One loop integration with hypergeometric series by using recursion relation

N.Watanabe(KEK)
Collaborated with T.Kaneko(KEK)
ACAT 2013, Beijing, 2013 May 16 - May 21

Introduction

- The N-loop n-point loop integrals are expressed in GKZ(Gelfand-Zelevinsky-Karpanov) hypergeometric functions.

  ⇒Hypergeometric functions will be useful!

- GKZ hypergeometric funcs. too general to calculate
- We interested in that 1-loop, 2-loop, 3-loop, ... integrals are corresponding to what subclass of GKZ.
  - Analytical properties?
  - How to calculate?

Introduction

- Classes of hypergeometric function
  - Gauss' hypergeometric function F (1 variable)
  - Generalized hypergeometric function ${}_pF_q$ (1 variable)
  - Appell's function F_1, F_2.. (2 variables)
  - Lauricella's function F_D (n variables)
  - Aomoto, Gelfand, ..., etc
- A rich identity of these functions are known from the power series expansion, integral representation, differential equation.

Introduction

- Representation with hypergeometric functions
  - Regge(1969,a class of generalized hypergeometric equations)
  - Tarasov et al., Davydychev, Kalmykov,...(1-, 2-loop, ...)
  - Duplančić and Nižić, Kurihara, (1-loop, for massless QCD)
- This work

The one loop n-point functions are exactly expressed in terms of some set of hypergeometric functions.

Introduction

Motivations

- We want to know the general expressions for one loop integrals.
  - Dimensions are treated as the parameter
  - Masses of the propagators are expressed arbitrary
  - Momenta of external legs are also arbitrary
  - N-point and tensor integrals can be treated
- What is the analytical properties of one loop?
- We also want to know how to treat these function by analytical or numerical method.

One-loop integrals

- Consider the (n+1)-point function
  - Denominator of the loop integral is

$$\sum_j [(l+r_j)^2 - m_j^2] x_j \ = |x|\Big(l+\sum_j r_j \frac{x_j}{|x|}\Big)^2 - \frac{1}{|x|}\Big[-\frac{1}{2}\sum_{j,k}(r_j-r_k)^2 x_j x_k + |x|\sum_j m_j^2 x_j\Big]$$

l:loop momentum, r:external momenta
m:masses, x:Feynman propagators

Loop momentum shift and integrate

- One-loop functions is expressed

$$I_{n+1}(s) = \int_{\Delta_n} d^n x\, \mathcal{D}_n^s$$

$$\mathcal{D}_n(x_1,...,x_n) = \frac{1}{2}\sum_{i,j=1}^{n} A_{ij}x_i x_j + \sum_i B_i x_i + C \ = \frac{1}{2}(Ax,x) + (B,x) + C$$

Bernstein theorem

- This theorem say that *P* and *b* functions are existed for arbitrary polynominal function

$$P(s)f^{s+1}=b(s)f^{s}$$

operator **polynominal**

- For one-loop integral case

F. V. Tkachov (1997)

$$P=-(s+1)\frac{1}{E_n}+\frac{1}{2E_n}(A^{-1}\partial\mathcal{D}_n,\partial)$$

P-operator

where $E_n=\frac{1}{2}(A^{-1}B,B)-C$

$$\boxed{P\mathcal{D}_n^{s+1}=(s+1)\mathcal{D}_n^{s}}$$

b-functions

Formulation for calculation

- Integral *I(s)* is related to different dimension *I(s+1)* by using Bernstein theorem

$$I_{n+1}(s) = \frac{1}{s+1}\int_{\Delta_n} d^n x \; P\mathcal{D}_n^{s+1}$$

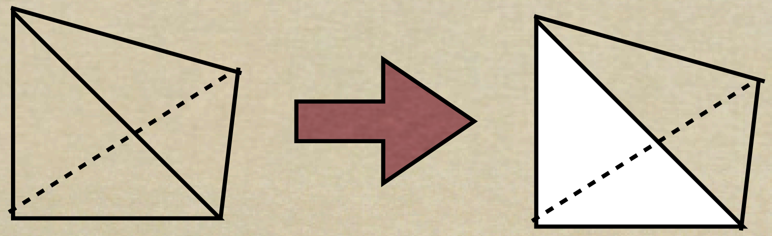
$$= \underline{\frac{1}{2(s+1)E_n} F_n(s)} \underline{- \frac{(s+n/2+1)}{(s+1)}\left(-\frac{1}{E_n}\right) I_{n+1}(s+1)}$$

Repeat again

$\vdots$ **m times operations**

$$= \frac{1}{2(s+1)E_n}\sum_{j=0}^{m-1}\frac{(s+n/2+1)_j}{(s+2)j}\left(-\frac{1}{E_n}\right)^j F_n(s+j) + \frac{(s+n/2+1)_m}{(s+1)_m}\left(-\frac{1}{E_n}\right)^m I_{n+1}(s+m)$$

Cont.



- The surface term F

$$F_n(s) := \int_{\Delta_n^n} d^n x \sum_j \partial_j \{ (A^{-1}\partial \mathcal{D}_n)_j \mathcal{D}_n^{s+1} \}$$

$$= -\sum_{j=0} \underline{h_j} I_{n,j} \qquad \textit{where} \;\; I_{n,j} := \int_{\Delta_{n;j}} \mathcal{D}_n^{s+1}(x_j = 0)\, d\eta_j$$

The coefficients h are determined by only kinematics

- The n+1-pt func. can be related to infinite sum of the n-pt func.

$$I_{n+1}(s) = \frac{1}{2} \sum_{j=0}^{\infty} \frac{(s+n/2+1)_j}{(s+1)_{j+1}} \left(-\frac{1}{E_n} \right)^{j+1} \sum_{k=0}^{n} h_k I_{n,k}(s+j+1)$$

$$\underline{+ \frac{(s+n/2+1)_m}{(s+1)_m} \left(-\frac{1}{E_n} \right)^m I_{n+1}(s+m) \Big|_{m\to\infty}}$$

Integration part vanishes

Cont.

- Iterating operations, n+1-pt func. can be reduced to 1-pt func.

kinematical factor

$$I_{\sigma_n}(s) = \boxed{\frac{1}{2^n}\sum_{k_n=0}^{n}\sum_{k_{n-1}=0}^{n-1}\cdots\sum_{k_1=0}^{1}\mathcal{D}(\rho(\sigma_n;k_n,k_{n-1},...,k_1))^s h_{\sigma_n,k_n}h_{\rho(\sigma_n;k_n),k_{n-1}}\cdots h_{\rho(\sigma_n;k_n,...,k_2),k_1}}$$

$$\boxed{\begin{gathered}\sum_{j_n=0}^{\infty}\sum_{j_{n-1}=0}^{\infty}\cdots\sum_{j_1=0}^{\infty}\frac{(s+n/2+1)_{j_n}}{(s+1)_{j_n+1}}\frac{(s+j_n+(n-1)/2+2)_{j_{n-1}}}{(s+j_n+2)_{j_{n-1}+1}}\cdots\frac{(s+j_n+\cdots+j_2+1/2+n)_{j_1}}{(s+j_n+\cdots+j_2+n)_{j_1+1}} \\ \left(-\frac{\mathcal{D}(\rho(\sigma_n;k_n,k_{n-1},...,k_1))}{E_{\sigma_n}}\right)^{j_n+1}\left(-\frac{\mathcal{D}(\rho(\sigma_n;k_n,k_{n-1},...,k_1))}{E_{\rho(\sigma_n;k_n)}}\right)^{j_{n-1}+1}\cdots\left(-\frac{\mathcal{D}(\rho(\sigma_n;k_n,k_{n-1},...,k_1))}{E_{\rho(\sigma_n;k_n,...,k_2)}}\right)^{j_1}\end{gathered}}$$

Hypergeometric function

- The n-point func. is denoted by a linear combination of n! these hypergeometric functions.

Another approach

- In another approach, we can prove this method.

$$\frac{d}{dz} z^{\gamma-1} F(1,\beta,\gamma;z) = (\gamma-1) z^{\gamma-2} F(1,\beta,\gamma-1;z)$$

differential eq.

$$\longrightarrow \quad \gamma = s+2 \qquad z := -\frac{D_n}{E_n} = 1 - \frac{(A^{-1}\partial D_n, \partial D_n)}{2E_n}$$

$$\sum_k \partial_k[(A^{-1}\partial D_n)_k f(z)] = -2\left[(1-z)\frac{d}{dz} - \frac{n}{2}\right] f(z)$$

kinematical relation

$$\boxed{D_n^s = \frac{1}{2(s+1)E_n} \sum_k \partial_k \left[(A^{-1}\partial D_n)_k D_n^{s+1} F\left(1, s+\frac{n}{2}+1, s+2; -\frac{D_n}{E_n}\right)\right]}$$

- Denominators of integrand are written in a total differential

↓

Same formulae are obtained

G function

- Definition of G function

$$\boxed{G_n(\alpha,\beta;\gamma;x) = \sum_{j_n=0}^{\infty}\sum_{j_{n-1}=0}^{\infty}\cdots\sum_{j_1=0}^{\infty}\frac{\prod_{i=1}^{n}(\alpha_i)_{j_i}\prod_{k=1}^{n}(\sum_{\ell=k}^{n}\beta_\ell)_{\sum_{\ell=k}^{n}j_\ell}}{(\gamma)_{\sum_{i=1}^{n}j_i}\prod_{k=1}^{n}(\sum_{\ell=k}^{n}\beta_\ell)_{\sum_{\ell=k+1}^{n}j_\ell}\prod_{i=1}^{n}j_i!}\prod_{i=1}^{n}x_i^{j_i}}$$

  - If these functions are evaluable, all one loop integrations are calculable in any physical parameters
  - These functions have information of not only physical meaning but also numerical stabilities.

⇒What are these functions?

G function

- Identities for G functions can be constructed
  - Integration form ⟹One kind of Aomoto-Gelfand hypergeometric

$$G_n(\alpha,\beta;\gamma;x)=\frac{\Gamma(\gamma)}{\prod_{j=1}^{n}\Gamma(\alpha_j)\Gamma(\gamma-\sum_{k=1}^{n}\alpha_k)}\int_{\Delta_n}d^nu\,\prod_{k=1}^{n}u_k^{\alpha_k-1}\left(1-\sum_{j=1}^{n}u_j\right)^{\gamma-\sum_{k=1}^{n}\alpha_k-1}\underbrace{\prod_{k=1}^{n}\left(1-\sum_{j=1}^{k}x_ju_j\right)^{-\beta_k}}_{\text{linear equations}}$$

  - Recursion relations

$$G_n(\alpha,\beta;\gamma;x)=\frac{\Gamma(\gamma)}{\Gamma(\alpha_1)\Gamma(\gamma-\alpha_1)}\int_0^1dw\,w^{\alpha_1-1}(1-w)^{\gamma-\alpha_1-1}(1-x_1w)^{-\sum_{j=1}^{n}\beta_j}G_{n-1}(\alpha',\beta';\gamma';x')$$

  - Differential

$$\frac{\partial}{\partial x_\ell}G_n(\alpha,\beta;\gamma;x)=\sum_{k=\ell}^{n}\frac{\alpha_\ell\beta_k}{\gamma}G_n(\alpha+e_\ell,\beta+e_k;\gamma+1;x)$$

  - Differential equations ⟹information of singularities

G function

- Tensor integrals are covered in these functions.

$$G_n(\alpha,\beta;\gamma;x)=\frac{\Gamma(\gamma)}{\prod_{j=1}^{n}\Gamma(\alpha_j)\Gamma(\gamma-\sum_{k=1}^{n}\alpha_k)}\int_{\Delta_n}d^n u\prod_{k=1}^{n}u_k^{\alpha_k-1}\Bigl(1-\sum_{j=1}^{n}u_j\Bigr)^{\gamma-\sum_{k=1}^{n}\alpha_k-1}\prod_{k=1}^{n}\Bigl(1-\sum_{j=1}^{k}x_j u_j\Bigr)^{-\beta_k}$$

Tensor integral is appear from differentiating by masses

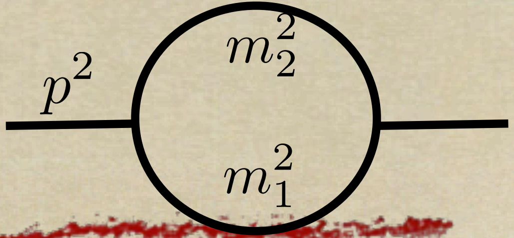
$$x_j D_n^s = \frac{1}{s+1}\frac{\partial}{\partial m_j^2}D_n^{s+1}$$

additional x appear

$$\alpha = (1,...,1) \Leftrightarrow \alpha \neq (1,...,1)$$

Scalar integral ⇔ Tensor integral

Example: 2-point function



- 2-point function is expressed in *G₁* function

$$G_1(\alpha, s+3/2; s+2; x_k) = \sum_{j=0}^{\infty} \frac{(\alpha)_j (s+3/2)_j}{(s+2)_j} \frac{x_k^j}{j!}$$

$\alpha = 1$ Scalar integral

$\alpha \neq 1$ Tensor integral

$$= F(\alpha, s+3/2; s+2; x_k)$$

α can be reduced to 1 by using some identities of hypergeometric func.

- G₁ functions can be expand around ε by multiple polylogarithm(MPL)

$$F(1, s+3/2; s+2; x_k) = (1-x_k)^{-1}(1+z_k)F(1,-s;s+2;z_k) \qquad z_k = \frac{x_k}{2-x_k+2\sqrt{1-x_k}}$$

⟷ identity

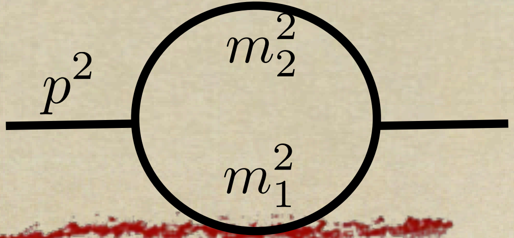
$$\boxed{s = -\epsilon \Longleftrightarrow d = 4-2\epsilon}$$

i.e.

$$F(1,\epsilon,2-\epsilon,z) = 1 + \epsilon\Big(\frac{z-(z-1)\log(1-z)}{z}\Big)$$
$$+\epsilon^2\Big(\frac{2(z-1)\mathrm{Li}_{1,1}(z,1) + (z-1) + \mathrm{Li}_2(z) + 2z + z(-\log(1-z)) + \log(1-z)}{z}\Big) + \mathcal{O}(\epsilon^3)$$

xsummer: S. Moch - P. Uwer(2006)

Example: 2-point function



- Scalar integral case

$$I_2(s) = \sum_{k=1}^{2} h_k G_1(1+s+3/2, s+1, x_k)$$

$$\vdots$$

$$x_k = \frac{p^2 m_k^2}{E}, \quad E = p^4 - 2p^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2$$

$$= (m_1^2)^s F_1(1, -s, -s; 2; 1/x_+, 1/x_-)$$

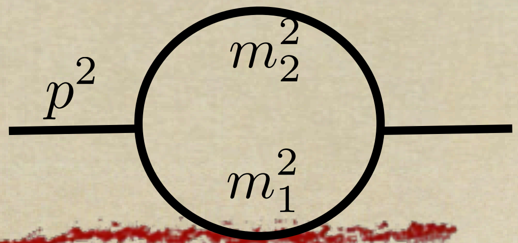
using identities: i.e.

$$F_1(1, \beta_1, \beta_2, 2 : x_1, x_2) = \frac{1}{1-\beta_2}\frac{1}{x_2} F\left(1, \beta_1, 2-\beta_2; \frac{x_1}{x_2}\right)$$
$$+ \frac{1}{1-\beta_2}\frac{x_2 - 1}{x_2}(1-x_1)^{-\beta_1}(1-x_2)^{-\beta_2} F\left(1, \beta_1, 2-\beta_2; \frac{x_1(x_2-1)}{x_2(x_1-1)}\right)$$

Result is consistent with direct analytical calculation $I_2(s) = \int_0^1 dx D^s$

Our method seems working well

Example: 2-point function



- Massless limit $\boxed{m_2^2 \to 0 \Rightarrow x_- \to 0, x_+ \to 1}$

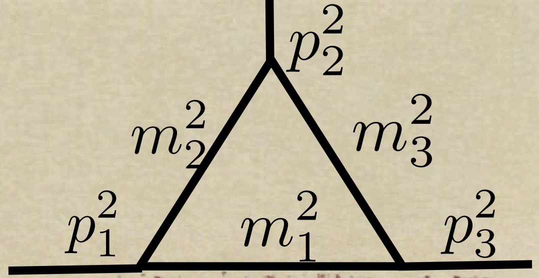
$$I_2(s) = \underbrace{\frac{x_-}{s+1}(m_1^2)^s F\left(1,-s,s+2;\frac{x_-}{x_+}\right)}_{\text{goes to zero}} + \underbrace{\frac{1-x_-}{s+1}(m_1^2)^s \left(\frac{(x_+-1)(x_--1)}{x_+x_-}\right)^s F\left(1,-s,s+2;\frac{1-x_-}{1-x_+}\right)}_{\text{not well defined}}$$

$$= \underbrace{\frac{1}{s+1}(m_2^2-p^2)^s F\left(s+1,-s,s+2;\frac{p^2}{p^2-m_2^2}\right)}_{\text{transformation of 2nd term, then massless limit can be taken}}$$

$$= (-p^2)^s B(s+1,s+1)$$

- We can select the good representation by using some identities.

Example: 3-point function

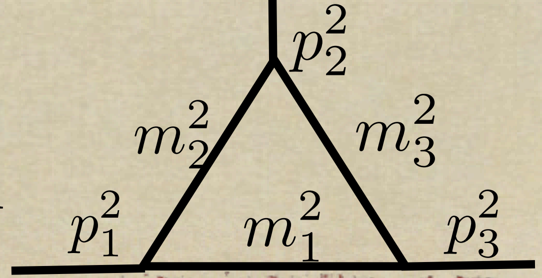


- The 3-point function can be written in a linear combination of G_2 functions.
- In general, G_2 function is expressed as Appell's F_3.

$$
\begin{aligned}
G_2((\alpha_1,\alpha_2),(1/2,s{+}2),s{+}3;x_1,x_2) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\alpha_1)_j(\alpha_2)_k(s+5/2)_{j+k}(s+2)_k}{(s+3)_{j+k}(s+5/2)_k j!\,k!}\,x_1^j x_2^k \\
&= (1-x_1)^{-\alpha_1}F_3\left(\alpha_1,\alpha_2,1/2,s+2,s+3;\frac{x_1}{x_1-1},x_2\right)
\end{aligned}
$$

  - A rich identity for F_3 can be used.

Example: 3-point function



- The scalar 3-point functions are written as

$$I_3(s) = \frac{1}{(s+1)(s+2)} \sum_{k_1=1}^{2} \sum_{k_2=1}^{3} \underline{h_{k_1,k_2}}\, G_2((1,1),(1/2,s+2);s+3,x_{(1,k_1,k_2)},x_{(2,k_1,k_2)})$$

kinematical factor

- Using some identities, G₂ can be expanded by MPL.

$$G_2((1,1),(1/2,s+2),s+3;x_1,x_2) = \frac{1}{1-x_2} F_1\left(1, s+\frac{3}{2}, 1, s+3; x_1, \frac{x_1-x_2}{1-x_2}\right)$$

new variables are related to original ones *x₁, x₂*

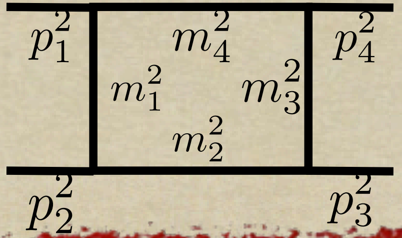
$$= \frac{1+z}{1-x_2} F_D\left(1; -s-1, 2(s+1), 1, 1; s+3; z, \frac{2z}{1+z}, \frac{1}{v_+}, \frac{1}{v_-}\right)$$

$$= \text{const.} \left\{ w_1\ F_1\left(s+2, -s-1, 1, s+3, \frac{z}{1+z}, w_1\right) \right.$$

$$\underline{\left. -w_2\ F_1\left(s+2, -s-1, 1, s+3, \frac{z}{1+z}, w_2\right) \right\}}$$

expanded by MPL

Example: 4-point function



- The 4-point functions are denoted by

$$G_3(\alpha,\beta;\gamma;x)=\sum_{j_3=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_1=0}^{\infty}\frac{(\beta_1+\beta_2+\beta_3)_{j_1+j_2+j_3}(\beta_2+\beta_3)_{j_2+j_3}(\beta_3)_{j_3}(\alpha_1)_{j_1}(\alpha_2)_{j_2}(\alpha_3)_{j_3}}{(\gamma)_{j_1+j_2+j_3}(\beta_1+\beta_2+\beta_3)_{j_2+j_3}(\beta_2+\beta_3)_{j_3}j_1!j_2!j_3!}\,x_1^{j_1}x_2^{j_2}x_3^{j_3}$$

- Construct many relation for G_3 (Integral form, recursion, ...etc)
- In the scalar case

$s=-2-\epsilon$ ⟷ $d=4-2\epsilon$

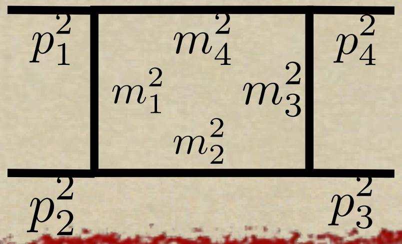
$$\alpha=(1,1,1),\quad \beta=\left(\frac{1}{2},\frac{1}{2},s+\frac{5}{2}\right),\quad \gamma=s+4,\quad s=-2-\epsilon$$

Integration form

$$G_3\left(\{1,1,1\},\left\{\frac{1}{2},\frac{1}{2},s+\frac{5}{2}\right\},s+4;x_1,x_2,x_3\right)=\frac{\Gamma(s+4)}{\Gamma(s+1)}\int_{\Delta_3}d^3u(1-u_1-u_2-u_3)^{s+1}$$

$$\times(1-x_1\ u_1)^{1/2}(1-x_2\ u_2)^{1/2}(1-x_3\ u_3)^{s+5/2}$$

Example: 4-point function



- The 4-point functions are denoted by

$$G_3(\alpha,\beta;\gamma;x) = \int_0^1 dw(s+3)(1-w)^{s+2}\left(1-wx_1\right)^{-s-\frac{7}{2}}$$

$$G_2\Big((1,1),(1/2,s+5/2);s+3;\Big(\frac{(1-w)x_2}{1-x_1w},\frac{(1-w)x_3}{1-x_1w}\Big)$$

$$= \int_0^1 dw\frac{\epsilon(1-w)^{-\epsilon}\left(1-wx_1\right)^{\epsilon-\frac{1}{2}}}{1-x_2}\Big(1-\frac{w\left(x_1-x_2\right)}{1-x_2}\Big)^{-1} F_1\left(1-2\epsilon;1,1;2-\epsilon;\frac{1+\sqrt{\eta}}{2},\frac{1-\sqrt{\eta}}{2}\right)$$

$$+\frac{\sqrt{\pi}(\epsilon-1)x_3^{\epsilon}\Gamma(1-\epsilon)}{(x_2-1)\sqrt{1-x_3}\Gamma\left(\frac{1}{2}-\epsilon\right)}F_1\left(1;1,\frac{1}{2};2;\frac{x_2-x_1}{x_2-1},\frac{x_3-x_1}{x_3-1}\right)$$

- General expressions are difficult, we can calculate up to order ε

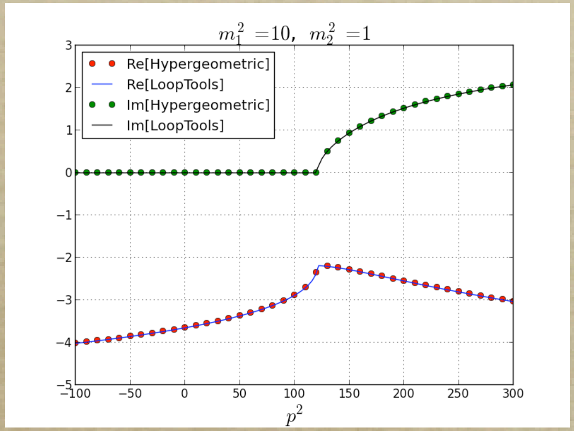
$$= \mathrm{const.}\left(F_D(1,\{1,-2\epsilon,-2\epsilon,2\epsilon,2\epsilon,1\};2;z_i(x_1,x_2,x_3))+\cdots\right)+\mathcal{O}(\epsilon^2)$$

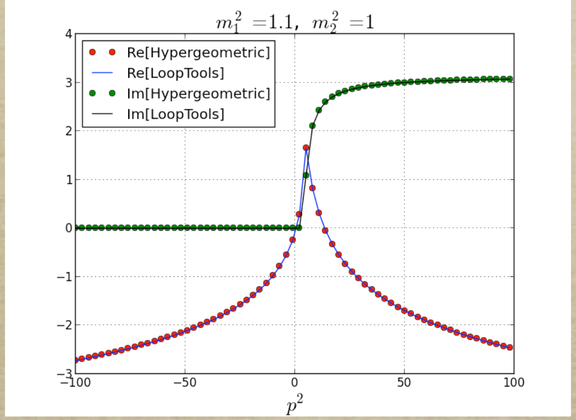
$$+\frac{2\sqrt{\pi}(\epsilon-1)x_3^{\epsilon}\Gamma(1-\epsilon)}{(x_2-1)\left(\sqrt{\frac{x_1-1}{x_3-1}}+1\right)\sqrt{1-x_3}\Gamma\left(\frac{1}{2}-\epsilon\right)}\ \log[(x_1,x_2,x_3)]$$

Cross check

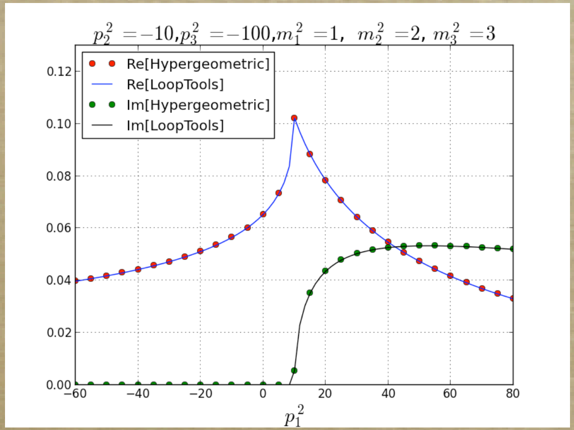
- Results are compared with LoopTools T. Hahn-M.Perez-Victoria (1999)

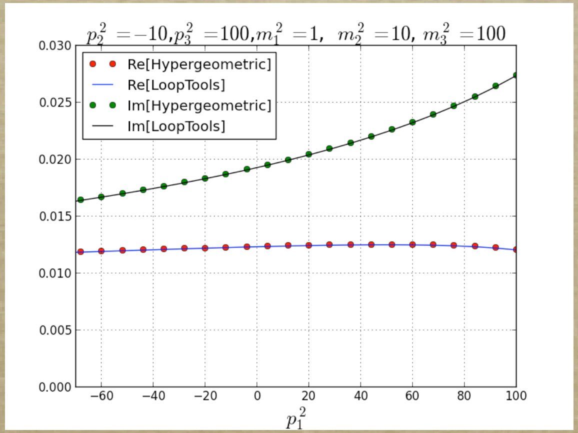
2-pt func.





3-pt func.





4-pt func. is not compared yet.

Summary and Outlook

- The general one loop integral with any physical parameter are written in a linear combination of Aomoto-Gelfand hypergeometric functions
  - Original integrand is quadratic eq. ⇒ product of linear eq.
- Many identities of hypergeometric funcs. are developed
- Our method is consistent with well-known results.

Outlook

- The general expressions for n≧4-point functions are constructed
- Extend this method for massless QCD loop with IR singularity.

Thank you !