Multi-Loop Integrand Reduction with Computational Algebraic Geometry

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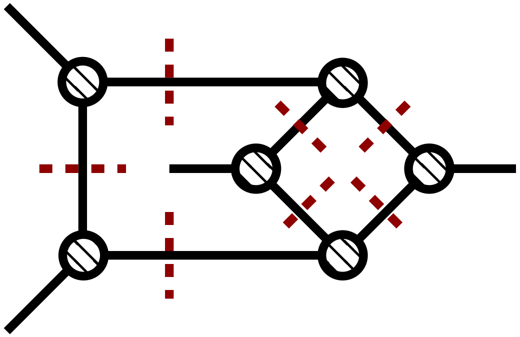
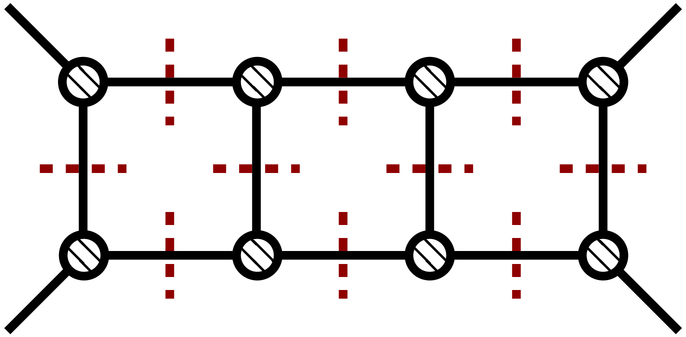
18$^{\mathrm{th}}$ May 2013

ACAT 2013, Beijing

Outline

- Integrand reduction and generalized unitarity beyond one-loop
- Multi-loop integral coefficients via computational algebraic geometry
  [Zhang arXiv:1206.5707 JHEP 1209:042 (2012)]
- Two-loop hepta-cuts : planar and non-planar
  [SB, Frellesvig, Zhang arXiv:1202.2019 JHEP 1204:055 (2012)]
- Three-loop maximal cuts : triple box
  [SB, Frellesvig, Zhang arXiv:1207:2976 JHEP 1208:065 (2012)]
- D-dimensional cuts at two-loops [SB, Frellesvig, Zhang (in progress)]





Background

- One-loop techniques: [Ossola,Papadopoulos,Pittau (2006)]

  [Ellis,Giele,Kunszt,Melnikov (2007-2008)]

  [Bern,Dixon,Dunbar,Kosower (1994)][Britto,Cachazo,Feng (2004)]

$\Rightarrow$ Automation of NLO predictions for LHC phenomenology

[see talks of Hahn, Heinrich, Kosower, Ossola, Maierhöfer, Yundin]

- NNLO predictions in QCD would be extremely valuable!

  Experimental precision will likely reach $\sim 1 - 2\%$ for a large number of processes

- Recent progress in extensions to two-loops:
  - OPP reduction at two-loops [Mastrolia, Ossola arXiv:1107.6041]

    [Mastrolia, Mirabella, Ossola, Peraro arXiv:1205.7087, arXiv:1209.4319]

    [Kleiss, Malamos, Papadopoulos, Verheyen arXiv:1206.4180]

    [Feng, Huang arXiv:1209.3747]
  - Maximal unitarity [Kosower, Larsen arXiv:1108.1180]

    [Larsen arXiv:1205.0297], [Larsen, Caron-Huot arXiv:1205.0801]

    [Johansson, Kosower, Larsen arXiv:1208.1754]

Background

- Feynman diagrams and integration-by-parts reduction

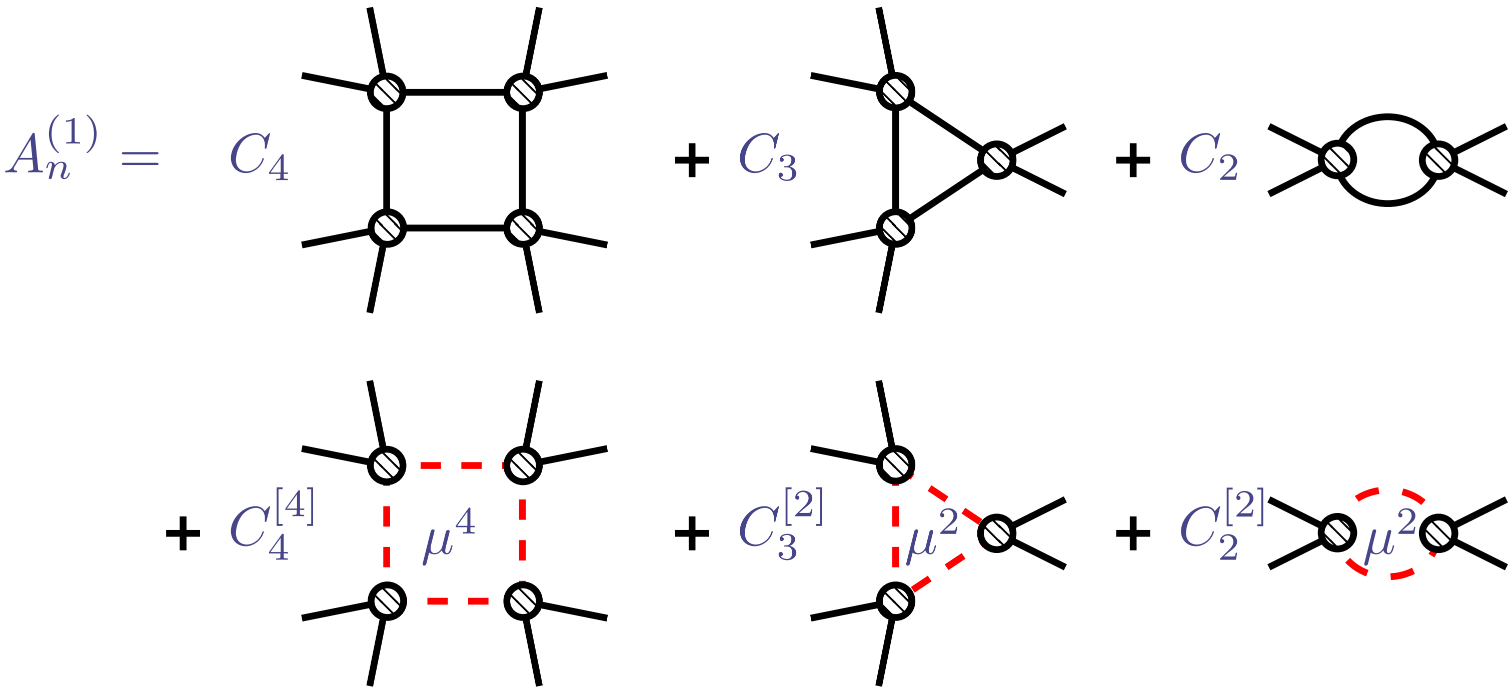
  current state-of-the-art for QCD corrections

  - $2 \to 2$ scattering amplitudes:
    - massless QCD [Anastasiou, Glover, Tejeda-Yeomans, Oleari (2000-2002)]

      [Bern, Dixon, Kosower (2000)][Bern, De-Frietas ,Dixon (2002)]
    - $pp \to W + j/e^+e^- \to 3j$ [Garland, Gehrmann, Glover, Koukoutsakis, Remiddi (2002)]
    - $pp \to H + 1j$ [Gehrmann, Jaquier, Glover, Koukoutsakis (2011)]
  - Full NNLO predictions for $2 \to 2$ processes (with IR subtractions)
    - $e^+e^- \to 3j$ [Gehrmann-De Ridder, Gehrmann, Glover, Heinrich (2007)]
    - $p\bar{p} \to t\bar{t}$ [Bernreuther, Czakon, Mitov (2012)] [Czakon, Fiedler, Mitov (2013)]
    - $gg \to gg$ [Gehrmann-De Ridder, Gehrmann, Glover, Pires (2013)]
    - $gg \to Hg$ [Boughezal, Caola, Melnikov, Petriello, Schulze (2013)]
- On-shell methods for higher multiplicity at two loops?
- Make more use of the progress in understanding $\mathcal{N} = 4$ SYM amplitudes

One-Loop Overview

- Scalar integral ≤ 4-point functions form a basis with rational coefficients



$k = \bar{k} + \mu$
$\bar{k}^2 = -\mu^2$

- Integrand representation (OPP) : $\Delta_4(k \cdot \omega) = C_4 + \tilde{C}_4 k \cdot \omega$
- 2 solutions to $\{l_i^2 = 0\}$:

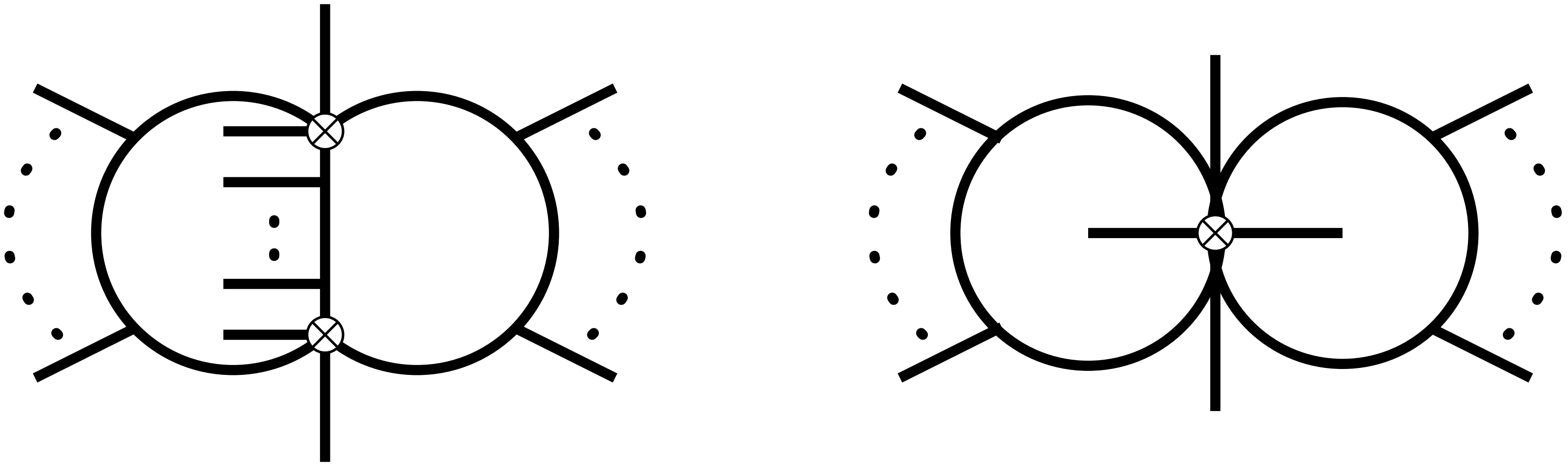
$$2C_4 = \Delta_4(k^{(1)} \cdot \omega) + \Delta_4(k^{(2)} \cdot \omega)$$

Two-Loop Integral Bases

- Complete basis of scalar integrals unknown
- Progress in understanding the planar case

[Gluza, Kosower, Kajda arXiv:1009.0472]

[Schabinger arXiv:1111.4220]



- No longer just scalar integrals, also tensor integrals in basis

A Two-Loop Integrand Basis

- Integrand is polynomial in irreducible scalar products (ISPs)
  spanned by indep. ext. moms. : $\{p_1, \ldots, p_k\}$ and spurious vecs. : $\{\omega_1, \ldots, \omega_j\}$.
- Gram matrix gives (non-linear) constraints on the polynomial form.

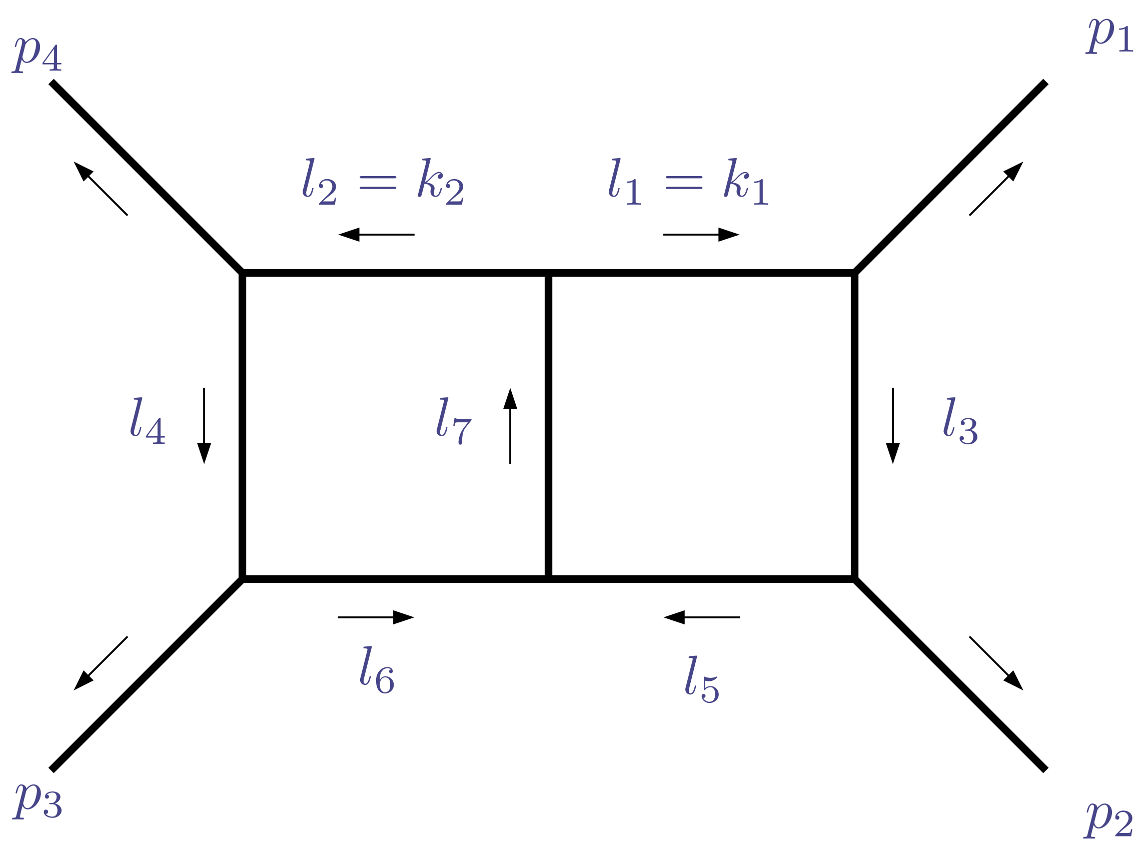
$$G\begin{pmatrix} v_1 \ldots v_n \\ r_1 \ldots r_n \end{pmatrix}, G_{ij} = v_i \cdot r_j$$

- Important to identify spurious terms which integrate to zero.

$$A_n^{(2)} = \int\int \frac{d^D k_1}{(4\pi)^{D/2}} \frac{d^D k_2}{(4\pi)^{D/2}} \sum_{p=3}^{11} \sum_{T_p \in \text{topologies}} \frac{\Delta_{p,T_p}(\{k_i \cdot p_j, k_i \cdot \omega_j\}, \epsilon)}{\prod_{i=1}^{p} l_i(k_1, k_2)}$$

1. **Determine parametrization for the integrand** $\Delta_{p,T_p}(\{k_i \cdot p_j, k_i \cdot \omega_j\}, \epsilon)$
2. **Fit coefficients of ISPs by sampling solutions of** $\{l_i^2 = 0\}$

Example : Planar Double Box



RSPs :

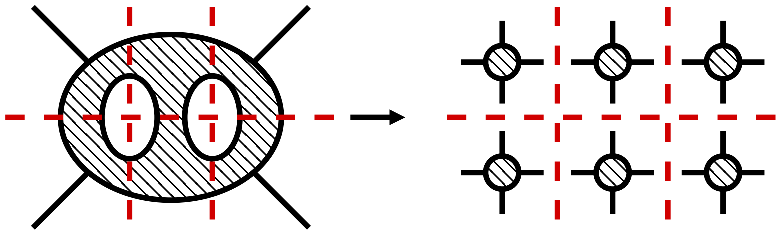
$$2k_1 \cdot p_1 = -(l_1 - p_1)^2 + l_1^2$$
$$= -l_3^2 + l_1^2$$

We will find :
32 coefficients in Δ
6 solutions to $\{l_i^2 = 0\}$

- ISPs $= \{k_1 \cdot p_4, k_2 \cdot p_1, k_1 \cdot \omega, k_2 \cdot \omega\}$
- $\Delta = c_0 + c_1 k_1 \cdot p_4 + \cdots + c_{16} k_1 \cdot \omega + \ldots$

Generalized Unitarity Cuts

- For $\{l_i^2 = 0\}$ the integrand factorizes into on-shell tree-level amplitudes



Algorithm to fit a generic integrand

- Parametrize the full set of on-shell solutions, $l_i^{(s)}(\tau_1, \ldots, \tau_p)$
- Identify the ISPs on each solution:

$$k_i \cdot p_j = f_{ij}(\tau_1, \ldots, \tau_p)$$

- Construct and solve the resulting linear system:

$$\Delta^{(s)}(\tau_1, \ldots, \tau_p) = \sum d_a \tau_1 \ldots \tau_p$$

$$\boxed{\mathbf{M} \cdot \vec{c} = \vec{d}}$$

Integrand Reduction

- Top-down approach
- Subtract previously determined poles, e.g.

$$\Delta_{6;\text{tri}|\text{box}} = \prod_{i=1}^{5} A_i^{(0)} - \frac{\Delta_{7;\text{box}|\text{box}}}{(k_1 - p_1)^2} = \sum_{i,j} d_{ij} \tau_i \tau_j$$

- Fitting can be done numerically or analytically
- Total number of topologies is still very large. . . .
- Towards automation:
  Solving the non-linear integrand constraints using algebraic geometry
  [Zhang arXiv:1205.5705]
- Public Mathematica code `BasisDet` [http://www.nbi.dk/~zhang/BasisDet.html]

An Algorithm for the Integrand Basis

- $B = \{v_1, v_2, v_3, v_4\}$, $[G_4]_{ij} = v_i \cdot v_j$, $P = \{l_1^2, \ldots, l_p^2\}$
- Gram matrix $[G_4]_{ij} = v_i \cdot v_j$. to re-write scalar products:

$$a \cdot b = (a \cdot v_1 \;\; a \cdot v_2 \;\; a \cdot v_3 \;\; a \cdot v_4) \;\; G_4^{-1} \begin{pmatrix} b \cdot v_1 \\ b \cdot v_2 \\ b \cdot v_3 \\ b \cdot v_4 \end{pmatrix} \tag{1}$$

- Re-write P using (1) $\Rightarrow$ set of equations for the scalar products.
- $\{P_i = 0\}$ has linear parts (RSPs) non-linear parts : ISP constraints $= I$
- Construct general ISP polynomial using renormalization constraints $= R$
- Remove I from R $(R/I) \Rightarrow$ Integrand Basis $= \Delta(ISPs)$.
  - Carried out using Gröbner bases and polynomial division [Buchberger (1976)]

Solving the On-Shell Constraints

- **primary decomposition of ideals** to identify all on-shell solutions

  [Lasker-Noether theorem (1905,1921)]

- Decompose $Z(I) \sim \{I = 0\}$ into a finite number of irreducible components
  - e.g. consider $I = \{x^2 - y^2\}$
    $I = \{x + y\} \cup \{x - y\} \Rightarrow Z(I) = \{x + y = 0\} \cup \{x - y = 0\}$
- Available in the public `Macaulay2` program [http://www.math.uiuc.edu/Macaulay2/]

  [Mathematica interface by Yang Zhang https://bitbucket.org/yzhphy/mathematicam2]

All of this applies to higher loops as well!

- Useful for studying the geometric structure of complicated multi-loop topologies

  [Huang, Zhang arXiv:1302.1023 JHEP 1304:080 (2013)]

BasisDet and M2

```
<< "/Users/simon/gitrepos/BasisDet/BasisDet-1-02.m"
<< "/Users/simon/gitrepos/mathematicaM2/mathematicaM2.m"

MathematicaM2 package, version 0.97, by Yang Zhang

Macaulay2 path: /Applications/Macaulay2-1.4/bin/M2

Path of temporary files: /Users/simon/gitrepos/mathematicam2/tmp

L = 2; (* Two-Loops *)
Dim = 4; (* 4 Dimensions*)
n = 4; (* 4 external legs *)
ExternalMomentaBasis = {p1, p2, p4}; (* vectors for real space*)
Kinematics = {
   p1^2 → 0, p2^2 → 0, p4^2 → 0, ω1^2 → -t*(s+t)/s,
   p1*p2 → s/2, p1*p4 → t/2, p2*p4 → -(s+t)/2};
numeric = {s → 11, t → 7}; (* Numerical replacements rules *)
Props = {l1, l1 - p1, l1 - p1 - p2, l2, l2 - p4, l2 + p1 + p2, l1 + l2};
RenormalizationCondition = {{{1, 0}, 4}, {{0, 1}, 4}, {{1, 1}, 6}};

GenerateBasis[1]
```

BasisDet and M2

```
Physical spacetime basis is {p1, p2, p4, ω1}
Number of irreducible scalar products: 4
Irreducible Scalar Products:{x14, x24, x13, x21}
Cut equations for ISP are listed in the variable 'CutEqnISP'
Possible renormalizable terms: 160

The basis contains 32 terms, which are listed in the variable 'Basis'
The explicit form of the integrand is listed in the variable 'Integrand'
Number of spurious terms: 16, listed in the variable 'SpuriousBasis'
Number of non-spurious terms: 16, listed in the variable 'NSpuriousBasis'
Time used: 1.39874 seconds
```

ISP (* List of ISPs *)

{x14, x24, x13, x21}

CutEqnISP (* Reduced on-shell equations for ISPs *)

$$\{-t^2+4\,t\,x13-4\,x13^2+4\,x14^2,\ -t^2+4\,t\,x21-4\,x21^2+4\,x24^2,$$
$$-4\,t\,x13\,x21+s\left(-x13^2+x14^2-2\,x13\,x21-x21^2+2\,x14\,x24+x24^2\right)\}$$

BasisDet and M2

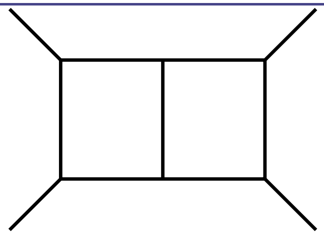
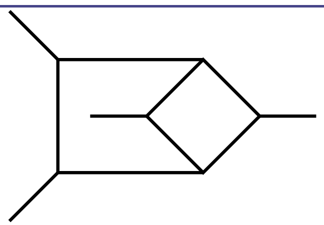
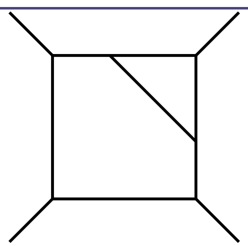
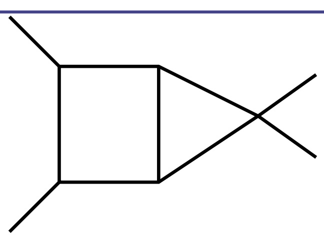
```
PrimaryDecomposition[CutEqnISP /. numeric, ISP, NumberField → "QQ"]
Print["Number of branches = ", Length[%]]

{{-7 + 2 x13 + 2 x14, -77 + 22 x13 + 22 x21 + 4 x13 x21, -7 + 2 x21 + 2 x24},
 {7 - 2 x13 + 2 x14, -77 + 22 x13 + 22 x21 + 4 x13 x21, 7 - 2 x21 + 2 x24},
 {x13, 7 + 2 x14, -7 + 2 x21 + 2 x24}, {x13, -7 + 2 x14, 7 - 2 x21 + 2 x24},
 {7 - 2 x13 + 2 x14, x21, -7 + 2 x24}, {-7 + 2 x13 + 2 x14, x21, 7 + 2 x24}}

Number of branches = 6
```

BasisDet and M2

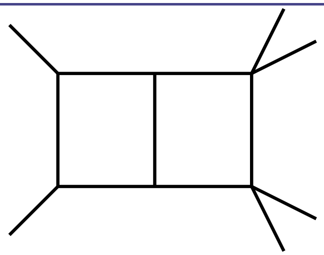
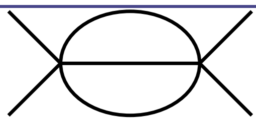
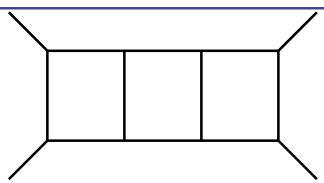
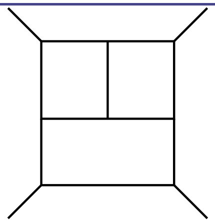
```
L = 3;
Dim = 4;
n = 4;
ExternalMomentaBasis = {p1, p2, p4};
Kinematics = {p1^2 → 0, p2^2 → 0, p4^2 → 0, p1 p2 → s / 2, p1 p4 → t / 2, p2 p4 → - (s + t) / 2,
   ω1^2 → -t * (s + t) / s};
numeric = {s → 11, t → 3};
Props = {l1, l1 - p1, l1 - p1 - p2, l3 + p1 + p2, l2 + p1 + p2, l2 - p4, l2, l3, l1 + l3, l2 - l3};
RenormalizationCondition = {{{1, 0, 0}, 4}, {{0, 1, 0}, 4}, {{0, 0, 1}, 4}, {{0, 1, 1}, 6},
   {{1, 1, 0}, 6}, {{1, 1, 1}, 8}};
GenerateBasis[0]
```

```
Physical spacetime basis is {p1, p2, p4, ω1}

Number of irreducible scalar products: 7

Irreducible Scalar Products:{x14, x24, x34, x13, x33, x21, x31}

Cut equations for ISP are listed in the variable 'CutEqnISP'

Possible renormalizable terms: 3385


The basis contains 398 terms, which are listed in the variable 'Basis'

The explicit form of the integrand is listed in the variable 'Integrand'

Number of spurious terms: 199, listed in the variable 'SpuriousBasis'

Number of non-spurious terms: 199, listed in the variable 'NSpuriousBasis'

Time used: 51.761 seconds
```

A Few Examples

Topology	ISPs (non-spurious+spurious)	$\|\Delta\|$(non-sp.+sp.)	#branches(dimension)
	2+2	$32(16+16)$	$6(1)$
	2+2	$38(19+19)$	$8(1)$
	2+1	$20(10+10)$	$2(2)$
	1+4	$69(18+51)$	$4(2)$

A Few Examples

Topology	ISPs (non-spurious+spurious)	$\|\Delta\|$(non-sp.+sp.)	#branches(dimension)
	2+2	$32(16+16)$	$4(1)$
	2+6	$42(12+30)$	$1(5)$
	4+3	$398(199+199)$	$14(2)$
	5+3	$584(292+292)$	$12(2)+4(3)$

Further Reduction to MIs via IBPs

- The integrand representation contains hundreds of integrals
- From this form we can apply further identities from conventional IBPs

[Tkachov, Chetyrkin (1980)]

- Public codes :

[AIR: Anastasiou, Lazopoulos (2004)]

[FIRE: Smirnov ,Smirnov (2008)][FIRE4: Smirnov ,Smirnov (2013)]

[Reduze2: Studerus, von Manteuffel (2009-2011)][LiteRed: Lee (2012)]

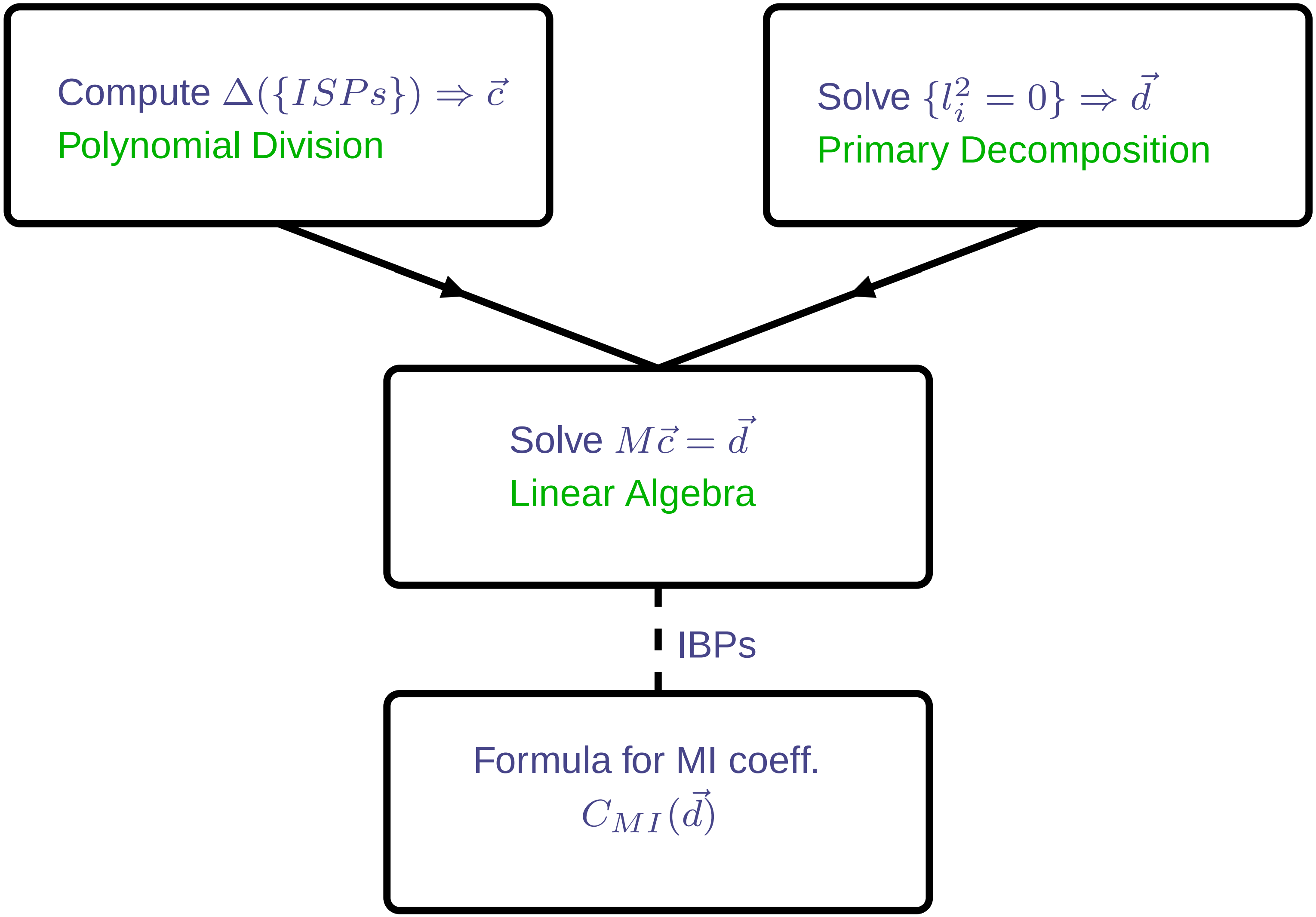
$$A_n^{(2)} = \int\int \frac{d^4k_1}{(4\pi)^2}\frac{d^4k_2}{(4\pi)^2}\frac{\vec{C}\cdot\vec{B}}{\prod_{i=1}^{n} l_i(k_1,k_2)}$$

solution to system of IBPs :

$$\int\int \vec{B} = M_{IBP}\cdot\int\int \vec{B}'$$

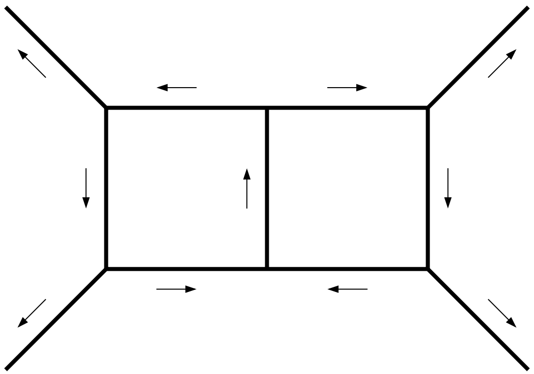
$$A_n^{(2)} = \vec{C}\cdot M_{IBP}\cdot\int\int \frac{d^4k_1}{(4\pi)^2}\frac{d^4k_2}{(4\pi)^2}\frac{\vec{B}'}{\prod_{i=1}^{n} l_i(k_1,k_2)}$$

Integrand Reduction Procedure

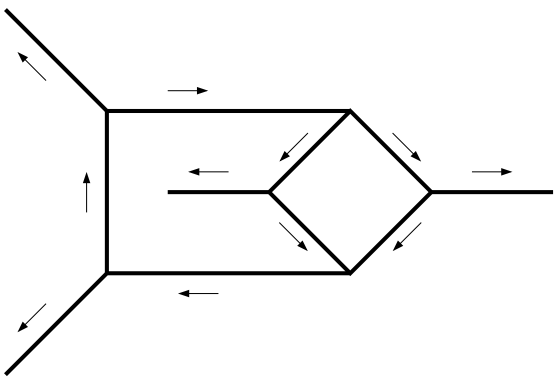


Applications and Tests

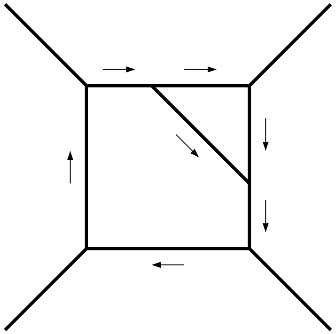
- Two-loop Hepta-cuts: planar and non-planar [SB, Frellesvig, Zhang arXiv:1202.2019]
- IBPs with `FIRE` [AV Smirnov,VA Smirnov]
- General analytic formulae for the MI coefficients
  [c.f. planar double box Kosower, Larsen arXiv:1108.1180]
- Check $gg \to gg$ scattering with adjoint fermions and scalars
  [Full agreement with Bern, De-Freitas, Dixon (2002)]



38×32 system, 2 MIs



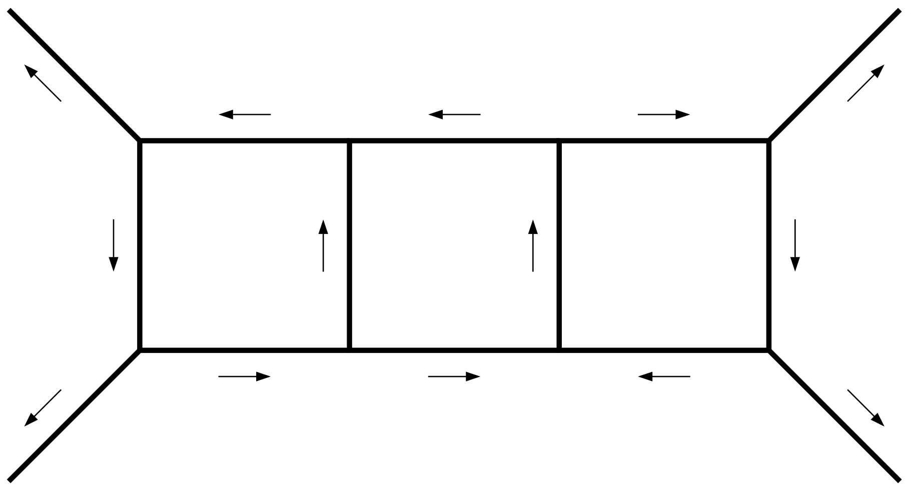
48×38 system, 2 MIs



20×20 system, no MIs

Application at Three Loops

- Planar triple box [SB, Frellesvig, Zhang arXiv:1207.2796]
- IBPs with Reduze2 [Studerus, von Manteuffel]
- General analytic formulae for the MI coefficients
- 14 branches of the on-shell solutions
- New results valid in non-supersymmetric theories (QCD)



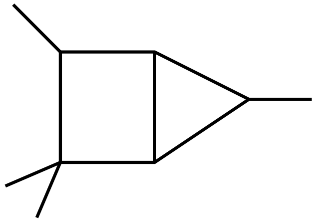
622×398 system, 3 MIs(!)

Complications with the 4D Basis

- Generalized unitarity for integrand reduction requires $I = \sqrt{I}$

> The ideal must be radical: e.g. $I = \{(x-y)^2\} \neq \sqrt{I} = \{x - y\}$
> $Z(I) = \{x - y = 0\} \Rightarrow x(\tau) = y(\tau) = \tau$
> let $f = c_0 + c_1(x-y) + c_2(x-y)^2,\ f \in R$
> and $[f] = c_0 + c_1(x-y)^2,\ [f] \in R/I$
> $\Rightarrow c_1$ cannot be extracted on the cut solution.

- $I \neq \sqrt{I} \Rightarrow \mathrm{rank}(M) > \dim(\vec{d})$
- Occurs in 4D systems e.g. tri|box

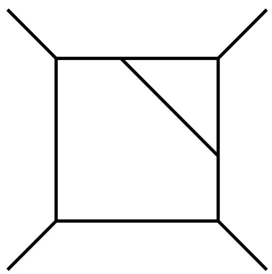


$$I = \{-x_{13}^2 + x_{14}^2,\ -x_{23}^2 + x_{24}^2,$$
$$-x_{13}^2 + x_{14}^2 - x_{23}^2 + x_{24}^2 - 2\,(x_{13}x_{23} - x_{14}x_{24})\}$$
$$\sqrt{I} = \{-x_{13}^2 + x_{14}^2,\ -x_{23}^2 + x_{24}^2, x_{14}x_{23} - x_{13}x_{24}, -x_{13}x_{23} + x_{14}x_{24}\}$$

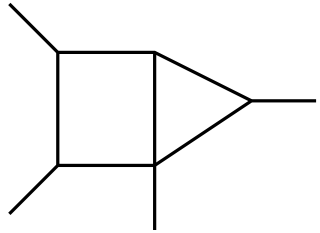
system fails at rank 2

Complications with the 4D Basis

- Loop parametrizations can degenerate over different multiplicities e.g. tri|pentagon and tri|box topologies



$$I_7 = I_{7;1} \bigcap I_{7;2} \qquad \dim(I_7; k) = 2$$



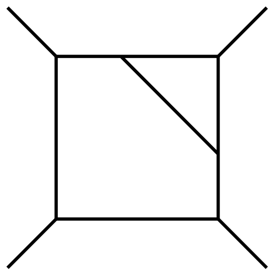
$$I_6 = I_{6;1} \bigcap I_{6;2} \bigcap I_{6;3} \bigcap I_{6;4} \bigcap I_{7;1} \bigcap I_{7;2}$$

On branches 5 and 6 we cannot form a polynomial on the cut integrand

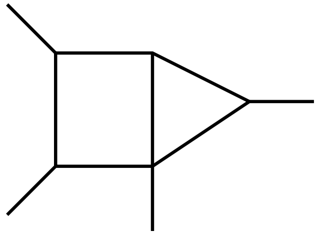
$$\Delta_6(k_1^{(s)}, k_2^{(s)}) = \prod_{i=1}^{5} A^{(0)}(k_1^{(s)}, k_2^{(s)}) - \frac{\Delta_7(k_1^{(s)}, k_2^{(s)})}{D_7^{(s)}} \to \infty$$

Complications with the 4D Basis

- Loop parametrizations can degenerate over different multiplicities e.g. tri|pentagon and tri|box topologies



$$I_7 = I_{7;1} \bigcap I_{7;2} \qquad \dim(I_7; k) = 2$$



$$I_6 = I_{6;1} \bigcap I_{6;2} \bigcap I_{6;3} \bigcap I_{6;4} \bigcap I_{7;1} \bigcap I_{7;2}$$

On branches 5 and 6 we cannot form a polynomial on the cut integrand

$$\Delta_6(k_1^{(s)}, k_2^{(s)}) = \prod_{i=1}^{5} A^{(0)}(k_1^{(s)}, k_2^{(s)}) - \frac{\Delta_7(k_1^{(s)}, k_2^{(s)})}{D_7^{(s)}} \to \infty$$

D-dimensional Cuts

- Issues in 4D can be circumvented in $4 - 2\epsilon$ dimensions
- Separate 4D from extra dimensions:

$$k_i^\nu = \bar{k}_i^\nu + \mu_i^\nu \qquad \int d^{4-2\epsilon} k_i = \int d\mu_i^{-2\epsilon} \int d^4 \bar{k}_i$$

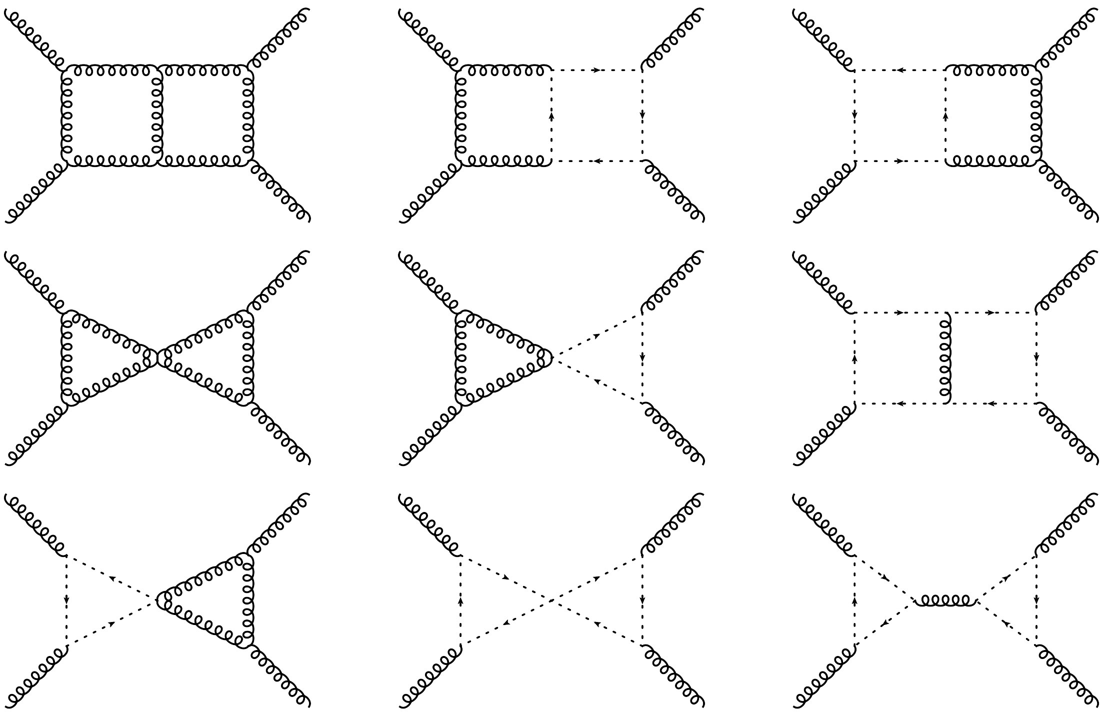
- At 2-loops we have 3 additional parameters:

$$\mu_{11} = \mu_1^2,\ \mu_{22} = \mu_2^2,\ \mu_{12} = 2\mu_1 \cdot \mu_2.$$

- D-dimensional system is larger but simpler to solve:
  - All ideals are prime $\Rightarrow I = \sqrt{I}$ and the is only one branch.
  - All linear systems are maximum rank: $\mathrm{rank}(M) = \dim(\vec{d})$
  - All on-shell systems have $11 - \#(\text{propagators})$ free parameters

Example: $4g$ $++++$ Amplitude

[Bern, Dixon, Kosower (2000)]



$D_1 = k_1^2$

$D_2 = (k_1 - p_1)^2$

$D_3 = (k_1 - p_{12})^2$

$D_4 = k_2^2$

$D_5 = (k_2 - p_4)^2$

$D_6 = (k_2 + p_{12})^2$

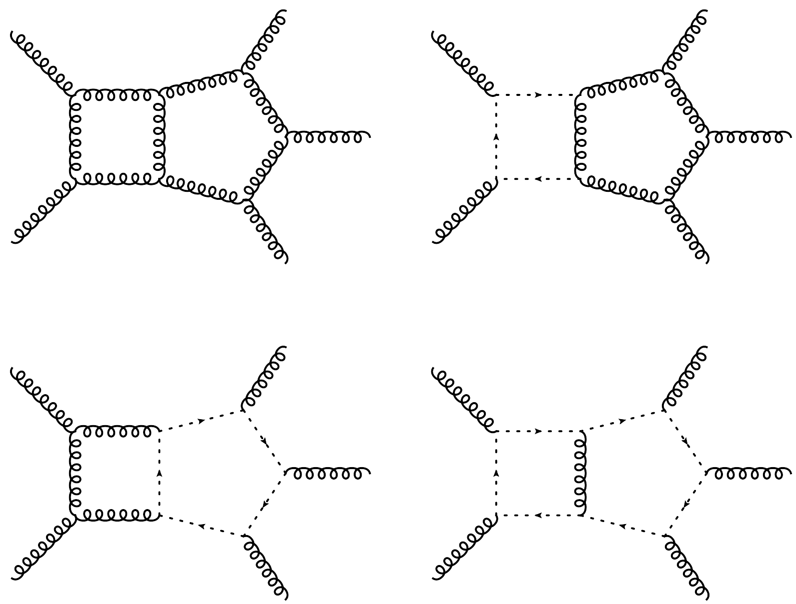
$D_7 = (k_1 + k_2)^2$

$$\mathcal{I}^{(2)}(1^+, 2^+, 3^+, 4^+)(k_1, k_2) = \frac{i}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{\Delta_{7;12*34*} + D_7 \Delta_{6;12[*,*]34}}{D_1 D_2 D_3 D_4 D_5 D_6 D_7}$$

$$\Delta_{7;12*34*} = -s_{12}^2 s_{23} \left((D_s - 2) \left(\mu_{11}\mu_{22} + \mu_{11}\mu_{33} + \mu_{22}\mu_{33} \right) + 4 \left(\mu_{12}^2 - 4\mu_{11}\mu_{22} \right) \right)$$

$$\Delta_{6;12[*,*]34} = -2(D_s - 2) s_{12} s_{23} \left(\mu_{11} + \mu_{22} \right) \mu_{12} - (D_s - 2)^2 s_{23} \left(\mu_{11}\mu_{22} k_1 \cdot k_2 + s_{12} \right)$$

Progress Towards $5g$ $+++++$ Amplitude



$$\mu_{33} = \mu_{11} + \mu_{22} + \mu_{12}$$

$$\mathcal{I}^{(2)}(1^+,2^+,3^+,4^+,5^+)(k_1,k_2) = \frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\,\frac{\Delta_{8;12*34*}+\ldots}{D_1 D_2 D_3 D_4 D_5 D_6 D_7 D_8}$$

$$\Delta_{8;123*45*} = \left((D_s-2)\left(\mu_{11}\mu_{22}+\mu_{11}\mu_{33}+\mu_{22}\mu_{33}\right)+4\left(\mu_{12}^2-4\mu_{11}\mu_{22}\right)\right)$$

$$\times\left(\left(s_{12}s_{23}s_{45}+\frac{s_{12}(s_{23}-s_{15})-s_{23}s_{34}+(s_{15}+s_{34})s_{45}}{\mathrm{tr}_5(1234)}\right)(k_1\cdot p_5)-\frac{s_{12}s_{23}s_{34}s_{45}^2 s_{51}}{\mathrm{tr}_5(1234)}\right)$$

Outlook

- A few small steps towards automated multi-loop amplitudes
- Computational algebraic geometry for integrand reduction
  - Efficient tools for solving unitarity cut equations
  - Generalizes easily to D-dimensional systems

[http://www.nbi.dk/~zhang/BasisDet.html]

- Full computations for $2 \to 3/4$ process should be feasible
- We didn't address the evaluation of the Master Integrals
- IBPs with many scales are quite challenging:
  - massive amplitudes, higher multiplicity,. . .