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Modulus Stabilization in Jordan Frame Supergravity For Modular Flavor Model

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Based on Fei Wang, Ying Kai Zhang 2601.09542 & 2604.XXXX

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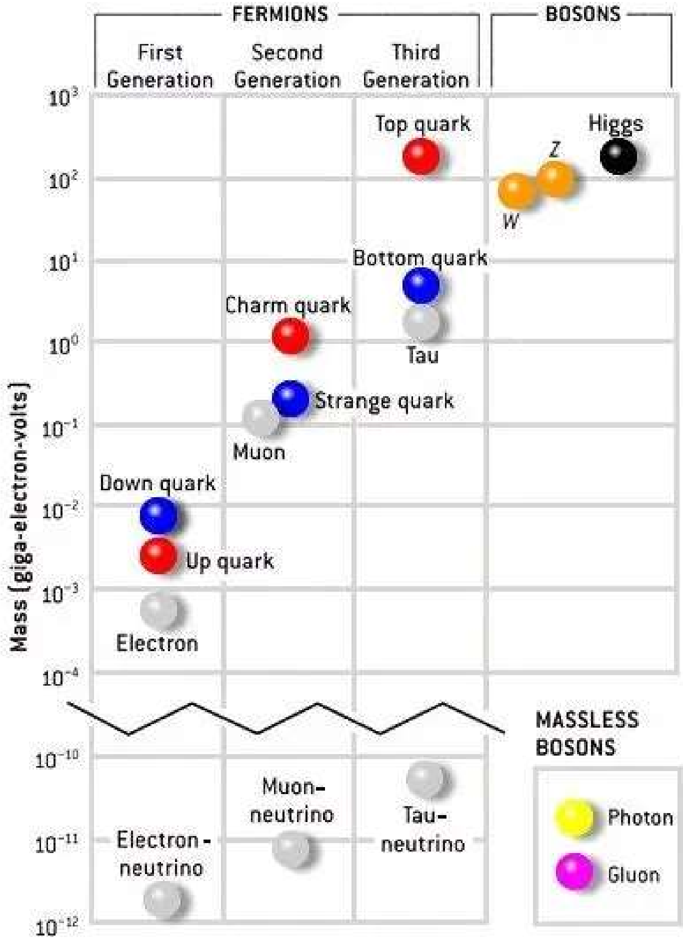
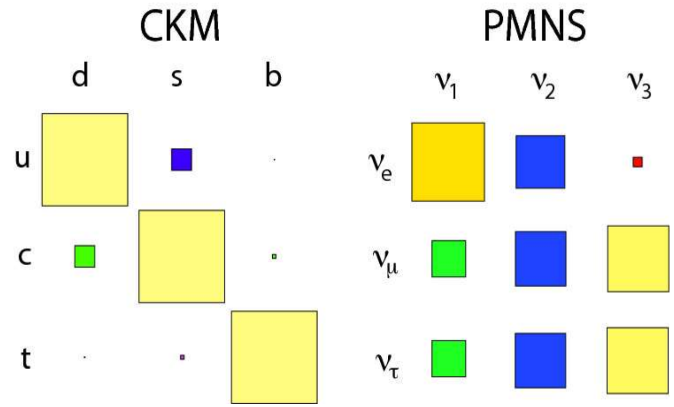
Modulus Stabilization in Jordan Frame Supergravity

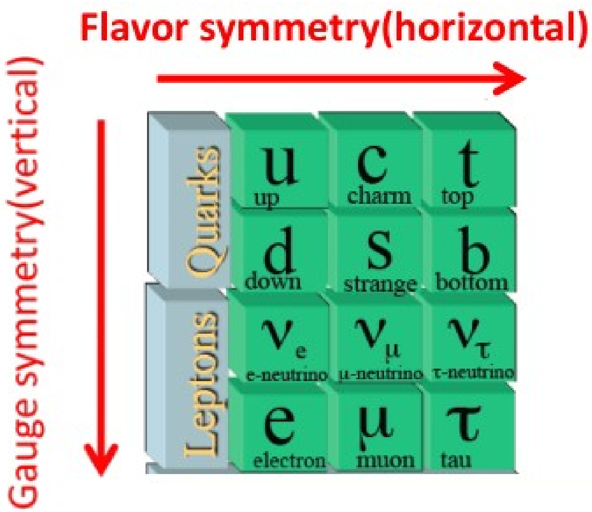
Why Modular Flavor Model



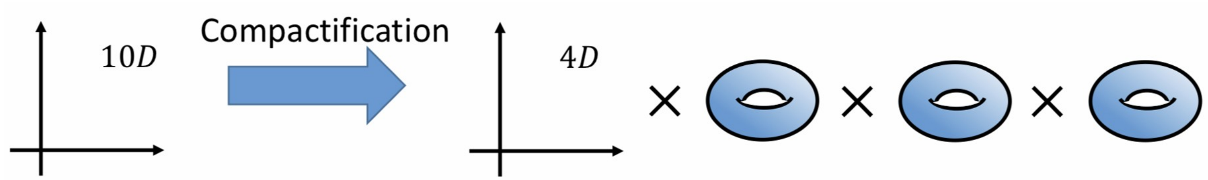


GUT ?

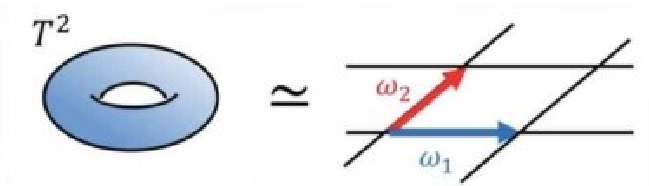
Needs addtional structures to explain the flavor structure



Modular Symmetry



- Avoid some difficulties relatd to ordinary flavor symmetry.
- Modular flavor symmetries are interesting approaches to tackle the origin of fermion mass hierarchies and mixing angles.
- The modular symmetry is a geometrical symmetry of T^2 and T^2/Z_2, and corresponds to the change of their cycle basis.
- Matter modes transform non-trivially under the modular symmetry. That is, the modular symmetry is a flavor symmetry.
- The modular flavor symmetry can also control higher dimensional operators in unbroken SUSY limit.



Lattice left invariant by modular transformations

$$\tau = \frac{e_2}{e_1}$$

$$\begin{pmatrix} e_2' \\ e_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_2 \\ e_1 \end{pmatrix}$$

$$\tau \longrightarrow \tau' = \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

Modular Symmetry

modular group $\Gamma \equiv SL(2, Z)$ can act on the upper half plane

$$\gamma : \tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \text{for } \gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z), \quad \mathtt{Im}(\tau) > 0\,.$$

inhomogeneous modular group is $\bar{\Gamma} = \Gamma / \{I, -I\}$

principal congruence subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathtt{mod}\ N) \right\}$$

inhomogeneous finite modular group $\Gamma_N \cong \overline{\Gamma} / \overline{\Gamma}(N)$

$\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ and $\Gamma_5 \simeq A_5$

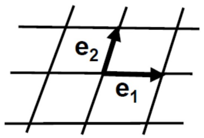
The group $\overline{\Gamma}$ is generated by S and T

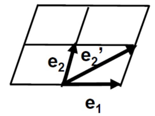
$$S^2 = \mathbb{1}\ , \quad (ST)^3 = \mathbb{1}$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

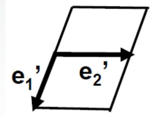
$$S : \tau \to -\frac{1}{\tau} \qquad T : \tau \to \tau + 1$$

$$\overline{\Gamma}(N) = \begin{cases} \Gamma(N) \text{ for } N > 2 \\ \Gamma(N)/\{I, -I\} \text{ for } N = 2 \end{cases}$$





corresponds to T



corresponds to S

Modular forms of weight k and level N

holomorphic functions $f(\tau)$ transforming under the action of $\Gamma(N)$

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)\,, \quad \gamma \in \bar{\Gamma}$$

$\rho(\gamma)_{ij}$ is a unitary matrix under Γ_N

Modular forms of weight $2k$ and level N form a linear space $\mathcal{M}_{2k}(\Gamma(N))$ finite dimension

Modular Flavor Model

Modular invariant action in SUSY case

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta}\ K(\Phi, \bar{\Phi}) + \int d^4x d^2\theta\ w(\Phi) + h.c.$$

the supermultiplets $\varphi^{(I)}$ transform in representation $\rho^{(I)}$ of a quotient group Γ_N

$$\begin{cases} \tau \to \dfrac{a\tau + b}{c\tau + d} \\ \varphi^{(I)} \to (c\tau + d)^{-k_I} \rho^{(I)}(\gamma)\varphi^{(I)} \end{cases} \qquad \gamma \text{ an element of } \Gamma_N.$$

The invariance of the action $\mathcal{S}$ requires

$$\begin{cases} w(\Phi) \to w(\Phi) \\ K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + f(\Phi) + f(\overline{\Phi}) \end{cases}$$

invariance of the Kahler potential

$$K(\Phi, \bar{\Phi}) = -h \log(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\varphi^{(I)}|^2$$

invariance of the superpotential $w(\Phi)$ under the modular group

$$w(\Phi) = \sum_n Y_{I_1 \ldots I_n}(\tau)\ \varphi^{(I_1)} \ldots \varphi^{(I_n)}$$

the functions $Y_{I_1 \ldots I_n}(\tau)$ modular forms of weight $k_Y(n)$ transforming in the representation ρ of Γ_N

$$k_Y(n) = k_{I_1} + \ldots + k_{I_n}$$

The product $\rho \times \rho^{I_1} \times \ldots \times \rho^{I_n}$ contains an invariant singlet

Modular flavor symmetry can combine with GUT models:

1. Explain the origin of Gauge & flavor structure together.
2. Few input parameters-very predictive.
3. String origin?

For example:
Xiao Kang Du, Fei Wang, JHEP01(2023)036; modular A_4+flipped SU(5)

Xiao Kang Du, Fei Wang, JHEP02(2021)221; modular S_3+SU(5)

Symmetric Points

No value of τ which preserves the full modular symmetry.

Specific residual symmetries remain at certain symmetric points

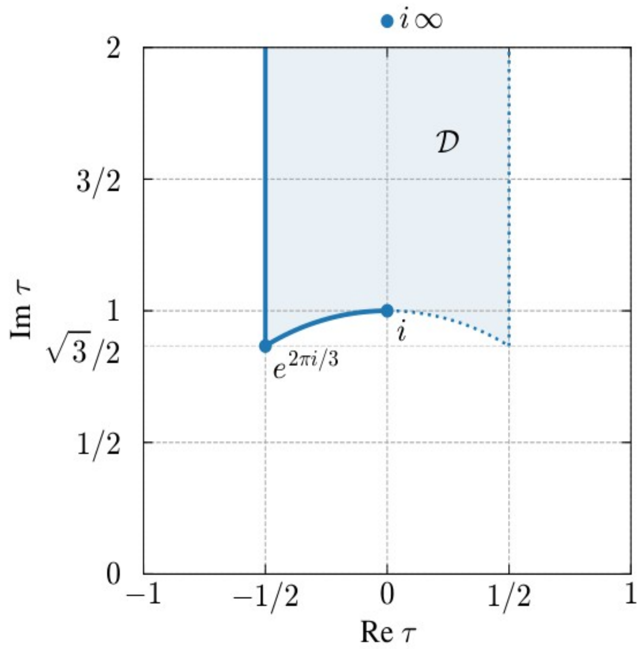
- $\tau_{\rm sym} = i$, invariant under S, preserving $\mathbb{Z}_4^S$ (note that $S^2 = R$);
- $\tau_{\rm sym} = i\infty$, invariant under T, preserving $\mathbb{Z}_N^T \times \mathbb{Z}_2^R$; and
- $\tau_{\rm sym} = \omega = \exp(2\pi i/3)$ (the "left cusp"), invariant under ST, preserving $\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R$.

At the fixed point τ_0

$$Y_{\mathbf{r}}^{(k)}(\tau_0) = Y_{\mathbf{r}}^{(k)}(\gamma_0\tau_0) = (c_0\tau_0 + d_0)^k \rho_{\mathbf{r}}(\gamma_0) Y_{\mathbf{r}}^{(k)}(\tau_0)\,,$$

⇨ $$\rho_{\mathbf{r}}(\gamma_0) Y_{\mathbf{r}}(\tau_0) = J_k^{-1}(\gamma_0, \tau_0) Y_{\mathbf{r}}(\tau_0)$$

Modular multiplets $Y_{\mathbf{r}}(\tau_0)$ at the fixed point is actually the eigenvector of the representation matrix $\rho_{\mathbf{r}}(\gamma_0)$ with eigenvalue $J_k^{-1}(\gamma_0, \tau_0)$.



The fundamental domain of the modular group

Near the Symmetric Points

Hierarchical fermion mass matrices can arise solely due to the proximity of the modulus τ to a symmetric point.

Feruglio argues that universality behavior emerges near the self-dual symmetric point $\tau = i$

For example, in the vicinity of ω, the degree of suppression for a bilinear $\psi_i^c M(\tau)_{ij} \psi_j$ is proven to be given by O(ϵ^l), with l = 0, 1, 2.

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger$$

More generally, the suppression determined by the representations of the residual symmetry group.

How to stabilize the modulus field near symmetric points?

P. P. Novichkov; J. T. Penedo b; S. T. Petcov, arXiv: 2102.07488

Independently from details of the theory, such as the finite modular group acting on the lepton multiplets, the weights of the matter multiplets and even the form of the kinetic terms, which are not required to be neither minimal nor flavour blind.

Ferruccio Feruglio, arXiv: 2211.00659

Modulus Stabilization

- Important task in string theory:
- ✓ moduli must be massive otherwise mediating unobserved long-range fifth forces
- ✓ Couplings etc of 4D theory depending on moduli

Type IIB string theory, flux-induced superpotential to fix the complex structure and dilaton moduli. The Kaher moduli remain not stabilized. Giddings-Kachru-Polchinski (GKP)

1. KKLT: fixed Kahler moduli with Kahler moduli dependent superpotential induced by non-perturbative effects- gaugino condensation etc. & add uplifting potential by anti-D branes.
2. Large Volume Scenario:

Multiple modulus.

The α′ and non-perturbative corrections compete and determine the structure of the scalar potential

Tasks:
Fix all the modulus field and obtain dS type vacua with tiny vacuum energy?

- Modular flavor model

1. Modulus VEV fixed the fitting parameter.
2. Trigger CP breaking.
3. Avoid new long-range forces.

Explain the origion of flavor violation.

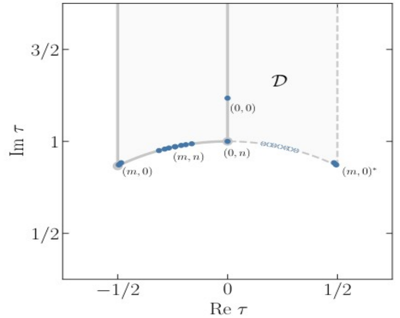
Modulus Stabilization

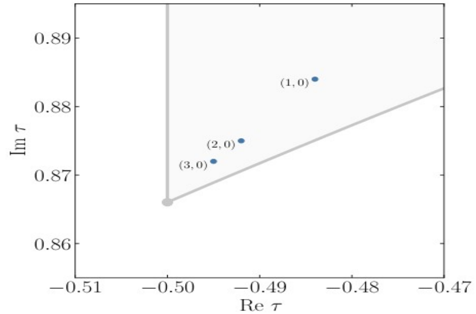
$$\begin{cases} K(\tau,\bar{\tau}) = -\Lambda_K^2 \log(2\,\mathrm{Im}\,\tau)\,, \\ W(\tau) = \Lambda_W^3 \dfrac{H(\tau)}{\eta(\tau)^{2\mathrm{n}}} \end{cases}$$

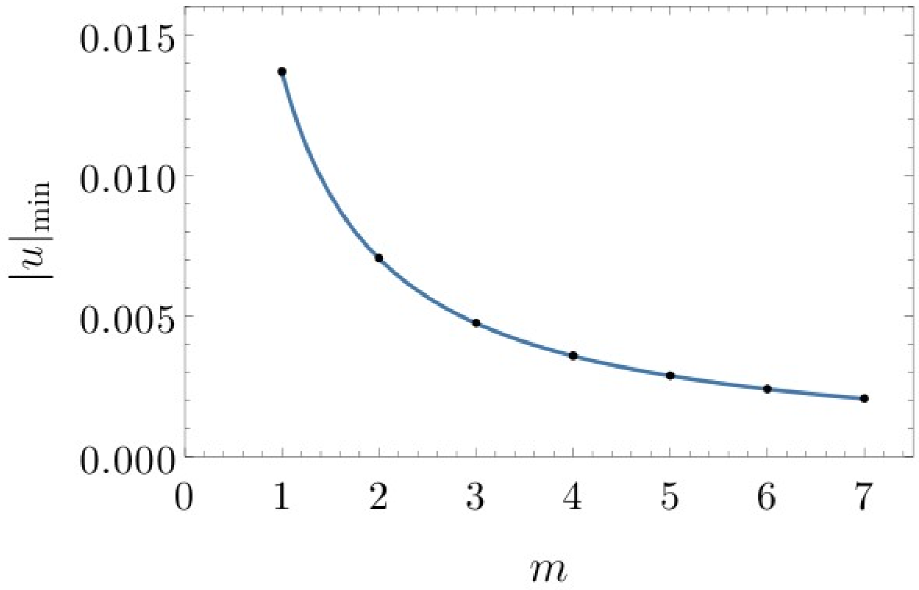
It was conjected that:
all extrema of V(τ) would correspond to CP-conserving values of τ, lie
1. either on the boundary of the fundamental domain *D*
2. or on the imaginary axis.

In fact, stabilized value of τ can lie within the fundamental domain.

Not possible for the point i∞ because scalar potential diverges there.







Deviation of the minimum of $V_{m,0}(\tau,\bar{\tau})$ from the left cusp $\tau = \omega$

P. P. Novichkov, J. T. Penedo, S. T. Petcov,
arXiv: 2201.02020

Jordan frame supergravity

By applying superconformal approach to supergravity after gauge fixing all extra symmetries in the superconformal algebra

The scalar-gravity part of the $\mathcal{N}=1$, $d=4$ supergravity in a generic Jordan frame with frame function $\Phi(z,\bar{z})$, a Kähler potential $\mathcal{K}(z,\bar{z})$ independent on the frame function, and superpotential $W(z)$ is,

$$\mathcal{L}_J^{\mathrm{scalar-grav}} = \sqrt{-g_J}\left[\Phi\left(-\frac{1}{6}R(g_J)+\mathcal{A}_\mu^2(z,\bar{z})\right)+\left(\frac{1}{3}\Phi g_{\alpha\bar{\beta}}-\frac{\Phi_\alpha\Phi_{\bar{\beta}}}{\Phi}\right)\hat{\partial}_\mu z^\alpha\hat{\partial}^\mu\bar{z}^{\bar{\beta}}-V_J\right].$$

Here

$$\Phi_\alpha \equiv \frac{\partial}{\partial z^\alpha}\Phi(z,\bar{z})\,, \qquad \Phi_{\bar{\beta}} \equiv \frac{\partial}{\partial \bar{z}^{\bar{\beta}}}\Phi(z,\bar{z}) = \overline{\Phi}_{\bar{\beta}}\,,$$
$$g_{\alpha\bar{\beta}} = \frac{\partial^2\mathcal{K}(z,\bar{z})}{\partial z^\alpha\partial\bar{z}^{\bar{\beta}}} \equiv \mathcal{K}_{\alpha\bar{\beta}}(z,\bar{z})\,,$$

and $\mathcal{A}_\mu$ is the purely bosonic part of the on-shell value of the auxiliary field A_μ. On shell it depends on scalar fields as follows:

$$\mathcal{A}_\mu(z,\bar{z}) \equiv -\frac{\mathrm{i}}{2\Phi}\left(\hat{\partial}_\mu z^\alpha\partial_\alpha\Phi - \hat{\partial}_\mu\bar{z}^{\bar{\alpha}}\partial_{\bar{\alpha}}\Phi\right).$$

See 1008.2942 by Ferrara et al

Einstein frame

The frame function and the Kahler potential have the following relation:

$$\mathcal{K}(\Phi(z,\bar{z})) = -3\log(-\tfrac{1}{3}\Phi(z,\bar{z}))\,, \qquad \Omega^2 = -\tfrac{1}{3}\Phi(z,\bar{z})$$

Origin from superconformal approach to supergravity after gauge fixing

D-gauge $\quad \mathcal{N} = \Phi(z,\bar{z})\,,$

$U(1)$-gauge $\quad y = \bar{y}\,,$

$$\mathcal{K}(z,\bar{z}) = -3\ln\left[-\tfrac{1}{3}Z^I(z)G_{I\bar{J}}(z,\bar{z})\bar{Z}^{\bar{J}}(\bar{z})\right] = -3\log\left(-\frac{1}{3}\frac{\Phi(z,\bar{z})}{Y(z,\bar{z})}\right),$$

$Y(z,\bar{z}) \equiv y(z,\bar{z})\bar{y}(z,\bar{z})$ takes unity value.

Kähler metric $\quad g_{\alpha\bar{\beta}} = \partial_\alpha\partial_{\bar{\beta}}\mathcal{K}(z,\bar{z})\,.$

Einstein frame potential

$$V_E = V_E^F + V_E^D = \mathrm{e}^{\mathcal{K}}\left(-3W\overline{W} + \nabla_\alpha W g^{\alpha\bar{\beta}}\nabla_{\bar{\beta}}\overline{W}\right) + \tfrac{1}{2}(\mathrm{Re}\, f)^{-1\,AB}P_A P_B$$

Modular invariant frame function

$$\Phi(\tau,\bar{\tau}) = 3\left[i(\tau-\bar{\tau})\eta^2(\tau)\overline{\eta^2(\tau)}\right]\left[\overline{H(\tau)}H(\tau)\right]\exp\left\{\xi\left[H(\tau)+\overline{H(\tau)}\right]\right\} \ .$$

Note that:

1. On must ensure that the Kahler metric is positive definite. Very stringent constraints.
2. This is not the most general possibility.
   For example, additional non-holomorphic Eisenstein series.

The modular invariant holomorphic function H(τ) is given by

$$H(\tau)_{m,n} = (j(\tau)-1728)^{\frac{m}{2}}\, j(\tau)^{\frac{n}{3}}\mathcal{P}\left(j(\tau)\right),$$
$$\equiv \left(\frac{G_6(\tau)}{[\eta(\tau)]^{12}}\right)^m \left(\frac{G_4(\tau)}{[\eta(\tau)]^{12}}\right)^n \mathcal{P}\left(j(\tau)\right) \ ,$$

We can parameterize the general modular invariant form of frame function as

$$\tilde{F}[G_0(\tau,\bar{\tau}), H(\tau), \overline{H(\tau)}],$$

$$G_0(\tau,\bar{\tau}) \equiv -i(\tau-\bar{\tau})\eta^2(\tau)\overline{\eta^2(\tau)}$$

The Dedekind η-function is a modular form with modular weight $1/2$ under modular transformation up to a phase. Its transformation under γ is given by

$$\eta(\gamma\tau) = \epsilon_\eta(\gamma)(c\tau+d)^{1/2}\eta(\tau)\ , \quad \forall\gamma\in\overline{\Gamma}\ , \tag{A.1}$$

where $\epsilon_\eta(\gamma)$ depends on γ (but not on τ) and can be expressed as

$$\epsilon_\eta(\gamma) = \exp\left[\frac{\pi i}{12}\omega(a,b,c,d) - i\frac{\pi}{4}\right], \tag{A.2}$$

that satisfies $\epsilon_\eta(\gamma)^{24}=1$ [83]. Here

$$\omega(a,b,c,d) = \left(\frac{a+d}{c} + 12s(-d,c)\right), \tag{A.3}$$

is an integer and $s(-d,c)$ is a Dedekind sum.

Constraints on modular invariant frame function:

The scaling of the metric

$$g^E_{\mu\nu} = \Omega^2 g^J_{\mu\nu} \ , \quad \Omega^2 = -\frac{1}{3}\Phi(\tau,\bar{\tau}) > 0 \ ,$$

with positive Ω^2 constrains $\Phi(\tau,\bar{\tau})$ to be real negative.

Rule out many choices of frame function

We can change into the Einstein frame by the previous reparameterization with

$$\sqrt{-g_J} = \Omega^{-4}\sqrt{-g_E},$$

$$\mathcal{R}_J = \Omega^2\left(\mathcal{R}_E + 6\Box_E \ln\Omega - 6 g_E^{\mu\nu}\partial_\mu(\ln\Omega)\partial_\nu(\ln\Omega)\right)$$

$$\Box_E \ln\Omega = \frac{1}{\sqrt{-g_E}}\partial_\mu\left(\sqrt{-g_E}\, g_E^{\mu\nu}\partial_\nu \ln\Omega\right) ,$$

The Lagrangian in the Einstein frame

$$\frac{\mathcal{L}}{\sqrt{-g_E}} = \frac{1}{2}\mathcal{R}_E - g_{\alpha\bar{\beta}}\, g_E^{\mu\nu}\hat{\partial}_\mu z^\alpha \hat{\partial}_\nu \bar{z}^{\bar{\beta}} - V_E$$

Relation between the Jordan and Eeinstein frame potential is

$$V_J = \frac{\Phi^2}{9} V_E$$

Modulus Stabilization in Einstein Frame

The modular invariant Kahler potential

$$K = -3\left\{\ln\left[-i(\tau-\bar{\tau})\eta^2(\tau)\overline{\eta^2(\tau)}\right] + \ln\left[\overline{H(\tau)}H(\tau)\right] + \xi\left[H(\tau)+\overline{H(\tau)}\right]\right\}.$$

The modular invariant superpotential can take the form

$$W = \Lambda^3\tilde{H}(\tau) = \tilde{c}_0^3 M_{Pl}^3\left(j(\tau)-1728\right)^{\tilde{m}/2} j(\tau)^{\tilde{n}/3}\tilde{\mathcal{P}}\left(j(\tau)\right),$$

with the most general form of $\tilde{\mathcal{P}}(j(\tau))$ given by

$$\tilde{\mathcal{P}}(j(\tau)) = \sum_{k=0}^{\tilde{N}} c_k\left[j(\tau)\right]^k \ ,$$

After the scaling with $g^E_{\mu\nu} = \Omega^2 g^J_{\mu\nu}$, the kinetic term of τ, in addition to contributions from $\mathcal{A}^2_\mu$, is no-longer canonical with

$$\frac{\mathcal{L}_E}{\sqrt{-g_E}} \supseteq \frac{M_P}{2}R + K_{\tau\bar{\tau}}g_E^{\mu\nu}\partial_\mu\tau\partial_\nu\bar{\tau} + \frac{1}{\Omega^3}i\bar{\Psi}_A\gamma^a e^\mu_a(D_\mu + \frac{1}{8}\omega_{\mu ab}[\gamma^a,\gamma^b])\Psi_A$$
$$+\frac{1}{\Omega^2}\left[(D_\mu H)^\dagger(D^\mu H)\right] - \frac{1}{\Omega^4}Y^{(k)}_{\mathbf{r};AB}(\tau)\bar{\Psi}_A H\Psi_B - \frac{1}{\Omega^4}V_{scalar} \ .$$

Yukawa couplings with canonical matter and Higgs fields no longer need any scale factor.

We do not need the normalized modulus field because the modular forms depend on τ

Scalar Potential in Supergravity

The supergravity scalar potential in the Einstein frame is given by

$$V_E = e^{K/M_P^2}\left[K_{\tau\bar\tau}^{-1}\left|\partial_\tau W + \frac{K_\tau}{M_P}W\right|^2 - 3\frac{|W|^2}{M_P^2}\right] ,$$

$$= \frac{\Lambda^6\left[K_{\tau\bar\tau}^{-1}\left|\tilde H'(\tau) + C_1(\tau,\bar\tau)\tilde H(\tau)\right|^2 - 3\left|\tilde H(\tau)\right|^2\right]}{\left[-i(\tau-\bar\tau)\eta^2(\tau)\overline{\eta^2(\tau)}\right]^3\left[\overline{H(\tau)}H(\tau)\right]^3\exp\left\{3\xi\left[H(\tau)+\overline{H(\tau)}\right]\right\}} .$$

Obviously, $e^K \propto \Phi^{-3}$. From the Kahler potential, we have

$$K_\tau \equiv C_1(\tau,\bar\tau)$$
$$= -\frac{3\left[i\eta^2(\tau)\overline{\eta^2(\tau)} + 2i(\tau-\bar\tau)\eta(\tau)\eta'(\tau)\overline{\eta^2(\tau)}\right]}{\left[i(\tau-\bar\tau)\eta^2(\tau)\overline{\eta^2(\tau)}\right]} - 3\frac{H'(\tau)}{H(\tau)} - 3\xi H'(\tau) ,$$

and $K_{\tau\bar\tau}^{-1} = 1/K_{\tau\bar\tau}$ is given via

$$K_{\tau\bar\tau} = \frac{3}{[-i(\tau-\bar\tau)]^2}.$$

$$\eta(\tau) = q^{1/24}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2},$$
$$= q^{1/24}\left[1+\sum_{n=1}^{\infty}(-1)^n\left(q^{n(3n-1)/2}+q^{n(3n+1)/2}\right)\right],$$
$$= q^{1/24}\left[1-q-q^2+q^5+q^7-q^{12}-\cdots\right], \qquad q\equiv e^{i2\pi\tau} .$$

$$j(\tau) \equiv \frac{3^6 5^3}{\pi^{12}}\frac{G_4(\tau)^3}{[\eta(\tau)]^{24}} = \left[\frac{72}{\pi^2\eta^6(\tau)}\left(\frac{\eta'(\tau)}{\eta^3(\tau)}\right)'\right]^3 ,$$
$$\approx 744 + q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \mathcal{O}(q^4) ,$$

Additional terms in the denominator can change the asympotic behavior of scalar potential near i∞

Note: The Kahler metric is not changed in this case.

Vacuum Structure of the Scalar Potential

The minimum conditions of the scalar potential

$$\frac{\partial}{\partial \tau} V_E(\tau, \bar{\tau}) = \frac{\partial}{\partial \bar{\tau}} V_E(\tau, \bar{\tau}) = 0$$

Given the form of the frame function, the derivative V_τ^E can be calculated to be

$$V_{E;\tau} = \frac{-27 \left[K_{\tau\bar{\tau}}^{-1} \left| \nabla_\tau W \right|^2 - 3\Lambda^6 \left| \tilde{H}(\tau) \right|^2 \right]'}{\Phi^3} + \frac{81 \left[K_{\tau\bar{\tau}}^{-1} \left| \nabla_\tau W \right|^2 - 3\Lambda^6 \left| \tilde{H}(\tau) \right|^2 \right] \Phi_\tau}{\Phi^4},$$

$|\tilde{H}(\tau)|^2$ can be factored out from $V_{E;\tau}$ for $\tilde{H}(\tau) \neq 0$

with

$$\Phi_\tau = \left[i - \frac{G_2(\tau)}{2\pi} (\tau - \bar{\tau}) \right] \Phi + 3 \left[i(\tau - \bar{\tau}) \eta^2(\tau) \overline{\eta^2(\tau)} \right] \left[\overline{H(\tau)} H'(\tau) \right] \exp \left\{ \xi \left[H(\tau) + \overline{H(\tau)} \right] \right\} .$$
$$+ 3 \left[i(\tau - \bar{\tau}) \eta^2(\tau) \overline{\eta^2(\tau)} \right] \left[\overline{H(\tau)} H(\tau) \right] \exp \left\{ \xi \left[H(\tau) + \overline{H(\tau)} \right] \right\} \xi H'(\tau) ,$$

To determine if a local extremum is a minimum, we need to assess the positive definiteness of the Hessian matrix

$$(H_f)_{ij} \equiv \frac{\partial^2 V}{\partial x_i \partial x_j} , \quad \text{for} \quad x_i = \Re\tau, \Im\tau,$$

The point $i\infty$ can be the local extrema.

$\tau = i\infty$ is a solution of $V_{E;\tau} = 0$ when $\xi > 0$.

It can be proven that the $V_E(i\infty)$ vanished there.

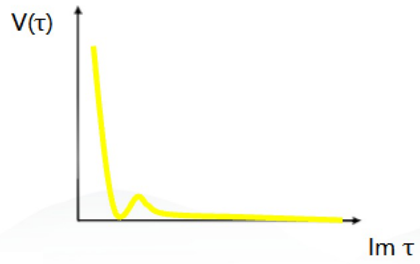
$\tau = i\infty$ is also a solution of $V_{E;\tau} = 0$ for

$$4\left(\frac{m}{2} + \frac{n}{3} + N\right) - \frac{1}{2} - 2\left(\frac{\tilde{m}}{2} + \frac{\tilde{n}}{3} + \tilde{N}\right) \geq 0$$

when $\xi = 0$,

assuming that $P(j(\tau))$ and $\tilde{P}(j(\tau))$ are degree N and $\tilde{N}$ polynomials in $j(\tau)$.

Vacuum Structure of the Scalar Potential



1. there is a run-away type local minimum at $\tau = i\infty$ for some input parameter choices, where the residual Z_N symmetry is unbroken.

2. We anticipate that quantum gravity effects may change the asymptotic behavior of the scalar potential, which can possibly stabilize the modulus field at certain point near $i\infty$ (for example, $\langle\tau\rangle = i\tau_0$ with large real τ_0).

3. Degenerate with other vacuums that satisfy $H(\tau)=0$ when the superpotential is nontrivial.

4. Additional SUSY breaking effects can possibly lift various Minkowski vacua to de Sitter vacua.

5. It is possible to stabilize the modulus fields within the fundamental domain to trigger both the modular symmetry breaking and CP breaking.

6. Possibly play a role in modulus fields driven inflation model.

Extreme Conditions

$$\left[K^{-1}_{\tau\bar{\tau}}\left|\nabla_\tau W\right|^2 - 3\Lambda^6\left|\tilde{H}(\tau)\right|^2\right]' \tag{3.6}$$
$$= \left(K^{-1}_{\tau\bar{\tau}}\right)_\tau \nabla_\tau W\overline{\nabla_\tau W} + K^{-1}_{\tau\bar{\tau}}(\nabla_\tau W)'\overline{\nabla_\tau W} + K^{-1}_{\tau\bar{\tau}}(\nabla_\tau W)(\overline{\nabla_\tau W})' - 3\Lambda^6\tilde{H}'(\tau)\overline{\tilde{H}(\tau)}\ .$$

Note that

$$(\nabla_\tau W)' = \Lambda^3\left[\tilde{H}'(\tau) + C_1(\tau,\bar{\tau})\tilde{H}(\tau)\right]'\ , \tag{3.7}$$
$$= \Lambda^3\left[\tilde{H}''(\tau) + [C_1(\tau,\bar{\tau})]_\tau\,\tilde{H}(\tau) + C_1(\tau,\bar{\tau})\tilde{H}'(\tau)\right]\ ,$$

and

$$(\nabla_\tau W)(\overline{\nabla_\tau W})' = \Lambda^3\left[\tilde{H}'(\tau) + C_1(\tau,\bar{\tau})\tilde{H}(\tau)\right]\left[\overline{C_1(\tau,\bar{\tau})}\right]_\tau\overline{\tilde{H}(\tau)}\ , \tag{3.8}$$

Local extrema needs vanishing H,H'and H" when H=0. (or other values when H≠0)

$$H'(i) = \begin{cases} -i\pi(12)^n\frac{[E_4(i)]^2}{[\eta(i)]^{12}}P(1728), & m=1 \\ 0, & m\neq 1 \end{cases}$$
$$H'(\omega) = \begin{cases} -\frac{i2\pi}{3}(-12)^{3m/2}\frac{E_6(\omega)}{[\eta(\omega)]^8}P(0), & n=1 \\ 0, & n\neq 1 \end{cases}\ . \tag{B.6}$$

and

$$H''(i) = \begin{cases} -\frac{2\pi^2}{[\eta(i)]^{24}}[E_4(i)]^4\left[\frac{n}{3}P(1728)(12)^{n-3} + (12)^n P'(1728)\right]\ , & m=0 \\ 0, & m=1 \\ -\frac{2\pi^2}{[\eta(i)]^{24}}[E_4(i)]^4(12)^n P(1728)\ , & m=2 \\ 0, & m>2 \end{cases}$$
$$H''(\omega) = \begin{cases} 0\ , & n\neq 2 \\ -\frac{8\pi^2}{9}\frac{[E_6(\omega)]^2}{[\eta(\omega)]^{16}}(-12)^{\frac{m}{2}}P(0)\ , & n=2 \end{cases} \tag{B.7}$$

See:
H.Fan, Fei Wang, Y.K.Zhang, Phys. Rev. D 112, 115040 (2025)
for detailed discussions.

Numerical Results

- Trivial superpotential, which amounts to $\tilde{H}(\tau) = 1$

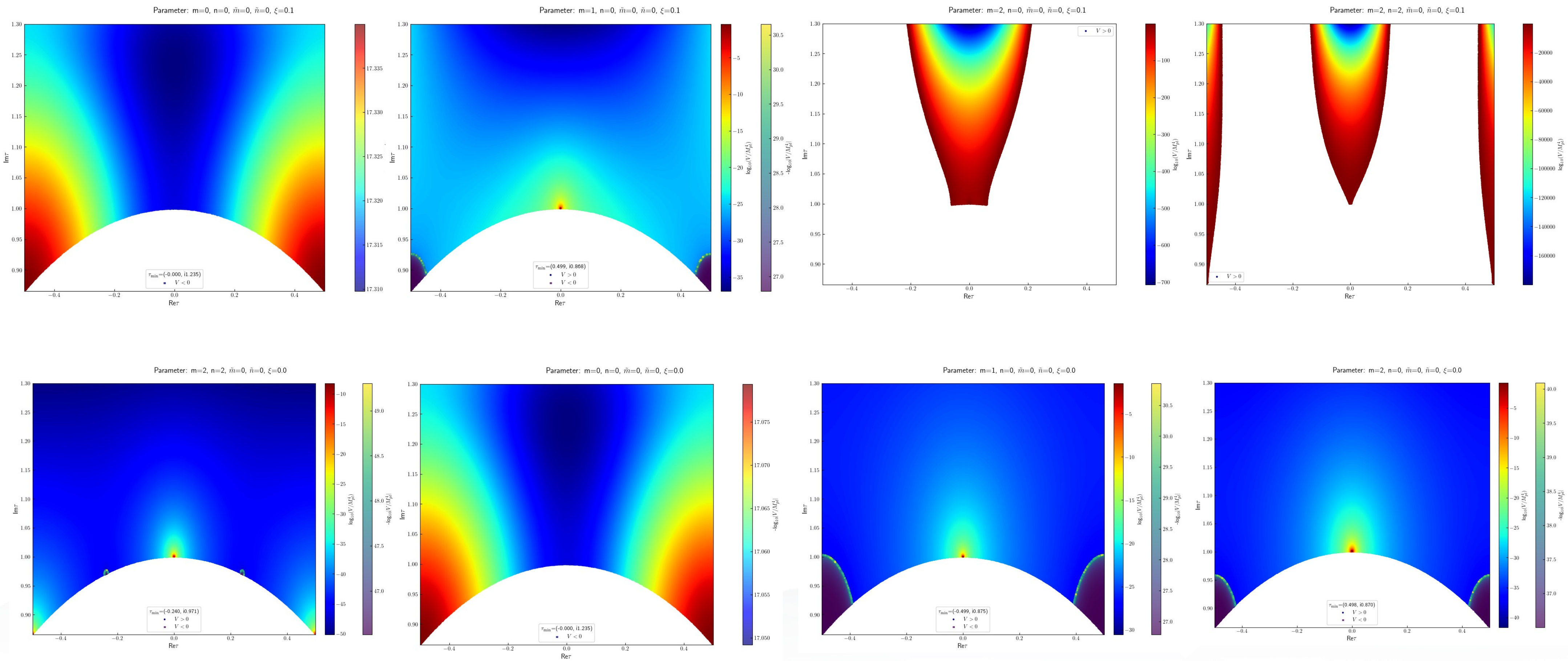


Figure 1. The values of $V(\tau, \tau^*)$ within the fundamental domain for different choices of (m, n) in $H(\tau)$ with $\tilde{m} = \tilde{n} = 0$ in the scenarios $\xi = 0.1$ (left panels) and $\xi = 0$ (right panels), respectively.

Numerical Results

- $\tilde{H}(\tau)$ takes the same form as $H(\tau)$

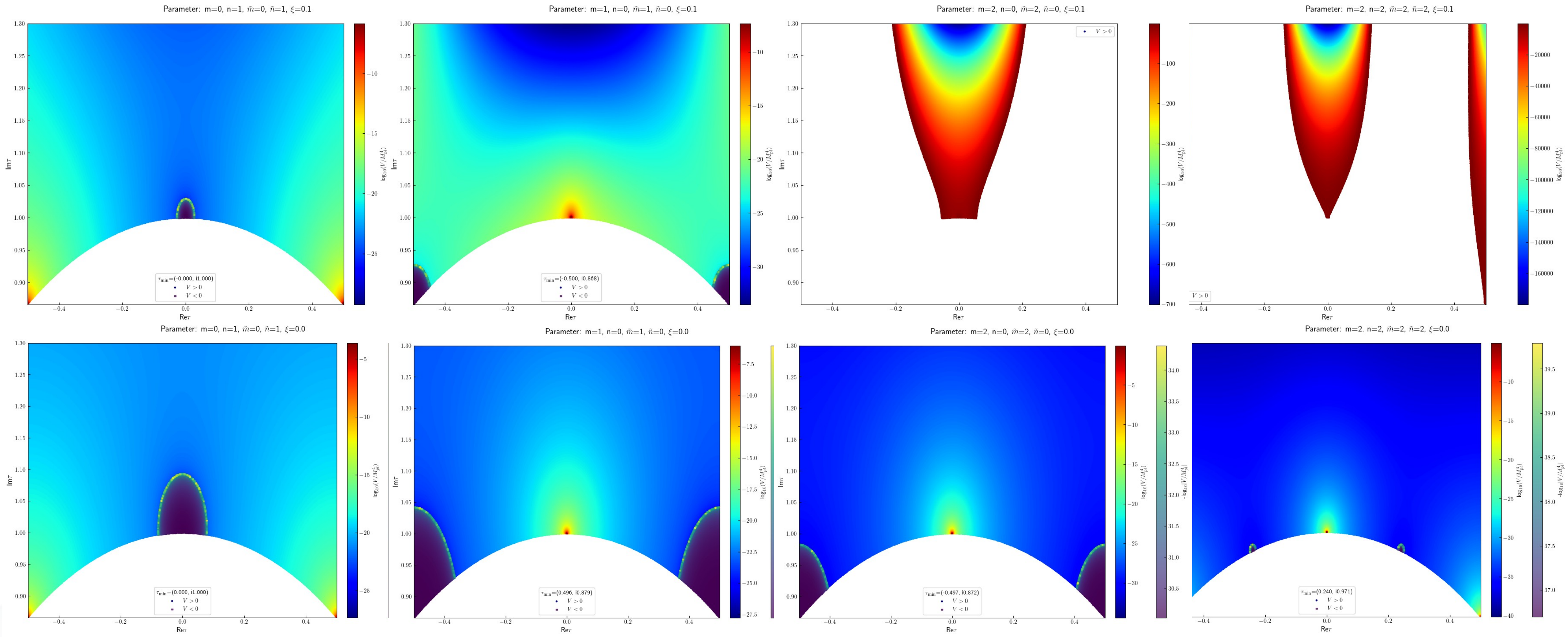


Figure 2. The values of $V(\tau, \tau^*)$ within the fundamental domain for the case $\tilde{H}(\tau)$ taking the same form as $H(\tau)$ in the scenarios $\xi = 0.1$ (left panels) and $\xi = 0$ (right panels), respectively.

Numerical Results

- General choice of $\tilde{H}(\tau)$ and $H(\tau)$, which adopts different choice of $(\tilde{m}, \tilde{n})$ and (m, n) (assuming $P(j(\tau)) = \tilde{P}(j(\tau)) = 1$).

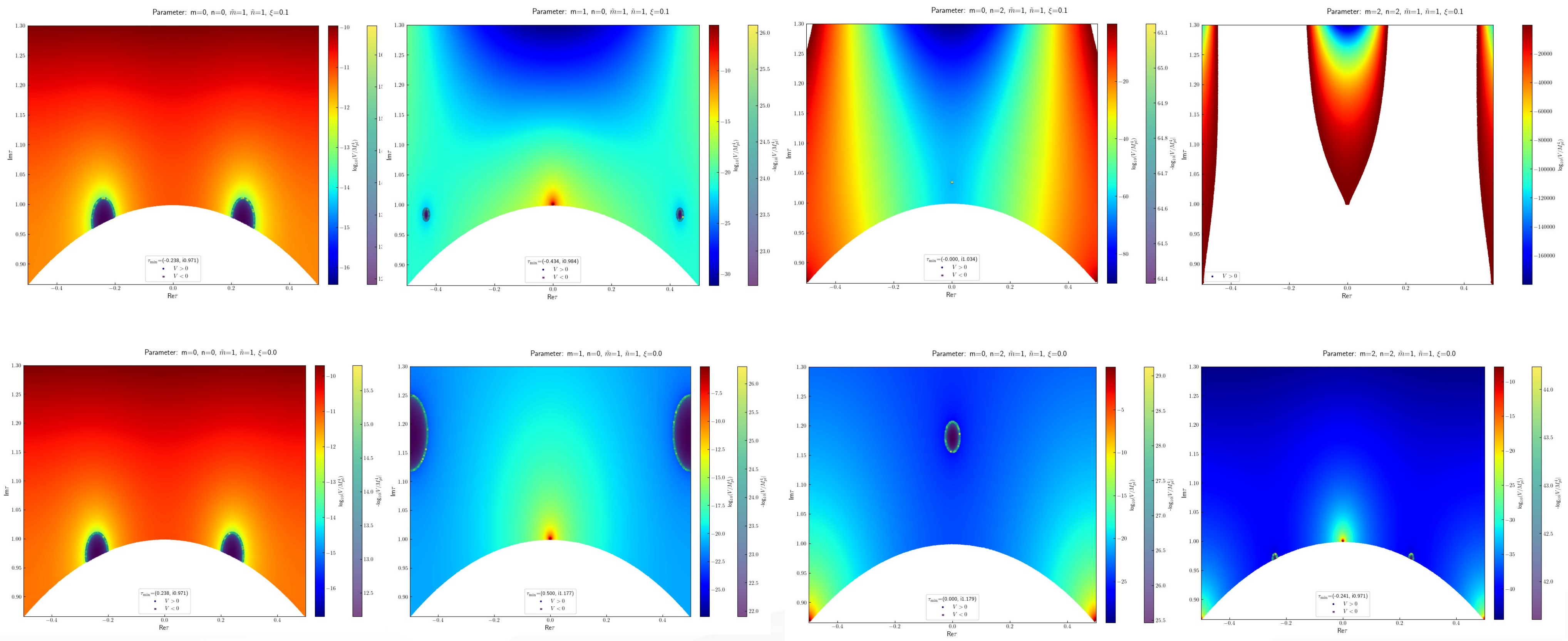


Figure 3. The values of $V(\tau, \tau^*)$ within the fundamental domain for the case $\tilde{H}(\tau)$ taking different form from $H(\tau)$ in the scenarios $\xi = 0.1$ (left panels) and $\xi = 0$ (right panels), respectively. Other settings are the same as that of the previous figures.

Numerical Results

- Most general choices of $\tilde{H}(\tau)$ and $H(\tau)$ with non-trivial $P(j(\tau))$ and $\tilde{P}(j(\tau))$.

m	n	$\tilde{m}$	$\tilde{n}$	ξ	$P(j(\tau))$	$\tilde{P}(j(\tau))$	$\tau_{\min}$	$V_{\min}/M_{\rm pl}^4$
0	1	0	1	0.1	$[j(\tau)-j(2i)]^2$	1	$0.448+1.223i$	0.0
1	0	0	0	0	1	$[j(\tau)-j(2i)]^2$	$-0.497+0.874i$	-1.103×10^{-5}
1	2	1	2	0.1	$[j(\tau)-j(0.1+2i)]^3$	$[j(\tau)-j(0.1+2i)]^3$	$0.1743+1.143i$	0.0

Table 1. The minimums of the scalar potential for some benchmark points with the most general choices of $\tilde{H}(\tau)$ and $H(\tau)$, adopting non-trivial $P(j(\tau))$ and $\tilde{P}(j(\tau))$.

When $\tilde{H}(\tau)=0$ and at the same time $H(\tau)\neq 0$, from the extrema condition of the scalar potential in (3.4), it is obvious that $\tilde{H}''(\tau)=\tilde{H}'(\tau)=0$ are sufficient conditions for $V_{E;\tau}=0$. We have the following discussions on such conditions (see our previous work for some relevant details)

The condition $H''(\tau)=0$ can be satisfied for $P'(j(\tilde{\tau}_i))=P''(j(\tilde{\tau}_i))=0$, which requires $\tau=\tilde{\tau}_i$ being a root of $P(j(\tau))$ with multiplicity three or greater. Therefore, the most general form of $P(j(\tau))$ should be

$$P(j(\tau))=\pm g(j(\tau))\prod_i(j(\tau)-j(\tilde{\tau}_i))^{k_i}, \qquad k_i\in\mathbb{N},$$

with $k_i\geq 3$ and $g(j(\tau))$ an arbitrary positive polynomial function of $j(\tau)$.

H.Fan, Fei Wang, Y.Zhang, Phys. Rev. D 112, 115040 (2025)

02

Factorable Multiple Modulus Fields Stabilization in the Jordan frame supergravity

Factoriabl Multiple Modular

finite modular transformations $\gamma_1, \ldots, \gamma_M$ in $\Gamma^1_{N_1} \times \Gamma^2_{N_2} \times \cdots \times \Gamma^M_{N_M}$

$$\gamma_J : \tau_J \to \gamma_J \tau_J = \frac{a_J \tau_J + b_J}{c_J \tau_J + d_J} ,$$

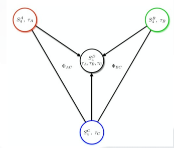
$$\phi_i(\tau_1, \ldots, \tau_M) \to \phi_i(\gamma_1\tau_1, \ldots, \gamma_M\tau_M) = \prod_{J=1,\ldots,M} (c_J\tau_J + d_J)^{-2k_{i,J}} \bigotimes_{J=1,\ldots,M} \rho_{I_{i,J}}(\gamma_J)\phi_i(\tau_1, \tau_2, \ldots, \tau_M)$$

$$Y_{(I_{Y,1},\ldots,I_{Y,M})}(\tau_1, \ldots, \tau_M) \to Y_{(I_{Y,1},\ldots,I_{Y,M})}(\gamma_1\tau_1, \ldots, \gamma_M\tau_M) = \prod_{J=1,\ldots,M} (c_J\tau_J + d_J)^{2k_{Y,J}} \bigotimes_{J=1,\ldots,M} \rho_{I_{Y,J}}(\gamma_J) Y_{(I_{Y,1},\ldots,I_{Y,M})}(\tau_1, \ldots, \tau_M) .$$

$$K(\phi_i, \overline{\phi}_i; \tau_1, \ldots, \tau_M, \overline{\tau}_1, \ldots, \overline{\tau}_M) = -\sum_{J=1,\ldots,M} h_J \log(-i\tau_J + i\overline{\tau}_J) + \sum_i \frac{\overline{\phi}_i \phi_i}{\prod_{J=1,\ldots,M} (-i\tau_J + i\overline{\tau}_J)^{2k_{i,J}}} ,$$

$$W(\phi_i; \tau_1, \ldots, \tau_M) = \sum_n \sum_{\{i_I, \cdots, i_n\}} \left(Y_{(I_{Y,1},\ldots,I_{Y,M})} \phi_{i_1} \cdots \phi_{i_n} \right)_{\mathbf{1}} ,$$

See King&Y.L.Zhou, 1906.02208
Du& F.Wang, JHEP01(2023)036



Can be broken into single modulus case via Higgs mechanism or by boundary conditions

Factoriabl Multiple Modulus Scenarios

For multiple modular scenarios, the modular invariant frame function can be written as

$$\tilde{\Phi}(\tau_k, \bar{\tau}_k) = -3 \prod_{k=1}^{n} \left[-i(\tau_k - \bar{\tau}_k)\eta^2(\tau_k)\overline{\eta^2(\tau_k)} \right] \prod_{l=1}^{n} \exp\left\{ -\frac{\lambda_{j;l}}{3}|j(\tau_l)|^2 - \frac{\lambda_{E;l}}{3} E(\tau_l, s_l) \right\}$$
$$\prod_{m=1}^{n} \exp\left\{ \xi_m \left[H_m + \overline{H_m} \right] \right\} \left| H_m \right|^{2t} .$$

with non-holomoprhic Eisenstein serie

$$E(\tau, s) = \sum_{(m,n)\neq(0,0)} \frac{y^s}{|m\tau + n|^{2s}}, \quad s > 1$$

It is a Maass form that satisfies

$$\Delta_L E(\tau, s) \equiv y^2 \partial_\tau \partial_{\bar{\tau}} E(\tau, s) = s(s-1)E(\tau, s)$$

Leading to Positive definite Kahler metric

$$j'(\tau) = -2\pi i \frac{E_6(\tau)E_4^2(\tau)}{[\eta(\tau)]^{24}}$$

The modular-invariant function $H(\tau)$ in the superpotential should be replaced by general form $H(\tau_i)$ with

$$\tilde{H}(\tau_1, \cdots, \tau_n) = \sum_{i=1}^{n} P_1(H(\tau_1)) \cdots P_i(H(\tau_i)) \cdots P_n(H(\tau_n)).$$

Factoriabl Multiple Modulus Scenarios

We choose

$$\tilde{\Phi}(\tau_k,\bar{\tau}_k) = -3\prod_{k=1}^{n}\left[-i(\tau_k-\bar{\tau}_k)\eta^2(\tau_k)\overline{\eta^2(\tau_k)}\right]\prod_{l=1}^{n}\exp\left\{-\frac{\lambda_{j;l}}{3}|j(\tau_l)|^2+\xi_m\left[H_m+\overline{H_m}\right]\right\}$$

and obtain

$$K = -3\sum^{n}\ln\left[-i(\tau_k-\bar{\tau}_k)\eta^2(\tau_k)\overline{\eta^2(\tau_k)}\right]+\sum_{l=1}^{n}\lambda_{j;l}|j(\tau_l)|^2$$

by the relations

$$\tilde{\Phi}(\tau_i,\bar{\tau}_i) = -3M_P^2 e^{-\frac{1}{3M_P^2}K(\tau,\bar{\tau})}$$

The superpotential is

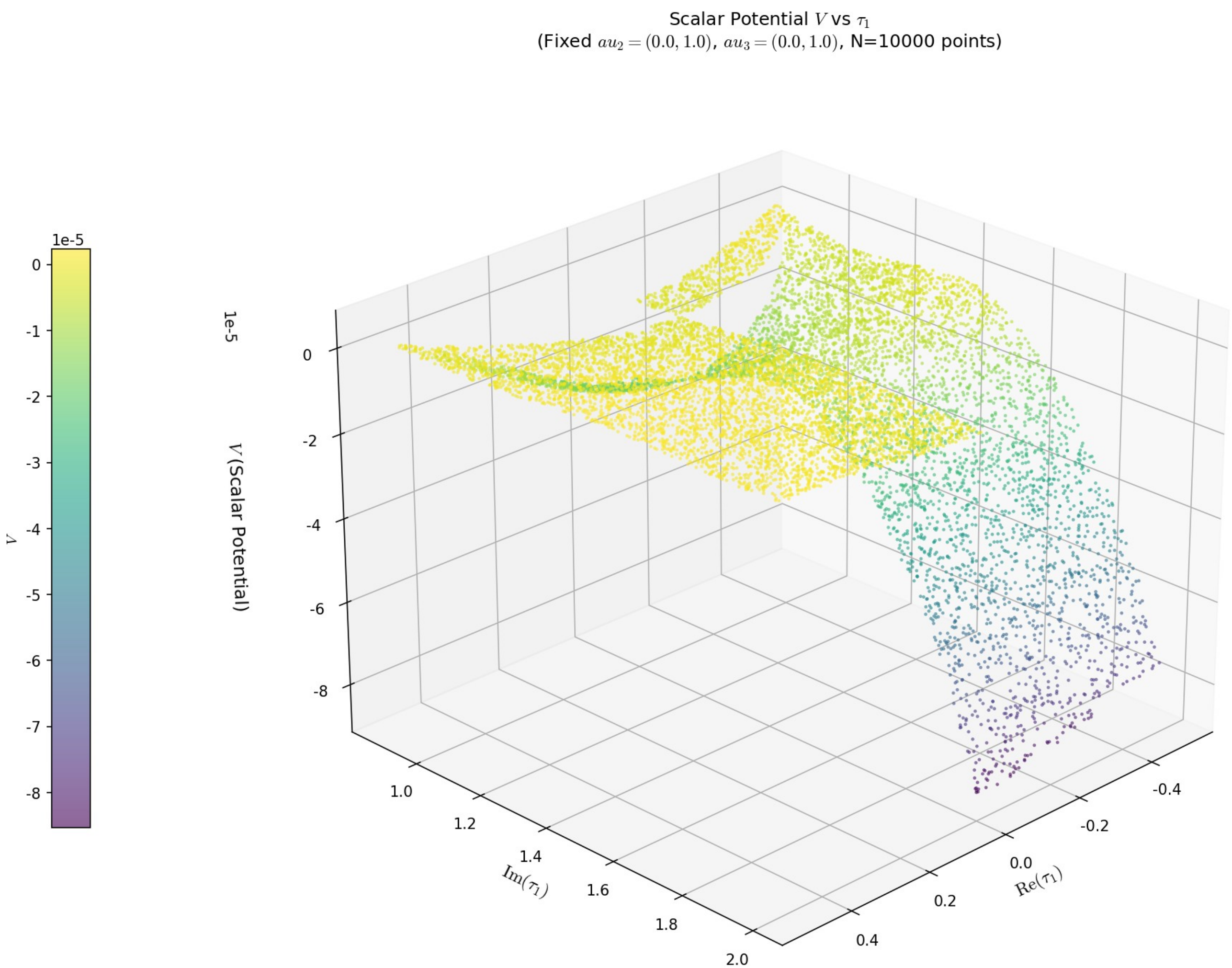
$$W = c_1^3\sum_{i=1}^{3}H(\tau_i)$$

The scalar potential can be calculated to be

$$V(\tau_i,\bar{\tau}_i) = c_1^6\prod_{i=1}^{3}\left[(2\Im\tau_i)^{-3}|\eta(\tau_i)|^{-12}e^{\lambda_{j;i}|j(\tau_i)|^2}\right]\left[\sum_{i=1}^{3}\left(\frac{3}{4(\Im\tau_i)^2}+\lambda_{j;i}|j'(\tau_i)|^2\right)^{-1}|\mathcal{D}_i|^2-3\left|\sum_{m=1}^{3}H(\tau_m)\right|^2\right]$$

with

$$\mathcal{D}_i = H'(\tau_i)+\left(\sum_{m=1}^{3}H(\tau_m)\right)\left(-\frac{3}{\tau_i-\bar{\tau}_i}-\pi i E_2(\tau_i)+\lambda_{j;i}j'(\tau_i)\overline{j(\tau_i)}\right)$$



A slice of the scalar potential

V(τ1,τ2,τ3)

THE END

Thanks