- Developments: Kinetic theory; Gluon

Thank you!