Topological susceptibility and Ward–Takahashi identities in QCD

Mamiya Kawaguchi

Anhui University of Science and Technology

2026/5/17

Outline

1. Introduction

- Topological susceptibility
- Ward–Takahashi identities
- Indicator of symmetry breaking

2. Topological susceptibility under extreme conditions

- Finite temperature
- Finite density
- Other situation

3. Summary

1. Introduction

- Topological susceptibility
- Ward–Takahashi identities
- Indicator of symmetry breaking

Topological susceptibility

$$\chi_{\rm top} = -i \int d^4x \langle 0 | T Q(x) Q(0) | 0 \rangle$$

Topological charge density

$$Q = (g^2/64\pi^2)\epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

- U(1) axial anomaly
- Hadron mass

  (e.g., Witten-Veneziano relation: $m_{\eta'} \leftrightarrow \chi_{\rm top}^{\rm (YM)}$)
- Chiral symmetry breaking and vacuum structure
  → QCD phase structure at finite T and μ_B
- Axion physics

Actively studied using lattice QCD simulations at T

- **Kanamori et al. / JLQCD (2026)**
  - Physical-point $N_f = 2+1$domain-wall study
  - Discretization and fermion-action effects in $\chi_{\rm top}(T)$
- **Kotov, Lombardo, Trunin (2025)**
  - Physical $N_f = 2+1+1$ Wilson twisted-mass fermions
  - θ-dependence, and DIGA behavior in high- T
- **Brandt et al. (2024)**
  - Physical $N_f = 2+1$stout-improved staggered quarks
  - Strong magnetic field effects on $\chi_{\rm top}(T)$
- **Aoki et al. / JLQCD (2024)**
  - $N_f = 2+1$Möbius domain-wall fermions
  - Dirac spectrum, $U(1)_A$ susceptibility, and $\chi_{\rm top}$
- **Athenodorou et al. (2022)**
  - High-temperature $\chi_{\rm top}$using staggered spectral projectors.

⋮ (Many other studies in lattice QCD and other approaches)

Topological susceptibility

$$\chi_{\rm top} = -i \int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

Topological charge density

$$Q = (g^2/64\pi^2)\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}$$

- ✓ $\chi_{\rm top} = 0$ in chiral limit
- ✓ Prove of $U(1)_A$ symmetry restoration
- ✓ Closely related to SU(2) chiral symmetry

Capture features of $\chi_{\rm top}$ from simple low-energy perspectives.

In this talk…
we investigate these points using effective model approaches.

Topological susceptibility

$$\chi_{\text{top}} = -i \int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

Topological charge density

$$Q = (g^2/64\pi^2)\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}$$

- ✓ $\chi_{\text{top}} = 0$ in chiral limit
- ✓ Prove of $U(1)_A$ symmetry restoration
- ✓ Closely related to SU(2) chiral symmetry

Important for understanding χ_{top}
in terms of hadronic degrees of freedom.

1. Introduction

- Topological susceptibility
- Ward–Takahashi identities
- Indicator of symmetry breaking

Connection with U(1)$_A$ anomaly

$$\chi_{\rm top} = -i \int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

Related to U(1) axial anomaly

$$\partial_\mu j^\mu_A = 2m_l \bar{\psi} i\gamma_5 \psi + 2N_f Q$$

$$\mathcal{L}_{\rm QCD} = \bar{\psi}(i\gamma^\mu D_\mu - m_l)\psi - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a} + \theta Q$$

U(1)$_A$ transformation

$$\mathcal{L}_{\rm QCD} = \bar{\psi} i\gamma^\mu D_\mu \psi - m_l \bar{\psi} \exp\left(i\theta/2\, \gamma_5\right) \psi - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a}$$

$$\chi_{\rm top} = -\int d^4x \frac{\delta^2 \Gamma_{\rm QCD}}{\delta\theta(x)\delta\theta(0)}\bigg|_{\theta=0}$$

- Consider the two-flavor case.
- Introduce θ-parameter.
- Quarks have m_l.

Ward–Takahashi identities

$$\chi_{\rm top} = -i \int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

Related to U(1) axial anomaly

$$\partial_\mu j_A^\mu = 2m_l \bar{\psi} i\gamma_5 \psi + 2N_f Q$$

Using the following relation

$$\langle \bar{q}q \rangle = -im_l \chi_\pi$$

$$\chi_{\rm top} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

Meson susceptibilities

$$\begin{cases} \chi_\eta = \int d^4x \langle 0|T(i\bar{\psi}\gamma_5\psi)(x)(i\bar{\psi}\gamma_5\psi)(0)|0\rangle \\ \chi_\pi \delta^{ab} = \int d^4x \langle 0|T(i\bar{\psi}\gamma_5\tau_f^a\psi)(x)(i\bar{\psi}\gamma_5\tau_f^b\psi)(0)|0\rangle \end{cases}$$

$$\mathcal{L}_{\rm QCD} = \bar{\psi}(i\gamma^\mu D_\mu - m_l)\psi - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a} + \theta Q$$

U(1)$_A$ transformation

$$\mathcal{L}_{\rm QCD} = \bar{\psi} i\gamma^\mu D_\mu \psi - m_l \bar{\psi} \exp(i\theta/2\,\gamma_5)\,\psi - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a}$$

$$\chi_{\rm top} = -\int d^4x \frac{\delta^2 \Gamma_{\rm QCD}}{\delta\theta(x)\delta\theta(0)}\bigg|_{\theta=0}$$

Ward–Takahashi identities and chiral limit

$$\chi_{\rm top} = -i \int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

Related to U(1) axial anomaly

$$\partial_\mu j^\mu_A = 2m_l \bar{\psi} i\gamma_5 \psi + 2N_f Q$$

$$\mathcal{L}_{\rm QCD} = \bar{\psi}(i\gamma^\mu D_\mu - m_l)\psi - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a} + \theta Q$$

U(1)$_A$ transformation

$$\mathcal{L}_{\rm QCD} = \bar{\psi} i\gamma^\mu D_\mu \psi - m_l \bar{\psi} \exp(i\theta/2\,\gamma_5)\,\psi - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a}$$

$$\chi_{\rm top} = -\int d^4x \frac{\delta^2 \Gamma_{\rm QCD}}{\delta\theta(x)\delta\theta(0)}\bigg|_{\theta=0}$$

$$\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$$

$\chi_{\rm top}$ is proportional to m_l^2.

→ In the chiral limit, $\chi_{\rm top} = 0$.

(θ-parameter becomes redundant.)

$\chi_{\rm top}$ in 2 flavor Linear Sigma Model (LSM)

Let's look at typical results from LSM analyses.

LSM is constructed by
- Chiral field: $\Phi = (S^a + iP^a)T^a \quad (a = 0, 1, 2, 3)$
- VEV of Φ : $\bar{\Phi} = \frac{1}{2}\sigma_0^{\rm vev}\, 1_{2\times 2}$

U(1)$_{\rm A}$ anomaly is described by det Φ term:

$$\mathcal{L}_{\rm int}^{\rm (anom)} = B(\det\Phi + \det\Phi^\dagger)$$

U(1) axial anomaly:

$$\partial_\mu j_A^\mu = (m_l \text{ part}) - 2iN_f B\big(\det\Phi - \det\Phi^\dagger\big)$$

Introduce θ-parameter

$$\mathcal{L}_{\rm int}^{\rm (anom)} = B(e^{i\theta}\det\Phi + e^{-i\theta}\det\Phi^\dagger)$$

$$\chi_{\rm top} = \lim_{\theta\to 0}\frac{d^2 V_{\rm eff}}{d\theta^2} = \frac{B}{3\sqrt{6}}(\sigma_0^{\rm vev})^3$$

In the chiral limit, $\chi_{\rm top} \neq 0$…

$\neq$

In the chiral limit, $\chi_{\rm top} = 0.$

???

χ_{top} in 2 flavor Linear Sigma Model (LSM)

Let's look at typical results from LSM analyses.

LSM is constructed by
- Chiral field: $\Phi = (S^a + iP^a)T^a \quad (a = 0, 1, 2, 3)$
- VEV of Φ : $\bar{\Phi} = \frac{1}{2}\sigma_0^{\text{vev}}\, 1_{2\times 2}$

U(1)$_A$ anomaly is described by det Φ term:

$$\mathcal{L}_{\text{int}}^{(\text{anom})} = B(\det\Phi + \det\Phi^\dagger)$$

U(1) axial anomaly:

$$\partial_\mu j_A^\mu = (m_l \text{ part}) - 2iN_f B\left(\det\Phi - \det\Phi^\dagger\right)$$

Introduce θ-parameter and perform U(1)$_A$ transformation $(\Phi \to e^{i\alpha}\,\Phi)$

$$\mathcal{L}_{\text{int}}^{(\text{anom})} + \mathcal{L}_{\text{int}}^{(\text{SB})} = B\left(\det\Phi + \det\Phi^\dagger\right) + c m_l \text{tr}\left[e^{-i\theta/2}\Phi + e^{i\theta/2}\Phi^\dagger\right]$$

$$\chi_{\text{top}} = \frac{f_\pi^2 m_\pi^4}{2}\left(\frac{1}{m_\pi^2} - \frac{1}{m_\eta^2}\right) = \chi_{\text{top}} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$$

Flavor-singlet nature and flavor dependence

$$\chi_{\text{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

This holds for any N_f.

*Consequence of flavor-singlet nature of χ_{top}.

$$\chi_{\text{top}} = -i\int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

At low- energy

For 2 flavor

$$\chi_{\text{top}} = \frac{f_\pi^2 m_\pi^4}{2}\left(\frac{1}{m_\pi^2} - \frac{1}{m_\eta^2}\right)$$

For 2+1 flavor, $\eta - \eta'$ mixing makes it somewhat complicated.

Does $\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$ really work?

$$\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$$

This holds for any N_f.

- 3 flavor NJL: $\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta) = 0.025/{\rm fm}^4$
  Chuan-Xin Cui, Jin-Yang Li, Shinya Matsuzaki, M.K., Akio Tomiya, *PRD* 105 (2022) 11, 114031
- 3 flavor ChPT: $\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta) = 0.0224/{\rm fm}^4$
  S. Borsanyi et al., Nature 539, no. 7627, 69 (2016).
- 3 flavor LSM: $\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta) = 0.0263/{\rm fm}^4$
  M. K., S. Matsuzaki and A. Tomiya, PRD 103 (2021) 054034.

$$\chi_{\rm top} = -i \int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

2+1+1 flavor lattice QCD: $\chi_{\rm top} = 0.0245/{\rm fm}^4$
S. Borsanyi et al., Nature 539, no. 7627, 69 (2016).

Does $\chi_{\text{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$ really work?

$$\chi_{\text{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

$$\chi_{\text{top}} = \frac{f_\pi^2 m_\pi^4}{2}\left(\frac{1}{m_\pi^2} - \frac{1}{m_\eta^2}\right)$$

$$\chi_{\text{top}} = -i\int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

This expression is useful in effective models.

Topological properties characterized by Q are reflected in meson properties.

1. Introduction

- Topological susceptibility
- Ward–Takahashi identities
- Indicator of symmetry breaking

χ_{top} and symmetry breaking/restoration

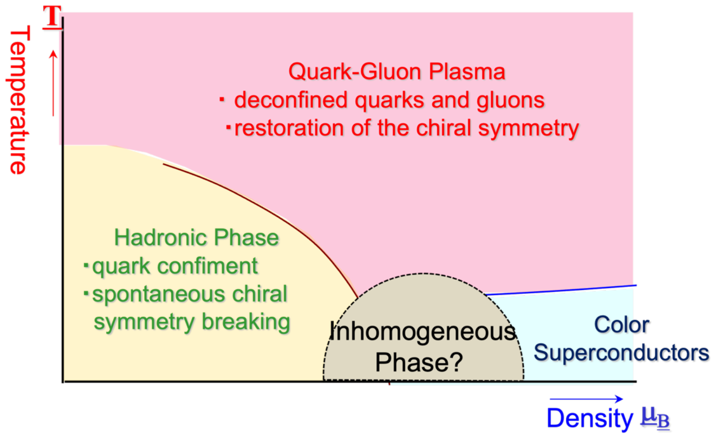
$$\chi_{\text{top}} = -i \int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

Topological charge density

$$Q = (g^2/64\pi^2)\epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

- ✓ $\chi_{\text{top}} = 0$ in chiral limit
- ✓ Prove of $U(1)_A$ symmetry restoration
- ✓ Closely related to SU(2) chiral symmetry

$$\chi_{\text{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$



- How does χ_{top} behave at finite T and μ_B?
- Is χ_{top} related to SU(2) chiral restoration?

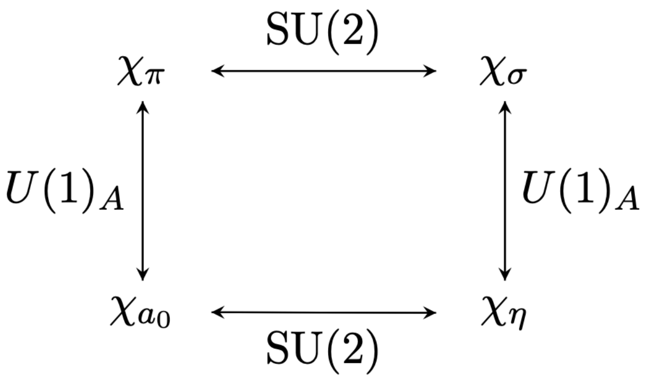
$\chi_{\rm top/meson}$ and symmetry breaking

$$\chi_{\rm top} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

← $\chi_{\rm top}$ is written by meson susceptibilities.

$$\begin{pmatrix} \chi_\eta = \int d^4x \langle 0|T(i\bar{\psi}\gamma_5\psi)(x)(i\bar{\psi}\gamma_5\psi)(0)|0\rangle \\ \chi_\pi\delta^{ab} = \int d^4x \langle 0|T(i\bar{\psi}\gamma_5\tau_f^a\psi)(x)(i\bar{\psi}\gamma_5\tau_f^b\psi)(0)|0\rangle \end{pmatrix}$$

Meson susceptibilities are transformed into each other under $SU(2)_L \times SU(2)_R$ and $U(1)_A$ transformation: $\chi_\pi \leftrightarrow \chi_\eta$



At the vacuum,
chiral SU(2) and $U(1)_A$ are broken: $\chi_\pi \neq \chi_\sigma \neq \chi_{a0} \neq \chi_\eta$

$\chi_{\rm meson}$ is sensitive probe of symmetry breaking.

Indicator of symmetry breaking/restoration

$$\chi_{\rm top} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

← $\chi_{\rm top}$ is written by meson susceptibilities.

$$\left(\begin{aligned} \chi_\eta &= \int d^4x \langle 0|T(i\bar{\psi}\gamma_5\psi)(x)(i\bar{\psi}\gamma_5\psi)(0)|0\rangle \\ \chi_\pi\delta^{ab} &= \int d^4x \langle 0|T(i\bar{\psi}\gamma_5\tau_f^a\psi)(x)(i\bar{\psi}\gamma_5\tau_f^b\psi)(0)|0\rangle \end{aligned} \right)$$

If QCD system undergoes…

- $\chi_\pi = \chi_\sigma$ → Chiral SU(2) symmetry is restored.
- $\chi_\sigma = \chi_\eta$ → $U(1)_A$ symmetry is restored.

At finite T and μ_B chiral restoration occurs.
→ Is $U(1)_A$ symmetry also restored?

$\chi_{\rm top}$ is indicator for measuring effective restoration of $U(1)_A$ symmetry.

Symmetry restoration probed by $\chi_{\rm top}$

$$\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$$

$\chi_{\rm top}$ is indicator for measuring
effective restoration of $U(1)_A$ symmetry.

- How does $\chi_{\rm top}$ behave at finite T and μ_B?
- Is $\chi_{\rm top}$ related to SU(2) chiral restoration?

Which symmetry is restored first:
chiral SU(2) symmetry or $U(1)_A$ symmetry?

Let's move onto hot and dense systems.

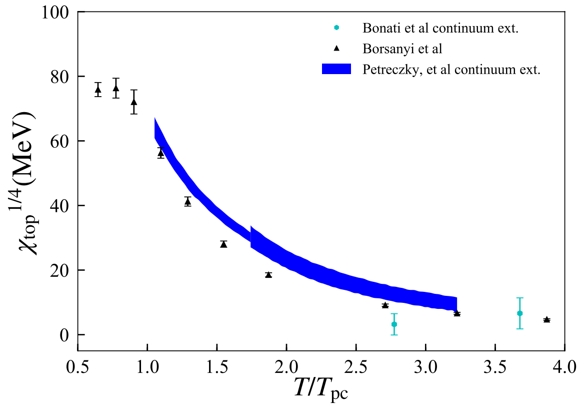
- Hot QCD matter
- Dense QC_2D matter

2. Topological susceptibility under extreme conditions

- Finite temperature
- Finite density
- Other situation

Lattice observations (T and μ_q dep.)

In hot QCD matter

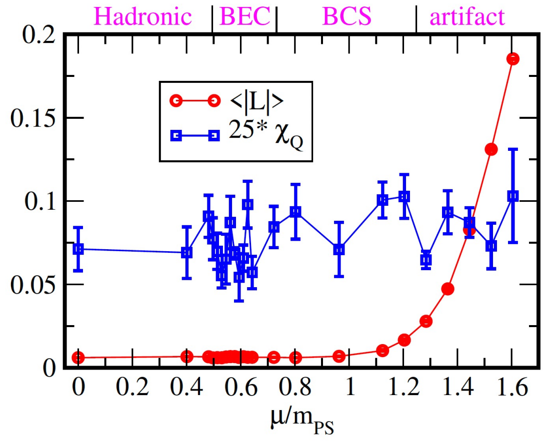


- C. Bonati et al, JHEP 11, 170 (2018), 1807.07954.
- *S. Borsanyi et al., Nature 539, no. 7627, 69 (2016).*
- P. Petreczky et al, Phys. Lett. B 762, 498-505 (2016)

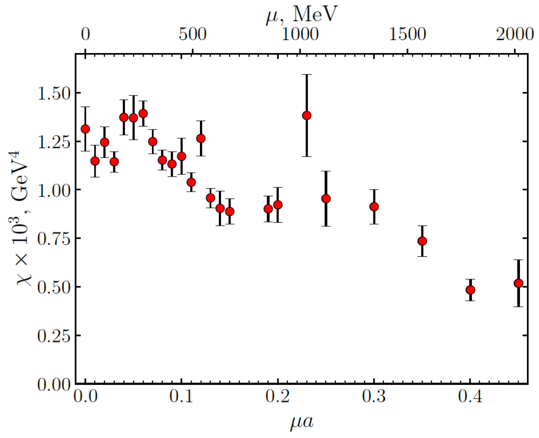
χ_{top} becomes smaller at high T.

Suppression of $U(1)_A$ anomaly

In dense QC_2D matter



Itou et al. JHEP01(2020)181



N. Astrakhantsev et al.
PRD 102 (2020) 7, 074507

μ_B does not affect χ_{top}.

χ_{top} is suppressed.

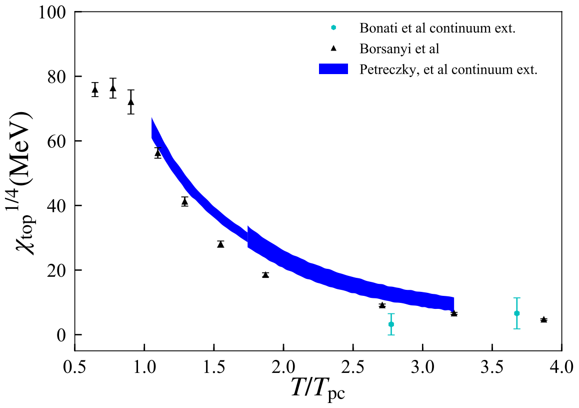
?

?

Fate of χ_{top} at high density is controversial …

Lattice observations (T) and effective model

In hot QCD matter



- C. Bonati et al, JHEP 11, 170 (2018), 1807.07954.
- *S. Borsanyi et al., Nature 539, no. 7627, 69 (2016).*
- P. Petreczky et al, Phys. Lett. B 762, 498-505 (2016)

$\chi_{\rm top}$ becomes smaller at high T.

Suppression of $U(1)_A$ anomaly

NJL model (linear sigma model)

$$\mathcal{L} = \bar{q}(i\gamma_\mu \partial^\mu - \mathbf{m})q + \mathcal{L}_{4f} + \mathcal{L}_{\rm KMT}$$

$$\mathcal{L}_{4f} = \frac{g_s}{2}\sum_{a=0}^{8}[(\bar{q}\lambda^a q)^2 + (\bar{q}i\gamma_5\lambda^a q)^2]$$

$$\mathcal{L}_{\rm KMT} = g_D[\det_{i,j} \bar{q}_i(1+\gamma_5)q_j + \text{h.c.}]$$

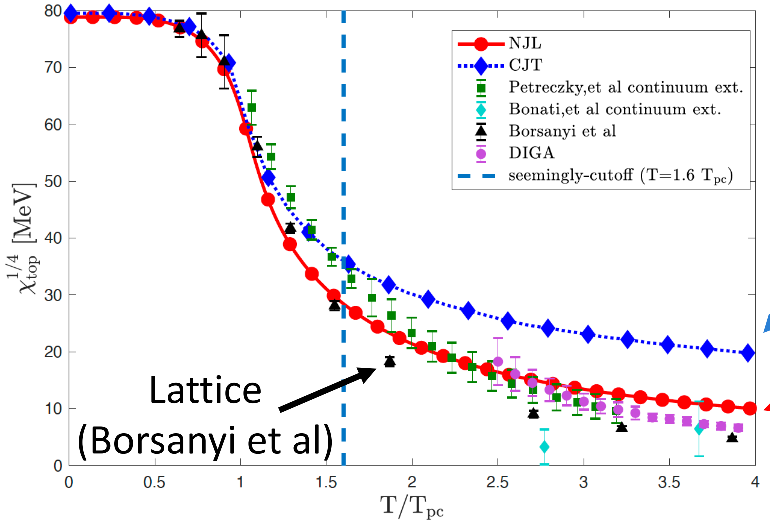
*Perform analysis within mean-field approximation.

*Model parameters are fixed to provide physical meson masses.

- Reproduce the WT identity of underlying QCD.
- Evaluate meson susceptibilities.

$$\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$$

Model results and lattice observations



LSM (CJT approach) w/ $\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$

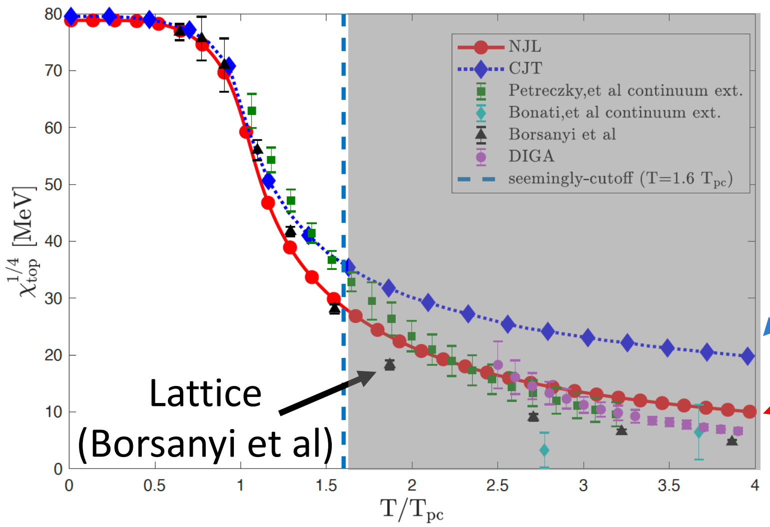
M.K., S. Matsuzaki, A. Tomiya,
PRD 103 (2021) 5, 054034L

NJL mean-field approach w/ $\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$

Chuan-Xin Cui, Jin-Yang Li, Shinya Matsuzaki,
M.K., Akio Tomiya, *PRD* 105 (2022) 11, 114031

Model results and lattice observations

Model results are in good agreement with lattice observations.



LSM (CJT approach) w/ $\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$

M.K., S. Matsuzaki, A. Tomiya,
PRD 103 (2021) 5, 054034L

NJL mean-field approach w/ $\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$

Chuan-Xin Cui, Jin-Yang Li, Shinya Matsuzaki,
M.K., Akio Tomiya, *PRD* 105 (2022) 11, 114031

χ_π contribution dominates T dependence: $\chi_\pi \gg \chi_\eta$.

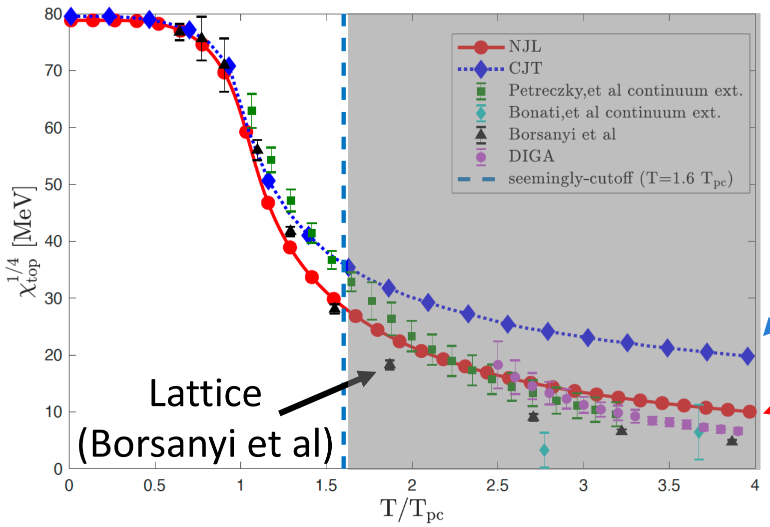
Using the WT identity
$\langle \bar{q} q \rangle = -i m_l \chi_\pi$

$$\chi_{\rm top} \cong \frac{i m_l^2}{4}\chi_\pi = -\frac{m_l}{4}\langle \bar{q} q \rangle$$

Suppression of $\chi_{\rm top}$ is linked with chiral restoration.

χ_{top} and chiral restoration at T

Model results are in good agreement with lattice observations.



LSM (CJT approach) w/ $\chi_{\text{top}} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$

M.K., S. Matsuzaki, A. Tomiya, PRD 103 (2021) 5, 054034L

NJL mean-field approach w/ $\chi_{\text{top}} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$

Chuan-Xin Cui, Jin-Yang Li, Shinya Matsuzaki, M.K., Akio Tomiya, *PRD* 105 (2022) 11, 114031

- ✓ $U(1)_A$ is effectively restored.
- ✓ Related to chiral restoration.

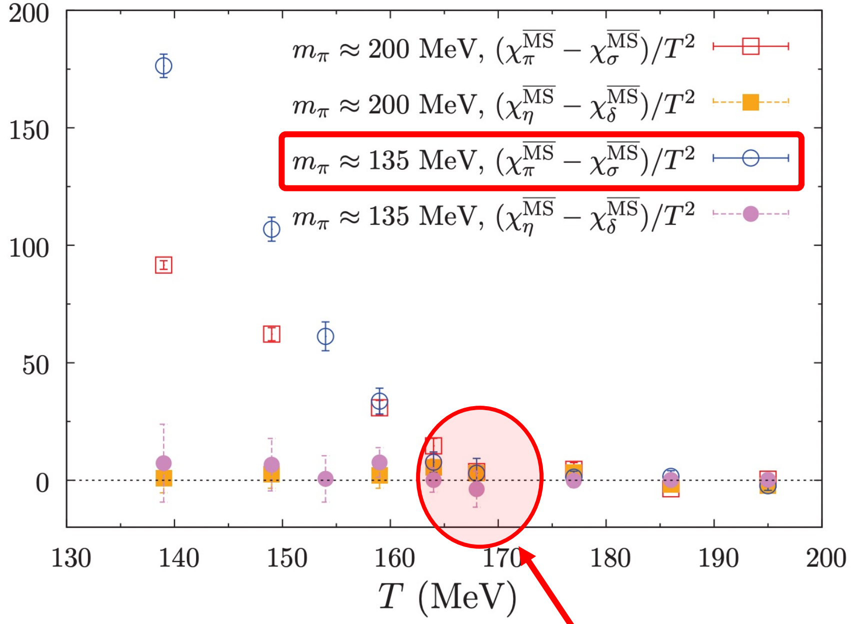
χ_π contribution dominates T dependence: $\chi_\pi \gg \chi_\eta$.

$$\chi_{\text{top}} \cong \frac{i m_l^2}{4}\chi_\pi = -\frac{m_l}{4}\langle \bar{q} q\rangle$$

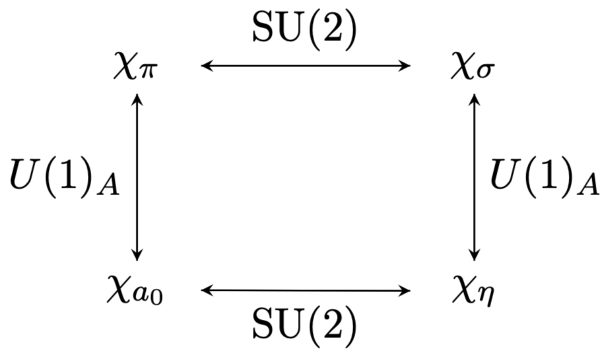
Suppression of χ_{top} is linked with chiral restoration.

Side remark: symmetry restoration in χ_{meson}

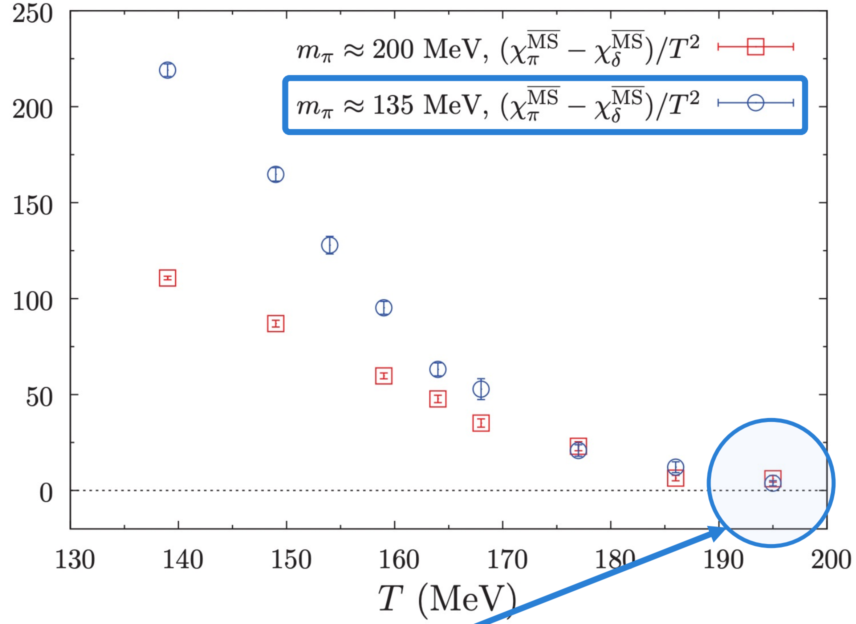
Chiral partner: $\chi_\pi - \chi_\sigma$



T. Bhattacharya et al.,
PRL 113 (2014) 8, 082001



U(1)$_A$ partner: $\chi_\pi - \chi_{a0}$



$\chi_\pi - \chi_\sigma = 0$ at $T \cong 170$ MeV

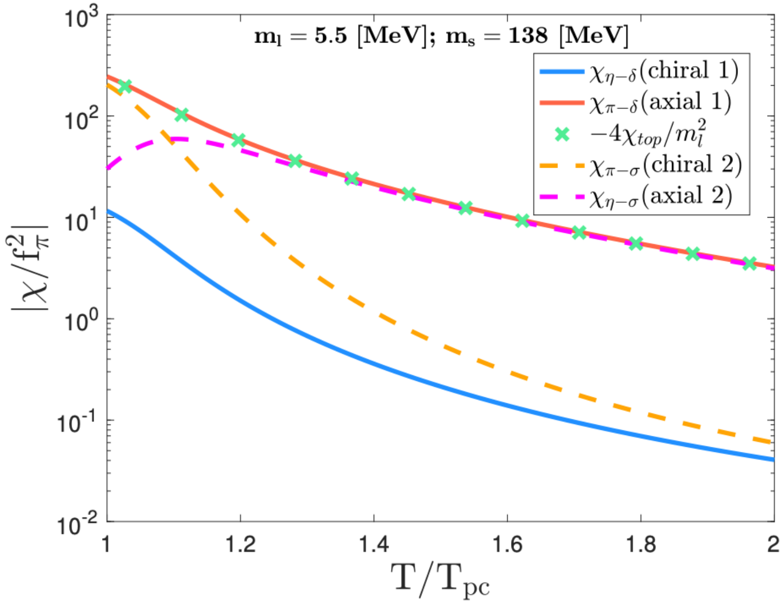
$\chi_\pi - \chi_\delta = 0$ at $T \cong 195$ MeV

Meson susceptibilities provide…

Chiral symmetry is restored faster than U(1)$_A$ symmetry.

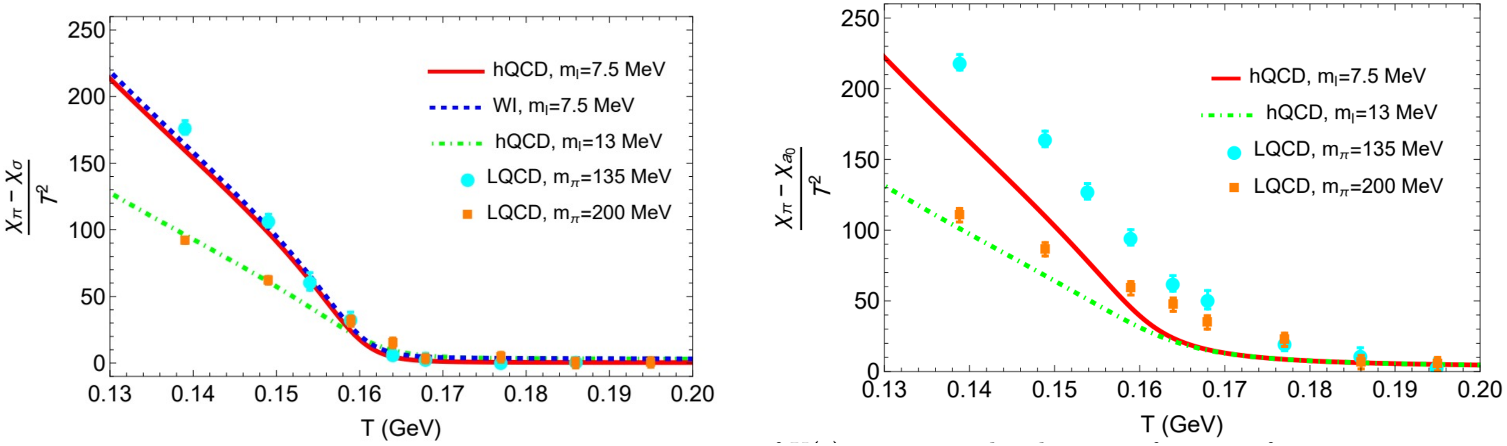
Side remark: symmetry restoration in χ_{meson}

NJL model



C. X. Cui, J. Y. Li, S. Matsuzaki, M. Kawaguchi and A. Tomiya, Particles 7 (2024) no.1, 237-263

Soft-wall holographic QCD model



H. A. Ahmed, D. Li, M. Kawaguchi and M. Huang, arXiv:2603.12911 [hep-ph]

Also in other approaches

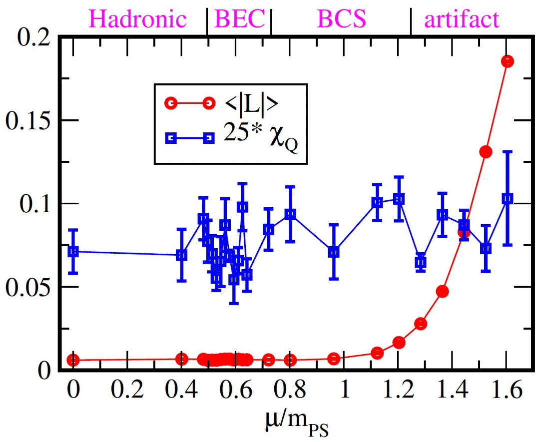
Chiral symmetry is restored faster than $U(1)_A$ symmetry.

2. Topological susceptibility under extreme conditions

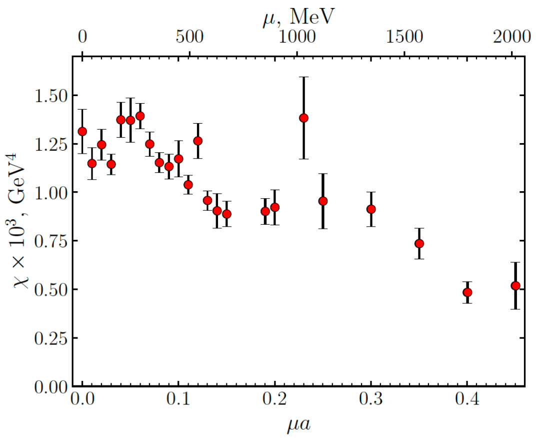
- Finite temperature
- Finite density
- Other situation

Lattice observations (μ_q)

$\chi_{\rm top}$ in dense QC_2D lattice simulations



Itou et al. JHEP01(2020)181



N. Astrakhantsev et al.
PRD 102 (2020) 7, 074507

μ_B does not affect $\chi_{\rm top}$.

$\chi_{\rm top}$ is suppressed.

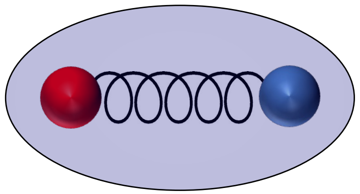
Fate of $\chi_{\rm top}$ at high density is controversial …

Effective model analysis provides useful benchmark.

Unique feature of dense QC_2D

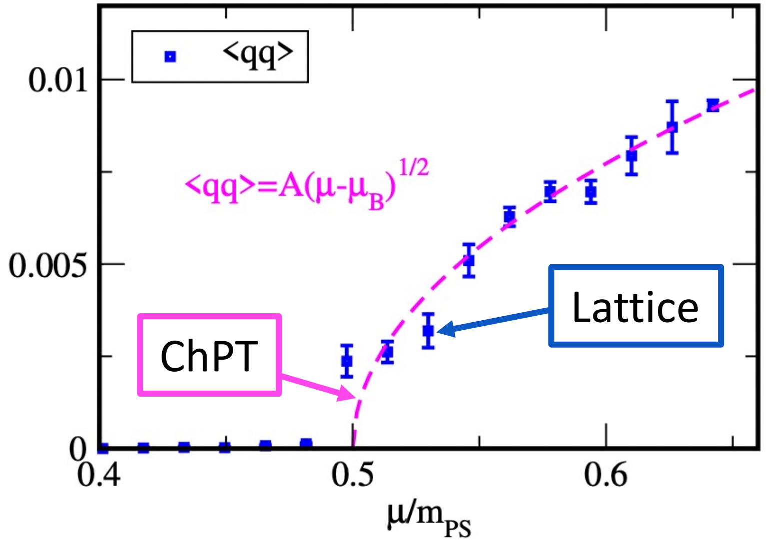
2-color QCD

Baryon is formed from two quarks.



"Bosonic baryon"

Baryon superfluid phase transition



K. Iida, E. Itou and T. G. Lee, JHEP 01, 181 (2020)

This feature can be described by …

$$\Sigma = \frac{1}{2}\begin{pmatrix} 0 & -B' + iB & \frac{\sigma - i\eta + a^0 - i\pi^0}{\sqrt{2}} & a^+ - i\pi^+ \\ B' - iB & 0 & a^- - i\pi^- & \frac{\sigma - i\eta - a^0 + i\pi^0}{\sqrt{2}} \\ -\frac{\sigma - i\eta + a^0 - i\pi^0}{\sqrt{2}} & -a^- + i\pi^- & 0 & -\bar{B}' + i\bar{B} \\ -a^+ + i\pi^+ & -\frac{\sigma - i\eta - a^0 + i\pi^0}{\sqrt{2}} & \bar{B}' - i\bar{B} & 0 \end{pmatrix}$$

Linear sigma model:

$$\mathcal{L}_{\rm LSM} = {\rm tr}[D_\mu \Sigma^\dagger D^\mu \Sigma] - V_0 - V_{\rm sp} - V_{\rm anom}$$

Transition to baryon superfluid phase

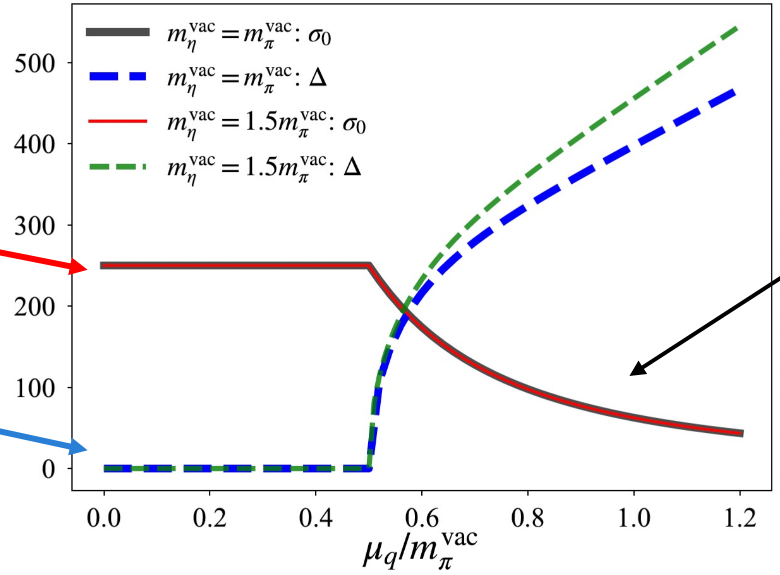
Linear sigma model:

$$\mathcal{L}_{\mathrm{LSM}} = \mathrm{tr}[D_\mu \Sigma^\dagger D^\mu \Sigma] - V_0 - V_{\mathrm{sp}} - V_{\mathrm{anom}}$$

Can describe phase transition from hadron phase to baryon superfluid phase.

Sigma meson:
$\sigma_0 = \langle \sigma \rangle \sim \langle \bar{q} q \rangle$

Positive parity baryon:
$\Delta = \langle B \rangle \sim \langle q q \rangle$



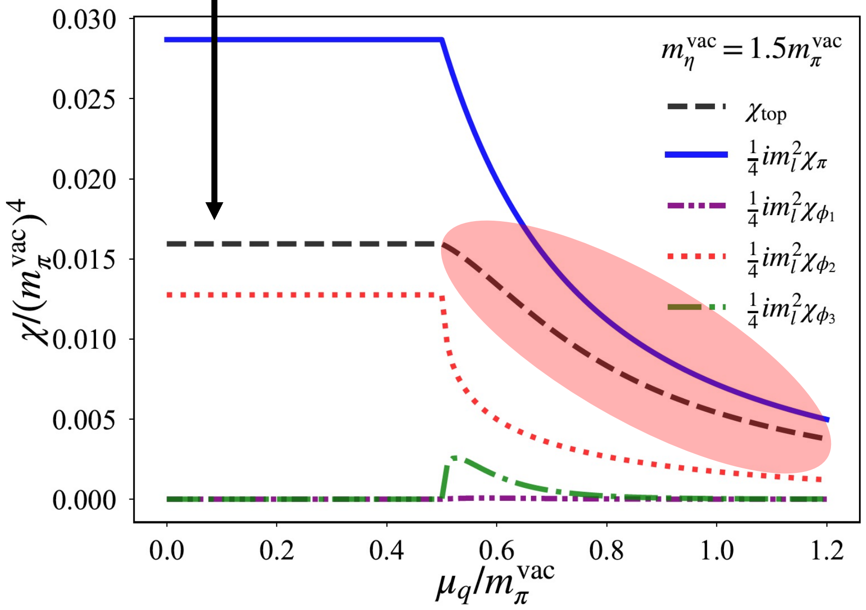
Chiral condensate scales with

$$\sigma_0 \sim \mu_q^{-2}$$

χ_{top} in superfluid phase

$$\chi_{\text{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

used in LSM analysis



M. K., D. Suenaga *JHEP 08 (2023) 189*

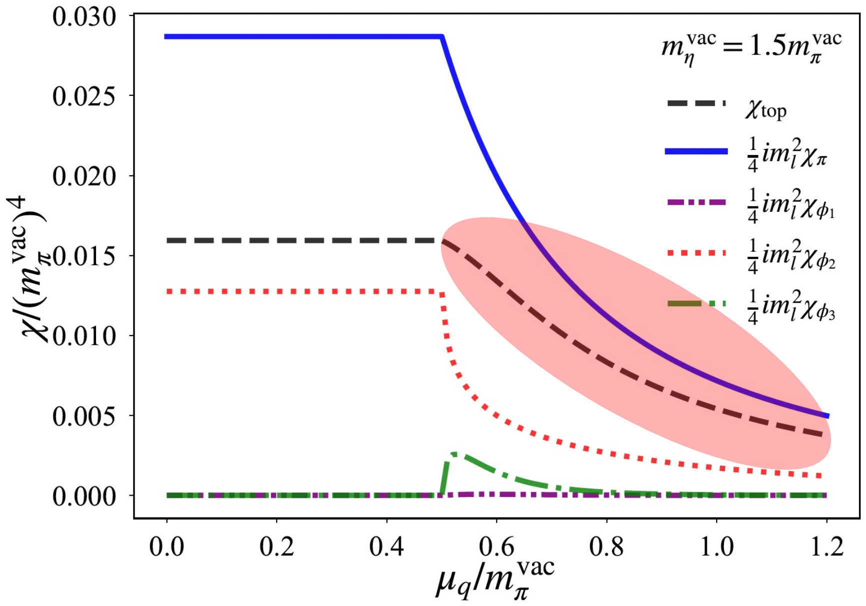
LSM result:

χ_{top} is suppressed in superfluid phase.

$\chi_{\rm top}$ in superfluid phase

$$\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$$

used in LSM analysis

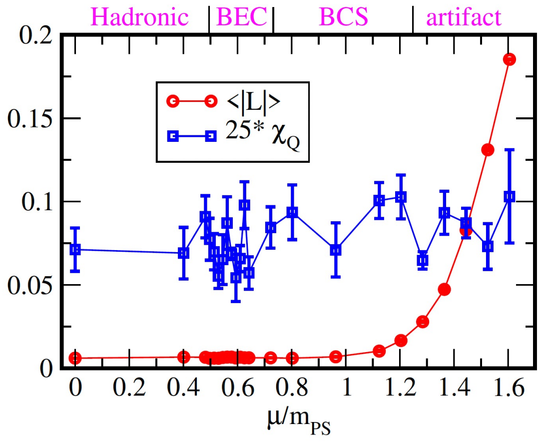


M. K., D. Suenaga *JHEP 08 (2023) 189*

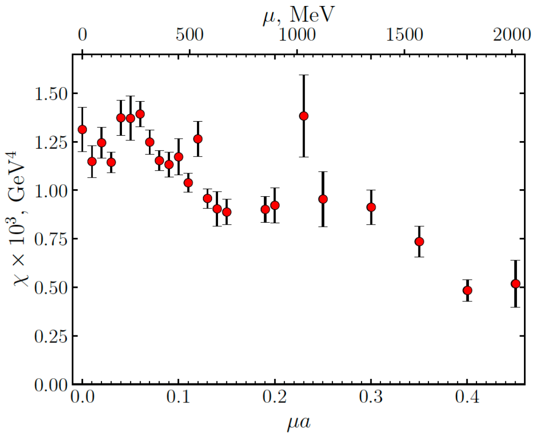
LSM result:

$\chi_{\rm top}$ is suppressed in superfluid phase.

Lattice observations



Itou et al. JHEP01(2020)181



N. Astrakhantsev et al.
PRD 102 (2020) 7, 074507

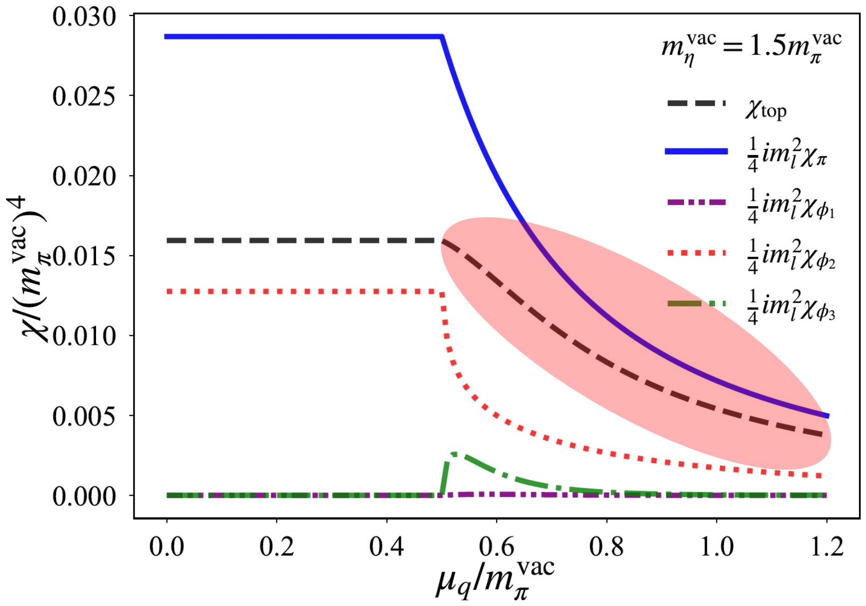
μ_B does not affect $\chi_{\rm top}$.

$\chi_{\rm top}$ is suppressed.

χ_{top} and chiral restoration at finite μ_B

$$\chi_{\text{top}} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$$

used in LSM analysis



M. K., D. Suenaga JHEP 08 (2023) 189

μ scaling coincides with chiral condensate.

$$\frac{\chi_{\text{top}}}{(m_\pi^{\text{vac}})^4} \sim \frac{(f_\pi^{\text{vac}})^2}{12} \mu_q^{-2} \quad \Longleftrightarrow \quad \sigma_0 \sim \mu_q^{-2}$$

At finite μ_B..

$$\chi_{\text{top}} \cong -\frac{m_l}{4}\langle \bar{q} q \rangle$$

✓ $U(1)_A$ is effectively restored.
✓ Related to chiral restoration.

χ_{top} and chiral restoration

$$\chi_{\text{top}} = -i\int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

Topological charge density

$$Q = (g^2/64\pi^2)\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}$$

- ✓ $\chi_{\text{top}} = 0$ in chiral limit
- ✓ Prove of $U(1)_A$ symmetry restoration
- ✓ Closely related to SU(2) chiral symmetry

Indicator/prove

$$\chi_{\text{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

At finite T/μ_B..

$$\chi_{\text{top}} \cong -\frac{m_l}{4}\langle\bar{q}q\rangle$$

- ✓ $U(1)_A$ is effectively restored.
- ✓ Related to chiral restoration.

2. Topological susceptibility under extreme conditions

- Finite temperature
- Finite density
- Other situation

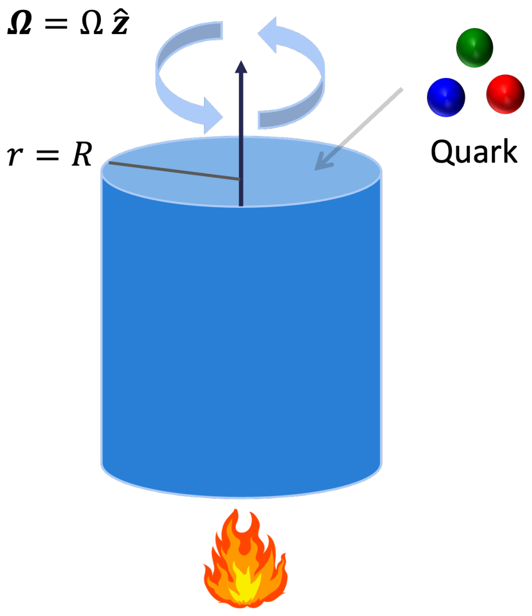
χ_{top} in other environments

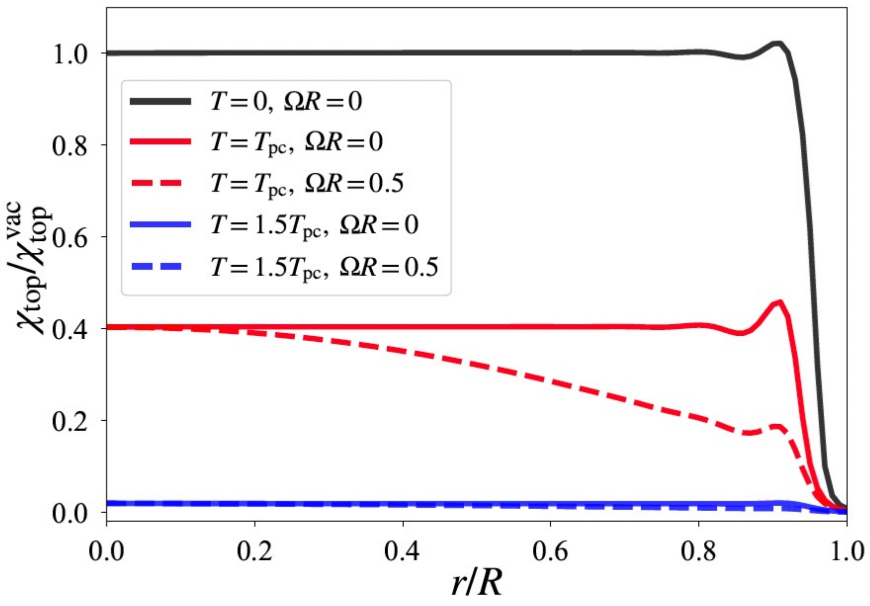
What happens in other environments?

- External magnetic field effect
  Lattice simulation in Brandt et al. JHEP 12 (2025), 228
- Rotational effect
  M. K. and K. Mameda, JHEP 11 (2025), 170

$$\chi_{\text{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

Also works in rotating systems.





There are…
- Rotational effect
- Boundary effect.

- These effects further decrease χ_{top}.
- Similar behavior to $\langle\bar{q}q\rangle$

$$\chi_{\text{top}} \cong -\frac{m_l}{4}\langle\bar{q}q\rangle$$

Summary

We have investigated $\chi_{\rm top}$ using effective models based on Ward–Takahashi identities.

$$\chi_{\rm top} = -i\int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

WT identity ⟺

$$\chi_{\rm top} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

At low- energy

$$\chi_{\rm top} = \frac{f_\pi^2 m_\pi^4}{2}\left(\frac{1}{m_\pi^2} - \frac{1}{m_\eta^2}\right)$$

for 2 flavor

At finite T/μ_B (rotating effect)

$$\chi_{\rm top} \cong -\frac{m_l}{4}\langle \bar{q}q\rangle$$

- ✓ $U(1)_A$ is effectively restored.
- ✓ Related to chiral restoration.

For quantitative studies and higher-T/μ regions, other approaches are needed. → Schwinger–Dyson and FRG analyses will be important.

Thank you

Back up

$\chi_{\rm top}$ in 2 flavor Linear Sigma Model (LSM)

U(1)$_A$ anomaly is described by det Φ term:

$$\mathcal{L}_{\rm int}^{\rm (anom)} = B(\det\Phi + \det\Phi^\dagger)$$

U(1) axial anomaly:

$$\partial_\mu j_A^\mu = (m_l\ \text{part}) - 2iN_f B\big(\det\Phi - \det\Phi^\dagger\big)$$

Introduce θ-parameter

$$\mathcal{L}_{\rm int}^{\rm (anom)} = B(e^{i\theta}\det\Phi + e^{-i\theta}\det\Phi^\dagger)$$

$$\chi_{\rm top} = \lim_{\theta\to 0}\frac{d^2V_{\rm eff}}{d\theta^2} = \frac{B}{3\sqrt{6}}(\sigma_0^{\rm vev})^3$$

In the chiral limit, $\chi_{\rm top} \neq 0$...

Why?

It corresponds to YM $\chi_{\rm top}$,

not the full $\chi_{\rm top}$.

In the chiral limit

$$\frac{B}{3\sqrt{6}}(\sigma_0^{\rm vev})^3 = \frac{f_\pi^2}{2N_f}m_{\eta'}^2 = \chi_{\rm top}^{\rm YM}$$

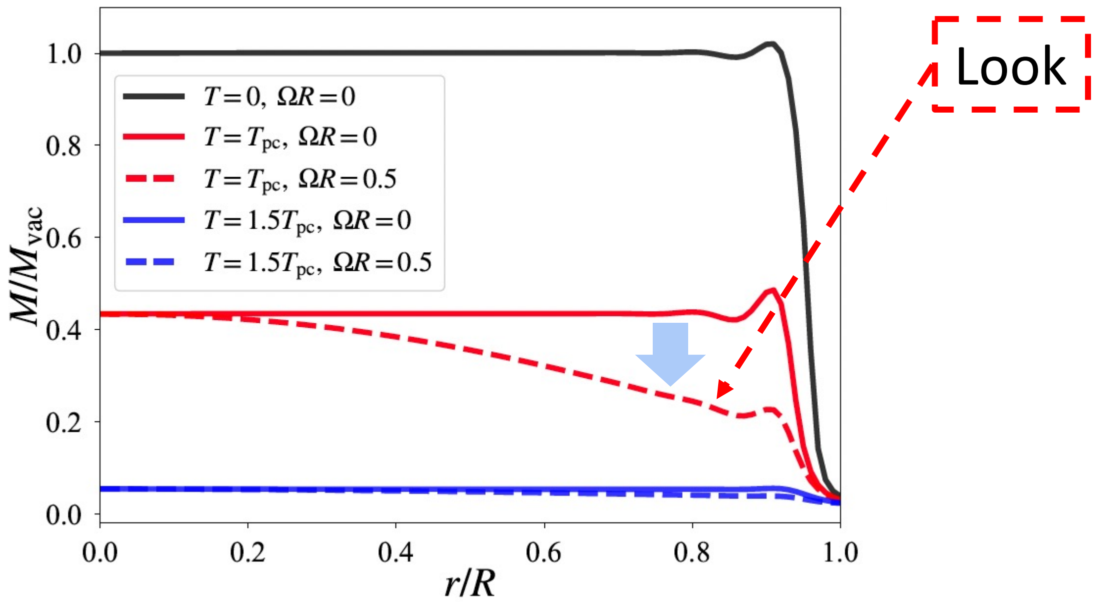
Witten--Veneziano relation

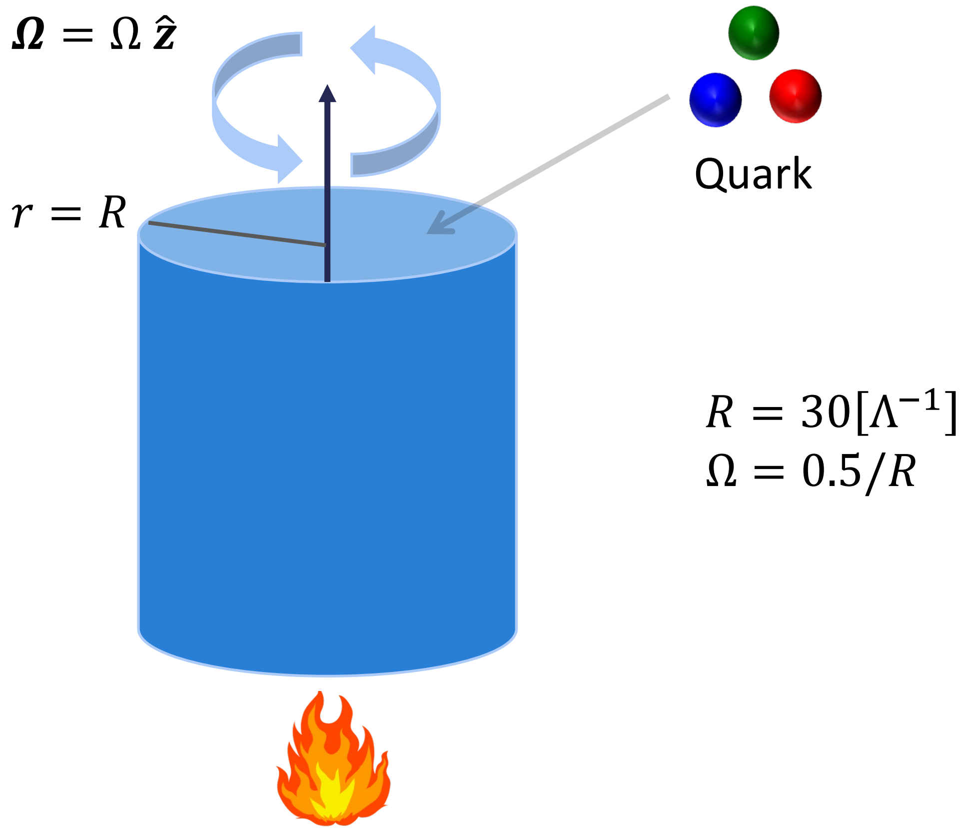
$$\chi_{\rm top} = \chi_{\rm top}^{\rm (gluon)} + \chi_{\rm top}^{\rm (quark)} = 0$$ in the chiral limit

$$\chi_{\rm top}^{\rm YM} = \chi_{\rm top}^{\rm (gluon)} = -\chi_{\rm top}^{\rm (quark)} = \frac{f_\pi^2}{2N_f}m_{\eta'}^2$$

Dynamical quark mass ($\sim\langle 0|\bar{q}q|0\rangle$)

r-dependence of dynamical quark mass

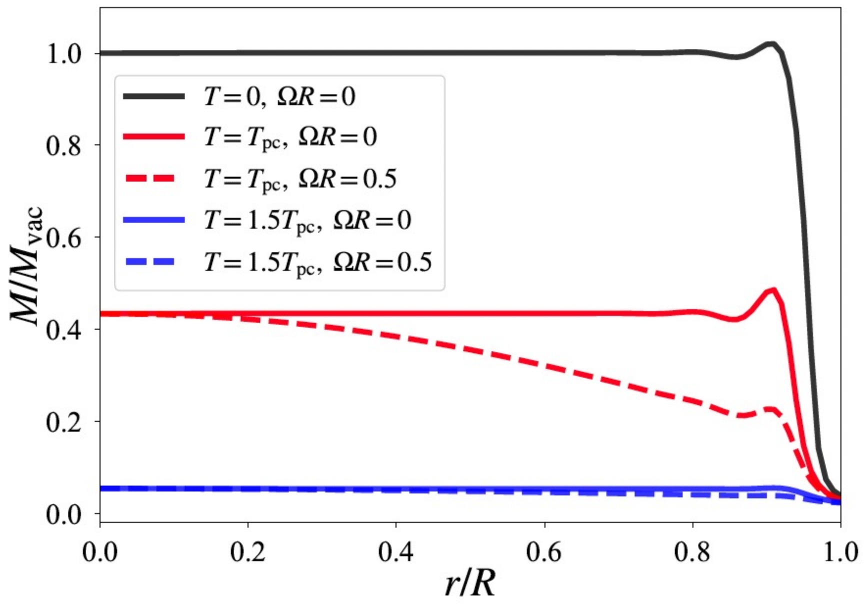




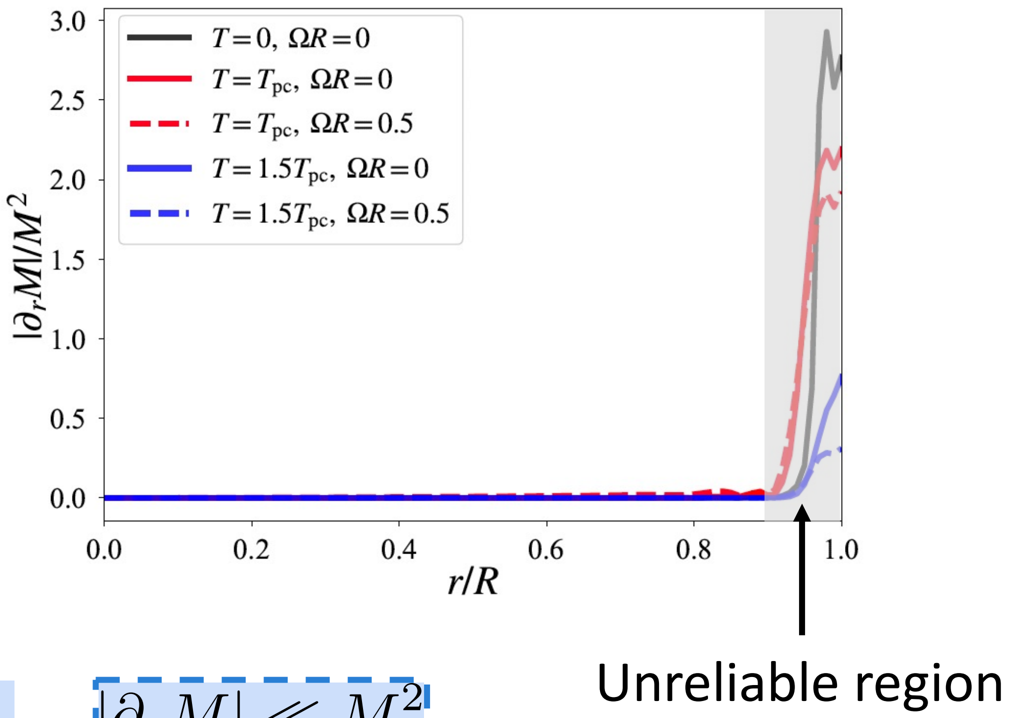
- M remains constant up to approximately boundary.
- M is suppressed at finite T.
- Rotational effect further decreases M.
- This effect is enhanced near boundary.

Validity check of our analysis

r-dependence of dynamical quark mass



Validity check of local density approximation



- M remains constant up to approximately boundary.
- M is suppressed at finite T.
- Rotational effect further decreases M.
- This effect is enhanced near boundary.

$|\partial_r M| \ll M^2$

It is satisfied for $r < 0.9R$:
For $r < 0.9R$, our result is reliable.

Effective model of QC_2D at low energy

Linear sigma model:

$$\mathcal{L}_{\rm LSM} = {\rm tr}[D_\mu \Sigma^\dagger D^\mu \Sigma] - V_0 - V_{\rm sp} - V_{\rm anom}$$

$$V_0 = m_0^2 {\rm tr}[\Sigma^\dagger \Sigma] + \lambda_1 ({\rm tr}[\Sigma^\dagger \Sigma])^2 + \lambda_2 {\rm tr}[(\Sigma^\dagger \Sigma)^2]$$

$$V_{\rm anom} = -c(\det \Sigma + \det \Sigma^\dagger)$$

$$V_{\rm sp} = -\bar{c}\, {\rm tr}[\zeta_{\rm sp}^\dagger \Sigma + \Sigma^\dagger \zeta_{\rm sp}]$$

Reproduce symmetry structure of QCD:

- $U(1)_A$ anomaly
- Chiral symmetry breaking.

$$\Sigma = \frac{1}{2}\begin{pmatrix} 0 & -B' + iB & \frac{\sigma - i\eta + a^0 - i\pi^0}{\sqrt{2}} & a^+ - i\pi^+ \\ B' - iB & 0 & a^- - i\pi^- & \frac{\sigma - i\eta - a^0 + i\pi^0}{\sqrt{2}} \\ -\frac{\sigma - i\eta + a^0 - i\pi^0}{\sqrt{2}} & -a^- + i\pi^- & 0 & -\bar{B}' + i\bar{B} \\ -a^+ + i\pi^+ & -\frac{\sigma - i\eta - a^0 + i\pi^0}{\sqrt{2}} & \bar{B}' - i\bar{B} & 0 \end{pmatrix}$$

Baryon is boson, which is embedded into Σ.

$$D_\mu \Sigma = \partial_\mu \Sigma - i\mu_q \delta_{\mu 0}(J\Sigma + \Sigma J^T)$$

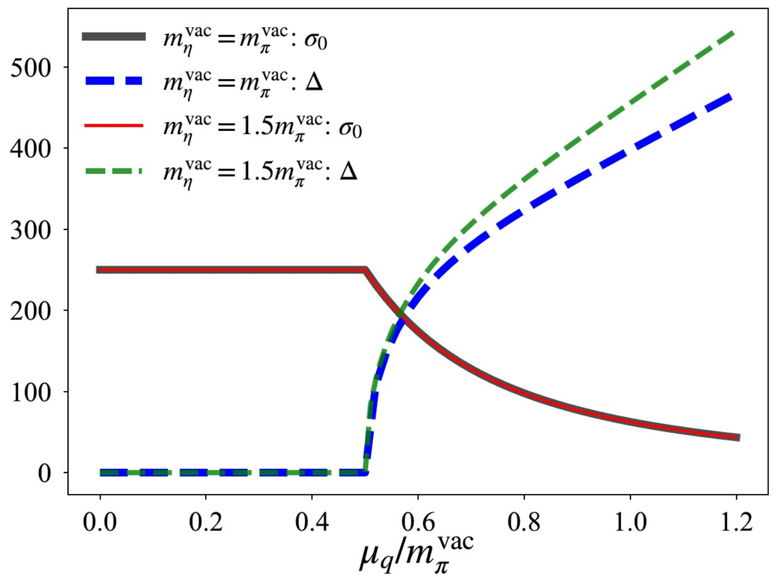
LSM gets μ_B dependence.

Baryon superfluid and mixing

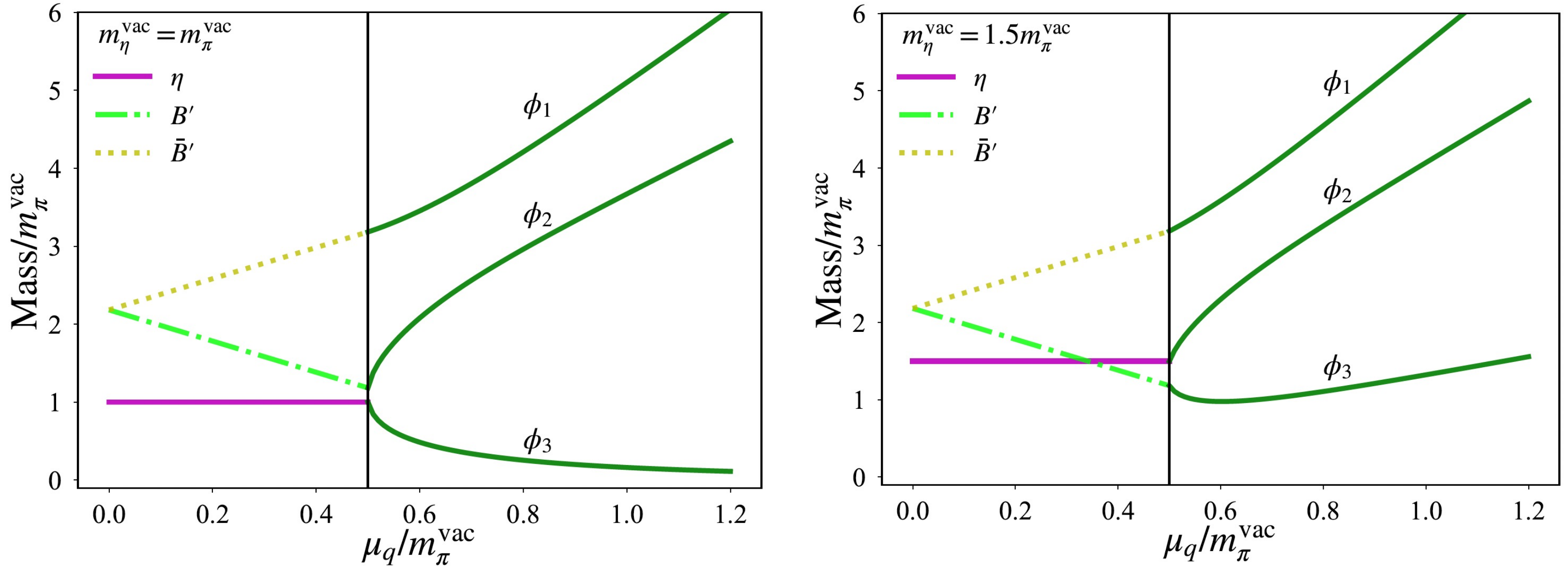
Baryon condensation ($\Delta \neq 0$) happens at large μ.

System undergoes baryon superfluid phase transition.

Condensates



Mass mixing among η meson and baryons



Possibility of being constant

In lattice simulations, diquark source field is introduce to evaluate diquark condensate:

$$\mathcal{L}^{(\mathrm{mass})}_{\mathrm{QC_2D}} = -m_l \bar{\psi}\psi - j\left(-\frac{i}{2}\psi^T C\gamma_5 \tau_c^2 \tau_f^2 \psi + \mathrm{h.c.}\right)$$

j should be set to zero at final step of calculation.

However…

j would remain sizable: artificial effect.

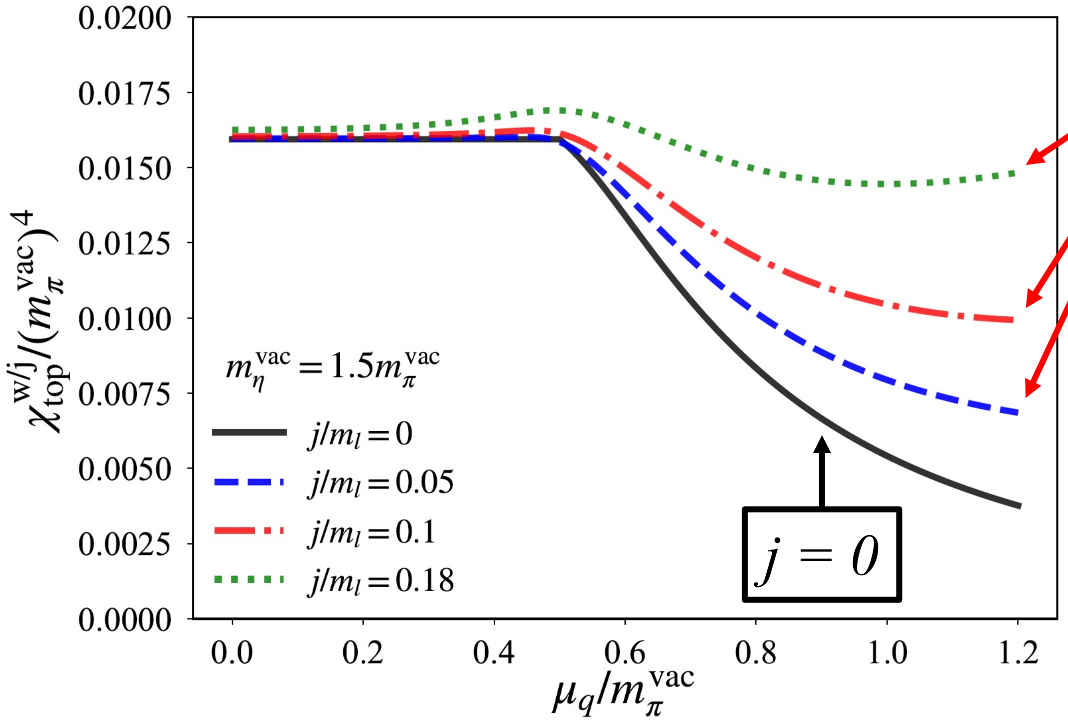
j contribution explicitly appears.

$$\chi_{\mathrm{top}}^{\mathrm{w/j}} = \chi_{\mathrm{top}} + \delta\chi_{\mathrm{top}}$$

Does j contribution significantly affect topological susceptibility?

Source effect on topological susceptibility

$$\chi_{\rm top}^{\rm w/j} = \chi_{\rm top} + \delta\chi_{\rm top}$$



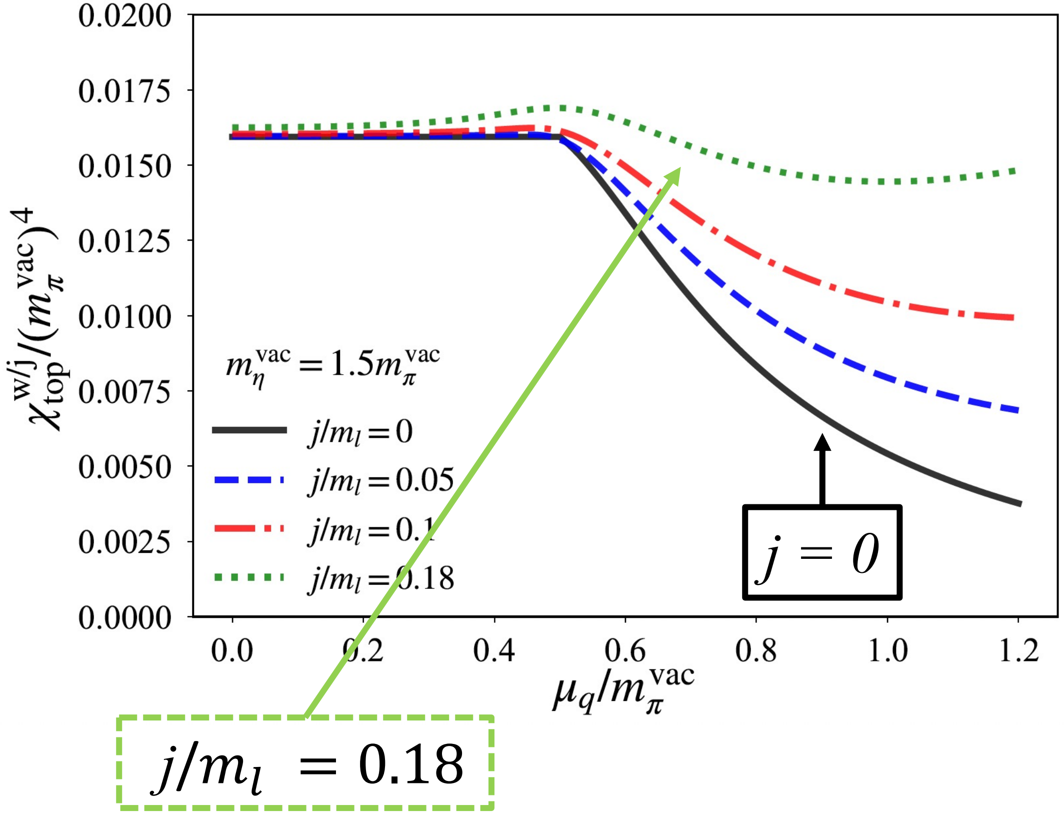
Source effect: $j \neq 0$

M. K., D. Suenaga *JHEP 08 (2023) 189*

Source effect on topological susceptibility

$$\chi_{\rm top}^{\rm w/j} = \chi_{\rm top} + \delta\chi_{\rm top}$$

Source field contaminates
the fate of topological susceptibility.

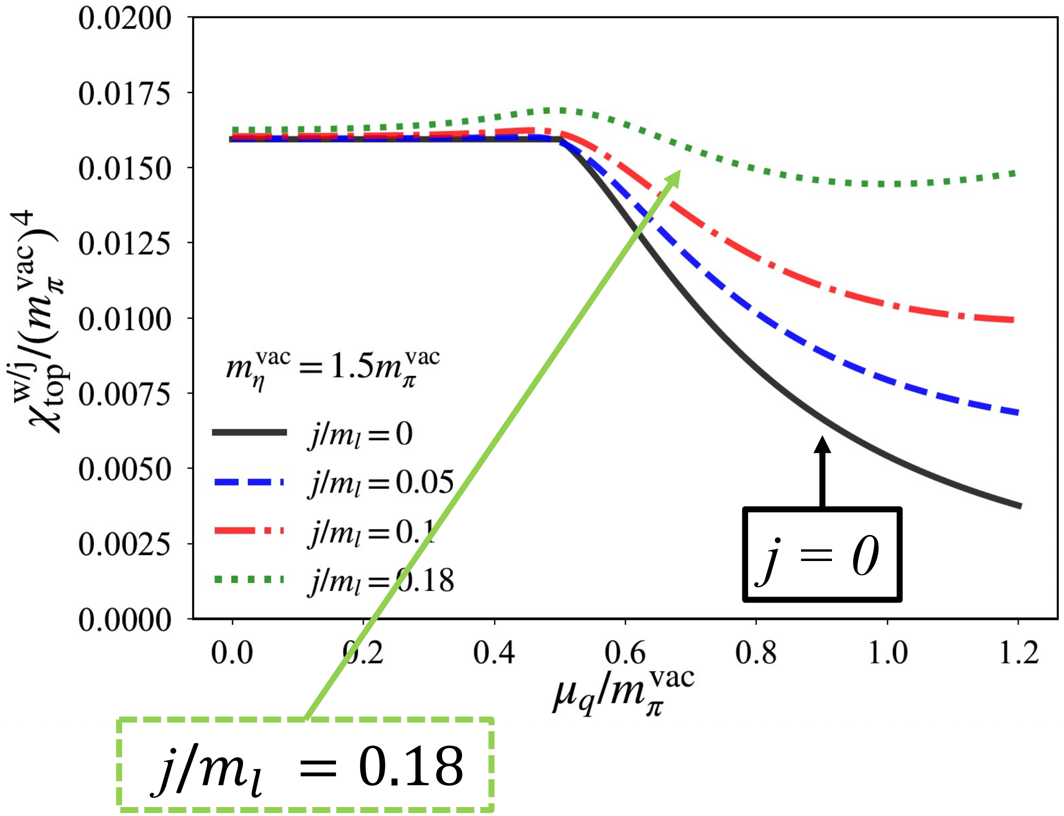


Even small j make significant contribution.

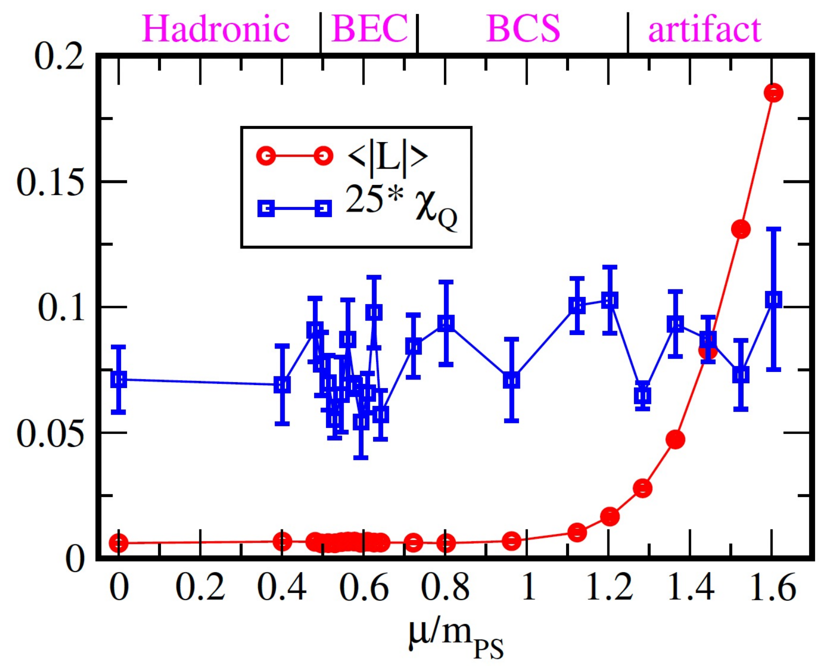
Source effect on topological susceptibility

$$\chi_{\rm top}^{\rm w/j} = \chi_{\rm top} + \delta\chi_{\rm top}$$

Source field contaminates the fate of topological susceptibility.



Constant behavior may be provided by j.



Itou et al. (2020)

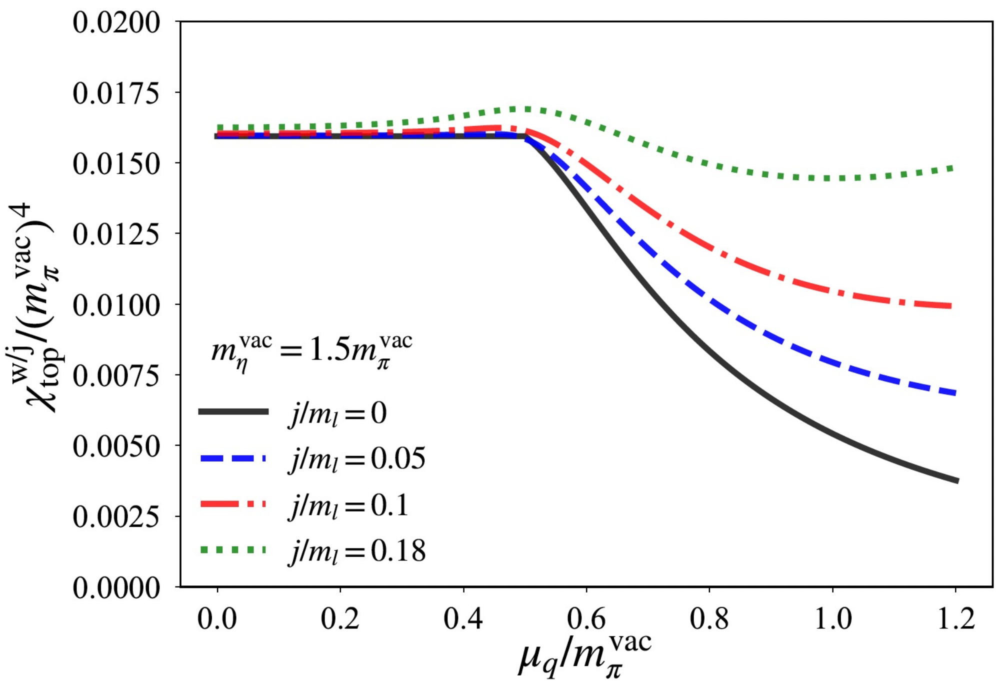
Even small j make significant contribution.

Our model results

We have investigated the topological susceptibility based on linear sigma model.

M. K., D. Suenaga *JHEP 08 (2023) 189*

LSM: $$\chi_{\mathrm{top}} = \frac{(f_\pi^{\mathrm{vac}})^2 (m_\pi^{\mathrm{vac}})^4}{2} \left(\frac{1}{m_\pi^2} - \frac{1}{m_\eta^2} \right)$$



$j = 0$

χ_{top} is suppressed. → N. Astrakhantsev et al. PRD 102 (2020) 7, 074507

$j \neq 0$

Suppression is contaminated. → Itou et al. JHEP01(2020)181

SU(2) pseudo-real property

- Fundamental representation: $\psi = 2$
- Complex conjugate multiplet: $\bar{\psi} = \bar{2}$

Pseudo-real property: $2 \cong \bar{2}$

$$T_c^a = -\tau_c^2 (T_c^a)^T \tau_c^2 \,, \quad \sigma^i = -\sigma^2 (\sigma^i)^T \sigma^2$$

Weyl representation: $$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}$$

Pseudo-real representation

$$\tilde{\psi}_R = \sigma^2 \tau_c^2 \psi_R^*, \quad \tilde{\psi}_L = \sigma^2 \tau_c^2 \psi_L^*$$

Grassmann property: $u_R^T u_R = 0$

$2 \cong \bar{2}$ means "fundamental representation" and "Pseudo-real representations" should satisfy $\tilde{u}_R^\dagger u_R = 0$.

Using fundamental and Pseudo-real representations… $$\Psi = \begin{pmatrix} \psi_R \\ \tilde{\psi}_L \end{pmatrix} = \begin{pmatrix} u_R \\ d_R \\ \tilde{u}_L \\ \tilde{d}_L \end{pmatrix}$$ Theory can have "SU(4) symmetry" .

Spontaneous symmetry breaking (SSB)

Using fundamental and
Pseudo-real representations…

$$\Psi = \begin{pmatrix} \psi_R \\ \tilde{\psi}_L \end{pmatrix} = \begin{pmatrix} u_R \\ d_R \\ \tilde{u}_L \\ \tilde{d}_L \end{pmatrix}$$

Theory can have "SU(4) symmetry" .

$$\Psi \to g\Psi \qquad g \in SU(4)$$

$$\langle \bar{\psi}\psi \rangle = -\frac{1}{2}\langle \Psi^T \sigma^2 \tau_c^2 E \Psi \rangle + \text{h.c.} \neq 0$$

Under SU(4) transformation, quark condensate is not invariant.
SSB happens: $SU(4) \to ?$.

Suppose that there is a subgroup h which is not broken.

$$\Psi \to h\Psi \qquad \langle \Psi^T \sigma^2 \tau_c^2 E \Psi \rangle \to \langle \Psi^T \sigma^2 \tau_c^2 (h^T E h) \Psi \rangle = \langle \Psi^T \sigma^2 \tau_c^2 E \Psi \rangle \qquad h^T E h = E$$

$$h \in Sp(4)$$

- Symmetry breaking pattern: $SU(4) \to Sp(4)$.