Applications of fiber bundles in gauge field theory

Qitong Fan ,1120231647

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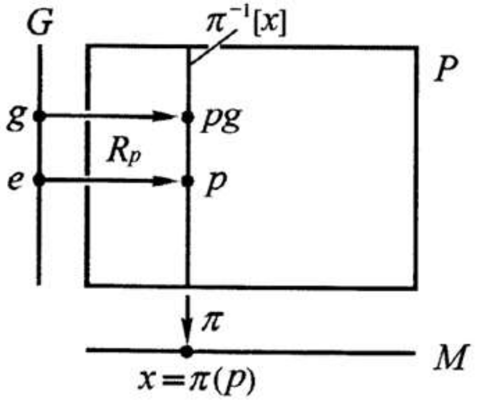
Fiber bundle

Principle Fiber bundle (P,M,G)
P: bundle manifold ,dim m + n
M: base manifold ,dim m
G: structure group(Lie group),dim n , right acts R : PxG->P
π： P->M, projective mapping ,reversible,$\pi^{-1}[x]$called fiber
σ (x):U in M ->P , section (if U= M, it is global)



Rignt acts: R : PxG->P, can be written as :$R_g: P \to P$, $or\ R(p): G \to P$
It don’t influent x =π（p)

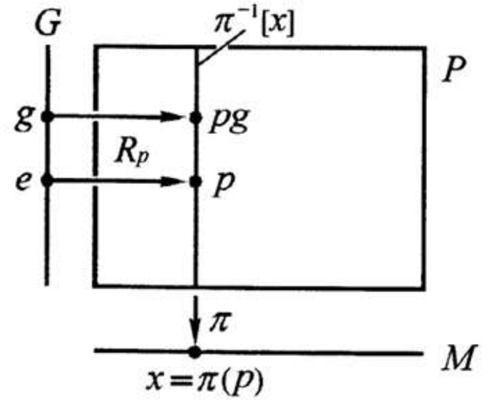
Fiber bundle

Lie algebra on G (especially SU(3))
exponential mapping: Exp: Ve->G , $\exp^{-1}(g^{\gamma}) = \gamma\, T_g{}^{a} \in V_e$
For Lie algebra of SU(n) (dim SU(n) = n^2-1),
We represent $g = \exp\left(\gamma\, T_g{}^{a}\right) = \exp\left(i\frac{\theta_a \lambda_a}{2}\right) = e^{i\theta_a T^a}$

$$SU(n) = \{U \in U(m) | detU = 1\}$$

$$LieAlgebra \quad \mathcal{SU}(n) = \{A \in \mathcal{U}(n), trA = 0\}$$

$$A^{+} = -A, \{m \times m \quad antihermit \quad matrices\}$$



Lie braket :$[T^a, T^b] = if^{abc}T_c$, $f^{abc}$is structure constant

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

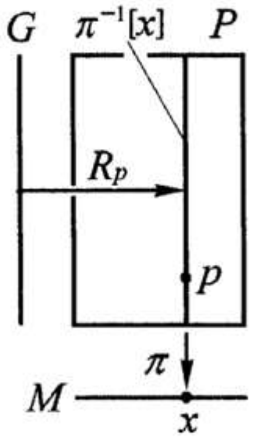
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

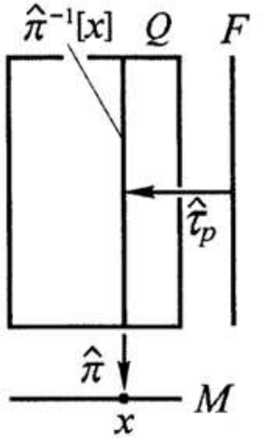
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

When m=3,A can be represented as these λ

Fiber bundle

fiber bundle associated to P(M,G) : (Q,M,F)
Q:$q \in Q, q = p \cdot f = pg \cdot g^{-1}f$
M: base manifold ,dim m
F: type fiber , a manifold left-invariant by g in G
$\hat{\pi}$: Q->M, projective mapping ,reversible
$\hat{\sigma}$:U in M ->P , section (if U= M, it is global)





Tangent space Tp :
Vertical subspace $V_p := \{ X \in T_pP \mid \pi_*(X) = 0 \}$
Horiziotal subspace $H_p \subset T_pP$,

$$(b)R_g * [H_p] = H_{pg}, \forall p \in P, g \in G, (a)T_pP = V_p \oplus H_p, \quad \forall p \in P,$$

$$(c)\ H_p\ of\ p\ are\ smooth$$

Connection in Principle fiber bundle :3 equivalent definition:
Horizontal space
a 1-form field:$\tilde{\omega} \in Lie\ Algebra\ of\ G$
Later will show in part 3

$$\tilde{\omega} \in \mathcal{G}, \quad \omega_p \quad is \quad C^\infty$$

$$(\mathbf{a})\tilde{\omega}_p(A_p^*) = A, \quad \forall A \in \mathcal{G}, p \in P,$$

$$(\mathbf{b})\tilde{\omega}_{pg}(R_{g*}X) = Ad_{g^{-1}}\tilde{\omega}_p(X), \quad \forall p \in P, g \in G, X \in \mathcal{T}_pP.$$

Yang-Mills gauge field(QCD)

quack 's quantum field : spinor field with 3 color
it is a inner space with transition U in SU(3)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix},$$

It's Lagrangian density :
If θ is not relevant with x: $A_\mu = 0$ is correct(global gauge invarience
Local gauge invariance:θ(x): introduce field translation $A_\mu \to A_\mu'$

$$\mathcal{L}_{\mathrm{q}} = \overline{\psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi,$$

$$\mathcal{L}_{\mathrm{q}} = \overline{\psi}(\mathrm{i}\gamma^\mu D_\mu - m)\psi, \quad D_\mu = \partial_\mu + \mathrm{i}qA_\mu,$$

$A_\mu$is gauge potential.(gauge field of SU(3), electromagnetic field of U(1)
In order to keep the Invariance of the Lagrangian:

$$\psi' = U\psi = e^{i\theta^a T_a}\psi$$

$$A_\mu = A_\mu{}^a T_a \ ,$$

$$A_\mu{}^{a\prime} = A_\mu{}^a - f^{abc}\theta^b A_\mu{}^c - \frac{1}{q}\partial_\mu \theta^a = A_\mu{}^a - \frac{1}{q} D_\mu^{ab}\theta^b$$

Description of gauge field in fiber bundle

Into fiber bundle theory :

Fiber bundle :M = Minkowski space

G = SU(3)

F =$V_G = C^3$, let $\rho(g): G \to \hat{G}, \rho(g) = \exp^{-1}\ (g)$

$\hat{G}$ is the representation group of G

transition between cross sections:

$$\sigma' \to \sigma: \quad \sigma'(x) = \sigma(x) g^{-1}(x)\ ,,$$

$$\phi(x) \in \hat{G}\ , \qquad \phi(x)' = \rho\big(g(x)\big)\phi(x)\ ,$$

$$f'(x) \ = \rho\big(g(x)\big) f(x)$$

There is an invariance :

$\Phi(x) = \sigma(x) \cdot f(x),\ \Phi(x) = p \cdot f \in Q$,

$\Phi(x)$ is a section of Q

In tangent bundle :

$$v(x) = \Big(x, e_\mu(x)\Big) \cdot f^\mu = e_\mu(x) f^\mu(x)$$

is covariant.

So we use $\Phi(x)$ to represent quack field ,$\sigma(x)$ represent gauge choice

Description of gauge field in fiber bundle

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How about connection?
Connection definition 3:
$\forall local\ trival\ \ T_U: \pi^{-1}[U] \to U \times G\ \ \ and\ T_V: \pi^{-1}[V] \to V \times G,\ \ (U \cap V \neq \emptyset)\ \ ,with\ \ g_{UV}: T_u \to T_V$
Given a 1-form field $\omega_U \in Lie\ Algebra\ of\ G\ \ ,fit$:

$$\omega_V(Y) = \mathcal{A}d_{g_{UV}(x)^{-1}}\omega_U(Y) + L^{-1}_{g_{UV}(x)*}g_{UV*}(Y), \quad \forall x \in U \cap V, \quad Y \in \mathcal{T}_x M$$

When $\sigma \to \sigma', \boldsymbol{\omega} \to \boldsymbol{\omega}': \sigma^* \widetilde{\boldsymbol{\omega}} \to \sigma'^* \widetilde{\boldsymbol{\omega}},$

For gauge potential :

$$A_\mu{}^{a'} = A_\mu{}^a - \frac{1}{q} D_\mu^{ab} \theta^b$$

Where$\boldsymbol{A} = A_\mu^a\, dx^\mu e^a,$ can be correctly fit defination3:
So we knew that k**A** is the connection **ω**

Description of gauge field in fiber bundle

Further more ,the 2-form **F** is correctly the
Cartan's second structure equation:

$$R_{\mu}^{\nu} \ = d\omega_{\mu}^{\nu} + \omega_{\mu}^{\lambda} \wedge \omega_{\lambda}^{\nu}$$

can be related with $\boldsymbol{F} = d\boldsymbol{A} + \boldsymbol{A}^{\mathbf{2}}$ (which will be proved later)
So we knew :F can be the curvature of fiber bundle

Differential form of gauge field and gluon

Proof of $\boldsymbol{F} = d\boldsymbol{A} + \boldsymbol{A^2}$

$$\boldsymbol{A} = A_\mu^a\, dx^\mu e^a, gauge\ transform\!: U, UU^+ = I,$$

$$\boldsymbol{A}' = U\boldsymbol{A}U^+ + UdU^+\ \ ,\left(dU^+ = \partial_\mu U^+ dx^\mu\ \ ,1 - form\right),$$

$$d\boldsymbol{A}' = Ud\boldsymbol{A}U^+ + dU\boldsymbol{A}U^+ - U\boldsymbol{A}dU^+ + dUdU^+\ \ ,$$

$$\boldsymbol{A'^2} = U\boldsymbol{A}^2U^+ + U\boldsymbol{A}dU^+ + UdU^+U\boldsymbol{A}U^+ + UdU^+UdU\ ,$$

Since the Lagrangian of the field needs to be integrated over the entire space, total differentials can be added to the equation without affecting the result.

$$\boldsymbol{A'^2} = U\boldsymbol{A}^2U^+ + U\boldsymbol{A}dU^+ - dU\boldsymbol{A}U^+ - dUdU^+\ \ ,$$

$$\boldsymbol{F} = \boldsymbol{A^2} + \boldsymbol{dA}\ ,where\ \boldsymbol{F}' = \boldsymbol{A'^2} + \mathrm{d}\boldsymbol{A}' = U\left(d\boldsymbol{A} + \boldsymbol{A^2}\right)U^+, covarience$$

When described in component form, the result is:

$$F_{\mu\nu} = \left[A_\mu, A_\nu\right] + \partial_\mu A_\nu - \partial_\nu A_\mu$$

Differential form of gauge field and gluon

When we write the Lagrangian of gauge fields, we obtain two terms: the self - energy term and the term interacting with quarks.
According to the QCD Feynman diagram rules, the gauge field's carrier particles are named as gluon:
(today we don't have to introduce the ghost field)

$$L_A = -\frac{1}{2} F_{\mu\nu}^{a} F_{b}^{\mu\nu} tr\left(T^a T^b\right) \ ,$$
$$L_I = -g\overline{\psi}\gamma^{\mu} A_{\mu}\psi,$$