Lepton Flavor Structure near a Modular Fixed Point

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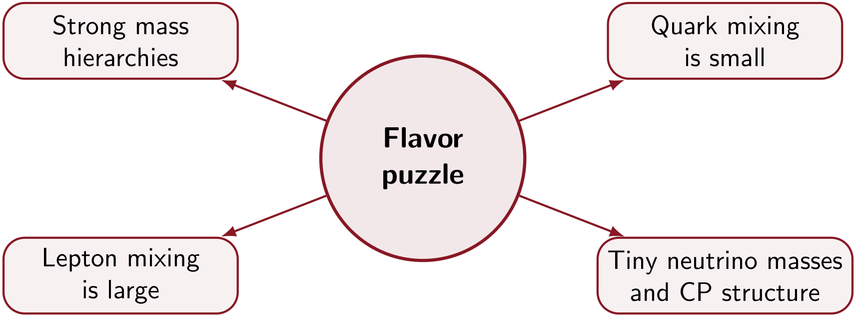
The 3rd Workshop on Grand Unified Theories:
Phenomenology and Cosmology (GUTPC 2026)

Work in progress

Flavor structure calls for an organizing principle



What we want to understand is not one isolated number, but the whole observed pattern.

A symmetry-based organizing principle is therefore desirable.

Traditional flavor symmetry and modular symmetry

Common goal: explain flavor structure by symmetry

Traditional flavor symmetry

Model input:
flavons and alignments usually add many free parameters

Typical structure:
Yukawa entries are often controlled more independently

Result:
more freedom, but usually less predictivity

Modular symmetry

Advantage 1:
modular invariance can greatly reduce the number of free parameters

Advantage 2:
shared modular forms can correlate different couplings

Result:
a more constrained and predictive structure

In modular models, the modulus τ can both reduce parameters and relate different couplings.

The modulus τ parametrizes torus geometry

A two-dimensional torus is characterized, up to overall rescaling and rotation, by a single complex modulus:

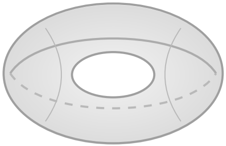
$$\tau = \frac{\omega_2}{\omega_1}, \qquad \mathrm{Im}\,\tau > 0.$$

Equivalent lattice bases describe the same torus, so τ is identified under

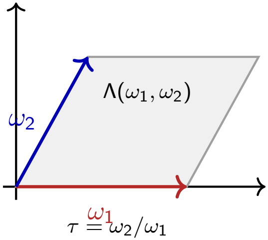
$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

The modular group is generated by

$$S : \tau \to -\frac{1}{\tau}, \qquad T : \tau \to \tau + 1.$$



A complex torus



A lattice basis (ω_1, ω_2) determines the torus shape.

The fixed point $\tau = i$ preserves residual S and CP

Residual modular symmetry

$$S : \tau \to -\frac{1}{\tau}, \qquad -\frac{1}{i} = i.$$

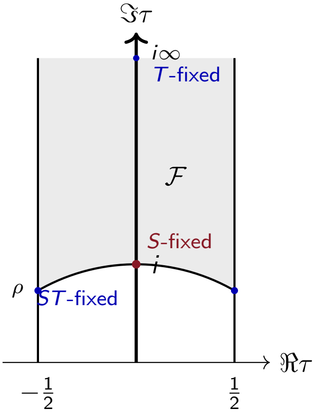
Hence $\tau = i$ is an S-fixed point.

Compatible generalized CP

In a symmetric basis, generalized CP may be taken as

$$\tau \to -\tau^*,$$

so exact i is also CP-preserving.



The point i is singled out by residual S and is also CP-preserving. *arXiv:1905.11970*.

At $\tau = i$, modular covariance becomes a direct constraint

A modular form $Y(\tau)$ of weight k transforms as

$$Y(\gamma\tau) = (c\tau + d)^k \, \rho(\gamma) \, Y(\tau).$$

Now take

$$\gamma = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad S : \tau \to -\frac{1}{\tau}.$$

Since $Si = i$, the fixed-point condition gives

$$Y(i) = Y(Si) = i^k \, \rho(S) \, Y(i).$$

Meaning

At a generic point, modular covariance is a transformation law. At a fixed point, it becomes an algebraic constraint on $Y(i)$ itself.

For an A_4 triplet

$$Y(i) = i^k \, \rho_3(S) \, Y(i).$$

So the allowed direction of $Y(i)$ is controlled by the eigenspaces of $\rho_3(S)$.

This is the exact-i starting point of the whole classification.

Exact-i induces a $D \oplus O$ decomposition

$$\rho_3(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad e = (1,1,1)^T, \qquad \rho_3(S)e = e.$$

$$P_D \equiv \frac{I_3 + \rho_3(S)}{2}, \qquad P_O \equiv \frac{I_3 - \rho_3(S)}{2}, \qquad \mathbb{C}^3 = D \oplus O, \quad D = \mathrm{Im}\, P_D, \; O = \mathrm{Im}\, P_O.$$

Residual-S eigenspaces

$$D = \mathrm{span}\{e\}, \qquad O = \{x \in \mathbb{C}^3 \mid \sum_i x_i = 0\}.$$

$$P_D = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad P_O = I_3 - P_D.$$

Weight branches at exact i

From

$$Y(i) = i^k \rho_3(S)\, Y(i),$$

one finds

$$k = 4n : \qquad Y^{(4n)}(i) \in D,$$

$$k = 4n+2 : \qquad Y^{(4n+2)}(i) \in O.$$

In the relevant exact-i branch, column alignment arises naturally

In minimal seesaw, what enters neutrino physics is the geometry of the actual Dirac columns after the contraction maps act:

$$d_a(\tau) = y_a\, C_a\, Y^{(a)}(\tau).$$

In the relevant exact-i branch, the fixed-point geometry drives the relevant columns into the same distinguished direction. Therefore, at exact i,

$$d_1(i) \parallel d_2(i).$$

This immediately implies

$$\mathrm{rank}\, M_D(i) = 1 \qquad \Longrightarrow \qquad \mathrm{rank}\, M_\nu^{(0)} = 1, \qquad \dim\ker M_\nu^{(0)} = 2.$$

So the leading neutrino structure leaves a two-dimensional unresolved light plane. This is precisely the kind of starting point we want for the later near-i analysis.

Structural meaning

The unresolved light plane is not imposed by hand; it is the natural exact-i starting point in the relevant branch.

A convenient local coordinate around i

Introduce the local coordinate

$$\omega \equiv \frac{\tau - i}{\tau + i}, \qquad \tau = i\,\frac{1+\omega}{1-\omega}.$$

$$\omega = 0 \quad \Longleftrightarrow \quad \tau = i.$$

In this variable, the residual action becomes especially simple:

$$S : \omega \to -\omega.$$

Notation

Below we use ε to denote the small near-i expansion parameter, schematically

$$\omega = O(\varepsilon).$$

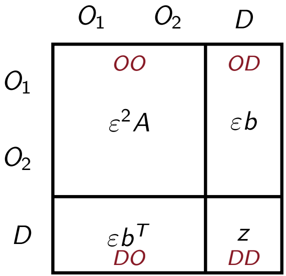
So the fixed point i is mapped to the origin, and the residual S action becomes linear in ω, which makes the small-deviation analysis more transparent.

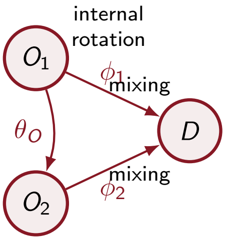
First key result: block hierarchy

At exact i, the leading neutrino structure is rank one, so the two light directions remain unresolved. For $\omega = O(\varepsilon)$, the mixings toward D appear at first order, while the induced dynamics inside the light plane starts only at second order.

$$M(\varepsilon) = \begin{pmatrix} \varepsilon^2 A & \varepsilon b \\ \varepsilon b^T & z \end{pmatrix} + \mathcal{O}(\varepsilon^3), \qquad M_{OO} = O(\varepsilon^2), \quad M_{OD} = O(\varepsilon), \quad M_{DD} = O(1).$$

So the light-plane rotation can become large, while the two mixings toward D remain suppressed.





Parent classes and canonical parent patterns

The parent classes are labeled by (r_O, n_{OD}), where r_O is the leading light-plane rank and n_{OD} is the number of unsuppressed O-D mixings.

$$A_{(1)}: \ \text{rank-1 } O\text{-plane block}, \qquad A_{(2)}: \ \text{rank-2 } O\text{-plane block}.$$

Class I $(1, 0)$

$$M \sim \begin{pmatrix} \varepsilon^2 A_{(1)} & \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix} \\ \begin{pmatrix} \varepsilon & \varepsilon \end{pmatrix} & 1 \end{pmatrix}$$

rank-one O-plane + two suppressed O-D mixings

Class III $(1, 1)$

$$M \sim \begin{pmatrix} A_{(1)} & \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \\ \begin{pmatrix} 1 & \varepsilon \end{pmatrix} & 1 \end{pmatrix}$$

rank-one O-plane + one unsuppressed O-D mixing

Class II $(2, 0)$

$$M \sim \begin{pmatrix} \varepsilon^2 A_{(2)} & \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix} \\ \begin{pmatrix} \varepsilon & \varepsilon \end{pmatrix} & 1 \end{pmatrix}$$

full-rank O-plane, but both O-D mixings are suppressed

Class IV $(2, 1)$

$$M \sim \begin{pmatrix} A_{(2)} & \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \\ \begin{pmatrix} 1 & \varepsilon \end{pmatrix} & 1 \end{pmatrix}$$

full-rank O-plane + one unsuppressed O-D mixing

Class III: generic and small-determinant cases

A minimal Class-III representative is

$$M_{III}^{(0)} = \begin{pmatrix} m & 0 & \beta \\ 0 & 0 & \varepsilon\gamma \\ \beta & \varepsilon\gamma & z \end{pmatrix}, \qquad m, \beta, \gamma, z = O(1),$$

with m, z the $O(1)$ scales of O_1, D, and β the large O-D mixing.
Define

$$\Delta \equiv mz - \beta^2,$$

which is the determinant of the enhanced (O_1, D) block.
If

$$\Delta = O(1),$$

this is the generic Class-III case, which we call **III-A**. Then that block still contains two $O(1)$ states, and the remaining O_2-D mixing is only $O(\varepsilon)$.
The promising special case is **III-B**, defined by

$$\boxed{\text{III-B}: \qquad mz - \beta^2 = O(\varepsilon).}$$

Then one state becomes light, so the residual O_2 coupling can generate the second large angle.

Class IV: generic and aligned cases

A canonical Class-IV representative is

$$M_{IV}^{(0)} = \begin{pmatrix} m & \mu & \beta \\ \mu & n & \varepsilon\gamma \\ \beta & \varepsilon\gamma & z \end{pmatrix}, \qquad A = \begin{pmatrix} m & \mu \\ \mu & n \end{pmatrix},$$

with A the O-plane block and $(\beta, \varepsilon\gamma)^T$ the O-D mixing vector before going to the principal basis.
Diagonalizing the O-plane block gives

$$R_{12}(\theta_O)^T A R_{12}(\theta_O) = \mathrm{diag}(a_1, a_2), \qquad \tan 2\theta_O = \frac{2\mu}{n - m},$$

where θ_O is the internal rotation inside the light plane and a_1, a_2 are the principal values of A.
Generically, both rotated O-D mixings are $O(1)$. This generic Class-IV case, which we call **IV-A**, therefore tends to make all three angles large.
The promising special case is **IV-B**, defined by

$$\boxed{\text{IV-B}: \qquad u = v_i + O(\varepsilon),}$$

where u is the mixing direction toward D in the O plane and v_i is one principal direction of A. Then one rotated O-D mixing remains small.

Structural prediction of the current classification

Class	Status	Reason
I	not viable	suppressed O-D hierarchy gives at most one large angle
II	usually not viable	richer O-plane, but both mixings toward D remain suppressed
III	needs refinement	generic case III-A is too weak; the promising case is III-B
IV	needs refinement	generic case IV-A usually gives all three angles large; the promising case is IV-B

Promising refined cases:

$$\text{III-B}: \qquad mz - \beta^2 = O(\varepsilon),$$

$$\text{IV-B}: \qquad u = v_i + O(\varepsilon).$$

Summary

1. The fixed point $\tau = i$ provides the adapted language
$$\mathbb{C}^3 = D \oplus O.$$

2. The first key result is
$$M_{OO} = O(\varepsilon^2), \qquad M_{OD} = O(\varepsilon), \qquad M_{DD} = O(1),$$
so this hierarchy can naturally generate at most one large angle.

3. Parent classes III and IV are not yet positive results by themselves: generic III-A is too weak, while generic IV-A usually gives all three angles large.

4. The current promising cases are
$$\text{III-B}: \quad mz - \beta^2 = O(\varepsilon), \qquad \text{IV-B}: \quad u = v_i + O(\varepsilon).$$

The next task is to construct explicit A_4 realizations of these promising cases.