Computation of multi-leg amplitudes with NJet

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ACAT 2013, 16–21 May 2013, IHEP Beijing

NLO results provide more accurate predictions and theoretical uncertainties for multi-jet backgrounds in new physics searches.

Hard process ingredients

$$\sigma^{\mathrm{NLO}} = \int_n \left(d\sigma_n^{\mathrm{B}} + \boxed{d\sigma_n^{\mathrm{V}}} + \int_1 d\sigma_{n+1}^{\mathrm{S}}\right) + \int_{n+1} \left(d\sigma_{n+1}^{\mathrm{R}} - d\sigma_{n+1}^{\mathrm{S}}\right)$$

$$\boxed{d\sigma_n^{\mathrm{V}}} = \frac{1}{2\hat{s}} \prod_{\ell=1}^{n} \frac{d^3 k_\ell}{(2\pi)^3 2E_\ell} \Theta_{\text{n-njet}} (2\pi)^4 \delta(P) \boxed{\left|\mathcal{M}_n(ij \to n)\right|^2}$$

QCD matrix elements

$$\boxed{\left|\mathcal{M}_n(ij \to n)\right|^2} = \sum_{\text{spin}} \sum_{\text{color}} \mathcal{A}_n^{\text{1-loop}} \times \mathcal{A}_n^{\text{tree}\dagger}$$

General solutions to virtual corrections

- Helac-NLO [public] SM [arXiv:1110.1499]
- GoSam [public] SM, MSSM, UFO [arXiv:1111.2034]
- NJet [public] jets [arXiv:1209.0100]
- BlackHat [semi-public] V+jets, jets [arXiv:0803.4180+. . .]
- MadLoop, SM, BSM [arXiv:1103.0621]
- Open Loops, QCD SM [arXiv:1111.5206]
- Recola, SM+EW [arXiv:1211.6316]
- Rocket, W+jets, WW+jets, $t\bar{t}$+jet [arXiv:0805.2152+. . .]
- MCFM [public] max. $2 \to 3$ [http://mcfm.fnal.gov]
- Feynman based approaches:
  VBFNLO [public], Denner et al., FeynCalc [public], Reina et al. . . .

NGluon: public C++ library for multi-parton primitive amplitudes via unitarity (now part of NJet) [arXiv:1011.2900]

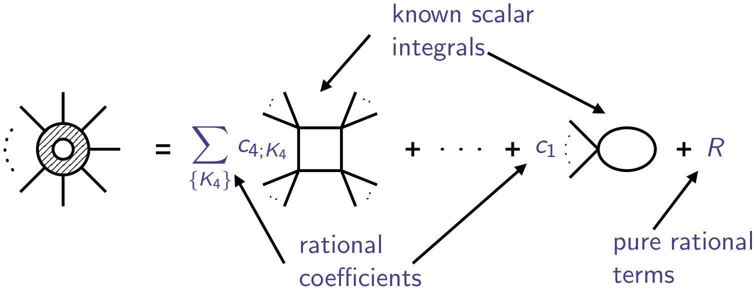
- Efficient tree amplitudes using Berends-Giele recursion.
- Rational terms from massive loop cuts.
- Extraction of integral coefficients via Fourier projections.
- Everything is in 4 dimensions (except loop integrals).

NJet: public C++ library for multi-parton matrix elements in massless QCD [https://bitbucket.org/njet/njet] [arXiv:1209.0100]

Features

- Full colour-summed amplitudes for up to 5 outgoing partons.
- Binoth Les Houches Accord interface for MC generators.

Structure of One-Loop Amplitudes



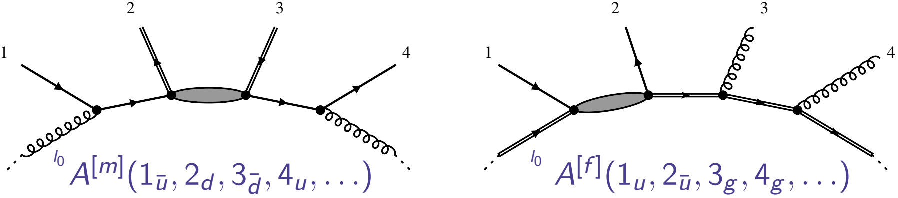
- Gauge theory amplitudes reduced to box topologies or simpler

  [Passarino,Veltman;Melrose]

- Isolate logarithms with cuts and exploit on-shell simplifications

  [Bern,Dixon,Kosower]

NParton computes arbitrary multi-fermion primitives.



All primitives are separated into two classes

- With **mixed** fermion and gluon loop content ($l_0 =$ gluon)
- With internal **fermion loops** ($l_0 =$ quark)

These two classes cover all partonic primitives in one loop QCD.

Colour decomposition of an L-loop amplitude:

$$\mathcal{A}_n^{(L)}(\{p_i\}) = \sum_c \underbrace{T_c(\{a_i\})}_{\text{colour basis}} \underbrace{\boxed{A_{n;c}^{(L)}(p_1,\ldots,p_n)}}_{\text{partial amplitudes}}$$

Partial amplitudes $\rightarrow$ squared matrix elements

$$\left|\mathcal{M}_n\right|^2 = \sum_{\text{hel}} \sum_{\text{col}} \mathcal{A}_n^{(L)} \mathcal{A}_n^{(0)\dagger} = \sum_{\text{hel}} \sum_{cc'} A_{n;c}^{(L)} \cdot \mathcal{C}_{cc'} \cdot A_{n;c'}^{(0)\dagger}$$

Colour matrix

$$\mathcal{C}_{cc'} = \sum_{\{a_i\}} T_c(\{a_i\}) T_{c'}(\{a_i\})$$

$$T_c(\{a_i\}) = T_{jk}^{a_1} \ldots \delta_{lm} \ldots$$

Colour decomposition of a 1-loop amplitude:

Partial amplitudes are linear combinations of primitive amplitudes.

$$A^{(1)}_{n;c} = \sum_k a_{k;c}\, A^{[m]}_n + N_f\, b_{k;c}\, A^{[f]}_n$$

Partial-Primitive decomposition for gluons and $q\bar{q}+$ gluons:

- Tree level: Kleiss-Kuijf basis of $(n-2)!$ primitives
- One-loop: a basis of $(n-1)!$ primitives.

[Kleiss,Kuijf], [Bern,Dixon,Dunbar,Kosower]

Partial-Primitive decomposition for multi-quark case:

No analytic formula. Reconstruct partials using diagram matching.

[Ellis,Kunszt,Melnikov,Zanderighi], [Ita,Ozeren], [NJet]

Outline of the algorithm

1. Generate all diagrams' topologies for the amplitude $\mathcal{A}_n$
2. Write primitives P_i as combinations of colour-stripped diagrams K_i using matching matrix M_{ij}
3. Invert the system to get partial amplitudes in terms of **independent set** of primitives $\hat{P}$

$$\mathcal{A}_n = \sum_c T_c(\{a_i\}) \sum_{j=1}^{\hat{N}_{\text{pri}}} Q_{cj} \hat{P}_j$$

Ensure linearly independent set by capturing all relations between color-ordered diagrams.

Number of primitives in tree, mixed and fermion loop amplitudes

Process	$N^{[0]}_{\text{pri}}$	$N^{[m]}_{\text{pri}}$	$N^{[f]}_{\text{pri}}$
$4\,g$	2	3	3
$\overline{u}u + 2\,g$	2	6	1
$\overline{u}u\overline{d}d$	1	4	1

Process	$N^{[0]}_{\text{pri}}$	$N^{[m]}_{\text{pri}}$	$N^{[f]}_{\text{pri}}$
$5\,g$	6	12	12
$\overline{u}u + 3\,g$	6	24	6
$\overline{u}u\overline{d}dg$	3	16	3

Process	$N^{[0]}_{\text{pri}}$	$N^{[m]}_{\text{pri}}$	$N^{[f]}_{\text{pri}}$
$6\,g$	24	60	60
$\overline{u}u + 4\,g$	24	120	33
$\overline{u}u\overline{d}d + 2\,g$	12	80	13
$\overline{u}u\overline{d}d\overline{s}s$	4	32	4

Process	$N^{[0]}_{\text{pri}}$	$N^{[m]}_{\text{pri}}$	$N^{[f]}_{\text{pri}}$
$7\,g$	120	360	360
$\overline{u}u + 5\,g$	120	720	230
$\overline{u}u\overline{d}d + 3\,g$	60	480	75
$\overline{u}u\overline{d}d\overline{s}sg$	20	192	20

Process	$N^{[0]}_{\text{pri}}$	$N^{[m]}_{\text{pri}}$	$N^{[f]}_{\text{pri}}$
$8\,g$	720	2520	2520
$\overline{u}u + 6\,g$	720	5040	1800
$\overline{u}u\overline{d}d + 4\,g$	360	3360	712
$\overline{u}u\overline{d}d\overline{s}s + 2\,g$	120	1344	263
$\overline{u}u\overline{d}d\overline{s}s\overline{c}c$	30	384	65

Desymmetrized amplitudes

Squared amplitudes are totally symmetric over final state gluons

$$\left|\mathcal{A}(x, g_1, \ldots, g_n, y)\right|^2 = \left|\mathcal{A}(x, \sigma\{g_1, \ldots, g_n\}, y)\right|^2$$

Gluon phase space integration is a symmetric operator

$$\int F(g_1, \ldots, g_n)\, \mathrm{d}PS_n = \int F(\sigma\{g_1, \ldots, g_n\})\, \mathrm{d}PS_n$$

Could replace squared amplitudes with something simpler

$$\int \boxed{\left|\mathcal{A}_{x\to n(g)}\right|^2}\, \mathrm{d}PS_n = \int \boxed{\mathcal{A}^{\mathsf{dsym}}(g_1, \ldots, g_n)}\, \mathrm{d}PS_n$$

where $\sum_{P_n} \mathcal{A}^{\mathsf{dsym}}(g_1, \ldots, g_n) = n!\left|\mathcal{A}_{x\to n(g)}\right|^2, \quad P_n \in \sigma\{g_1, \ldots, g_n\}$

Example:

$$\iiint_a^b \boxed{(x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2)}\, dx\, dy\, dz = \iiint_a^b \boxed{6x^2y}\, dx\, dy\, dz$$

Special non-symmetric gluon colour sums

- Contain significantly fewer loop primitives
- Give original full colour sums after symmetrization

$$\sigma^{V}_{gg\to n(g)} = \int \mathrm{d}PS_n \boxed{A^{(0)\dagger} \cdot \mathcal{C}_{n!\times(n+1)!/2} \cdot A^{(1)}}$$
$$= (n-2)! \int \mathrm{d}PS_n \boxed{A^{(0)\dagger} \cdot \mathcal{C}^{\mathrm{dsym}}_{n!\times(n+1)} \cdot A^{(1),\mathrm{dsym}}}$$

$n!/2$ reduction of time per point[1]

	$gg \to 3g$	$gg \to 4g$	$gg \to 5g$
Standard sum	0.22 s	6.19 s	171.31 s
De-symmetrized	0.07 s	0.50 s	2.76 s
Speedup	$\times\, 3$	$\times\, 12$	$\times\, 60$

[1] Where n is the number of final state gluons

Sources of accuracy loss

- Accumulation of rounding errors – negligible
- Catastrophic large cancellations – significant in certain kinematic regions (small Gram determinants, etc)

Large cancellation

$$\frac{A}{C} - \frac{B}{C} \sim 1$$

$$\begin{array}{rl} & 1.1111111115495439 \\ - & 1.1111111112345678 \\ \hline = & \underbrace{0.000000000}_{\text{lost}}3149761\underbrace{00000000}_{\text{new tail}} \end{array}$$

If $C \to 0$ then $A \to B$

In finite precision machine arithmetic the tail is zero-extended.

Scaling test to determine accuracy

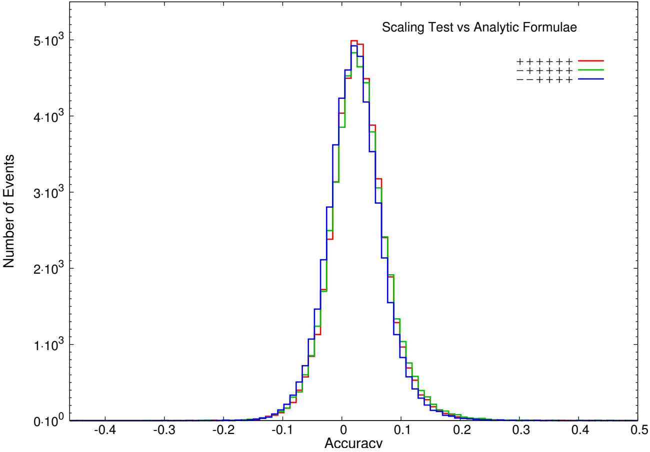
Scaling test

- Evaluate the amplitude several times using different “scaled” units (for instance: $1\times$GeV, $1.33\times$GeV, etc).
- Use known dimension of the amplitudes to scale them back to a common unit (GeV).
- The difference between obtained values is an error estimate.

Why it works?

$$
\begin{array}{lcl}
 & & 1.11111111\color{blue}{5495439} \\
 & - & 1.11111111\color{blue}{2345678} \\
\hline
A_1 & = & 0.00000000\color{red}{3149761}00000000 \\
\times 1.33 & & 1.11111111\color{green}{8228751}\not{4}\not{3} \leftarrow \text{round-off} \\
\times 1.33 & - & 1.11111111\color{green}{3913578}\not{8}\not{6} \leftarrow \text{round-off} \\
\hline
\times 1.33 & = & 0.00000000\color{green}{4315173}00000000 \\
A_2 & = & 0.00000000\color{red}{3149761}\underbrace{\color{green}{31386861}}_{\text{difference}}
\end{array}
$$

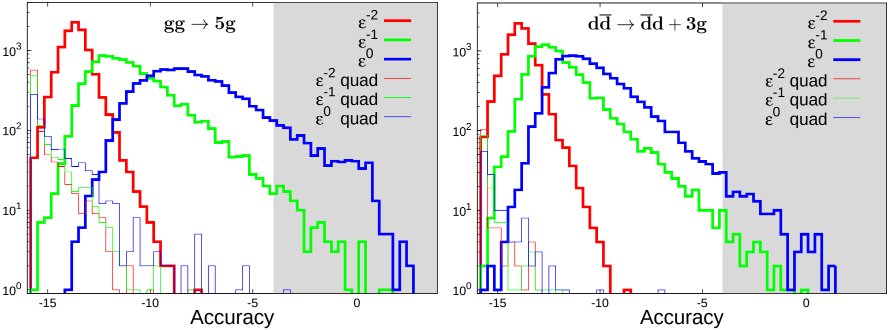
Testing the scaling test



$$\log\left(\frac{A_{\texttt{NGluon}} - A_{\text{analytic}}}{A_{\text{analytic}}}\right) - \log\left(\frac{2(A_{\texttt{NGluon}} - A^{\text{scaled}}_{\texttt{NGluon}})}{A_{\texttt{NGluon}} + A^{\text{scaled}}_{\texttt{NGluon}}}\right)$$

Reliable, but essentially statistical.
A safety margin of 2 digits is advised.

Left: 7 gluon squared amplitude. Right: 4 quarks + 3 gluons.



Thick lines – double precision.

Thin lines – fixed with quadruple precision.

Full colour and helicity sum time per point [clang, Xeon 3.30 GHz].

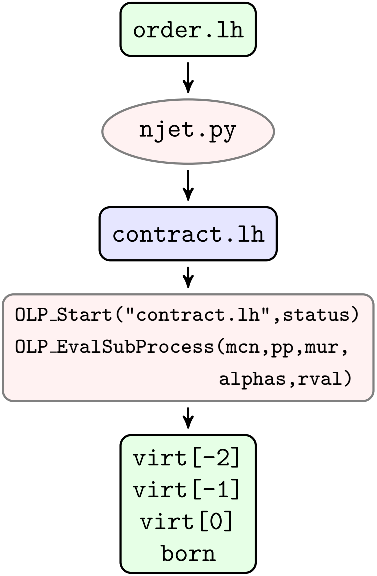
process	T_{sd}[s]	$T_{4\text{ dig.}}$[s] (%)
$4\,g$	0.030	0.030 (0.00)
$\bar{u}u{+}2g$	0.032	0.032 (0.00)
$\bar{u}u\bar{d}d$	0.011	0.011 (0.00)
$\bar{u}u\bar{u}u$	0.022	0.022 (0.00)

process	T_{sd}[s]	$T_{4\text{ dig.}}$[s] (%)
$5\,g$	0.22	0.22 (0.22)
$\bar{u}u{+}3g$	0.34	0.35 (0.06)
$\bar{u}u\bar{d}d{+}g$	0.11	0.11 (0.00)
$\bar{u}u\bar{u}u{+}g$	0.22	0.22 (0.03)

process	T_{sd}[s]	$T_{4\text{ dig.}}$[s] (%)
$6\,g$	6.19	6.81 (1.37)
$\bar{u}u{+}4g$	7.19	7.40 (0.38)
$\bar{u}u\bar{d}d{+}2g$	2.05	2.06 (0.08)
$\bar{u}u\bar{u}u{+}2g$	4.08	4.15 (0.21)
$\bar{u}u\bar{d}d\bar{s}s$	0.38	0.38 (0.00)
$\bar{u}u\bar{d}d\bar{d}d$	0.74	0.74 (0.00)
$\bar{u}u\bar{u}u\bar{u}u$	2.16	2.17 (0.02)

process	T_{sd}[s]	$T_{4\text{ dig.}}$[s] (%)
$7\,g$	171.3	276.7 (8.63)
$\bar{u}u{+}5g$	195.1	241.2 (3.25)
$\bar{u}u\bar{d}d{+}3g$	45.7	48.8 (0.88)
$\bar{u}u\bar{u}u{+}3g$	92.5	101.5 (1.29)
$\bar{u}u\bar{d}d\bar{s}sg$	7.9	8.1 (0.23)
$\bar{u}u\bar{d}d\bar{d}dg$	15.8	16.2 (0.29)
$\bar{u}u\bar{u}u\bar{u}ug$	47.1	48.6 (0.41)

All times include two evaluations for the scaling test.



Create an ‘order’ file

NJet takes an ‘order’ file
and returns a ‘contract’ file

Check that requested
options were accepted

Link with `libnjet.so`
Call ‘Start’ once to initialize
Call ‘EvalSubProcess’ to evaluate PS points

Use returned values to calculate XS
$1/\text{eps}^2$, $1/\text{eps}$, finite, born

Recent progress in fixed order NLO jet production

- $pp \to 2$ jets [Kunszt,Soper (1992)]
  [Giele,Glover,Kosower (1993)]
- $pp \to 3$ jets [gluons Trocsanyi (1996)]
  [gluons Giele,Kilgore (1997)]
  [Nagy NLOJET++ (2003)]
- $pp \to 4$ jets [Bern et al. BlackHat (2012)]
  [Badger, Biedermann, Uwer, VY (2013)]
- $pp \to 5$ jets [Badger, Biedermann, Uwer, VY (preliminary)]

Tools (linked together with BLHA interface)

- `NJet` — full colour virtual matrix elements
  scalar integrals — QCDLoop/FF [Ellis,Zanderighi,van Oldenborgh]
  extended precision — libqd [Hida,Li,Bailey]
- Sherpa/COMIX — trees, CS subtraction, PS integration
  [Hoeche,Gleisberg,Krauss,Kuhn,Soff,...]

ATLAS jet cuts

- anti-kt $R = 0.4$, $p_T^{\text{1st}} > 80$ GeV, $p_T^{\text{other}} > 60$ GeV, $|\eta| < 2.8$

Parameters

- $pp \to 2$, 3, 4 and 5 jets at 7 TeV
- $\mu_R = \mu_F = \hat{H}_T/2$, scale variations $\hat{H}_T/4$ and $\hat{H}_T$
- MSTW2008 PDF set, $\alpha_s(M_Z)$ from PDFs

NJet + Sherpa: total XS for 3, 4 and 5 jets at 7 TeV

	3 jets	4 jets	5 jets
σ_{LO} [nb]	$93.40(0.03)^{+50.4}_{-30.3}$	$9.98(0.01)^{+7.4}_{-3.9}$	$1.003(0.005)^{+0.94}_{-0.45}$
σ_{NLO} [nb]	$53.74(0.16)^{+2.1}_{-20.7}$	$5.61(0.13)^{+0.0}_{-2.2}$	$0.578(0.13)^{+0.0}_{-0.21}$

Reduced scale uncertainty

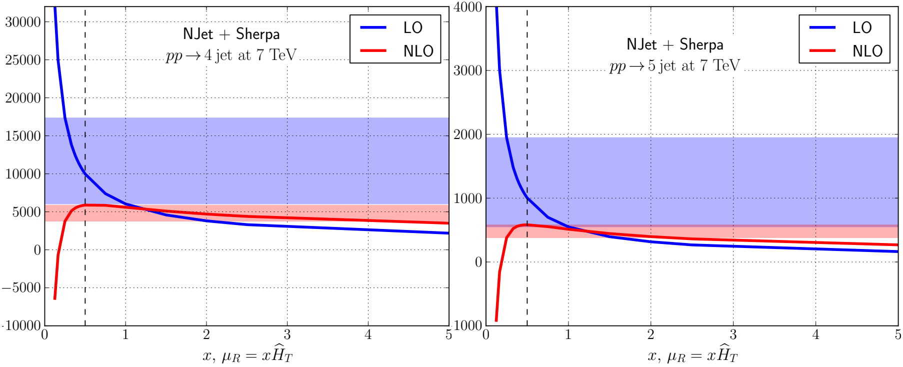
NLO scale variations are about 28%, 19% and 14% of the LO scale uncertainty for 3, 4 and 5 jet cross-sections respectively

Cross-check at 7 TeV

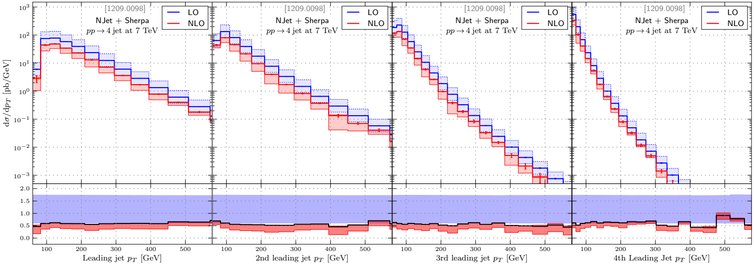
Agreement with 4 jet results by BlackHat collaboration
[Bern,Diana,Dixon,Febres Cordero,Hoeche,Kosower,Ita,Maitre,Ozeren]
[arXiv:1112.3940]

ROOT Ntuple format stores extra information, which allows to vary renormalization scale in the analysis

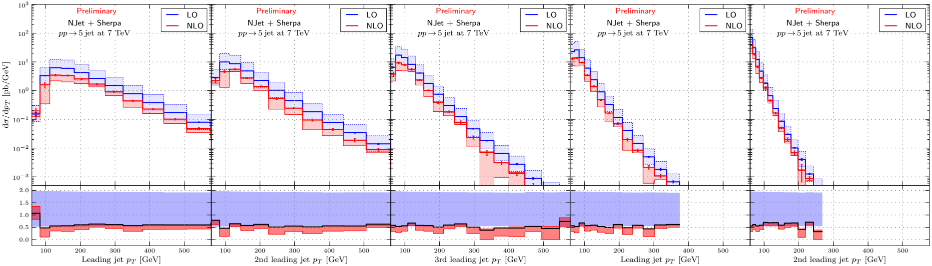
[SM and NLO Multileg WG Summary report 2010]



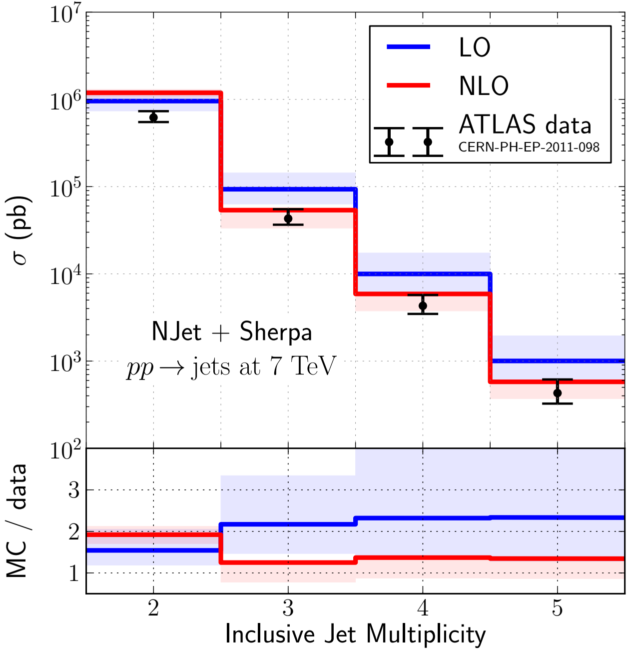
$pp \to 4$ jets



$pp \to 5$ jets



NJet + Sherpa: comparison with LHC jet measurements



Summary

- NJet: numerical evaluation of one-loop amplitudes in massless QCD.
- General construction for primitive and partial amplitudes.
- Full colour results for ≤ 5 jets
- Binoth Les Houches Accord interface.
- NJet+Sherpa: 3 and 4 jets at NLO at 7 and 8 TeV [arXiv:1209.0098]
- NJet+Sherpa: First results for 5 jets.
- Publicly available from the NJet project page `https://bitbucket.org/njet/njet`

Bonus material

1. Generate all diagrams[2] D_i for a given n-parton amplitude $\mathcal{A}_n$

$$\mathcal{A}_n = \sum_{i=1}^{N_{\text{dia}}} D_i = \sum_{i=1}^{\hat{N}_{\text{dia}}} C_i K_i \qquad C_i = \sum_c T_c F_{ci}$$

2. Write all possible primitives P_i as combinations of colour-stripped diagrams K_i using matching matrix M_{ij}

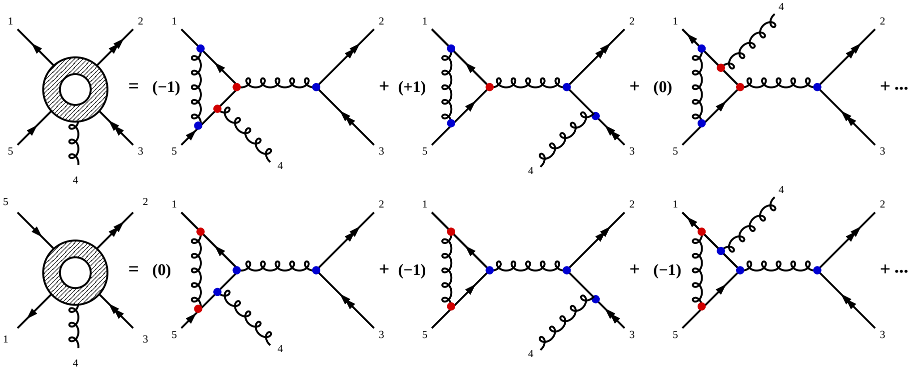
$$P_i = \sum_{j=1}^{\hat{N}_{\text{dia}}} M_{ij} K_j \qquad i \in \{1, 2, \ldots, N_{\text{pri}}\} \qquad N_{\text{pri}} = N_{\text{pri}}^{[m]} + N_{\text{pri}}^{[f]}$$

$$N_{\text{pri}}^{[f]} = (n-1)! \qquad N_{\text{pri}}^{[m]} = \begin{cases} (n-1)! & n_q = 0 \\ n_q(n-1)!/2 & n_q = 2, 4, \ldots \end{cases}$$

[2] only topologies are needed

Matching Matrix M_{ij}

$$\color{red}{P_i} = \sum_{j=1}^{\hat{N}_{\text{dia}}} \color{green}{M_{ij}}\color{blue}{K_j} \qquad M_{ij} \in \{0,\ 1,\ -1\}$$



Matching of $A_5^{[m]}(1_{\overline{d}}, 2_u, 3_{\overline{u}}, 4_g, 5_d)$ and $A_5^{[m]}(1_{\overline{d}}, 4_g, 3_{\overline{u}}, 2_u, 5_d)$.
Each vertex is either ordered or unordered with respect to the colour ordered Feynman rules and the primitive in question.

$$P_i = \sum_{j=1}^{\hat{N}_{\text{dia}}} M_{ij} K_j \qquad M_{ij} \in \{0,\ 1,\ -1\}$$

Number of **independent** primitive amplitudes (denoted $\hat{P}_j$)

$$\hat{N}_{\text{pri}} = \textbf{rank}\,\mathbf{M}$$

$$\hat{N}_{\text{pri}} \leq (N_{\text{pri}} = N_{\text{rows}}) \quad \hat{N}_{\text{pri}} \leq (\hat{N}_{\text{dia}} = N_{\text{cols}})$$

Reduced row echelon form of $\hat{\mathbf{M}} = [\mathbf{M}|-\mathbb{1}]$

- upper $\hat{N}_{\text{pri}}$ rows — **solution** of K_j in terms of $\hat{P}_i$
- lower $N_{\text{pri}} - \hat{N}_{\text{pri}}$ rows — left null space of $\mathbf{M}$ (relations)

$$K_i = \sum_{j=1}^{\hat{N}_{\text{pri}}} B_{ij} \hat{P}_j \qquad \{\hat{P}_j\}_{\hat{N}_{\text{pri}}} \subset \{P_j\}_{N_{\text{pri}}}$$

Putting everything together

- Colour factors in terms of the colour "trace basis"
- Kinematic factors in terms of independent primitives

$$C_i = \sum_c T_c F_{ci} \qquad K_i = \sum_{j=1}^{\hat{N}_{\text{pri}}} B_{ij} \hat{P}_j$$

$$\mathcal{A}_n = \sum_{i=1}^{\hat{N}_{\text{dia}}} C_i K_i = \sum_c T_c \sum_{j=1}^{\hat{N}_{\text{pri}}} \underbrace{\sum_{i=1}^{\hat{N}_{\text{dia}}} F_{ci} B_{ij}}_{Q_{cj}} \hat{P}_j$$

We obtain partial amplitudes in terms of a basis of independent primitive amplitudes $\hat{P}_j$ for a given class of primitives

$$\mathcal{A}_n = \sum_c T_c \boxed{\sum_{j=1}^{\hat{N}_{\text{pri}}} Q_{cj} \hat{P}_j}$$