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Sakai-Sugimoto Model in an Off-Shell: Chiral Lagrangian to All Orders

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Physical Motivation

Chiral Symmetry in QCD

In the massless quark limit QCD possesses approximate global symmetry,

$$SU(N_f)_L \times SU(N_f)_R.$$

This symmetry is spontaneously broken (CSB):

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V,$$

where quark condensate is the order parameter,

$$\langle \bar{q}q \rangle \neq 0.$$

Consequences:

- $N_f^2 - 1$ Goldstone bosons.
- For $N_f = 2$: $\pi^{\pm}, \pi^0$.

Small quark masses $\Rightarrow$ pseudo-Goldstone bosons.

Chiral Lagrangian

- Effective local low-energy theory of QCD.
  - In contrast, today we will obtain non-local effective action.
- Approximate chiral symmetry $SU(2)_L \times SU(2)_R$.
- Expansion in powers of π-meson momentum (derivatives).

$$\Sigma = \exp\left(\frac{i}{f_\pi}\pi^a\sigma^a\right) \in SU(2)$$

Up to 4$^{\text{th}}$ order in derivatives:

$$\mathcal{L} = -\frac{f_\pi^2}{4}\mathrm{Tr}\left(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right) + L_1\left(\mathrm{Tr}\left(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right)\right)^2 + \\ + L_2\left(\mathrm{Tr}\left(\partial_\mu\Sigma^\dagger\partial_\nu\Sigma\right)\right)^2 + \dots$$

f_π — pion decay constant, L_i — low-energy constants (LECs).

η'-meson

Pseudoscalar meson η':

- Acquires mass through $U(1)_A$ anomaly.
- Mass generation is subleading in the large N_c limit.

$$\mathcal{L}_{\eta'} = -\frac{1}{2}\partial_\mu \eta' \partial^\mu \eta'$$

η', π are massless today! $\Rightarrow$ $U(2)$ multiplet.

And Also Vector Mesons

A couple of ways to introduce (axial-)vector mesons:

- Resonance Chiral Theory (general chiral invariant Lagrangian describing couplings of meson resonances to pions).
- Hidden Local Symmetry approach (vector mesons are gauge fields of HLS).

Infinite tower of gapped vector modes with alternating parity:

$$\mathcal{L}_V = -\frac{1}{2}\sum_{n=1}^{\infty} \mathrm{Tr}\left(\partial_\mu V_{\nu,n}\partial^\mu V_n^\nu\right) - \frac{1}{2}\sum_{n=1}^{\infty} m_n^2 \mathrm{Tr}\left(V_{\mu,n}V_n^\mu\right)$$

In principle, the Lagrangian also involves interactions between the vector and pseudoscalar sectors.

Holography. . .

All of this can be studied using the holography.

Holographic principle (AdS/CFT-correspondence, gauge/gravity duality):

- Appeared in string theory.
- Relates a gravity theory (“bulk theory”) to a gauge theory on the boundary (“boundary theory”).
- Top-down models: start with string theory construction.

. . . and Chiral Lagrangian

(Witten-)Sakai-Sugimoto (SS) model — top-down holographic model that reproduces Chiral Lagrangian.

Our work: Study of the SS model in the off-shell formalism.

Our main results: Effective action for the boundary theory.
Its features:

- Non-linear contributions from higher order EOMs.
- All orders in derivatives.
- Non-local, but in the local approximation reproduces Chiral Lagrangian.
- N_f is arbitrary.

SS Model Recap

Spacetime

10D geometry is generated by N_c D4-branes, $N_c \gg 1$ (Witten background):

$$ds_{10}^2 = \left(\frac{U}{R}\right)^{3/2} \left(\eta_{\mu\nu} dx^\mu dx^\nu + f(U) d\tau^2\right) + \\ + \left(\frac{R}{U}\right)^{3/2} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right)$$

$$f(U) = 1 - \frac{U_{\mathsf{KK}}^3}{U^3}, \qquad \eta_{\mu\nu} = \mathrm{diag}\left\{+1, -1, -1, -1\right\}$$

$R^3 = \pi g_s N_c l_s^3$ — background curvature radius.

Coordinate U corresponds to the energy scale and is bounded from below, $U \geq U_{\mathsf{KK}}$.

Scale

Coordinate τ is compactified:

$$\tau \sim \tau + \delta\tau, \quad \delta\tau = \frac{2\pi}{M_{\mathrm{KK}}}, \quad M_{\mathrm{KK}} = \frac{3}{2}\frac{U_{\mathrm{KK}}^{1/2}}{R^{3/2}}$$

M_{KK} is an important scale in the model:

- Inversed radius of the compactified coordinate.
- For energies $< M_{\mathrm{KK}}$, the theory on $D4$ branes is effectively 4D.

Matter

"Matter": N_f pairs of D8 (probe) branes.

Non-abelian generalization of the Dirac-Born-Infeld (DBI) action:

$$S_{\rm DBI} = -T_8 \int_{\rm D8} d^9x\, e^{-\phi}\, {\rm STr} \sqrt{-\det\left(G_{MN} + 2\pi\alpha' F_{MN}\right)}$$

$$F_{MN} = \partial_M A_N - \partial_N A_M + i\left[A_M, A_N\right], \qquad A_M \in \mathfrak{u}(N_f)$$

T_8 is the string tension, $e^{-\phi}$ is the dilaton, $\alpha' = l_s^2$.
STr is the symmetrized trace.

DBI action is expanded in the low-energy limit.

Bulk Action

Spacetime geometry contains S_4 sphere.
Assume that A_M does not depend on S_4 coordinates and its components along the sphere are zero $\Rightarrow$ 5D bulk theory.

$$S_{\text{bulk}} = S_{\text{DBI}} + S_{\text{CS}}$$

S_{CS} is the Chern-Simons (CS) action: interaction term between A_M and the Ramond-Ramond field C_3 (F_4 is its field strength):

$$S_{\text{CS}} = \frac{1}{48\pi^3} \int_{\text{D8}} F_4 \wedge \omega_5 = \frac{N_c}{24\pi^2} \int_{\text{D8}\setminus S_4} \omega_5$$

CS 5-form:

$$\omega_5(A) = \text{STr}\left(AF^2 - \frac{i}{2} A^3 F - \frac{1}{10} A^5 \right)$$

(Leads to the Wess-Zumino-Witten term.)

Expansion of the DBI Action

To work with the DBI-action we consider the low-energy limit, $\alpha' = l_s^2 \to 0 \Rightarrow$ expansion in powers of F^2:

$$S_{\mathrm{DBI}} \simeq \int_{\mathrm{D8}\backslash S_4} d^4x\, dz\, u(z)^2 \left[N_f + \mathcal{O}\left(F^2\right) + \mathcal{O}\left(F^4\right) + \mathcal{O}\left(\alpha'^6\right)\right]$$

$$u(z) = \left(1 + z^2\right)^{1/3}$$

(S_{CS} is not expanded.)

- Most commonly, only leading order F^2 is considered.
- Recently, it was shown that higher, non-linear terms F^4 are important.

C. Hoyos, N. Jokela and D. Logares, *Revisiting the chiral effective action in holographic models*, Phys. Rev. D **107** (2023) no.2, 026017, [2203.05916]

Our Procedure

Gauge Fixing

5D $U(N_f)$ gauge symmetry of the model at $z \to \pm\infty$ boundaries is realized as $U(N_f)_L \times U(N_f)_R$ chiral symmetry.

5D $U(N_f)$ gauge freedom is partially fixed by imposing the "axial" gauge:

$$A_z = 0$$

Σ and η': the Wilson line of the z-component of the bulk field:

$$\Sigma = P \exp\left(-i \int_{-\infty}^{+\infty} dz\, \mathcal{A}_z(x^\mu, z)\right)$$

($\mathcal{A}_M$ is the non-abelian part of A_M.)
Σ and η' remain as the dynamical fields!

Boundary Conditions

Σ and η' emerge as boundary conditions for the bulk field:

$$A_\mu(x,z) \underset{z\to+\infty}{\longrightarrow} B_\mu(x), \qquad A_\mu(x,z) \underset{z\to-\infty}{\longrightarrow} 0$$

$$B_\mu(x) = -\sqrt{\frac{2}{N_f}}\frac{1}{f_\pi}\partial_\mu\eta' \mathbb{1}_{N_f} - i\Sigma^{-1}\partial_\mu\Sigma$$

- B_μ will be the dynamical field of the boundary effective action.
- Residual gauge transformation was fixed to get zero condition at $z \to -\infty$.

In the bulk, fixing the axial gauge amounts to changing the integration variable from A_z to Σ and η' in the path integral,

$$Z_{\text{bulk}} = \int DA_\mu\, DA_z\, e^{iS} \sim \int DA_\mu\, D\Sigma\, D\eta'\, e^{iS[A_z=0]}$$

Bulk EOMs and Perturbative Expansion

In total, five equations of motion:

- Four components A_μ of the bulk field $\Rightarrow$ found by solving four dynamical EOMs.
- Fifth equation is a constraint $\Rightarrow$ on-shell dynamics of Σ and η'.
- Constraint is not solved $\Rightarrow$ **off-shell formalism!**

EOMs are non-linear $\Rightarrow$ solved perturbatively in a weak field approximation (different from $\alpha' \to 0$ expansion of the DBI action!):

$$A_M = \varepsilon A_M^{(1)} + \varepsilon^2 A_M^{(2)} + \varepsilon^3 A_M^{(3)} + \mathcal{O}(\varepsilon^4)$$

For quartic terms in the effective action only expansion up to ε^3 is necessary.

More on Solving Equations

- $A_\mu(x,z) \underset{z\to+\infty}{\to} B_\mu(x), \qquad A_\mu(x,z) \underset{z\to-\infty}{\to} 0$
  Boundary conditions are satisfied by the solution for $A_\mu^{(1)}$.

- A_μ and corresponding EOMs are split into longitudinal and transverse:

$$A_\mu = \frac{q_\mu}{q^2} A_l + A_{\mu,t}, \qquad q^\mu A_{\mu,t} = 0$$

- Higher order EOMs are solved using Green functions.

Transverse Equation

EOMs for $A_{\mu,t}^{(1)}$ = dynamical equations for vector mesons $V_{\mu,n}$ (after proper identification).
Solving them integrates vector mesons out.

Instead, we leave these equations unsolved ⇒ "integrating in".

Transverse equation defines a spectral problem:

$$\partial_z \left(u(z)^3 \partial_z \psi_n(z) \right) = \frac{\lambda_n}{u(z)} \psi_n(z), \qquad \psi_n(z) \underset{z \to \pm\infty}{\to} 0$$

$$u(z) = \left(1 + z^2\right)^{1/3}$$

Eigenvalues λ_n determine the masses of vector modes.

Off-Shell Action

Solutions of the dynamical EOMs are substituted into the bulk action ⇒ "off-shell" action.

$$Z_{\text{bulk}} = \int DV_{\mu,n}\, D\Sigma\, D\eta'\, e^{iS_{\text{off-shell}}[\Sigma,\eta',V_{\mu,n}]}$$

Integrating over the bulk coordinate:

$$S_{\text{off-shell}} = \int d^4x\, dz\, \mathcal{L}_{\text{off-shell}}\,[z] = \int d^4x\, \mathcal{L}_{\text{eff}} = S_{\text{eff}}$$

produces the off-shell effective action.

The partition function of the boundary theory reads:

$$Z_{\text{boundary}} = \int DV_{\mu,n}\, D\Sigma\, D\eta'\, e^{iS_{\text{eff}}[\Sigma,\eta',V_{\mu,n}]}$$

Results

Effective Action

The full expression for the effective action (the main result!) is very involved: 21 different groups of terms, 8 pages!

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The full expression for the effective action (the main result!) is very involved: 21 different groups of terms, 8 pages!

- Degrees of freedom are B-field (containing π and η') and multiplets vector mesons $V_{\mu,n}$.
- Non-local expression.
- Interactions between B and $V_{\mu,n}$ up to the fourth combined order.

Quadratic Terms

These terms are local.

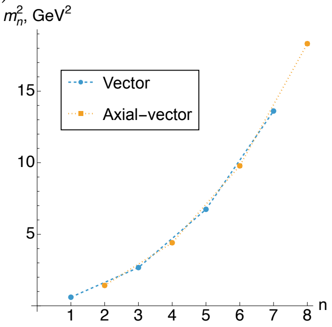
$$S_{\text{eff,quad}} = -\frac{\kappa}{\pi}\int d^4q\,\mathrm{STr}\left[B_\nu(q)\,B^\nu(-q)\right] - \\ -\sum_n \int d^4q\,\left(q^2-\lambda_n\right)\mathrm{STr}\left[V_{\nu,n}(-q)\,V_n^\nu(q)\right]$$

$$\kappa = \frac{\lambda_{\text{YM}} N_c}{6^3\pi^3}$$

Free Action

$$S_{\text{eff,free}} = -\frac{f_\pi^2}{4}\int d^4x\,\text{Tr}\left(\partial_\mu \Sigma^{-1}\partial^\mu \Sigma\right)$$
$$-\frac{1}{2}\int d^4x\,\partial_\mu \eta' \partial^\mu \eta'$$
$$-\int d^4x \sum_n \left(\text{STr}\left[\partial_\mu V_{\nu,n}\partial^\mu V_n^\nu\right] + m_n^2\,\text{STr}\left[V_{\mu,n}V_n^\mu\right]\right),$$

$$f_\pi^2 = \frac{N_c \lambda_{\text{YM}}}{54\pi^4} M_{\text{KK}}^2,$$
$$m_n^2 = -\lambda_n M_{\text{KK}}^2$$



Non-Locality

Important property of the effective action: non-locality, contained in the momentum-dependent interaction vertices.

<u>Example:</u> (CS-induced, $\pi\pi V$ vertex)

$$S_{\text{eff}} \sim i\varepsilon^{z\mu\nu\rho\sigma} \int d^4q\, d^4p \left(\frac{\delta^{\varkappa}_{\nu} - D^{\varkappa}_{\nu}(q)}{q^2 - \lambda_k} - \frac{D^{\varkappa}_{\nu}(q)}{\lambda_k} \right) \times \\ \times\, p_\mu\, \mathrm{STr}\left[B_{\varkappa}(-q)\, B_\rho(q-p)\, V_{\sigma,n}(p)\right]$$

$D^{\varkappa}_{\nu}(q)$ is the longitudinal projector:

$$D^{\nu}_{\mu}(q) = \frac{\eta^{\sigma\nu} q_\mu q_\sigma}{q^2}$$

λ_k are the eigenvalues of the spectral problem of the transverse equation.

Local Expansion

- To extract local contributions: local expansion of the vertex in powers of q^2 in $q \to 0$ limit.
- In coordinate space this corresponds to gradient expansion.
- Can be truncated at any order.

Example:

$$\frac{\delta_\nu^\varkappa - D_\nu^\varkappa(q)}{q^2 - \lambda_k} - \frac{D_\nu^\varkappa(q)}{\lambda_k} \underset{q\to 0}{\approx} -\frac{\delta_\nu^\varkappa}{\lambda_k}$$

$$S_{\text{eff}} \sim -\varepsilon^{z\mu\nu\rho\sigma} \int d^4x \, \mathrm{STr}\left[B_\nu B_\rho \partial_\mu V_{\sigma,n}\right]$$

Quartic Action for Pions, Aribtrary N_f

- In local approximation.
- Only single-trace operators.

$$S^{SU(N_f)}_{\text{eff,quart}} = \int d^4x \, L^{SU(N_f)}_3 \, \text{Tr}\left(\partial_\mu \Sigma^{-1} \partial^\mu \Sigma \partial_\nu \Sigma^{-1} \partial^\nu \Sigma\right) + \\ + \int d^4x \, \tilde{L}^{SU(N_f)}_3 \, \text{Tr}\left(\partial_\mu \Sigma^{-1} \partial_\nu \Sigma \partial^\mu \Sigma^{-1} \partial^\nu \Sigma\right)$$

$$L^{SU(N_f)}_3 = 2L^{(nl)} + L^{(l)}, \qquad \tilde{L}^{SU(N_f)}_3 = L^{(nl)} - L^{(l)}$$

$$L^{(l)} = -\frac{N_c \lambda_{\text{YM}}}{216\pi^3} C, \qquad L^{(nl)} = -\frac{9N_c}{128\pi\lambda_{\text{YM}}} a^{(3)}_{11}$$

$$C \approx 0.157, \qquad a^{(3)}_{11} \approx -0.013$$

$$L^{(l)} = \mathcal{O}\left(N_c \lambda_{\text{YM}}\right), \qquad L^{(nl)} = \mathcal{O}\left(\frac{N_c}{\lambda_{\text{YM}}}\right)$$

$L^{(l)}$, $L^{(nl)}$ — same order in N_c!

Quartic Action for Pions, $N_f = 2$

Reduces to familiar expression with double-trace operators:

$$S^{SU(2)}_{\text{eff,quart}} = \int d^4x\, L_1^{SU(2)} \left(\text{Tr}\left(\partial_\mu \Sigma^{-1} \partial^\mu \Sigma\right)\right)^2 + \\ + \int d^4x\, L_2^{SU(2)} \left(\text{Tr}\left(\partial_\mu \Sigma^{-1} \partial_\nu \Sigma\right)\right)^2$$

$$L_1^{SU(2)} = \frac{L_3^{SU(N_f)} - \tilde{L}_3^{SU(N_f)}}{2} = \frac{L^{(nl)}}{2} + L^{(l)}$$

$$L_2^{SU(2)} = \tilde{L}_3^{SU(N_f)} = L^{(nl)} - L^{(l)}$$

Outlook

- Elastic scattering amplitude for $N_f = 3 \Rightarrow$ to analyze decays of kaons into pions.
- Interactions between vector and pseudoscalar sectors.
  - CS-induced encodes interactions involving vector mesons of the same parities (such as $\omega \to \rho\pi$).
  - DBI-induced: opposite parities ($\rho(1450) \to (a_1(1260) + \pi)_{P\text{-wave}}$).
- Introduce photons $\Rightarrow$ radiative decays, muon anomalous magnetic moment (hadronic contributions: HVP and HLbL).

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Thank you!

Elastic Scattering

- $N_f = 2 \Rightarrow$ pions.
- $N_f = 3 \Rightarrow$ pions and kaons.
- $N_f > 3 \Rightarrow$ pions, kaons, ...

$$\pi^a(p_a) + \pi^b(p_b) \rightarrow \pi^c(p_c) + \pi^d(p_d)$$

$$S = I + iT, \quad T = (2\pi)^4 \delta\left(p_a + p_b - p_c - p_d\right) M$$

Its Amplitude

Low-energy limit of the tree-level scattering amplitude M:

$$M = \delta_{ab}\delta_{cd}A(s,t,u) + \delta_{ac}\delta_{bd}A(t,s,u) + \delta_{ad}\delta_{bc}A(u,t,s)$$
$$+\frac{N_f}{2}\left(A(s,t,u)d_{abe}d_{cde} + A(t,s,u)d_{ace}d_{bde} + A(u,t,s)d_{ade}d_{bce}\right)$$

$$A(s,t,u) = \frac{8\left(L_3^{SU(N_f)} - \tilde{L}_3^{SU(N_f)}\right)}{N_f f_\pi^4}s^2 + \frac{8\tilde{L}_3^{SU(N_f)}}{N_f f_\pi^4}\left(t^2+u^2\right)$$
$$s = -\left(p_a+p_b\right)^2, \quad t = -\left(p_a-p_c\right)^2, \quad u = -\left(p_a-p_d\right)^2$$

Vector meson exchange diagrams contribute only at $\mathcal{O}\left(p^6\right)$!