Dispersive analysis of $J/\psi \to \pi^0\gamma^\star$ transition form factor

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$J/\psi \to \pi^0 e^+ e^-$: motivation

- Sensitive to the transition form factor (TFF) of $J/\psi \to \pi^0 \gamma^\star$

- The branching fractions of $J/\psi \to P e^+ e^-$ $(P = \pi^0, \eta, \eta')$
  BESIII, PRD 89, 092008 (2014)

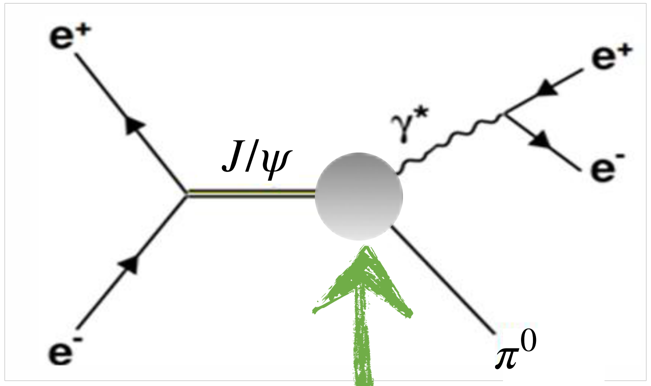
- Exp. v.s. VMD

  η and η' $\surd$; π^0 $\times$

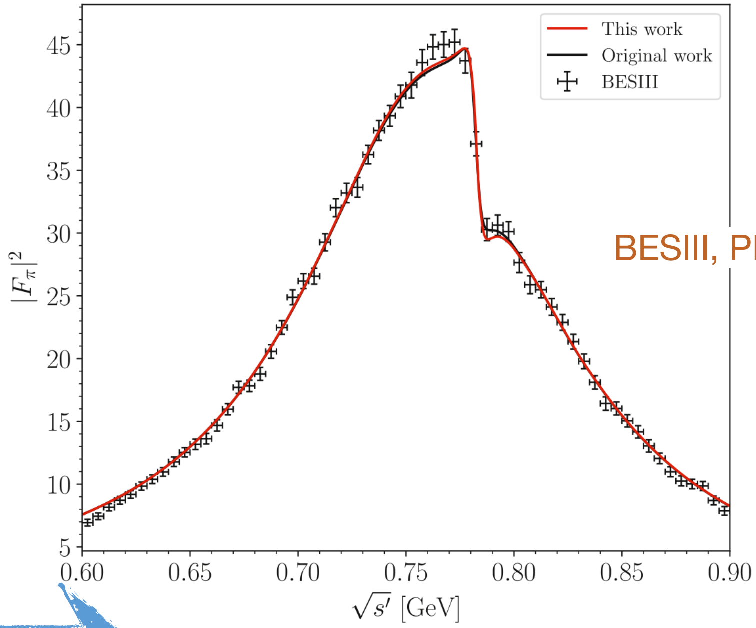
- VMD estimates:

  charmonium dominance,
  isospin-breaking transition

- Isospin-conserving manner, dominated by light-quark degrees of freedom ($2\pi\ldots$)



Transition form factor for $J/\psi \to \pi^0 \gamma^\star$



BESIII, PLB 753 (2016)

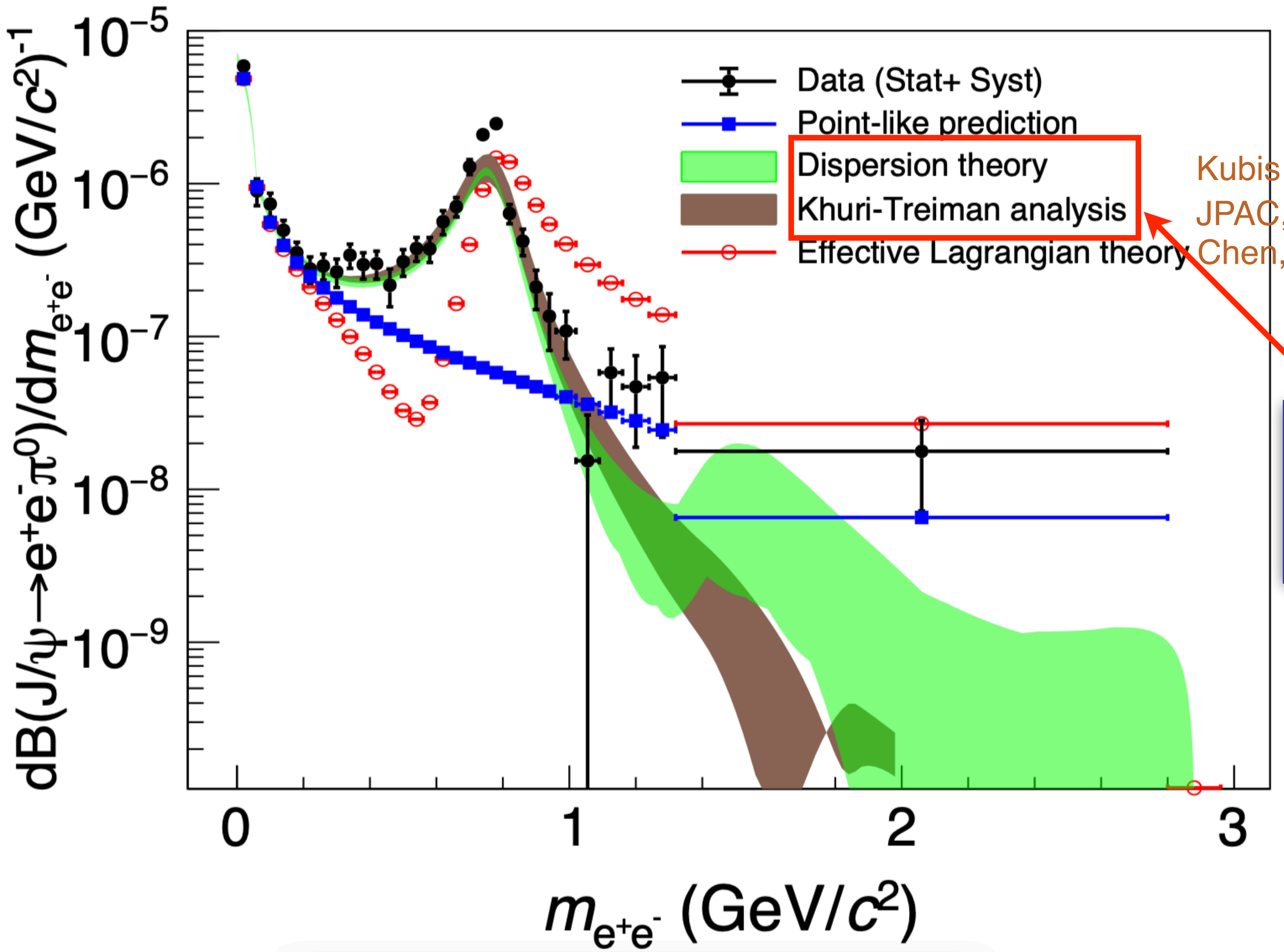
$\rho - \omega$ interference

BESIII result

BESIII, *PRD* 112, L011101 (2025)

- A significant resonant structure corresponding to ρ, ω (100 billion J/ψ events)

$$\frac{\mathrm{dBR}_{\psi\to\pi^0\ell^+\ell^-}}{\mathrm{BR}_{\psi\to\pi^0\gamma}\ \mathrm{d}s} = \frac{16\alpha}{3\pi}\left(1+\frac{2m_\ell^2}{s}\right)\frac{q_\ell(s)q^3_{\psi\pi^0}(s)}{\left(M_\psi^2-M_{\pi^0}^2\right)^3}\left|F_{\psi\pi^0}(s)\right|^2 , \quad F_{\psi\pi^0}(s) = \frac{f_{\psi\pi^0}(s)}{f_{\psi\pi^0}(0)}$$



Kubis and Niecknig, PRD (2015)

JPAC, PRD (2023)

Chen, Guo and Zou, PRD (2015)

Dispersive analyses using Khuri-Treiman equation

$J/\psi \to \pi^0\gamma^\star$ transition form factor

- The $J/\psi \to \pi^0\gamma^\star$ transition form factor: Landsberg, Phys. Rept. (1985)

$$\left\langle \pi^0(p) \middle| j_\mu(0) \middle| \psi\left(p_V, \lambda\right) \right\rangle = -i\epsilon_{\mu\nu\alpha\beta}\epsilon^\nu\left(p_V, \lambda\right) p^\alpha q^\beta f_{\psi\pi^0}(s)$$

- (P-wave) $\pi\pi$ intermediate state:

$$\frac{\text{disc}}{2i} f_{\psi\pi^0}^{(2\pi)}(s) = \frac{s\sigma_\pi^3(s)}{96\pi} F_\pi^{V*}(s) f_1(s)$$

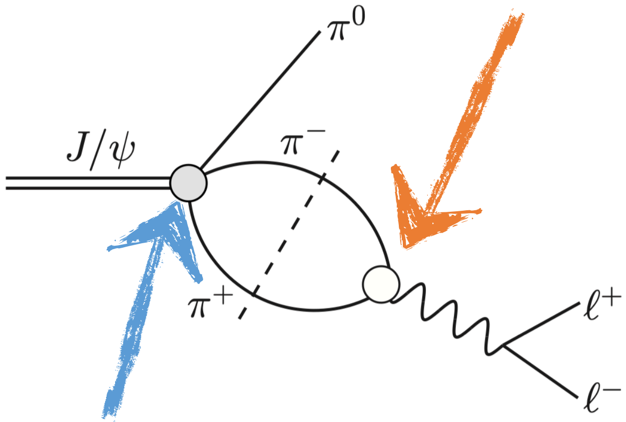
Koepp, PRD (1974)

- **Dispersion relation** (unsubtracted):

$$f_{\psi\pi^0}(s) = \frac{1}{2\pi i}\int_{4M_\pi^2}^{\infty} \text{d}s' \frac{\text{disc} f_{\psi\pi^0}(s')}{s' - s - i\epsilon},$$

π vector form factor

$F_\pi^V(s)$



$f_1(s)$

P-wave $J/\psi \to 3\pi$ decay amplitude

$\rho - \omega$ mixing

- F_π^V depends on $\rho - \omega$ mixing
- Naive approach leads to a spectral function in which the double discontinuities of 2π and 3π intermediate states may no longer cancel

$$F_\pi^V(s) \to \left(1 + \epsilon_{\rho\omega}\frac{s}{M_\omega^2 - s - iM_\omega\Gamma_\omega}\right)F_\pi^V(s) \quad \Rightarrow \quad \text{Im}\left[\frac{\text{disc}f_{\psi\pi^0}(s)\,|_{2\pi}}{2i}\right] \neq 0$$

- Two-potential formalism + Dispersion relation
  - Three channel system: $2\pi,\ 3\pi,\ \ell^+\ell^-$
  - Generalize to a dispersion relation including a consistent treatment $\rho - \omega$ mixing
  - P-wave $J/\psi \to 3\pi$ decay amplitude is construct from the **Khuri-Treiman** (KT) formalism Khuri and Treiman, Phys. Rev. (1960)…
  - Ensure unitarity, analyticity and crossing symmetry

Dispersive representation

- Modified dispersion relation (once sub.)

See also $\eta'\gamma^\star\gamma^\star$ TFF, Holz et.al., EPJC (2022)

$$
\begin{aligned}
f_{\psi\pi^0}^{(2\pi,3\pi)}(s) = f_{\psi\pi^0}^{(2\pi)}(0) \\
&+ \frac{s}{96\pi^2}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \frac{\sigma_\pi^3(s')F_\pi^{V*}(s')f_1(s')}{s'-s-i\epsilon}\left[1+\frac{\tilde{\epsilon}_{\rho\omega}s}{M_\omega^2-s-iM_\omega\Gamma_\omega}\right] \\
&+ \frac{w_{\psi\omega\pi}s}{M_\omega^2-s-iM_\omega\Gamma_\omega}\left[1+\frac{\tilde{\epsilon}_{\rho\omega}s}{48\pi^2 g_{\omega\gamma}^2}\int_{4M_\pi^2}^{\infty}\mathrm{d}s'\frac{\sigma_\pi^3(s')|F_\pi^V(s')|^2}{s'(s'-s-i\epsilon)}\right] \\
&+ \frac{f_{\psi\pi^0}^{2\pi,3\pi}(0)w_{\psi\phi\pi}s}{M_\phi^2-s-iM_\phi\Gamma_\phi},
\end{aligned}
$$

$I = 1$ "$\rho-\omega$" mixing

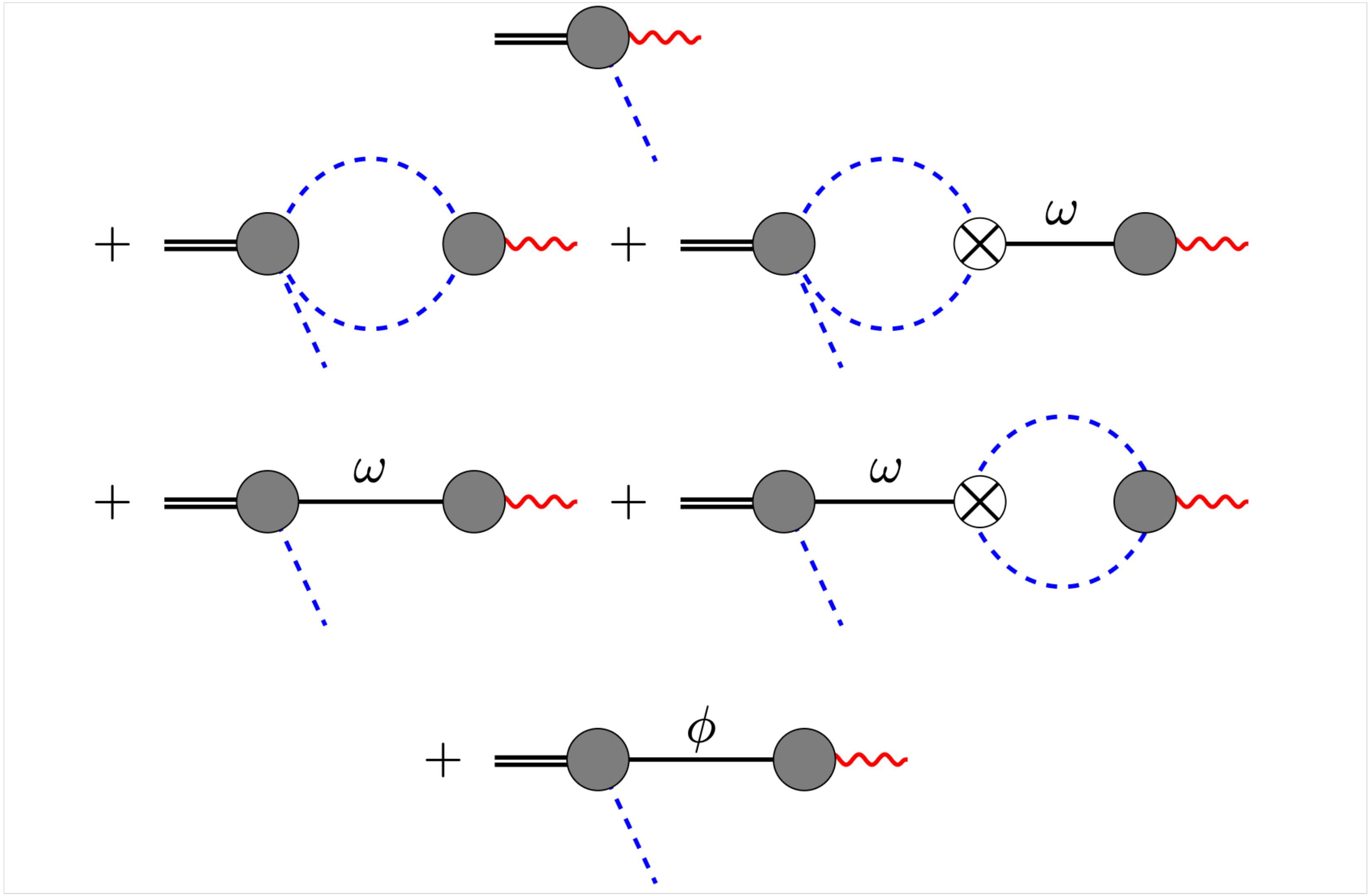
$I = 0$ "$\omega-\rho$" mixing

- $\rho-\omega$ mixing parameter $\tilde{\epsilon}_{\rho\omega} = \epsilon_{\rho\omega} - e^2 g_{\omega\gamma}^2 = 1.65(2)\times 10^{-3}$
- Weights $w_{\psi\omega(\phi)\pi}$
- Coupling $g_{\omega\gamma} = \sqrt{\frac{3\Gamma\left(\omega\to e^+e^-\right)}{4\pi\alpha^2 M_\omega}} = 0.0606(9)$ from VMD model
- P-wave $J/\psi \to 3\pi$ decay amplitude $f_1(s)$

Dispersive representation

- Modified dispersion relation (once sub.)

See also $\eta'\gamma^\star\gamma^\star$ TFF, Holz et.al., EPJC (2022)



Dispersive representation

- Modified dispersion relation (once sub.)

See also $\eta'\gamma^\star\gamma^\star$ TFF, Holz et.al., EPJC (2022)

$$
\begin{aligned}
f_{\psi\pi^0}^{(2\pi,3\pi)}(s) = f_{\psi\pi^0}^{(2\pi)}(0) \\
&+\frac{s}{96\pi^2}\int_{4M_\pi^2}^{\infty}\mathrm{d}s'\frac{\sigma_\pi^3(s')F_\pi^{V*}(s')f_1(s')}{s'-s-i\epsilon}\left[1+\frac{\tilde{\epsilon}_{\rho\omega}s}{M_\omega^2-s-iM_\omega\Gamma_\omega}\right] \\
&+\frac{w_{\psi\omega\pi}s}{M_\omega^2-s-iM_\omega\Gamma_\omega}\left[1+\frac{\tilde{\epsilon}_{\rho\omega}s}{48\pi^2 g_{\omega\gamma}^2}\int_{4M_\pi^2}^{\infty}\mathrm{d}s'\frac{\sigma_\pi^3(s')|F_\pi^V(s')|^2}{s'(s'-s-i\epsilon)}\right] \\
&+\frac{f_{\psi\pi^0}^{2\pi,3\pi}(0)w_{\psi\phi\pi}s}{M_\phi^2-s-iM_\phi\Gamma_\phi},
\end{aligned}
$$

$I=1$ "$\rho-\omega$" mixing

$I=0$ "$\omega-\rho$" mixing

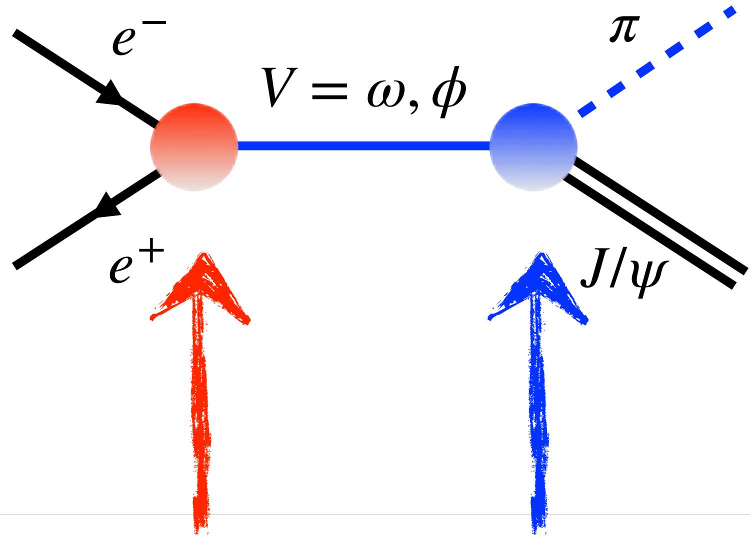
This nontrivial consistency condition is indeed satisfied

$$
\mathrm{Im}\left[\frac{1}{2i}\,\mathrm{disc}\,f_{\psi\pi^0}(s)\Big|_{2\pi}+\frac{1}{2i}\,\mathrm{disc}\,f_{\psi\pi^0}(s)\Big|_{3\pi}\right]=0
$$

!

Isoscalar-vector pole contribution

- Weights $w_{\psi\omega(\phi)\pi}$ are determined by the pole dominance model
- Matching the expressions of the $e^+e^- \to J/\psi\pi^0$ cross section at isoscalar-vector poles



$$f_{\psi\pi}(s)\Big|_{s\to M_V^2} = \frac{w_{\psi V\pi}s}{M_V^2 - s - iM_V\Gamma_V}$$

$$|w_{\psi V\pi}| = \sqrt{\frac{72M_\psi^3\Gamma_\psi\Gamma_\omega \mathcal{BR}(V\to e^+e^-)\mathcal{BR}(J/\psi\to V\pi^0)}{\alpha^2 M_V \lambda^{\frac{3}{2}}\left(M_\psi^2, M_V^2, M_\pi^2\right)}}$$

Fixed

$$= \begin{cases} 4.38(24)\times 10^{-5}\ \mathrm{GeV}^{-1}, & \mathrm{V} = \omega \\ 4.71(3)\times 10^{-6}\ \mathrm{GeV}^{-1} \text{ or } 8.61(5)\times 10^{-7}\ \mathrm{GeV}^{-1}, & \mathrm{V} = \phi \end{cases}$$

Khuri-Treiman representation of f_1

- Decay amplitude for $V(p_V) \to \pi^+(p_+)\,\pi^-(p_-)\,\pi^0(p_0)$:

$$\mathcal{M}(s,t,u) = i\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu}p_+^{\nu}p_-^{\alpha}p_0^{\beta}\mathcal{F}(s,t,u)$$

- s-channel partial-wave expansion of the amplitude

$$\mathcal{F}(s,t,u) = \sum_{J=0}^{\infty}(q_{\pi\pi}(s)q_{\psi\pi}(s))^{J-1}P'_J(z_s)\,f_J(s)$$

- KT decomposition of the amplitude via *reconstruction theorem*

Stern, Sazdjian and Fuchs, PRD (1993), for $\pi\pi$

$$\mathcal{F}(s,t,u) = \sum_{J=0}^{J_{\max}}(q_{\pi\pi}(s)q_{\psi\pi}(s))^{J-1}P'_J(z_s)\,\mathcal{F}_J(s) + (s\leftrightarrow t) + (s\leftrightarrow u)$$

- Consider only P-wave: $\mathcal{F}(s,t,u) = \mathcal{F}(s)+\mathcal{F}(t)+\mathcal{F}(u), \quad \mathcal{F}\equiv\mathcal{F}_1$

several complex variables → single complex variables

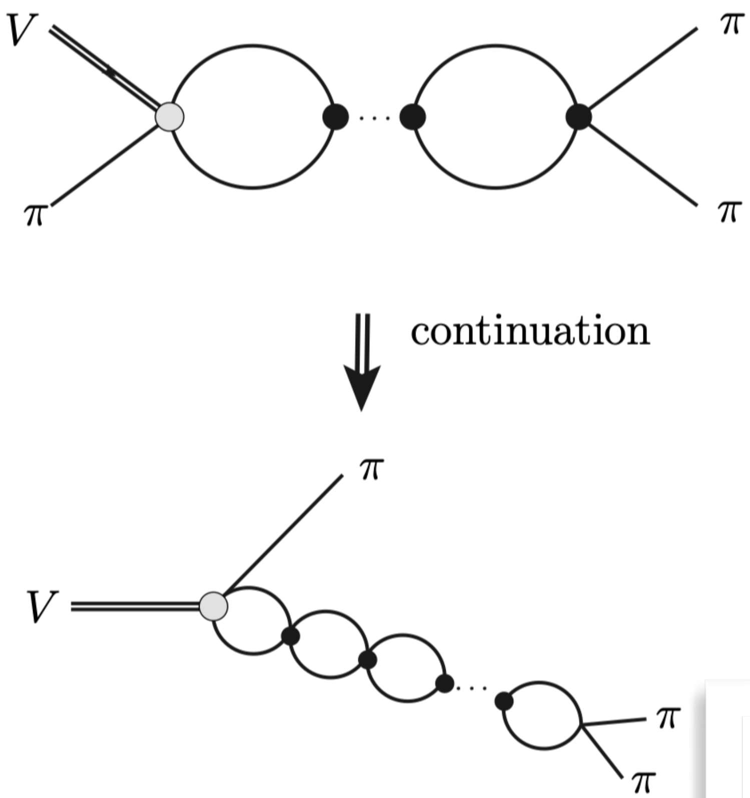
- Partial wave projection of the KT decomposition Khuri and Treiman, Phys. Rev. (1960)…

$$f_1(s) = \mathcal{F}(s)+\hat{\mathcal{F}}(s), \quad \hat{\mathcal{F}}(s) = 3\int_{-1}^{1}\frac{dz_s}{2}(1-z_s^2)\,\mathcal{F}\left(t(s,z_s)\right) \equiv 3\left\langle(1-z_s^2)\mathcal{F}\right\rangle$$

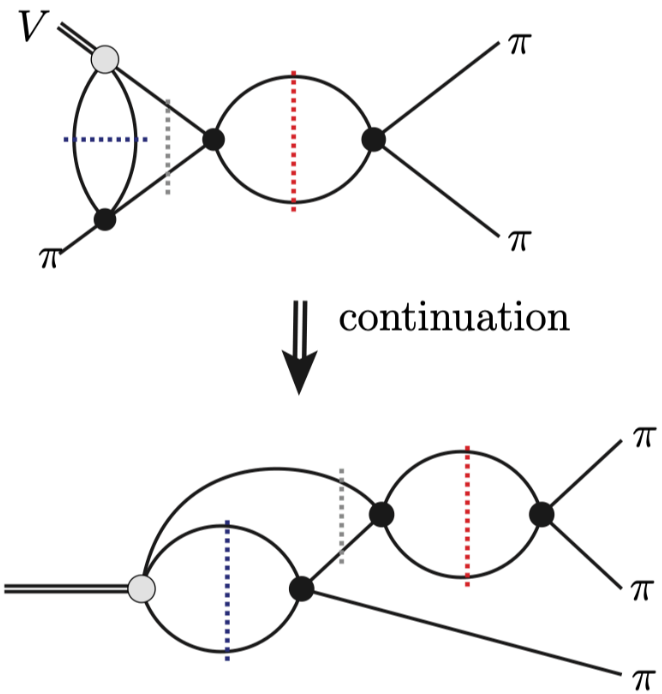
  - $\mathcal{F}(s)$: right-hand cut (RHC)
  - $\hat{\mathcal{F}}(s)$: left-hand cut (given by the RHC of the crossed channels $\mathcal{F}(t)$, $\mathcal{F}(u)$)

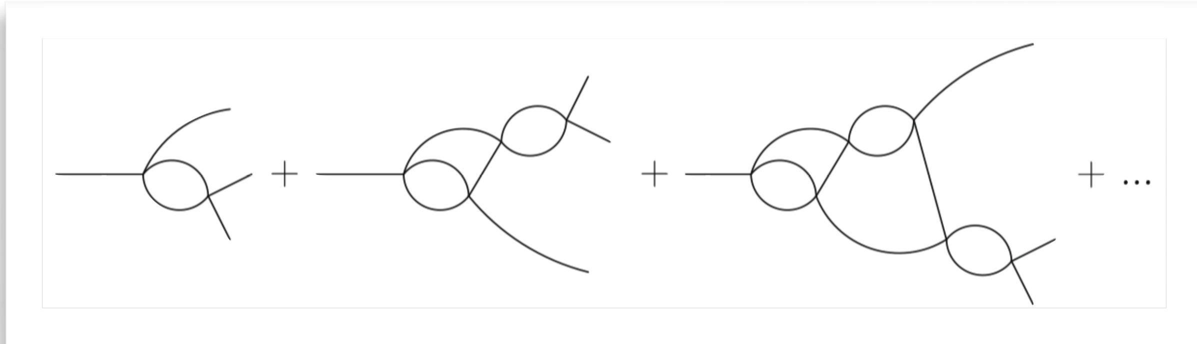
Three-particle decay dynamics

- In many decay processes one wants to take into account unitarity/FSI in the three possible channels



- Khuri-Treiman equations:
  - Include full (direct+crossed) rescattering effects





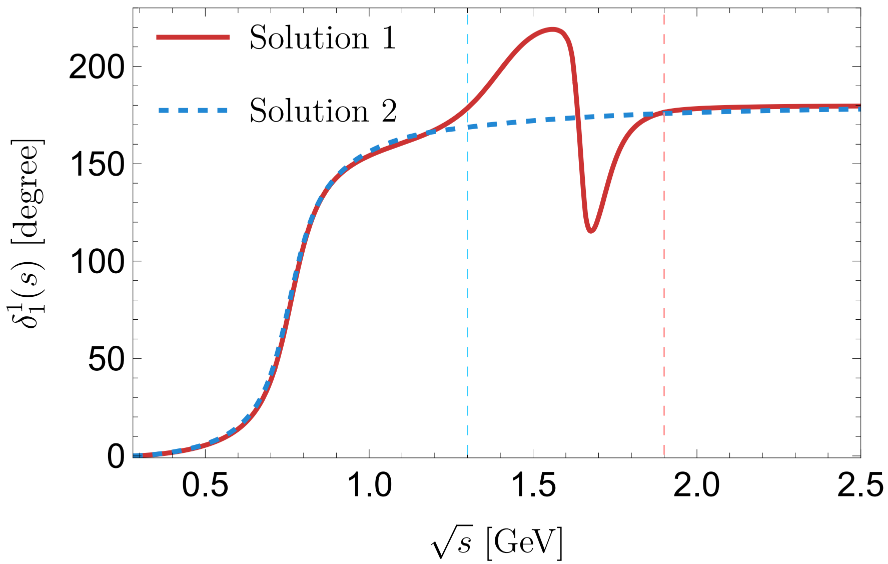
KT equation: phase shifts

- KT-type representation

$$\mathscr{F}(s) = a\Omega(s)\left\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta(s')\,\hat{\mathscr{F}}(s')}{\left|\Omega(s')\right|(s'-s)}\right\}, \quad \hat{\mathscr{F}}(s) = 3\left\langle(1-z_s^2)\mathscr{F}\right\rangle$$

$$\Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)}\right\}$$

- $\pi\pi$ P-wave phase-shift δ taken as input
  - Solution 1: Schneider, Kubis and Niecknig, PRD (2012)
  - Solution 2: Pelaez, Raban and Ruiz de Elvira, PRD (2025) (fit-1)
  - The deviation in these solutions: **theoretical uncertainty**

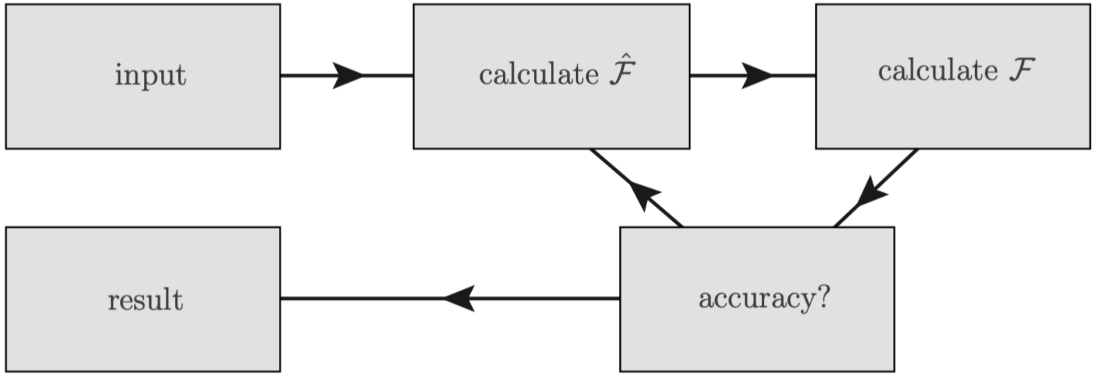


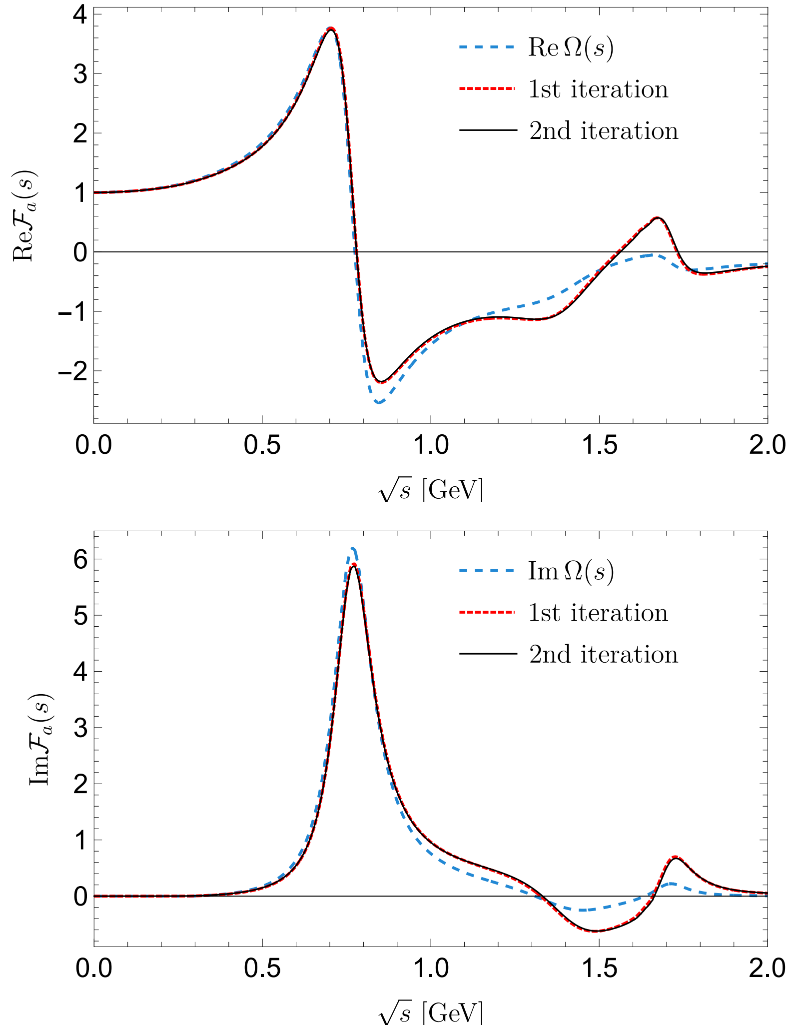
KT equations: solutions

- KT-type representation

$$\mathscr{F}(s) = a\Omega(s)\left\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}s'}{s'}\frac{\sin\delta(s')\,\hat{\mathscr{F}}(s')}{\left|\Omega(s')\right|(s'-s)}\right\}, \quad \hat{\mathscr{F}}(s) = 3\left\langle(1-z_s^2)\mathscr{F}\right\rangle$$

- Solution by numerical **iteration**
- Initial input: $\mathscr{F}(s) = \Omega(s)$





$J/\psi \to \pi^0\gamma^\star$ TFF: 2π

- Dispersive representation (unsubtracted)

$$f^{(2\pi)}_{\psi\pi^0}(s) = \frac{1}{48\pi^2}\int_{4M_\pi^2}^{\infty} \mathrm{d}x \frac{x\sigma_\pi^3(x)F_\pi^{V*}(x)f_1(x)}{x-s}$$

$$F_\pi^V(s) = \Omega(s)$$

- Sum rule

$$f^{(2\pi)}_{\psi\pi^0}(0) = \frac{1}{48\pi^2}\int_{4M_\pi^2}^{\infty} \mathrm{d}x\ \sigma_\pi^3(x)F_\pi^{V*}(x)f_1(x)$$

  - sum rule results in $\left|f^{2\pi}_{\psi\pi^0}(0)\right| = (4.8 \pm 0.2)\times 10^{-4}\mathrm{GeV}^{-1}$
  - $\left|f_{\psi\pi^0}(0)\right|$ from real photon width, $\Gamma_{\psi\to\pi^0\gamma} = \dfrac{\alpha\left(M_\psi^2 - M_{\pi^0}^2\right)^3}{24M_\psi^3}\left|f_{\psi\pi^0}(0)\right|^2$

$$\left|f_{\psi\pi^0}(0)\right| = (6.0 \pm 0.3)\times 10^{-4}\mathrm{GeV}^{-1}$$

  - two-pion intermediate state alone saturates the sum rule for the TFF normalization to about **80%**

$J/\psi \to \pi^0\gamma^\star$ TFF: 4π (ρ')

- The branching fractions of the J/ψ into multipion are actually larger
- From data on $e^+e^- \to$ [hadrons] $_{I=1}$, the most important inelastic intermediate state of isospin $I = 1$ ought to be 4π, which is approximate to effective ρ' pole

$$\frac{\text{disc}}{2i} f^{(\rho')}_{\psi\pi^0}(s) = \frac{M^2_{\rho'} g_{J/\psi\to\rho'\pi} g_{\rho'\gamma} \sqrt{s}\,\Gamma_{\rho'}(s)}{\left(M^2_{\rho'} - s\right)^2 + s\Gamma^2_{\rho'}(s)}$$

$$\Gamma^{(4\pi)}_{\rho'}(s) = \theta\left(s - 16M^2_\pi\right) \frac{\gamma_{\rho'\to 4\pi}(s)}{\gamma_{\rho'\to 4\pi}\left(M^2_{\rho'}\right)} \Gamma_{\rho'}, \quad \gamma_{\rho'\to 4\pi}(s) = \frac{\left(s - 16M^2_\pi\right)^{\frac{9}{2}}}{s^2}$$

- $\left|g_{\rho'\gamma}\right| = 0.0752$ from VMD estimate Zanke, Hoferichter and Kubis, JHEP (2021)
- Effective coupling:

$\mathrm{BR}(J/\psi \to \rho'\pi)\mathrm{BR}(\rho' \to 2\pi) = 2.2(1.2) \times 10^{-4}$ PDG

$\mathrm{BR}(\rho' \to 2\pi) = 6\,\%$

Zanke, Hoferichter and Kubis, JHEP (2021)

→

$\mathrm{BR}(J/\psi \to \rho'\pi) = 3.7(1.8) \times 10^{-3}$

$\left|g_{J/\psi\to\rho'\pi}\right| = 2.73^{+0.61}_{-0.80} \times 10^{-3}\ \mathrm{GeV}^{-1}$

$\left|f^{(\rho')}_{\psi\pi^0}(0)\right| = 1.3^{+0.3}_{-0.4} \times 10^{-4}\mathrm{GeV}^{-1}$

20%?

Destructive interference implied by large N_c QCD

$J/\psi \to \pi^0\gamma^\star$ TFF: charmonium

- We adopt the simple monopole ansatz Fu et al., Mod.Phys.Lett.A (2012)

$$f^{c\bar{c}}_{\psi\pi^0}(s) = \frac{\left|f^{(c\bar{c})}_{\psi\pi^0}(0)\right| e^{i\delta^{c\bar{c}}}}{1 - s/\Lambda^2}$$

- The effective pole mass Λ is is fixed to the mass of the lowest 1^{--} charmonium J/ψ

- Constraint: $\left|f_{\psi\pi^0}(0)\right| = (6.0 \pm 0.3) \times 10^{-4}\ \text{GeV}^{-1}$

$$\left|f^{(c\bar{c})}_{\psi\pi^0}(0)\right| = -f^{(2\pi,\rho')}_{\psi\pi^0}(0)\cos\delta^{c\bar{c}} + \sqrt{\left|f_{\psi\pi^0}(0)\right|^2 - \left(f^{(2\pi,\rho')}_{\psi\pi^0}(0)\right)^2 \sin^2\delta^{c\bar{c}}}$$

Bin averaged fits

- We fix the sign of the weight factor for ϕ: $w_{\psi\phi\pi} = -|w_{\psi\phi\pi}|$
- Subtracted constant $f^{(2\pi)}_{\psi\pi^0}(0)$ is fixed by the sum rule
- Only two fit parameters

$\delta_{\psi\omega\pi}$ and $\delta^{c\bar{c}}$

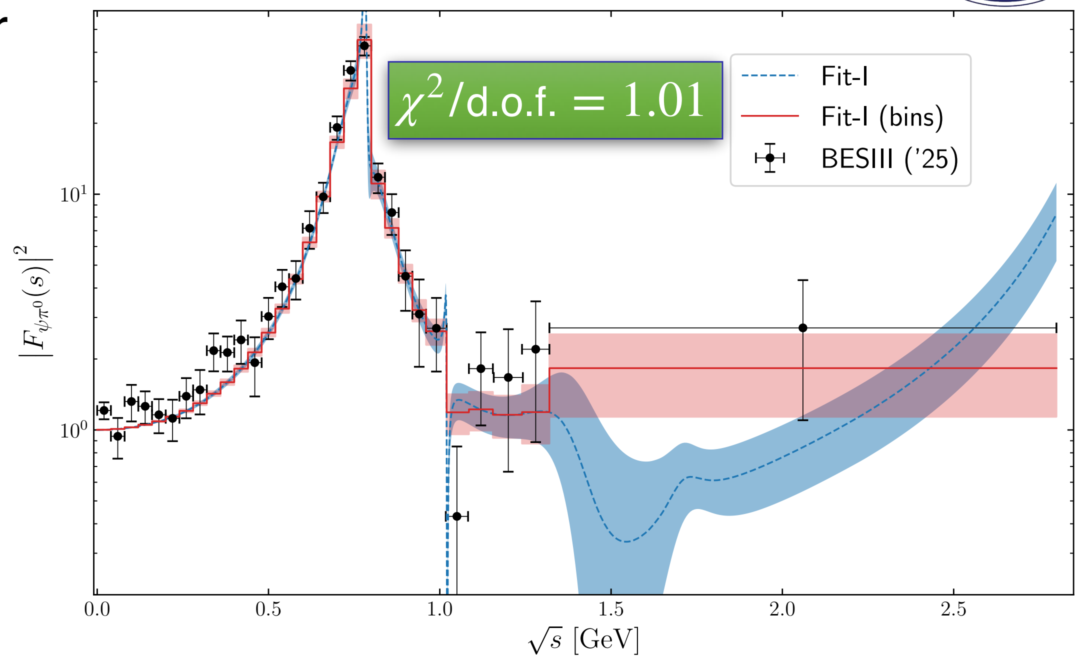
Fit-I & Fit-II

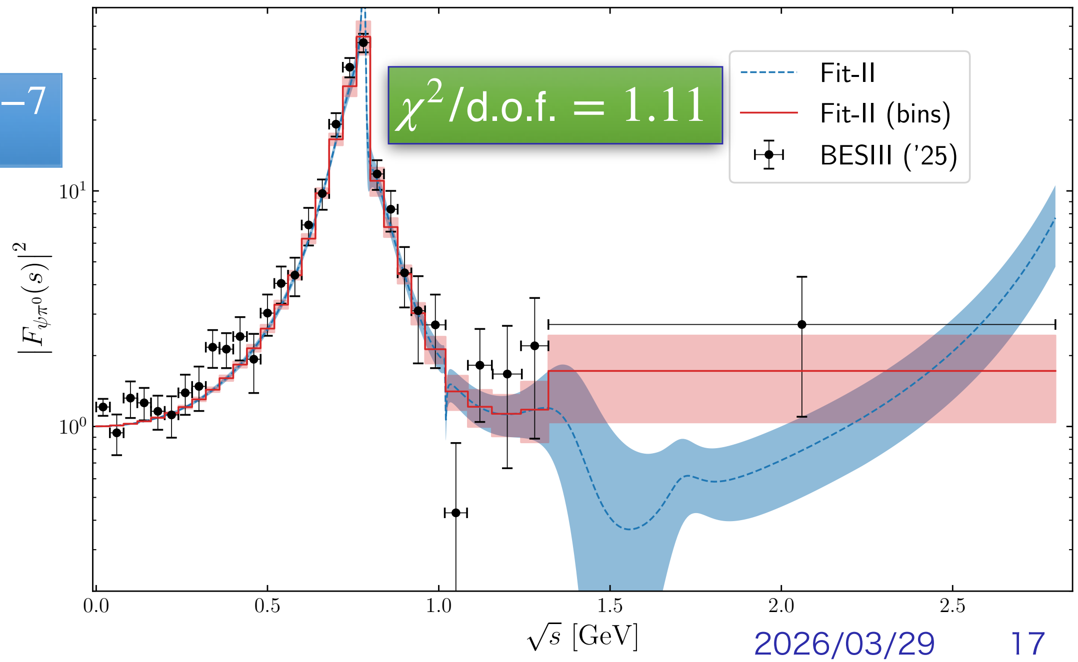
PDG

$\mathrm{BR}\left(J/\psi \to \phi\pi^0\right) = 3 \times 10^{-6}$ or 1×10^{-7}

Multi-solution ambiguity

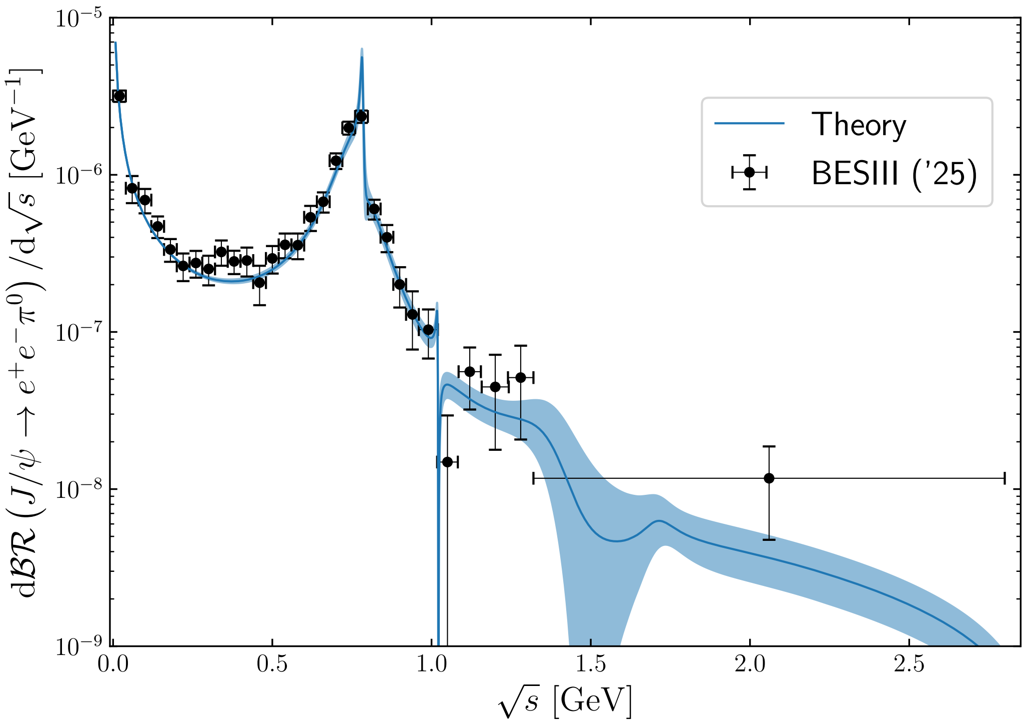
- Significant cusp structure corresponding to ω $(\phi?)$
- Nontrivial dip structure corresponding to $\rho'(1450)$

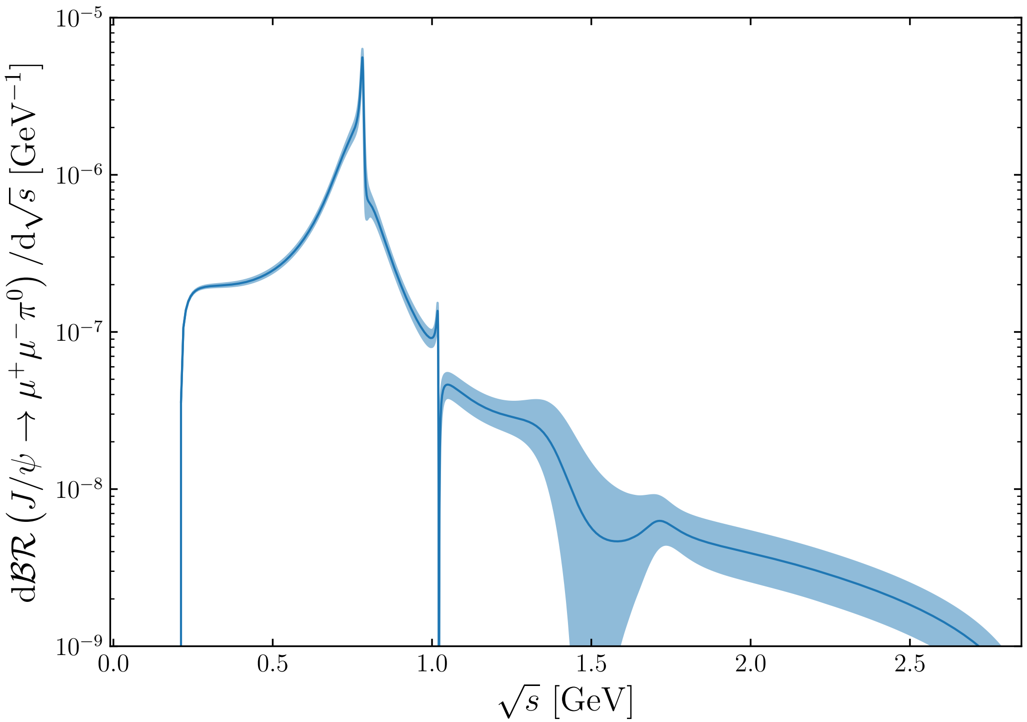




Branching fractions

- Differential branching fractions



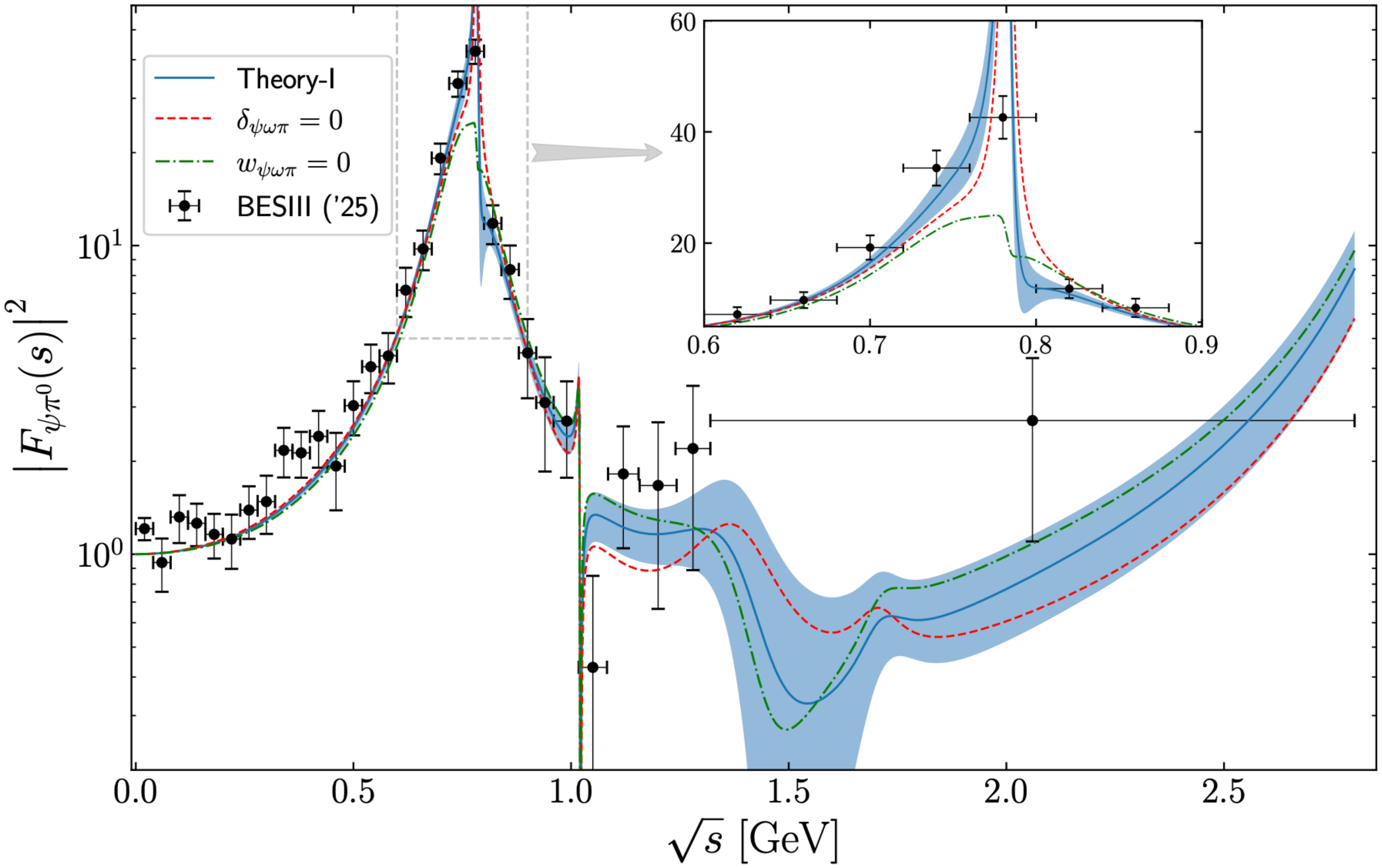


- Branching fractions ($\times\, 10^{-7}$)

	Exp [15]	This Work	DR [16]	RChT [8]	RChT [7]	VMD [5]
$J/\psi \to \pi^0 e^+e^-$	$8.06 \pm 0.31(\text{stat}) \pm 0.38(\text{syst})$	$7.03^{+0.62}_{-0.60}$	$(5.5 \ldots 6.4)$	12.94 ± 0.44	11.91 ± 1.38	$3.89^{+0.37}_{-0.33}$
$J/\psi \to \pi^0 \mu^+\mu^-$	—	$4.15^{+0.58}_{-0.50}$	$(2.7 \ldots 3.3)$	3.04 ± 0.10	2.80 ± 0.32	$1.01^{+0.10}_{-0.09}$

Nonzero $\delta_{\psi\omega\pi}$?

$$f_{\psi\pi}(s)\Big|_{s\to M_V^2} = \frac{w_{\psi V\pi}s}{M_V^2 - s - iM_V\Gamma_V}$$



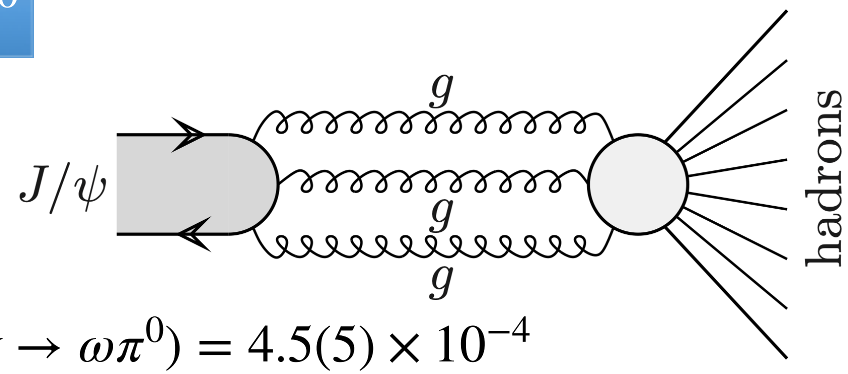
Phase between strong and em. amplitudes

- $J/\psi \to \pi^0\gamma^\star$ transition through two dominant modes: *isoscalar/isovector*

$\rho^0\pi^0$ and $\omega\pi^0$

- One-photon em. current:

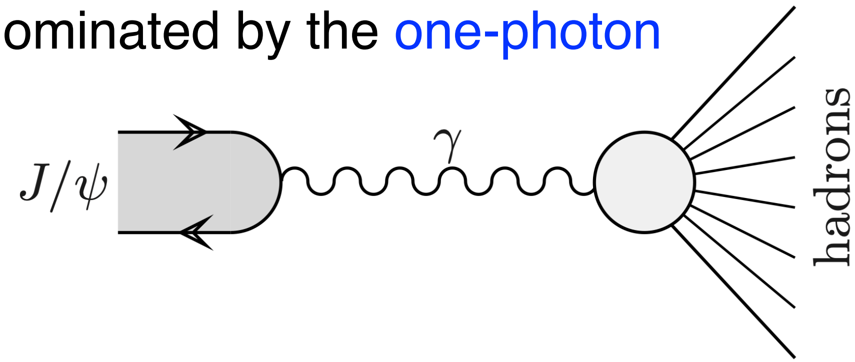
$$\frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d = \frac{1}{6}J_\mu^s + \frac{1}{2}J_\mu^v$$



- PDG:

$$\mathrm{Br}(J/\psi \to \rho^0\pi^0) = 6.2(6)\times 10^{-3} \gg \mathrm{Br}(J/\psi \to \omega\pi^0) = 4.5(5)\times 10^{-4}$$

- $J/\psi \to \rho^0\pi^0$ should proceed mainly through strong interaction
- The isospin-breaking $J/\psi \to \omega\pi^0$ mode is dominated by the one-photon mechanisms



☑ Phase between $3g$ and 1γ contributions in J/ψ hadronic decays can be extracted as

$(62 \pm 21)^\circ$ deviates slightly (1.5σ) from 90°

☆ global fit to $J/\psi \to VP$: $(72 \pm 17)^\circ$ hep-ph/9902300

Summary and outlook

- We update the results for $J/\psi \to e^+e^-\pi^0$ by incorporating $\rho-\omega$ interference with the dispersive method ensuring unitarity, analyticity and crossing symmetry

- $\mathrm{BR}\left(J/\psi \to \pi^0 e^+ e^-\right) = 7.03^{+0.62}_{-0.60} \times 10^{-7}, \quad \mathrm{BR}\left(J/\psi \to \pi^0 \mu^+ \mu^-\right) = 4.15^{+0.58}_{-0.50} \times 10^{-7}$

- Relative phase between the strong and the one-virtual-photon (electromagnetic) modes in hadronic decays of J/ψ as $(62 \pm 21)^\circ$

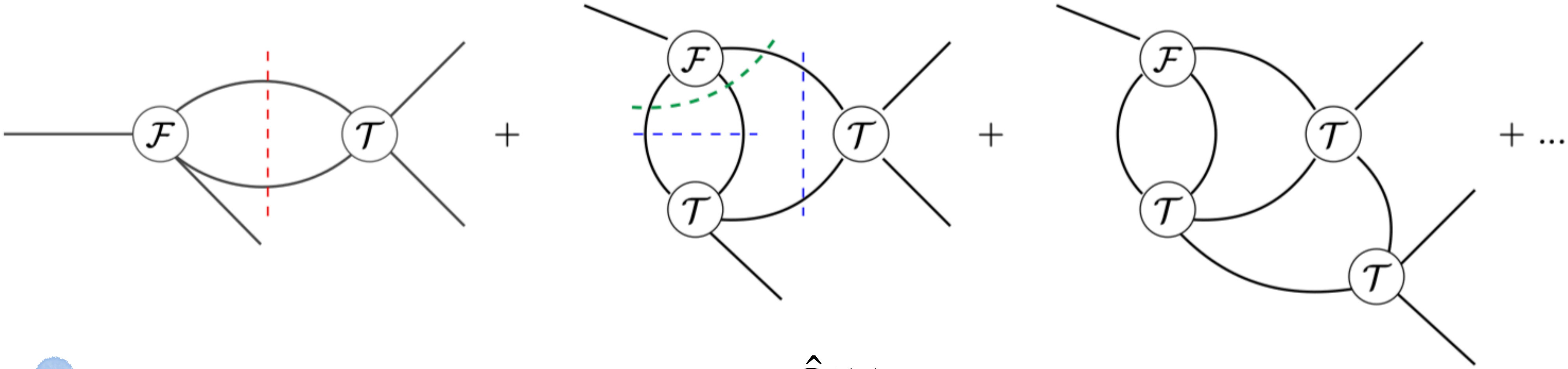
- Extend to $\omega, \phi \to e^+e^-\pi^0$, where the $\rho-\omega$ mixing effect was ignored in previous dispersive analyses JPAC, 2505.15309

Thank you for your attention!

Backup

Unitarity and analyticity

- Unitarity relation for the single variable amplitude $\mathscr{F}(s)$



- Complications: integration contour for $\hat{\mathscr{F}}(s)$
- **“Pinocchio” method** (**triangle topology** continuation) Bronzan and Kacser, Phys. Rev. (1963)
  - deform path of angular integral to avoid crossing branch cuts
- Gasser-Rusetsky method Gasser and Rusetsky, EPJC (2018)
  - deform path of dispersion integral
- Pasquier inversion R. Pasquier and J.Y. Pasquier, Phys. Rev. (1968, 1969)
  - interchange of the order of integrations to obtain integral equations in one variable

Analytic continuation of the inhomogeneities

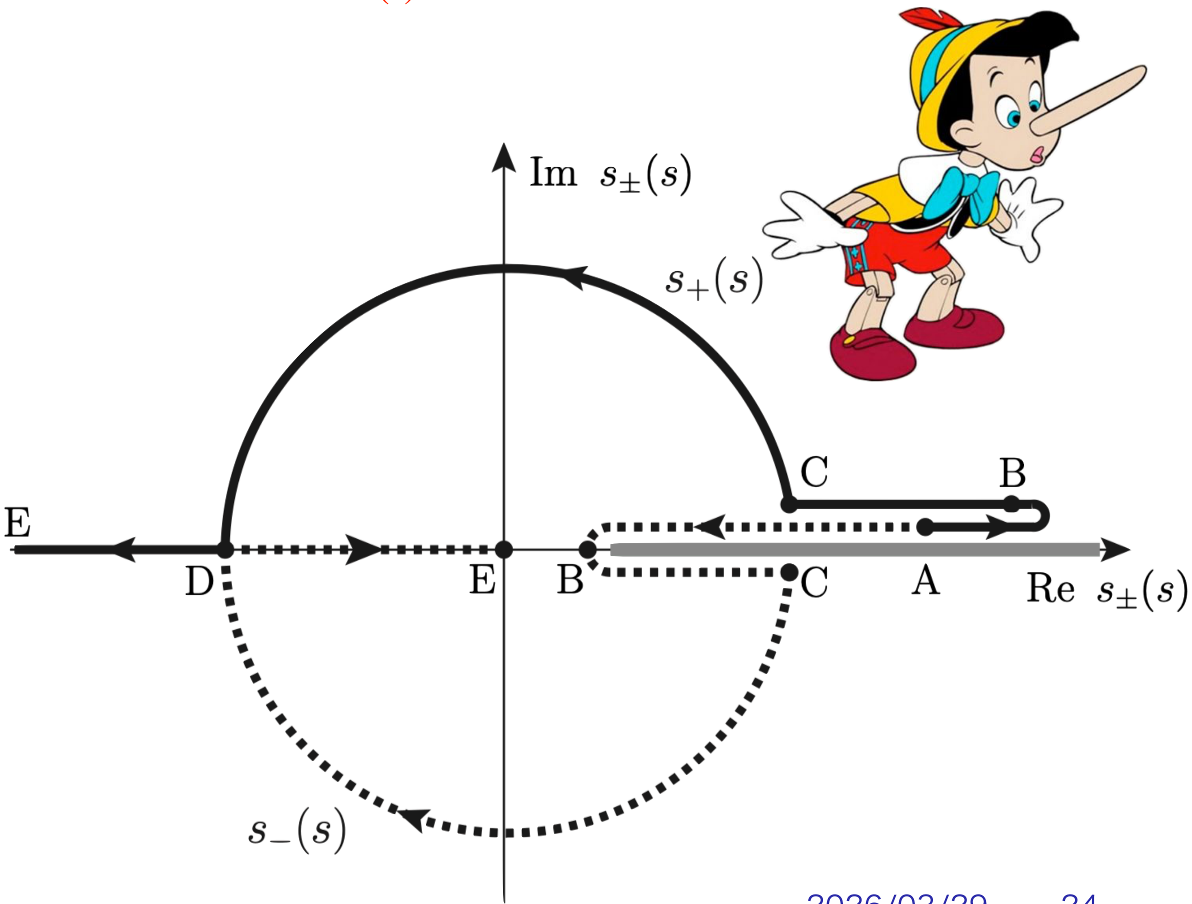
- $i\epsilon$ prescription: give the vector-meson mass an infinitesimal positive imaginary part, $M_V^2 + i\epsilon$ if $M_V > 3M_\pi$ Bronzan and Kacser, Phys. Rev. (1963)
- Angular average integral ($s + t + u = 3M_\pi^2 + M_V^2 \equiv = 3s_0$)

$$\langle z^n \mathscr{F} \rangle \equiv \frac{1}{2}\int_{-1}^{1} z^n \mathscr{F}\left(\frac{3s_0 - s + z\kappa(s)}{2}\right) \mathrm{d}z = \frac{1}{\kappa(s)}\int_{s_-(s)}^{s_+(s)} \mathrm{d}s' \left(\frac{2s' - 3s_0 + s}{\kappa(s)}\right)^n \mathscr{F}(s')$$

The trajectories of $s_\pm(s)$

$$2s_+(s) = \begin{cases} 3s_0 - s + |\kappa(s)| + i\epsilon, s \in \left[4M_\pi^2, M_-^2\right], \\ 3s_0 - s + i|\kappa(s)|, \quad s \in [M_-^2, M_+^2], \\ 3s_0 - s - |\kappa(s)|, \quad s \in [M_+^2, \infty), \end{cases}$$

$$2s_-(s) = \begin{cases} 3s_0 - s - |\kappa(s)| + i\epsilon, s \in \left[4M_\pi^2, \frac{M_V^2 - M_\pi^2}{2}\right], \\ 3s_0 - s - |\kappa(s)| - i\epsilon, s \in \left[\frac{M_V^2 - M_\pi^2}{2}, M_-^2\right], \\ 3s_0 - s - i|\kappa(s)|, \quad s \in [M_-^2, M_+^2], \\ 3s_0 - s + |\kappa(s)|, \quad s \in [M_+^2, \infty), \end{cases}$$



$M_\pm^2 = (M_V \pm M_\pi)^2$

Analytic continuation of the inhomogeneities

$$\langle z^n \mathscr{F} \rangle \equiv \frac{1}{2}\int_{-1}^{1} z^n \mathscr{F}\left(\frac{3s_0 - s + z\kappa(s)}{2}\right) \mathrm{d}z = \frac{1}{\kappa(s)}\int_{s_-(s)}^{s_+(s)} \mathrm{d}s' \left(\frac{2s' - 3s_0 + s}{\kappa(s)}\right)^n \mathscr{F}(s')$$

a singularity-free function $\tilde{\mathscr{F}}(s)$

$$\tilde{\mathscr{F}}(s) \equiv \kappa^3(s)\hat{\mathscr{F}}(s) = 3\int_{s_-(s)}^{s_+(s)} ds' \left(\kappa^2(s) - \left(2s' - 3s_0 + s\right)^2\right) \mathscr{F}(s')$$

$$\mathscr{F}(s) = a\Omega(s)\left\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} \frac{\mathrm{d}s'}{s'} \frac{\sin\delta(s')\,\tilde{\mathscr{F}}(s')}{\left|\Omega(s')\right| \kappa^3(s')(s'-s)}\right\}$$