Systematic analysis of the form factors of B_c to $P(D)$ -wave charmonia and corresponding weak decays

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J. Lu, D. Y. Chen, G. L. Yu and Z. G. Wang, Phys. Lett. B 872, 140057 (2026).

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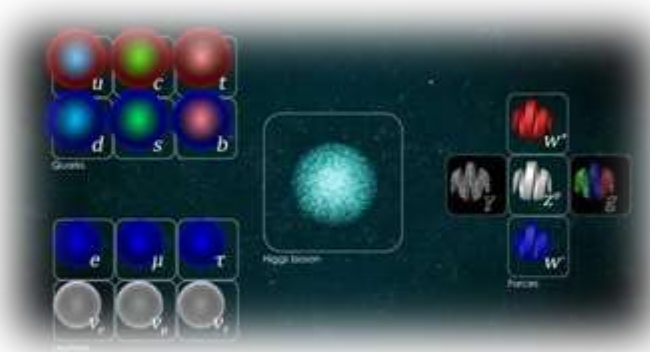

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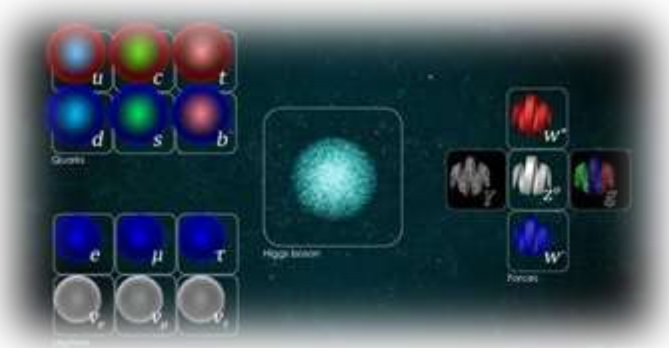


1. Introduction

➢ Uniqueness of the B_c Meson:

◆ **Composition:** The only quarkonium in the Standard Model with two different heavy flavors (b and $\bar{c}$).

◆ **Decay Properties:**

- Cannot annihilate via strong or electromagnetic interactions.
- Decays exclusively through weak interaction ($b \to c$ or $\bar{c} \to \bar{s}$).
- Provides an ideal laboratory for studying both weak decays and non-perturbative QCD.

◆ **Experimental Prospects:**

- Large production cross-section at LHCb ($\sim 1\ \mu$b at $\sqrt{s} = 14$ TeV).
- $\mathcal{O}(10^9)$ B_c mesons expected per fb^{-1}, enabling detailed studies of rare decay channels.
- Some weak decay channels have been successively observed by LHCb in recent years.

■ **F. Abe et al. (CDF), Phys. Rev. D 58, 112004 (1998).**
■ **Y. N. Gao, J. He, P. Robbe, M. H. Schune, and Z. W. Yang, Chin. Phys. Lett. 27, 061302 (2010).**
■ **S. Navas et al. (Particle Data Group), Phys. Rev. D 110, 030001 (2024).**

1. Introduction

➢ Charmonium as a probe of QCD:

- **Charmonium ($c\bar{c}$):** A perfect system for studying QCD in the non-perturbative regime.
- ***P*-wave charmonia ($L = 1$):**
  - Spin-triplet $\chi_{cJ}(J = 0,1,2)$ ($,^3P_0, ^3P_1, ^3P_2$) and spin-singlet $h_c(J = 1)$ ($,^1P_1$).
  - Well-established experimentally; masses and decay constants are known.
- ***D*-wave charmonia ($L = 2$):**
  - Spin-triplet ψ_J ($J = 1,2,3$) ($,^3D_1, ^3D_2, ^3D_3$) and spin-singlet η_{c2} ($J = 2$) ($,^1D_2$).
  - Experimental status: ($\psi_1(3770)$) observed; ψ_2 observed, ψ_3 (X(3842)) recently observed by LHCb; η_{c2} not yet firmly established.

- **J. J. Aubert et al. (E598), Phys. Rev. Lett. 33, 1404 (1974).**
- **W. M. Tanenbaum et al., Phys. Rev. Lett. 35, 1323 (1975).**
- **J. S. Whitaker et al., Phys. Rev. Lett. 37, 1596 (1976).**
- **R. Brandelik et al. (DASP), Phys. Lett. B 76, 361 (1978).**
- **R. Partridge et al., Phys. Rev. Lett. 45, 1150 (1980).**
- **...**

1. Introduction

➢ Motivation and research goals:

- **Previous work:** Form factors for $B_c \to S$-wave and P-wave charmonia have been studied using three-point QCD sum rules.
- **This work:**
  - **Corrected** mistakes in previous P-wave Calculations.
  - **Extend** the systematic analysis to D-wave charmonia ($\psi_1, \psi_2, \psi_3, \eta_{c2}$).
- **Primary goals:**
  - Calculate **vector, axial-vector, and tensor form factors** for $B_c \to P$-wave and D-wave transitions.
  - Predict **semileptonic decay branching ratios** ($B_c \to X l \nu_l$).
  - (P-wave paper) Predict **nonleptonic decay branching ratios** ($B_c \to XP/V$).
  - Provide theoretical guidance for future LHCb measurements.

- **K. Azizi, H. Sundu, and M. Bayar, Phys. Rev. D 79, 116001 (2009).**
- **K. Azizi, Y. Sarac, and H. Sundu, Eur. Phys. J. C 73, 2638 (2013).**
- **B. Wu, G. L. Yu, J. Lu, and Z. G. Wang, Phys. Lett. B 859, 139118 (2024).**

2. Theoretical approach

➢ Three-points QCD sum rules:

- **Framework:** A powerful non-perturbative method to study hadronic transition matrix elements.
- **Three-point correlation function:**

$$\Pi(p,p') = i^2 \int d^4x d^4y e^{ip'x} e^{i(p-p')y} \times \langle 0| T\{J_X(x)\tilde{J}(y)J^+_{B_c}(0)\} |0\rangle$$

$$\Pi(p,p') = i^2 \int d^4x d^4y e^{ip'\cdot z} e^{i(p-p')\cdot y} e^{-ip\cdot x} \times \langle 0| \mathcal{T}\{J_X(z)\tilde{J}(y)J^\dagger_{B_c}(x)\} |0\rangle |_{z\to 0},$$

- **Key steps:**
  - **Phenomenological side:** Insert complete set of hadronic states → form factors.
  - **QCD side:** Compute using OPE (perturbative + gluon condensate) → QCD spectral densities.
  - **Matching:** Perform Borel transformation and apply quark-hadron duality to extract form factors.
  - **Extrapolation:** Use z-series expansion to extend results from spacelike ($Q^2 > 0$) to timelike ($Q^2 < 0$) region.

2. Theoretical approach

➢ Interpolating and transition currents:

- ***B_c* current:** $J_{B_c}(0) = \bar{c}(0)i\gamma_5 b(0)$
- **Transition currents:** $\tilde{J}(y) = \bar{c}(y)\Gamma b(y)$ where $\Gamma = \gamma_\mu, \gamma_\mu\gamma_5$ and $\sigma_{\mu\nu}$ or $\sigma_{\mu\nu}\gamma_5$
- ***P* and *D*-wave charmonia currents:**

$$
\begin{aligned}
J^{\chi_{c0}}(x) &= \bar{c}(x)c(x) \\
J^{\chi_{c1}}_{\mu}(x) &= \bar{c}(x)\gamma_\mu\gamma_5 c(x) \\
J^{h_c}_{\mu\nu}(x) &= \bar{c}(x)\sigma_{\mu\nu}c(x) \\
J^{\chi_{c2}}_{\mu\nu}(x) &= \frac{i}{2}\bar{c}(x)(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu - \frac{1}{2}g_{\mu\nu}\overleftrightarrow{\not{D}})c(x)
\end{aligned}
$$

$$
\begin{aligned}
J^{\psi_1}_{\alpha}(z) &= \bar{c}(z)\overleftrightarrow{D}_{\alpha'}\overleftrightarrow{D}_{\beta'}\gamma_\rho\left(g^{\alpha'\beta'}g^{\rho\alpha} + g^{\alpha'\rho}g^{\beta'\alpha} + g^{\rho\beta'}g^{\alpha'\alpha}\right)c(z), \\
J^{\psi_2}_{\alpha\beta}(z) &= \bar{c}(z)\Big(\gamma_\alpha\gamma\cdot\overleftrightarrow{D}\overleftrightarrow{D}_\beta + \gamma_\alpha\overleftrightarrow{D}_\beta\gamma\cdot\overleftrightarrow{D} + \gamma_\beta\gamma\cdot\overleftrightarrow{D}\overleftrightarrow{D}_\alpha \\
&\quad + \gamma_\beta\overleftrightarrow{D}_\alpha\gamma\cdot\overleftrightarrow{D} - g_{\alpha\beta}\gamma\cdot\overleftrightarrow{D}\gamma\cdot\overleftrightarrow{D}\Big)\gamma_5 c(z), \\
J^{\eta_{c2}}_{\alpha\beta}(z) &= \bar{c}(z)\left(\overleftrightarrow{D}_\alpha\overleftrightarrow{D}_\beta + \overleftrightarrow{D}_\beta\overleftrightarrow{D}_\alpha - \frac{1}{2}g_{\alpha\beta}\overleftrightarrow{D}\cdot\overleftrightarrow{D}\right)c(z), \\
J^{\psi_3}_{\alpha\beta\gamma}(z) &= \bar{c}(z)\left(\overleftrightarrow{D}_\alpha\overleftrightarrow{D}_\beta\gamma_\gamma + \overleftrightarrow{D}_\gamma\overleftrightarrow{D}_\alpha\gamma_\beta + \overleftrightarrow{D}_\beta\overleftrightarrow{D}_\gamma\gamma_\beta\right)c(z), \qquad (2)
\end{aligned}
$$

$\overleftrightarrow{D}_\mu = \frac{1}{2}(\overrightarrow{D}_\mu - \overleftarrow{D}_\mu)$ $\overrightarrow{D}_\mu = \overrightarrow{\partial}_\mu - ig_s t^a G^a_\mu$ and $\overleftarrow{D}_\mu = \overleftarrow{\partial}_\mu + ig_s t^a G^a_\mu$

■ Q. Xin and Zhi-Gang Wang, Int. J. Mod. Phys. A 41 2650052 (2026).

2. Theoretical approach

➢ **Phenomenological side:**

◆ **Three-point correlation function in hadron level:**

$$\Pi^{\text{phy}}(p,p') = \frac{\langle 0|\, J_X(0)\, |X(p')\rangle \langle B_c(p)|\, J^{\dagger}_{B_c}(0)\, |0\rangle}{(m^2_{B_c}-p^2)(m^2_X-p'^2)} \times \langle X(p')|\, \tilde{J}(0)\, |B_c(p)\rangle + \dots,$$

◆ **Double Borel transformation:** $p^2 \to -P^2$, $p'^2 \to -P'^2$ and $q^2 \to -Q^2$

➢ **Decay constants and projection operators:**

◆ **Decay constants:**

$$\langle 0|\, J^{\chi_{c0}}(0)\, |\chi_{c0}(p')\rangle = f_{\chi_{c0}} m_{\chi_{c0}}$$

$$\langle 0|\, J^{\chi_{c1}}_{\mu}(0)\, |\chi_{c1}(p')\rangle = f_{\chi_{c1}} m_{\chi_{c1}} \epsilon^{\chi_{c1}}_{\mu}$$

$$\langle 0|\, J^{\chi_{c1}}_{\mu}(0)\, |\eta_c(p')\rangle = i f_{\eta_c} p'_{\mu}$$

$$\langle 0|\, J^{h_c}_{\mu\nu}(0)\, |h_c(p')\rangle = i f_{h_c} \varepsilon_{\mu\nu\alpha\beta} \epsilon^{h_c}_{\alpha} p'_{\beta}$$

$$\langle 0|\, J^{h_c}_{\mu\nu}(0)\, |J/\psi(p')\rangle = i f_{J/\psi}(p'_{\mu}\epsilon^{J/\psi}_{\nu} - p'_{\nu}\epsilon^{J/\psi}_{\mu})$$

$$\langle 0|\, J^{\chi_{c2}}_{\mu\nu}(0)\, |\chi_{c2}(p')\rangle = f_{\chi_{c2}} m^3_{\chi_{c2}} \epsilon^{\chi_{c2}}_{\mu\nu}$$

$$\langle B_c(p)|\, J^{+}_{B_c}(0)\, |0\rangle = \frac{f_{B_c} m^2_{B_c}}{m_b + m_c}$$

2. Theoretical approach

➢ Decay constants and projection operators:

◆ Decay constants:

$$\begin{aligned}
\langle 0| J_\alpha^{\psi_1}(0) |\psi_1(p')\rangle &= f_{\psi_1}\epsilon_\alpha^{\psi_1}, \\
\langle 0| J_\alpha^{\psi_1}(0) |\chi_{c0}(p')\rangle &= f_{\chi_{c0}} p'_\alpha, \\
\langle 0| J_{\alpha\beta}^{\psi_2[\eta_{c2}]}(0) |\psi_2(p')\rangle &= f_{\psi_2[\eta_{c2}]}\epsilon_{\alpha\beta}^{\psi_2[\eta_{c2}]}, \\
\langle 0| J_{\alpha\beta}^{\psi_2[\eta_{c2}]}(0) |\chi_{c1}(p')\rangle &= f_{\chi_{c1}}(p'_\alpha\epsilon_\beta^{\chi_{c1}} + p'_\beta\epsilon_\alpha^{\chi_{c1}}), \\
\langle 0| J_{\alpha\beta}^{\psi_2[\eta_{c2}]}(0) |\eta_c(p')\rangle &= f_{\eta_c} p'_\alpha p'_\beta,
\end{aligned}$$

$$\begin{aligned}
\langle 0| J_{\alpha\beta\gamma}^{\psi_3}(0) |\psi_3(p')\rangle &= f_{\psi_3}\epsilon_{\alpha\beta\gamma}^{\psi_3}, \\
\langle 0| J_{\alpha\beta\gamma}^{\psi_3}(0) |\chi_{c2}(p')\rangle &= f_{\chi_{c2}}(p'_\alpha\epsilon_{\beta\gamma}^{\chi_{c2}} + p'_\gamma\epsilon_{\alpha\beta}^{\chi_{c2}} + p'_\beta\epsilon_{\gamma\alpha}^{\chi_{c2}}), \\
\langle 0| J_{\alpha\beta\gamma}^{\psi_3}(0) |J/\psi[\psi_1](p')\rangle &= f_{J/\psi[\psi_1]}(p'_\alpha p'_\beta\epsilon_\gamma^{J/\psi[\psi_1]} \\
&\quad + p'_\gamma p'_\alpha\epsilon_\beta^{J/\psi[\psi_1]} + p'_\beta p'_\gamma\epsilon_\alpha^{J/\psi[\psi_1]}), \\
\langle 0| J_{\alpha\beta\gamma}^{\psi_3}(0) |\chi_{c0}(p')\rangle &= f_{\chi_{c0}} p'_\alpha p'_\beta p'_\gamma,
\end{aligned}$$

◆ Projection operators:

$$\begin{aligned}
\langle 0| J_\alpha^{\psi_1}(0) |\psi_1(p')\rangle &= f_{\psi_1}\epsilon_\alpha^{\psi_1}, \\
\langle 0| J_\alpha^{\psi_1}(0) |\chi_{c0}(p')\rangle &= \boxed{f_{\chi_{c0}} p'_\alpha,}
\end{aligned}$$

$$\begin{aligned}
\tilde{\Pi}_{1\xi\mu}^{\psi_1\mathrm{QCD}}(p,p') &= \boxed{\left(g^{\xi\alpha} - \frac{p'^\xi p'^\alpha}{p'^2}\right)}\Pi_{1\alpha\mu}^{\psi_1\mathrm{QCD}} \\
&= \Pi_1^{\psi_1\mathrm{QCD}}\varepsilon_{\xi\mu\lambda\tau}p^\lambda p'^\tau,
\end{aligned}$$

2. Theoretical approach

➢ Form factors:

◆ Form factors (*P*-wave):

$$\langle \chi_{c0}(p')|\bar{c}(0)\gamma_\mu\gamma_5 b(0)|B_c(p)\rangle = F_0^{B_c\to\chi_{c0}}(q^2)\frac{m_{B_c}^2-m_{\chi_{c0}}^2}{q^2}q_\mu$$
$$+F_1^{B_c\to\chi_{c0}}(q^2)\left(p_\mu+p'_\mu-\frac{m_{B_c}^2-m_{\chi_{c0}}^2}{q^2}q_\mu\right) \quad (5)$$

$$\langle \chi_{c1}[h_c](p')|\bar{c}(0)\gamma_\mu b(0)|B_c(p)\rangle$$
$$= -i(m_{B_c}-m_{\chi_{c1}[h_c]})V_1^{B_c\to\chi_{c1}[h_c]}(q^2)\epsilon_\mu^{\chi_{c1}[h_c]*}$$
$$+iV_2^{B_c\to\chi_{c1}[h_c]}(q^2)\frac{\epsilon^{\chi_{c1}[h_c]*}\cdot P}{m_{B_c}-m_{\chi_{c1}[h_c]}}P_\mu$$
$$+2im_{\chi_{c1}[h_c]}\left[V_3^{B_c\to\chi_{c1}[h_c]}(q^2)-V_0^{B_c\to\chi_{c1}[h_c]}(q^2)\right]\frac{\epsilon^{\chi_{c1}[h_c]*}\cdot P}{q^2}q_\mu$$
$$\langle \chi_{c1}[h_c](p')|\bar{c}(0)\gamma_\mu\gamma_5 b(0)|B_c(p)\rangle$$
$$= -\frac{A^{B_c\to\chi_{c1}[h_c]}(q^2)}{m_{B_c}-m_{\chi_{c1}[h_c]}}\varepsilon_{\mu\alpha\beta\gamma}\epsilon_\alpha^{\chi_{c1}[h_c]*}P_\beta q_\gamma \quad (6)$$

$$\langle \chi_{c2}(p')|\bar{c}(0)\gamma_\mu b(0)|B_c(p)\rangle = h^{B_c\to\chi_{c2}}(q^2)\varepsilon_{\mu\beta\rho\sigma}\epsilon_{\beta\lambda}^{\chi_{c2}*}P^\lambda P^\rho q^\sigma$$
$$\langle \chi_{c2}(p')|\bar{c}(0)\gamma_\mu\gamma_5 b(0)|B_c(p)\rangle = -i\,k^{B_c\to\chi_{c2}}(q^2)\epsilon_{\mu\beta}^{\chi_{c2}*}P^\beta$$
$$-i\epsilon_{\beta\lambda}^{\chi_{c2}*}P^\beta P^\lambda\left[b_+^{B_c\to\chi_{c2}}(q^2)P_\mu+b_-^{B_c\to\chi_{c2}}(q^2)q_\mu\right] \quad (7)$$

$$\langle \chi_{c0}(p')|\bar{c}(0)\sigma_{\mu\nu}b(0)|B_c(p)\rangle = -\frac{T^{B_c\to\chi_{c0}}(q^2)}{m_{B_c}-m_{\chi_{c0}}}\varepsilon_{\mu\nu\alpha\beta}p^\alpha p'^\beta \quad (10)$$

$$\langle \chi_{c1}[h_c](p')|\bar{c}(0)\sigma_{\mu\nu}\gamma_5 b(0)|B_c(p)\rangle$$
$$= -T_0^{B_c\to\chi_{c1}[h_c]}(q^2)\frac{\epsilon^{\chi_{c1}[h_c]*}\cdot q}{(m_{B_c}-m_{\chi_{c1}[h_c]})^2}\varepsilon_{\mu\nu\alpha\beta}p^\alpha p'^\beta$$
$$+T_1^{B_c\to\chi_{c1}[h_c]}(q^2)\varepsilon_{\mu\nu\alpha\beta}p_\alpha\epsilon_\beta^{\chi_{c1}[h_c]*}$$
$$+T_2^{B_c\to\chi_{c1}[h_c]}(q^2)\varepsilon_{\mu\nu\alpha\beta}p'_\alpha\epsilon_\beta^{\chi_{c1}[h_c]*} \quad (11)$$

$$\langle \chi_{c2}(p')|\bar{c}(0)\sigma_{\mu\nu}b(0)|B_c(p)\rangle$$
$$= \frac{q^\tau}{m_{B_c}}\left[T_0^{B_c\to\chi_{c2}}(q^2)\frac{\epsilon_{\rho\tau}^{\chi_{c2}*}q^\rho}{(m_{B_c}-m_{\chi_{c2}})^2}\varepsilon_{\mu\nu\chi\delta}p^\chi p'^\delta\right.$$
$$\left.+T_1^{B_c\to\chi_{c2}}(q^2)\varepsilon_{\mu\nu\chi\delta}\epsilon_{\chi\tau}^{\chi_{c2}*}p_\delta+T_2^{B_c\to\chi_{c2}}(q^2)\varepsilon_{\mu\nu\chi\delta}\epsilon_{\chi\tau}^{\chi_{c2}*}p'_\delta\right]$$
$$(12)$$

2. Theoretical approach

➢ Form factors:

◆ Form factors (D-wave):

$$\begin{aligned}
\langle\psi_1(p')|\,\bar{c}(0)\gamma_\mu b(0)\,|B_c(p)\rangle &= g^{B_c\to\psi_1}(q^2)\varepsilon_{\mu\lambda\tau\sigma}\epsilon^{\psi_1*\lambda}P^\tau q^\sigma,\\
\langle\psi_1(p')|\,\bar{c}(0)\gamma_\mu\gamma_5 b(0)\,|B_c(p)\rangle &= -i\left\{f^{B_c\to\psi_1}(q^2)\epsilon_\mu^{\psi_1*}+\epsilon^{\psi_2*}\cdot P\left[a_+^{B_c\to\psi_1}(q^2)P_\mu+a_-^{B_c\to\psi_1}(q^2)q_\mu\right]\right\},\\
\langle\psi_1(p')|\,\bar{c}(0)\sigma_{\mu\nu}b(0)\,|B_c(p)\rangle &= i\left[T_0^{B_c\to\psi_1}(q^2)\frac{\epsilon^{\psi_1*}\cdot q}{(m_{B_c}+m_{\psi_1})^2}\varepsilon_{\mu\nu\lambda\tau}p^\lambda p'^\tau+T_1^{B_c\to\psi_1}(q^2)\varepsilon_{\mu\nu\lambda\tau}p^\lambda\epsilon^{\psi_1*\tau}+T_2^{B_c\to\psi_1}(q^2)\varepsilon_{\mu\nu\lambda\tau}p'^\lambda\epsilon^{\psi_1*\tau}\right],
\end{aligned}\tag{5}$$

$$\begin{aligned}
\langle\psi_2[\eta_{c2}](p')|\,\bar{c}(0)\gamma_\mu b(0)\,|B_c(p)\rangle &= i\left\{m^{B_c\to\psi_2[\eta_{c2}]}(q^2)\epsilon_{\mu\lambda}^{\psi_2[\eta_{c2}]*}P^\lambda+\epsilon_{\lambda\tau}^{\psi_2[\eta_{c2}]*}P^\lambda P^\tau\left[z_+^{B_c\to\psi_2[\eta_{c2}]}(q^2)P_\mu+z_-^{B_c\to\psi_2[\eta_{c2}]}(q^2)q_\mu\right]\right\},\\
\langle\psi_2[\eta_{c2}](p')|\,\bar{c}(0)\gamma_\mu\gamma_5 b(0)\,|B_c(p)\rangle &= -n^{B_c\to\psi_2[\eta_{c2}]}(q^2)\varepsilon_{\mu\tau\rho\sigma}\epsilon_{\tau\lambda}^{\psi_2[\eta_{c2}]*}P^\lambda P^\rho q^\sigma,\\
\langle\psi_2[\eta_{c2}](p')|\,\bar{c}(0)\sigma_{\mu\nu}\gamma_5 b(0)\,|B_c(p)\rangle &= -\frac{iq^\sigma}{m_{B_c}}\left[T_0^{B_c\to\psi_2[\eta_{c2}]}(q^2)\frac{\epsilon_{\rho\sigma}^{\psi_2[\eta_{c2}]*}q^\rho}{(m_{B_c}+m_{\psi_2[\eta_{c2}]})^2}\varepsilon_{\mu\nu\lambda\tau}p^\lambda p'^\tau+T_1^{B_c\to\psi_2[\eta_{c2}]}(q^2)\varepsilon_{\mu\nu\lambda\tau}\epsilon_{\lambda\sigma}^{\psi_2[\eta_{c2}]*}p_\tau\right.\\
&\left.+T_2^{B_c\to\psi_2[\eta_{c2}]}(q^2)\varepsilon_{\mu\nu\lambda\tau}\epsilon_{\lambda\sigma}^{\psi_2[\eta_{c2}]*}p'_\tau\right],
\end{aligned}\tag{6}$$

$$\begin{aligned}
\langle\psi_3(p')|\,\bar{c}(0)\gamma_\mu b(0)\,|B_c(p)\rangle &= y^{B_c\to\psi_3}(q^2)\varepsilon_{\mu\lambda\tau\rho}\epsilon_{\lambda\chi\sigma}^{\psi_3*}P_\chi P_\sigma P^\tau q^\rho,\\
\langle\psi_3(p')|\,\bar{c}(0)\gamma_\mu\gamma_5 b(0)\,|B_c(p)\rangle &= -i\left\{w^{B_c\to\psi_3}(q^2)\epsilon_{\mu\lambda\tau}^{\psi_3*}P^\lambda P^\tau+\epsilon_{\lambda\tau\rho}^{\psi_3*}P^\lambda P^\tau P^\rho\left[o_+^{B_c\to\psi_3}(q^2)P_\mu+o_-^{B_c\to\psi_3}(q^2)q_\mu\right]\right\},\\
\langle\psi_3(p')|\,\bar{c}(0)\sigma_{\mu\nu}b(0)\,|B_c(p)\rangle &= \frac{iq^\sigma q^\chi}{m_{B_c}^2}\left[T_0^{B_c\to\psi_3}(q^2)\frac{\varepsilon_{\rho\sigma\chi}^{\psi_3*}q^\rho}{(m_{B_c}+m_{\psi_3})^2}\varepsilon_{\mu\nu\lambda\tau}p^\lambda p'^\tau+T_1^{B_c\to\psi_3}(q^2)\varepsilon_{\mu\nu\lambda\tau}\epsilon_{\lambda\sigma\chi}^{\psi_3*}p_\tau\right.\\
&\left.+T_2^{B_c\to\psi_3}(q^2)\varepsilon_{\mu\nu\lambda\tau}\epsilon_{\lambda\sigma\chi}^{\psi_3*}p'_\tau\right],
\end{aligned}\tag{7}$$

2. Theoretical approach

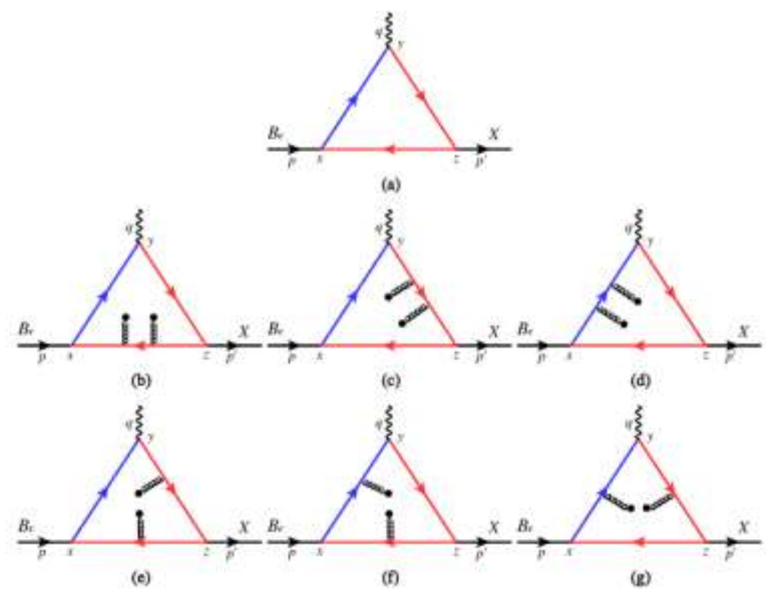
➢ QCD side:

◆ Wick's theorem:

$$
\begin{aligned}
\Pi^{\psi_1 \mathrm{QCD}}_{1\alpha\mu}(p,p') &= i\int d^4x d^4y e^{ip'\cdot z} e^{i(p-p')\cdot y} e^{-ip\cdot x} Tr[C^{km}(x-z)\Gamma_\alpha(z) C^{mn}(z-y)\gamma_\mu B^{nk}(y-x)\gamma_5]\Big|_{z\to 0},\\
\Pi^{\psi_1 \mathrm{QCD}}_{2\alpha\mu}(p,p') &= i\int d^4x d^4y e^{ip'\cdot z} e^{i(p-p')\cdot y} e^{-ip\cdot x} Tr[C^{km}(x-z)\Gamma_\alpha(z) C^{mn}(z-y)\gamma_\mu\gamma_5 B^{nk}(y-x)\gamma_5]\Big|_{z\to 0},\\
\Pi^{\psi_1 \mathrm{QCD}}_{\alpha\mu\nu}(p,p') &= i\int d^4x d^4y e^{ip'\cdot z} e^{i(p-p')\cdot y} e^{-ip\cdot x} Tr[C^{km}(x-z)\Gamma_\alpha(z) C^{mn}(z-y)\sigma_{\mu\nu} B^{nk}(y-x)\gamma_5]\Big|_{z\to 0},
\end{aligned}
$$

◆ Full propagator and Feynman diagrams:

$$
\begin{aligned}
Q^{ij}(x) &= \frac{i}{(2\pi)^4}\int d^4k e^{-ik\cdot x}\Big\{\frac{\delta^{ij}}{\not{k}-m_Q}\\
&-\frac{g_s G^n_{\alpha\beta} t^n_{ij}}{4}\frac{\sigma^{\alpha\beta}(\not{k}+m_Q)+(\not{k}+m_Q)\sigma^{\alpha\beta}}{(k^2-m_Q^2)^2}\\
&-\frac{g_s^2 (t^a t^b)_{ij} G^a_{\alpha\beta} G^b_{\mu\nu}(f^{\alpha\beta\mu\nu}+f^{\alpha\mu\beta\nu}+f^{\alpha\mu\nu\beta})}{4(k^2-m_Q^2)^5}+\ldots\Big\}.
\end{aligned}
$$



2. Theoretical approach

➢ QCD sum rules:

◆ Quark-hadron duality and QCD sum rules for form factors:

$$f^{B_c\to\psi_1}(Q^2) = \frac{m_b+m_c}{f_{\psi_1} f_{B_c} m_{B_c}^2} \exp\left(\frac{m_{B_c}^2}{T^2}+\frac{m_{\psi_1}^2}{kT^2}\right)\int_{s_{\min}}^{s_0}\int_{u_{\min}}^{u_0} ds du \rho_2^{\psi_1 \mathrm{QCD}}(s,u,Q^2) \exp\left(-\frac{s}{T^2}-\frac{u}{kT^2}\right),$$

$$a_+^{B_c\to\psi_1}(Q^2) = \frac{m_b+m_c}{2f_{\psi_1} f_{B_c} m_{B_c}^2} \exp\left(\frac{m_{B_c}^2}{T^2}+\frac{m_{\psi_1}^2}{kT^2}\right)\int_{s_{\min}}^{s_0}\int_{u_{\min}}^{u_0} ds du \left[\rho_3^{\psi_1 \mathrm{QCD}}(s,u,Q^2)+\rho_4^{\psi_1 \mathrm{QCD}}(s,u,Q^2)\right] \exp\left(-\frac{s}{T^2}-\frac{u}{kT^2}\right),$$

$$a_-^{B_c\to\psi_1}(Q^2) = \frac{m_b+m_c}{2f_{\psi_1} f_{B_c} m_{B_c}^2} \exp\left(\frac{m_{B_c}^2}{T^2}+\frac{m_{\psi_1}^2}{kT^2}\right)\int_{s_{\min}}^{s_0}\int_{u_{\min}}^{u_0} ds du \left[\rho_3^{\psi_1 \mathrm{QCD}}(s,u,Q^2)-\rho_4^{\psi_1 \mathrm{QCD}}(s,u,Q^2)\right] \exp\left(-\frac{s}{T^2}-\frac{u}{kT^2}\right), \quad \text{(A1)}$$

$$\rho_i^{\psi_1 QCD}(s,u,Q^2) = \frac{\operatorname{Im}_{\varepsilon_1\to 0,\varepsilon_2\to 0} \tilde{\Pi}_i^{\psi_1 QCD}(p^2+i\varepsilon_1, p'^2+i\varepsilon_2, Q^2)}{\pi^2}$$

- The threshold parameters **s0** and **u0** are determined by the calculations of two-point QCD sum rules.

2. Theoretical approach

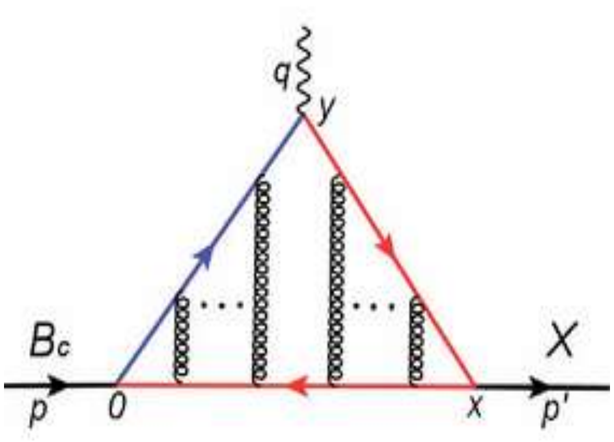
➢ The Coulomb-like correction (P-wave):

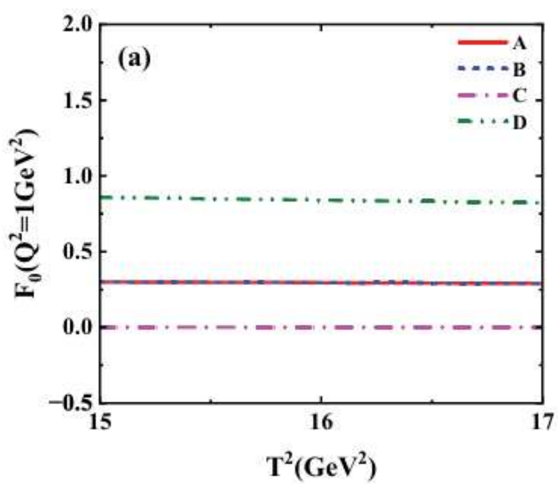
◆ QCD sum rules beyond leading order:

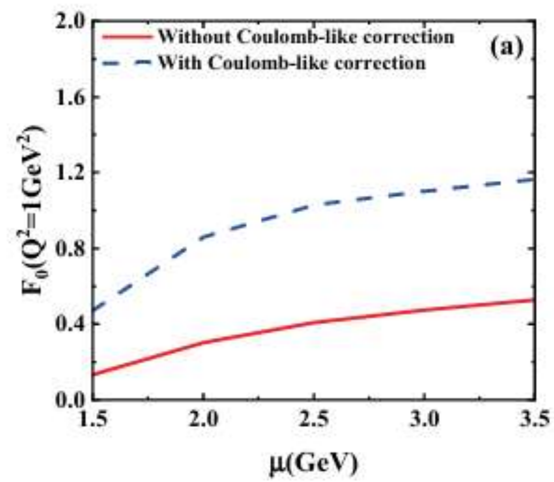
$$\rho_c^{pert}(s,u,Q^2) = \sqrt{\frac{4\pi\alpha_s^C}{3v_1}\left[1-\exp\left(-\frac{4\pi\alpha_s^C}{3v_1}\right)\right]^{-1}} \times \sqrt{\frac{4\pi\alpha_s^C}{3v_2}\left[1-\exp\left(-\frac{4\pi\alpha_s^C}{3v_2}\right)\right]^{-1}}\,\rho^{pert}(s,u,Q^2)$$

$$v_1 = \sqrt{1-\frac{4m_b m_c}{s-(m_b-m_c)^2}}$$

$$v_2 = \sqrt{1-\frac{4m_c^2}{u}}$$



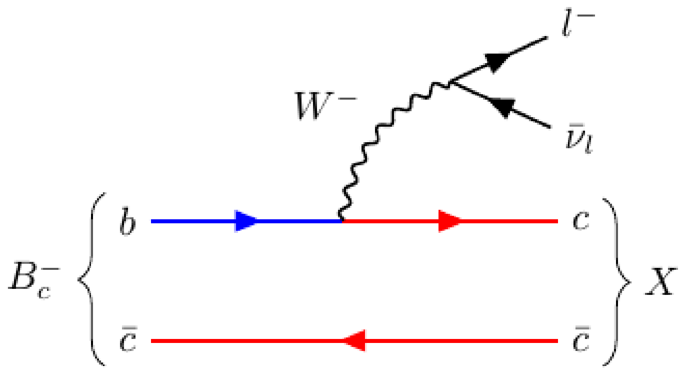


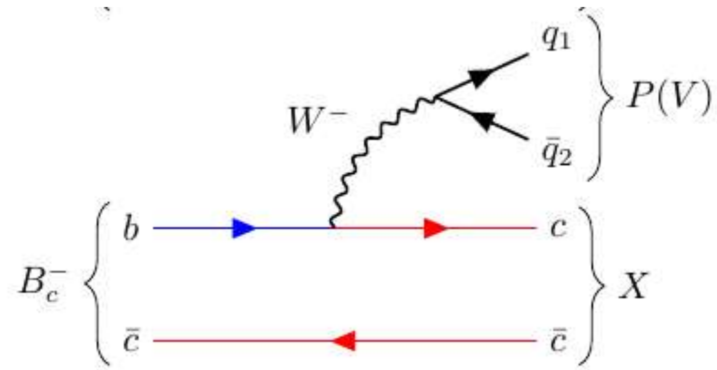


- **Coulomb correction** (α_s/v) increases form factors by a factor of 2-3, highlighting the need for rigorous NLO calculations.

2. Theoretical approach

- **The semileptonic and noleptonic decays:**





- **Effective Hamiltonian:**

$$H_{eff} = \frac{G_F}{\sqrt{2}} \sum_{l=e,\mu,\tau} V_{cb}\bar{c}\gamma_\mu(1-\gamma_5)b\bar{\nu}_l\gamma_\mu(1-\gamma_5)l$$

$$H_{eff} = \frac{G_F}{\sqrt{2}} \sum_{q=d,s} V_{cb}V^*_{uq}a_1\bar{c}\gamma_\mu(1-\gamma_5)b\bar{q}\gamma_\mu(1-\gamma_5)u$$

- **Transition matrix:**

$$\begin{aligned} T &= \langle X(p')l(l)\bar{\nu}_l(v)\big| H_{eff}\,|B_c(p)\rangle \\ &= \frac{G_F}{\sqrt{2}} V_{cb}\,\langle X(p')\big|\,\bar{c}\gamma_\mu(1-\gamma_5)b\,|B_c(p)\rangle \\ &\times \langle l(l)\bar{\nu}_l(v)|\,\bar{\nu}_l\gamma_\mu(1-\gamma_5)l\,|0\rangle \end{aligned}$$

$$\begin{aligned} T &= \langle X(p')P[V](q)\big| H_{eff}\,|B_c(p)\rangle \\ &= \frac{G_F}{\sqrt{2}} V_{cb}V^*_{uq}a_1\,\langle X(p')\big|\,\bar{c}\gamma_\mu(1-\gamma_5)b\,|B_c(p)\rangle \\ &\times \langle P[V](q)|\,\bar{q}\gamma_\mu(1-\gamma_5)u\,|0\rangle \end{aligned}$$

3. Numerical results

➢ Numerical Inputs:

IP	Values (GeV)	IP	Values
m_{B_c}	6.274 [73]	$f_{\chi_{c0}}$	0.343 GeV [74]
$m_{\chi_{c0}}$	3.414 [73]	$f_{\chi_{c1}}$	0.338 GeV [74]
$m_{\chi_{c1}}$	3.511 [73]	f_{h_c}	0.235 GeV [75]
m_{h_c}	3.525 [73]	$f_{\chi_{c2}}$	0.0111 ± 0.0062 [76]
$m_{\chi_{c2}}$	3.556 [73]	$\langle g_s^2 GG \rangle$	0.88 ± 0.15 GeV^4 [77–79]
f_{B_c}	0.371 ± 0.037 [44]		

IP	Values (GeV)	IP	Values
m_{B_c}	6.274 [65]	f_{B_c}	0.371 ± 0.037 GeV [58]
m_{ψ_1}	3.77 [65]	f_{ψ_1}	$13.82^{+2.18}_{-1.98}$ GeV^4 [48]
m_{ψ_2}	3.82 [65]	f_{ψ_2}	$29.29^{+4.00}_{-3.69}$ GeV^4 [48]
$m_{\eta_{c2}}$	3.83 [48]	$f_{\eta_{c2}}$	$11.64^{+1.72}_{-1.58}$ GeV^4 [48]
m_{ψ_3}	3.84 [65]	f_{ψ_3}	$14.94^{+2.11}_{-1.94}$ GeV^4 [48]

◆ Masses of *b* and *c* quarks:

$$m_Q(\mu) = m_Q(m_Q)\left[\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right]^{\frac{12}{33-2n_f}},$$

$$\alpha_s(\mu) = \frac{1}{b_0 t}\left[1 - \frac{b_1}{b_0^2}\frac{\log t}{t} + \frac{b_1^2(\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t^2}\right],$$

$$t = \log\left(\frac{\mu^2}{\Lambda_{QCD}^2}\right),\quad b_0 = \frac{33-2N_f}{12\pi},\quad b_1 = \frac{153-19N_f}{24\pi^2}$$

$$b_2 = \frac{2857-\frac{5033}{9}N_f+\frac{325}{27}N_f^2}{128\pi^3}.\quad \Lambda_{QCD} = 213\ \text{MeV}$$

$$m_c(m_c) = 1.275 \pm 0.025\ \text{GeV}$$

$$m_b(m_b) = 4.18 \pm 0.03\ \text{GeV}.$$

◆ Energy scale: $\mu = 2$ GeV, determined from NLO two-point sum rules for B_c.

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3. Numerical results

➢ Borel platform selection:

$$\Pi_{\text{pole}}^{\text{QCD}}(T^2) = \int_{s_{\min}}^{s_0}\int_{u_{\min}}^{u_0} ds du \rho^{\text{QCD}}(s,u,Q^2)\exp\left(-\frac{s}{T^2}-\frac{u}{kT^2}\right),$$

$$\Pi_{\text{cont.}}^{\text{QCD}}(T^2) = \int_{s_0}^{\infty}\int_{u_0}^{\infty} ds du \rho^{\text{QCD}}(s,u,Q^2)\exp\left(-\frac{s}{T^2}-\frac{u}{kT^2}\right).$$

◆ Criteria:

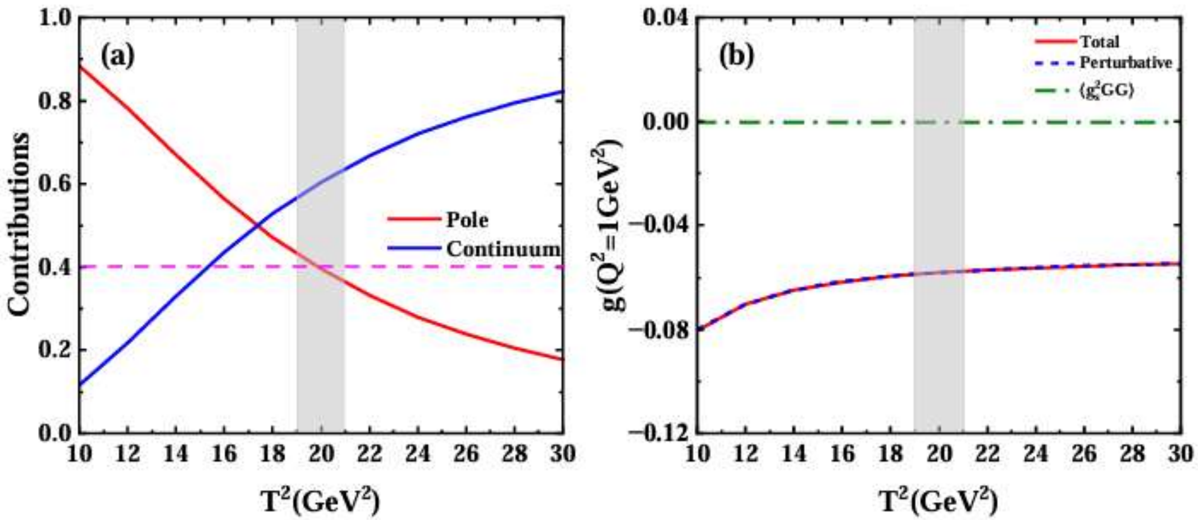
- **Pole dominance:** Pole > 40-50%
- **OPE convergence** (gluon condensate contribution small)

◆ Example: $B_c \to \psi_1$ form factor g at $Q^2 = 1$ GeV²

- Borel platform: $T^2 = 19 - 21$ GeV²
- Pole contribution: ~40%
- Gluon condensate contribution: < 1%

$$\text{Pole} = \frac{\Pi_{\text{pole}}^{\text{QCD}}(T^2)}{\Pi_{\text{pole}}^{\text{QCD}}(T^2)+\Pi_{\text{cont.}}^{\text{QCD}}(T^2)},$$

$$\text{Continuum} = \frac{\Pi_{\text{cont.}}^{\text{QCD}}(T^2)}{\Pi_{\text{pole}}^{\text{QCD}}(T^2)+\Pi_{\text{cont.}}^{\text{QCD}}(T^2)}.$$



3. Numerical results

➢ z-series expansion:

- **Method:** z-series expansion to extrapolate from spacelike ($Q^2 > 0$) to timelike ($Q^2 < 0$) region.

$$F(Q^2) = \frac{1}{1+Q^2/m_{\rm pole}^2} \times \sum_{k=0}^{N-1} b_k \left[z(Q^2,t_0)^k - (-1)^{k-N}\frac{k}{N}\, z(Q^2,t_0)^N \right]$$

$$z(Q^2,t_0) = \frac{\sqrt{t_+ + Q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ + Q^2} + \sqrt{t_+ - t_0}},$$

$t_\pm = (m_{B_c} \pm m_X)^2$, and $t_0 = t_+ - \sqrt{t_+(t_+ - t_-)}$.

- **Fitting Parameters (D-wave results):**

Mode	FF	b_0	b_1	b_2	$F(0)$
	g	-0.062	0.19	5.35	$-0.060^{+0.009}_{-0.010}$
	f	2.92	31.00	-589.80	$3.13^{+0.58}_{-0.49}$
	a_+	0.037	-1.73	21.78	$0.025^{+0.000}_{-0.001}$
$B_c \to \psi_1$	a_-	-0.054	3.35	-52.82	$-0.030^{+0.003}_{-0.005}$
	T_0	2.76	-66.32	733.27	$2.28^{+0.15}_{-0.17}$
	T_1	0.079	2.60	-40.91	$0.097^{+0.015}_{-0.013}$
	T_2	0.6	-1.62	-51.82	$0.62^{+0.11}_{-0.10}$
	m	-0.43	-2.44	102.41	$-0.44^{+0.06}_{-0.07}$
	z_+	-0.015	0.68	-10.70	$-0.010^{+0.001}_{-0.001}$
	z_-	-0.32×10^{-2}	-0.50	13.95	$(-0.63^{+0.10}_{-0.14}) \times 10^{-2}$
$B_c \to \psi_2$	n	0.020	-0.45	4.25	$0.017^{+0.002}_{-0.002}$
	T_0	2.57	-75.21	889.27	$2.05^{+0.21}_{-0.21}$
	T_1	-0.019	-4.33	92.54	$-0.046^{+0.018}_{-0.023}$
	T_2	-0.61	10.03	-78.23	$-0.54^{+0.05}_{-0.05}$
	m	-0.28	1.13	44.01	$-0.27^{+0.05}_{-0.07}$
	z_+	-0.026	1.04	-15.95	$-0.020^{+0.002}_{-0.001}$
	z_-	0.024	-1.51	30.34	$0.014^{+0.001}_{-0.000}$
$B_c \to \eta_{c2}$	n	0.56×10^{-2}	-0.07	-0.13	$(0.50^{+0.11}_{-0.09}) \times 10^{-2}$
	T_0	5.68	-204.32	3051.17	$4.31^{+0.32}_{-0.36}$
	T_1	-0.63×10^{-3}	0.85×10^{-2}	0.02	$(-0.56^{+0.05}_{-0.06}) \times 10^{-3}$
	T_2	-0.25	3.49	3.41	$-0.23^{+0.04}_{-0.05}$
	y	0.52×10^{-2}	-0.19	2.81	$(0.39^{+0.03}_{-0.03}) \times 10^{-2}$
	w	-0.25	5.82	-62.24	$-0.21^{+0.02}_{-0.02}$
	o_+	0.16×10^{-2}	-0.50×10^{-2}	-0.52	$(0.15^{+0.01}_{-0.01}) \times 10^{-2}$
$B_c \to \psi_3$	o_-	0.54×10^{-2}	-0.18	2.58	$(0.41^{+0.00}_{-0.00}) \times 10^{-2}$
	T_0	-0.097	-76.41	2601.08	$-0.52^{+0.13}_{-0.17}$
	T_1	-0.20	15.53	-324.08	$-0.11^{+0.01}_{-0.01}$
	T_2	2.48	-86.17	1281.06	$1.91^{+0.15}_{-0.17}$

3. Numerical results

➢ Form factors at $Q^2 = 0$:

◆ *P*-wave results and compared with other collaborations:

Modes	Form factors	This work	This work*	[9, 10]	[25]	[26]	[27]	[28]	[31]
	F_0	$0.29^{+0.03}_{-0.03}$	$0.84^{+0.06}_{-0.08}$	0.67	0.47	0.33	0.337	–	1.65
$B_c \to \chi_{c0}$	F_1	$0.29^{+0.03}_{-0.03}$	$0.83^{+0.07}_{-0.07}$	0.67	0.47	0.33	0.337	–	1.65
	T	$0.37^{+0.03}_{-0.04}$	$1.05^{+0.07}_{-0.08}$	–	–	–	–	–	–
	A	$0.20^{+0.02}_{-0.02}$	$0.57^{+0.03}_{-0.05}$	0.13	0.36	0.24	0.215	0.21	1.24
	V_0	$0.064^{+0.013}_{-0.012}$	$0.16^{+0.03}_{-0.03}$	0.03	0.13	0.15	0.025	0.07	0.17
	V_1	$0.50^{+0.05}_{-0.55}$	$1.40^{+0.12}_{-0.12}$	0.28	0.85	0.72	0.339	0.47	2.87
$B_c \to \chi_{c1}$	V_2	$0.091^{+0.005}_{-0.006}$	$0.27^{+0.01}_{-0.02}$	0.059	0.15	0.10	0.078	0.08	0.57
	T_0	$0.093^{+0.006}_{-0.008}$	$0.27^{+0.01}_{-0.02}$	–	–	–	–	–	–
	T_1	$-0.47^{+0.06}_{-0.06}$	$-1.31^{+0.11}_{-0.14}$	–	–	–	–	–	–
	T_2	$1.29^{+0.13}_{-0.14}$	$3.68^{+0.31}_{-0.35}$	–	–	–	–	–	–
	A	$0.081^{+0.010}_{-0.012}$	$0.21^{+0.02}_{-0.02}$	0.14	0.07	0.06	0.039	0.04	0.07
	V_0	$0.85^{+0.08}_{-0.07}$	$2.46^{+0.16}_{-0.18}$	0.03	0.64	0.41	0.390	0.35	2.11
	V_1	$2.59^{+0.24}_{-0.26}$	$7.47^{+0.57}_{-0.65}$	0.29	0.50	0.42	0.298	0.30	0.54
$B_c \to h_c$	V_2	$0.17^{+0.02}_{-0.02}$	$0.49^{+0.07}_{-0.05}$	0.059	−0.32	−0.18	−0.196	−0.16	−0.99
	T_0	$0.17^{+0.01}_{-0.02}$	$0.48^{+0.03}_{-0.03}$	–	–	–	–	–	–
	T_1	$-0.041^{+0.002}_{-0.002}$	$-0.14^{+0.00}_{-0.01}$	–	–	–	–	–	–
	T_2	$0.54^{+0.01}_{-0.03}$	$1.62^{+0.04}_{-0.05}$	–	–	–	–	–	–
	h	$0.034^{+0.002}_{-0.003}$	$0.095^{+0.005}_{-0.007}$	-1.70×10^{-4}	0.022	–	–	–	–
	k	$2.06^{+0.10}_{-0.21}$	$5.67^{+0.37}_{-0.38}$	0.18	1.27	–	–	–	–
	b_+	$-0.010^{+0.001}_{-0.004}$	$-0.031^{+0.002}_{-0.001}$	−0.038	−0.011	–	–	–	–
$B_c \to \chi_{c2}$	b_-	$0.011^{+0.001}_{-0.000}$	$0.035^{+0.001}_{-0.001}$	−0.056	0.020	–	–	–	–
	T_0	$-0.097^{+0.003}_{-0.015}$	$-0.24^{+0.01}_{-0.03}$	–	–	–	–	–	–
	T_1	$0.027^{+0.002}_{-0.003}$	$0.089^{+0.006}_{-0.007}$	–	–	–	–	–	–
	T_2	$2.18^{+0.14}_{-0.22}$	$6.02^{+0.38}_{-0.42}$	–	–	–	–	–	–

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3. Numerical results

➢ The semileptonic decay widths and branching ratios:

◆ *P*-wave results and compared with other collaborations:

TABLE VII: Decay widths (in 10^{-6}eV) and branching ratios (%) of B_c semileptonic decays. Branching ratios are calculated at τ_{B_c} = 0.51 ps [73]. The superscript star denotes the results obtained by considering Coulomb-like correction.

Decay channels	Decay widths		Branching ratios								
	This work	This work*	This work	This work*	[25]	[33]	[21]	[23]	[9, 10]	[27]	[28]
$B_c^- \to \chi_{c0} e \bar{\nu}_e$	$1.25^{+0.20}_{-0.24}$	$9.99^{+1.29}_{-1.44}$	$0.097^{+0.016}_{-0.019}$	$0.78^{+0.09}_{-0.12}$	0.21	0.12	0.17	0.11	0.182	0.11	–
$B_c^- \to \chi_{c0} \mu \bar{\nu}_\mu$	$1.24^{+0.20}_{-0.24}$	$9.93^{+1.26}_{-1.45}$	$0.096^{+0.016}_{-0.018}$	$0.77^{+0.10}_{-0.11}$	0.21	0.12	0.17	0.11	0.182	0.11	–
$B_c^- \to \chi_{c0} \tau \bar{\nu}_\tau$	$0.14^{+0.02}_{-0.02}$	$1.05^{+0.20}_{-0.08}$	$0.011^{+0.002}_{-0.002}$	$0.082^{+0.015}_{-0.007}$	0.024	0.017	0.013	0.013	0.049	0.0094	–
$B_c^- \to \chi_{c1} e \bar{\nu}_e$	$0.61^{+0.15}_{-0.10}$	$4.85^{+0.71}_{-0.83}$	$0.047^{+0.012}_{-0.007}$	$0.38^{+0.05}_{-0.07}$	0.14	0.15	0.092	0.066	0.146	0.030	0.052
$B_c^- \to \chi_{c1} \mu \bar{\nu}_\mu$	$0.60^{+0.15}_{-0.09}$	$4.82^{+0.70}_{-0.82}$	$0.047^{+0.011}_{-0.007}$	$0.37^{+0.06}_{-0.06}$	0.14	0.15	0.092	0.066	0.146	0.029	0.052
$B_c^- \to \chi_{c1} \tau \bar{\nu}_\tau$	$0.064^{+0.018}_{-0.007}$	$0.51^{+0.08}_{-0.08}$	$0.0049^{+0.0014}_{-0.0005}$	$0.039^{+0.007}_{-0.005}$	0.015	0.024	0.0089	0.0072	0.0147	0.0026	0.0057
$B_c^- \to h_c e \bar{\nu}_e$	$22.15^{+3.18}_{-3.43}$	$194.23^{+23.08}_{-24.12}$	$1.72^{+0.24}_{-0.27}$	$15.06^{+1.79}_{-1.87}$	0.31	0.18	0.27	0.17	0.142	0.11	0.105
$B_c^- \to h_c \mu \bar{\nu}_\mu$	$22.05^{+3.16}_{-3.41}$	$193.31^{+22.98}_{-23.96}$	$1.71^{+0.24}_{-0.27}$	$14.98^{+1.79}_{-1.85}$	0.31	0.18	0.27	0.17	0.142	0.11	0.104
$B_c^- \to h_c \tau \bar{\nu}_\tau$	$3.03^{+0.36}_{-0.46}$	$26.92^{+3.08}_{-2.66}$	$0.24^{+0.02}_{-0.04}$	$2.09^{+0.24}_{-0.21}$	0.022	0.025	0.017	0.015	0.0137	0.0051	0.0088
$B_c^- \to \chi_{c2} e \bar{\nu}_e$	$7.50^{+0.79}_{-0.81}$	$55.92^{+7.83}_{-6.64}$	$0.58^{+0.06}_{-0.06}$	$4.34^{+0.51}_{-0.52}$	0.17	0.19	0.17	0.13	0.130	–	–
$B_c^- \to \chi_{c2} \mu \bar{\nu}_\mu$	$7.43^{+0.78}_{-0.80}$	$55.38^{+7.74}_{-6.58}$	$0.57^{+0.07}_{-0.06}$	$4.29^{+0.60}_{-0.51}$	0.17	0.19	0.17	0.13	0.130	–	–
$B_c^- \to \chi_{c2} \tau \bar{\nu}_\tau$	$0.41^{+0.04}_{-0.01}$	$3.01^{+0.44}_{-0.29}$	$0.031^{+0.004}_{-0.001}$	$0.23^{+0.04}_{-0.02}$	0.0092	0.029	0.0082	0.0093	0.020	–	–

3. Numerical results

➢ The semileptonic decay widths and branching ratios:

◆ *D*-wave results:

TABLE III: Decay widths (in 10^{-7} eV) and branching ratios (10^{-3}) of B_c to D-wave charmonia semileptonic decays. Branching ratios are determined by $\tau_{B_c} = 0.51$ ps [65].

Decay channels	Decay widths	Branching ratios	Decay channels	Decay widths	Branching ratios
$B_c^- \to \psi_1 e^- \bar{\nu}_e$	$28.70^{+10.70}_{-7.90}$	$2.22^{+0.84}_{-0.61}$	$B_c^- \to \eta_{c2} e^- \bar{\nu}_e$	$3.02^{+0.96}_{-0.75}$	$0.23^{+0.08}_{-0.05}$
$B_c^- \to \psi_1 \mu^- \bar{\nu}_\mu$	$28.41^{+10.60}_{-7.82}$	$2.20^{+0.83}_{-0.60}$	$B_c^- \to \eta_{c2} \mu^- \bar{\nu}_\mu$	$2.97^{+0.95}_{-0.74}$	$0.23^{+0.07}_{-0.06}$
$B_c^- \to \psi_1 \tau^- \bar{\nu}_\tau$	$1.31^{+0.54}_{-0.38}$	$0.10^{+0.04}_{-0.03}$	$B_c^- \to \eta_{c2} \tau^- \bar{\nu}_\tau$	$0.027^{+0.011}_{-0.007}$	$0.21^{+0.08}_{-0.06} \times 10^{-2}$
$B_c^- \to \psi_2 e^- \bar{\nu}_e$	$3.32^{+0.89}_{-0.85}$	$0.26^{+0.07}_{-0.07}$	$B_c^- \to \psi_3 e^- \bar{\nu}_e$	$1.15^{+0.31}_{-0.28}$	$0.090^{+0.020}_{-0.022}$
$B_c^- \to \psi_2 \mu^- \bar{\nu}_\mu$	$3.27^{+0.88}_{-0.84}$	$0.25^{+0.07}_{-0.06}$	$B_c^- \to \psi_3 \mu^- \bar{\nu}_\mu$	$1.14^{+0.30}_{-0.28}$	$0.088^{+0.022}_{-0.021}$
$B_c^- \to \psi_2 \tau^- \bar{\nu}_\tau$	$0.039^{+0.012}_{-0.010}$	$0.30^{+0.09}_{-0.08} \times 10^{-2}$	$B_c^- \to \psi_3 \tau^- \bar{\nu}_\tau$	$0.98^{+0.26}_{-0.22} \times 10^{-2}$	$0.76^{+0.20}_{-0.17} \times 10^{-3}$

- The branching ratios for $\psi_1, \psi_2(\eta_{c2})$ *and* ψ_3 channels can reach the order of magnitude 10^{-3}, 10^{-4} and 10^{-5}, respectively, which can provide a valuable reference for future experimental measurements for B_c meson by LHCb collaboration.

3. Numerical results

➢ The nonleptonic decay widths and branching ratios:

◆ Only *P*-wave results:

TABLE VIII: Decay widths (in 10^{-7}eV) and Branching ratios (10^{-3}) of B_c nonleptonic decays. The superscript star denotes the results obtained by considering Coulomb-like correction.

Decay channels	Decay widths		Branching ratios							
	This work	This work*	This work	This work*	[22]	[23]	[26]	[17]	[31]	[16]
$B_c^- \to \chi_{c0}\pi^-$	$3.11^{+0.65}_{-0.58}$	$25.30^{+3.99}_{-4.07}$	$0.24^{+0.05}_{-0.04}$	$1.96^{+0.21}_{-0.30}$	0.21	0.26	0.66	1.6	6.47	9.8
$B_c^- \to \chi_{c0}K^-$	$0.24^{+0.05}_{-0.04}$	$1.96^{+0.31}_{-0.31}$	$0.019^{+0.004}_{-0.004}$	$0.15^{+0.03}_{-0.02}$	0.016	0.02	0.052	0.12	0.49	–
$B_c^- \to \chi_{c0}\rho^-$	$8.23^{+1.46}_{-1.67}$	$66.60^{+8.88}_{-11.31}$	$0.64^{+0.11}_{-0.13}$	$5.16^{+0.69}_{-0.87}$	0.58	0.67	1.69	5.8	–	33
$B_c^- \to \chi_{c0}K^{*-}$	$0.43^{+0.08}_{-0.09}$	$3.49^{+0.46}_{-0.59}$	$0.033^{+0.006}_{-0.006}$	$0.27^{+0.04}_{-0.04}$	0.04	0.037	0.096	0.33	–	–
$B_c^- \to \chi_{c1}\pi^-$	$0.14^{+0.06}_{-0.05}$	$0.91^{+0.31}_{-0.34}$	$0.011^{+0.005}_{-0.004}$	$0.070^{+0.025}_{-0.026}$	0.2	0.0014	0.13	0.51	0.064	0.089
$B_c^- \to \chi_{c1}K^-$	$0.011^{+0.004}_{-0.004}$	$0.070^{+0.024}_{-0.026}$	$0.00082^{+0.00038}_{-0.00028}$	$0.0054^{+0.0019}_{-0.0020}$	0.015	0.00011	0.010	0.038	0.0049	–
$B_c^- \to \chi_{c1}\rho^-$	$0.88^{+0.23}_{-0.23}$	$6.74^{+1.05}_{-1.74}$	$0.068^{+0.018}_{-0.018}$	$0.52^{+0.08}_{-0.13}$	0.15	0.10	0.43	2.8	–	4.6
$B_c^- \to \chi_{c1}K^{*-}$	$0.050^{+0.017}_{-0.010}$	$0.41^{+0.07}_{-0.10}$	$0.0041^{+0.0011}_{-0.0010}$	$0.032^{+0.005}_{-0.008}$	0.01	0.0073	0.027	0.18	–	–
$B_c^- \to h_c\pi^-$	$24.07^{+4.74}_{-3.74}$	$199.98^{+28.92}_{-25.76}$	$1.87^{+0.36}_{-0.29}$	$15.50^{+2.24}_{-1.99}$	0.46	0.53	0.96	0.54	9.73	16
$B_c^- \to h_cK^-$	$1.86^{+0.37}_{-0.29}$	$15.48^{+2.20}_{-1.59}$	$0.14^{+0.03}_{-0.02}$	$1.20^{+0.17}_{-0.15}$	0.035	0.041	0.075	0.043	0.74	–
$B_c^- \to h_c\rho^-$	$61.36^{+12.36}_{-9.43}$	$523.62^{+65.30}_{-78.40}$	$4.76^{+0.95}_{-0.73}$	$40.59^{+5.06}_{-6.08}$	1.0	1.3	2.42	2.3	–	53
$B_c^- \to h_cK^{*-}$	$3.44^{+0.68}_{-0.53}$	$29.20^{+3.64}_{-4.35}$	$0.27^{+0.05}_{-0.04}$	$2.26^{+0.29}_{-0.33}$	0.07	0.071	0.13	0.13	–	–
$B_c^- \to \chi_{c2}\pi^-$	$19.43^{+1.94}_{-3.97}$	$142.31^{+19.87}_{-20.76}$	$1.51^{+0.15}_{-0.31}$	$11.03^{+1.54}_{-1.61}$	0.38	0.22	–	4.0	4.37	8.3
$B_c^- \to \chi_{c2}K^-$	$1.45^{+0.15}_{-0.29}$	$10.63^{+1.50}_{-1.53}$	$0.11^{+0.01}_{-0.02}$	$0.82^{+0.12}_{-0.11}$	0.028	0.017	–	0.31	0.33	–
$B_c^- \to \chi_{c2}\rho^-$	$52.70^{+5.31}_{-9.90}$	$387.82^{+54.56}_{-53.93}$	$4.09^{+0.41}_{-0.77}$	$30.06^{+4.23}_{-4.18}$	1.10	0.65	–	16	–	33
$B_c^- \to \chi_{c2}K^{*-}$	$2.80^{+0.28}_{-0.51}$	$20.61^{+2.90}_{-2.82}$	$0.22^{+0.02}_{-0.04}$	$1.60^{+0.22}_{-0.22}$	0.074	0.038	–	0.96	–	–

- The theoretical results of different collaborations are not consistent well with each other.

[22] D. Ebert, R. N. Faustov, and V. O. Galkin, Phys. Rev. D 82, 034019 (2010). (RQM)

[23] E. Hernandez, J. Nieves, and J. M. Verde-Velasco, Phys. Rev. D 74, 074008 (2006). (NRQM)

3. Numerical results

➢ The nonleptonic decay widths and branching ratios:

◆ The recent LHCb results for $B_c \to \chi_J \pi$:

A study of $B_c^+ \to \chi_c \pi^+$ decays is reported using proton-proton collision data, collected with the LHCb detector at centre-of-mass energies of 7, 8, and 13 TeV, corresponding to an integrated luminosity of $9\,\text{fb}^{-1}$. The decay $B_c^+ \to \chi_{c2}\pi^+$ is observed for the first time, with a significance exceeding seven standard deviations. The relative branching fraction with respect to the $B_c^+ \to J/\psi\pi^+$ decay is measured to be

$$\frac{\mathcal{B}_{B_c^+\to\chi_{c2}\pi^+}}{\mathcal{B}_{B_c^+\to J/\psi\pi^+}} = 0.37 \pm 0.06 \pm 0.02 \pm 0.01\,,$$

where the first uncertainty is statistical, the second is systematic, and the third is due to the knowledge of the $\chi_{c2} \to J/\psi\gamma$ branching fraction. No significant $B_c^+ \to \chi_{c1}\pi^+$ signal is observed and an upper limit for the relative branching fraction for the $B_c^+ \to \chi_{c1}\pi^+$ and $B_c^+ \to \chi_{c2}\pi^+$ decays of

$$\frac{\mathcal{B}_{B_c^+\to\chi_{c1}\pi^+}}{\mathcal{B}_{B_c^+\to\chi_{c2}\pi^+}} < 0.49$$

is set at the 90% confidence level.

- Our prediction for the central value of this ratio is 0.16, which is consistent with the experimental result.

■ R. Aaij et al. (LHCb Collaboration), JHEP 02 (2024) 173 .

4. Summary

➢ Conclusions and Outlook

◆ Conclusions:

- This work provides the first systematic analysis of $B_c \to D$-wave charmonia form factors using three-point QCD sum rules.
- Together with previous studies on S-wave and P-wave, a complete picture of B_c transitions to low-lying charmonia is emerging.
- Predicted branching ratios offer valuable guidance for experimental searches at LHCb.

◆ Outlook:

□ Theory:

- Perform rigorous NLO QCD corrections to give more precise prediction.
- Study non-leptonic decays using more advanced factorization methods.
- Extend to higher orbital excitations and radial excitations.

□ Experiment:

- Search for more B_c decays to P and D-wave charmonia states at LHCb.
- Use precise branching ratio measurements to test and refine theoretical models.

Thanks for your listening!