Multiloop QCD: 20 years of continuous development

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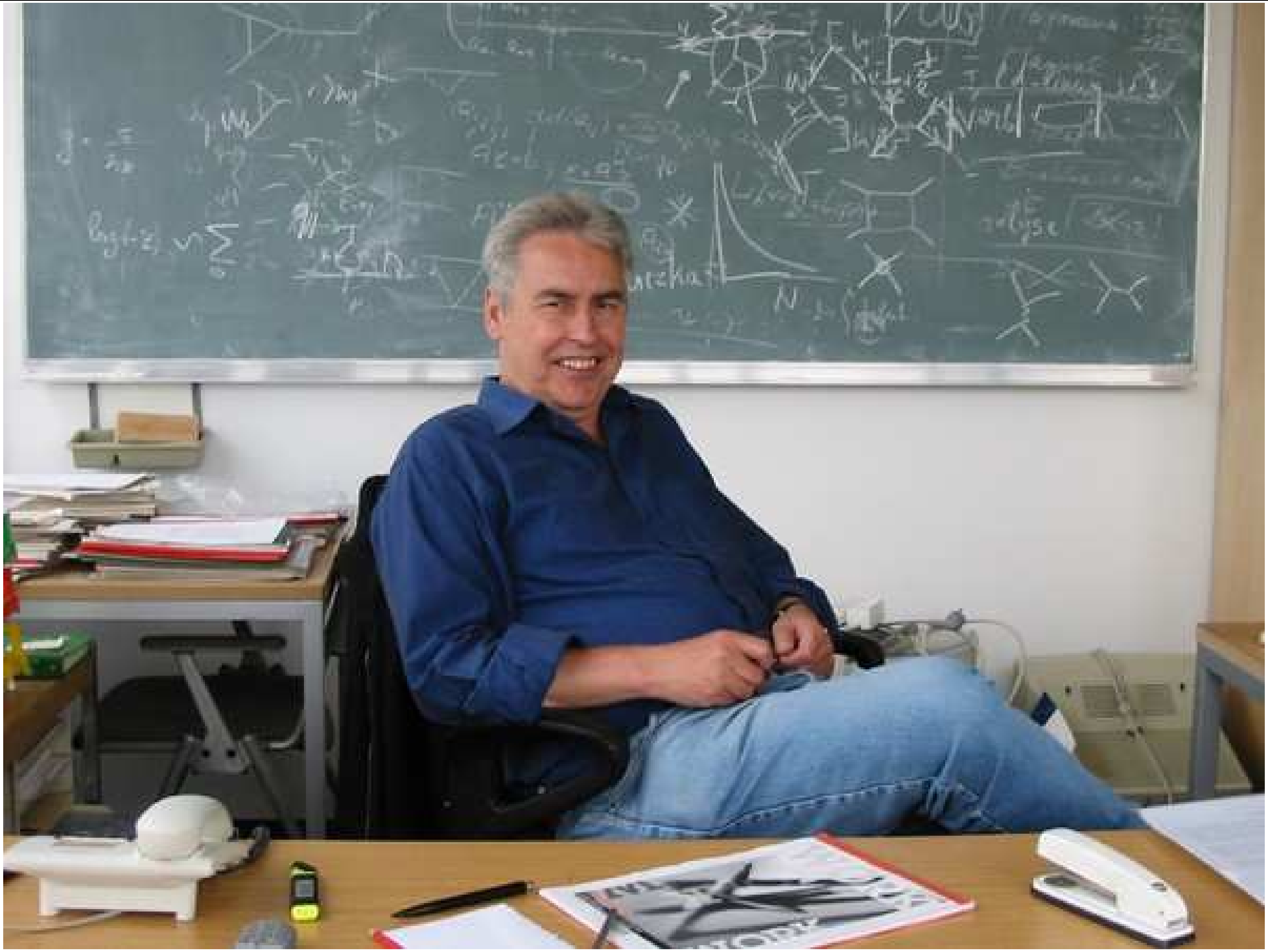
Possible and impossible in multiloop renormalization group

K.G. Chetyrkin

Talk at the 3rd International Workshop on Software Engineering, Artificial Intelligence and Expert systems for High-energy and Nuclear Physics, Oberammergau, Germany, 4-8 Oct 1993

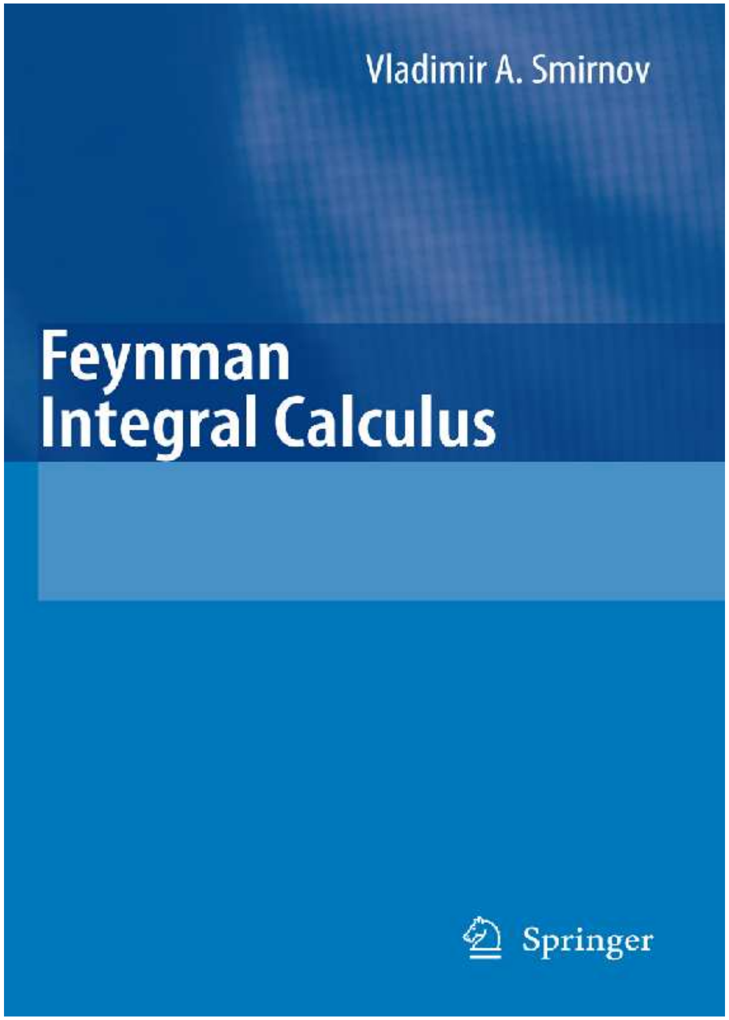
Prof. Dr. Jochem Burkhard Fleischer, 17 December 1937 - 1 April 2013

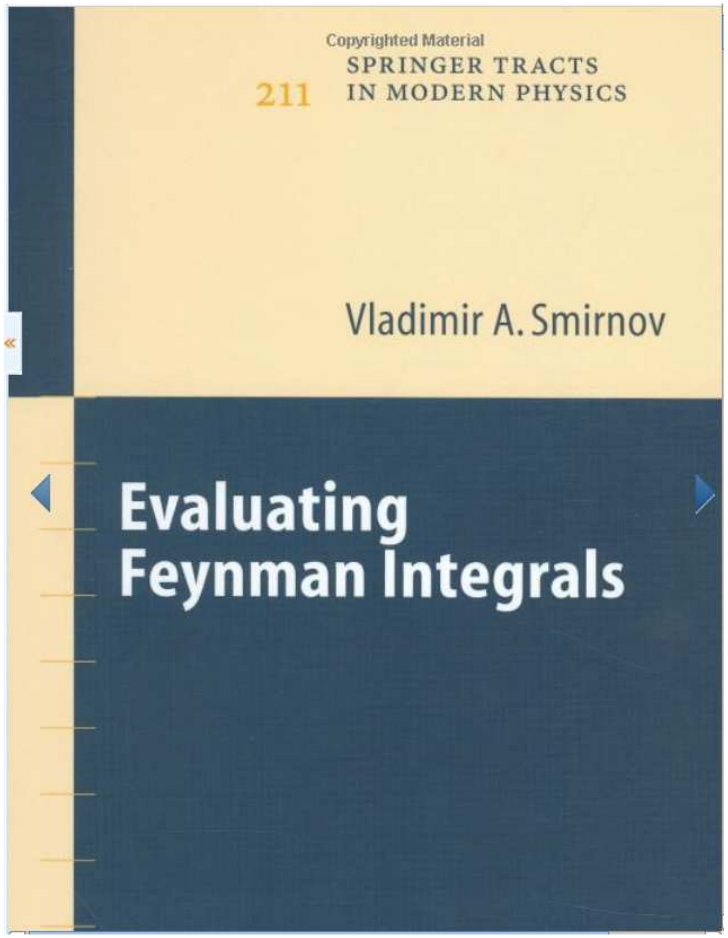


Last 30 years $\rightarrow$ revolution in our ability to deal with multiloop Feynman Integrals: Main (but not all!) ingredients:

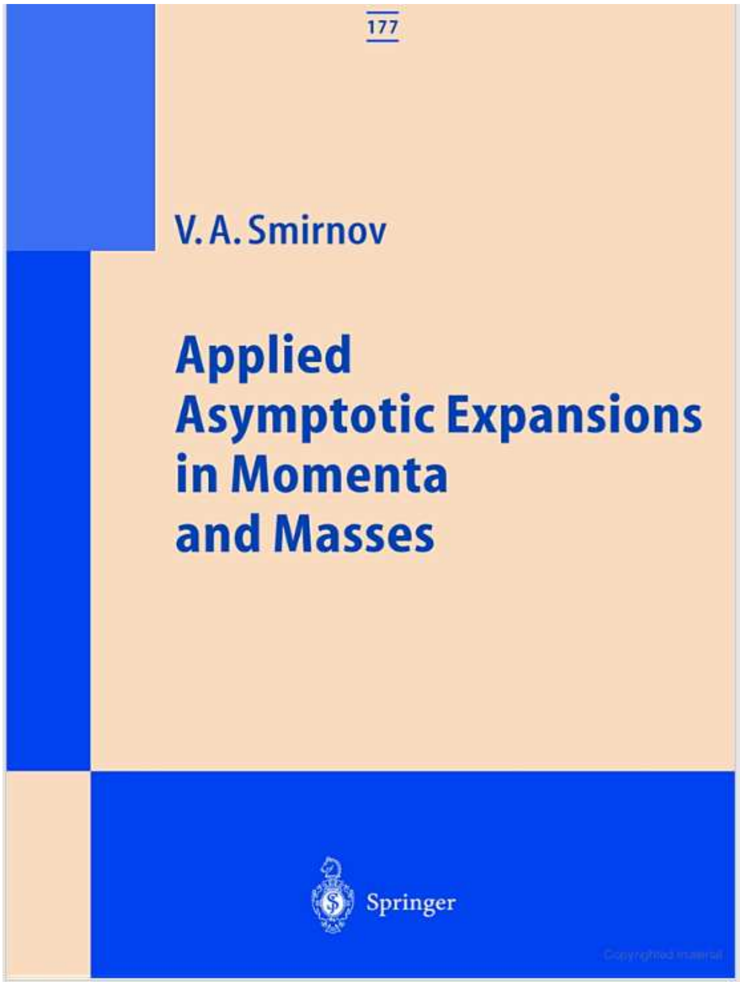
- Dim. Reg. /G. t' Hooft & M. Veltman (72)/
- "IBP" (Integration by Parts) method (see below)
- effective theories, various expansion of FI's, methods of regions /M. Beneke & V. Smirnov (1998) + . . . / and the extension principle /F. Tkachov (1982) . . . /
- "Mellin-Barns" /N. Ussyukina, A. Davyduchev . . . V. Smirnov . . . B. Tausk . . . /
- shifts and recurrence relations in the space-time dimension D /O. Tarasov . . . P. Baikov . . . R. Lee/
- "IR-reduction" $\longrightarrow$ most useful trick to automatically reduce # of loops by one in computing Z-factors (read **any** β-function and anomalous dimension in **any** theory) /see below/
- summations & special functions /see talks by C. Schneider & J. Blümlein/
- Computer Algebra: from legendary SCHOONSCHIP (M. Veltman) to Mathematica and, especially, FORM (J. Vermaseren), see the talk by T. Ueda
- numerical methods: "Páde approach" (J. Flescher . . . O. Tarasov . . .) sector & slicing techniques (see the talk by G. Heinrich)

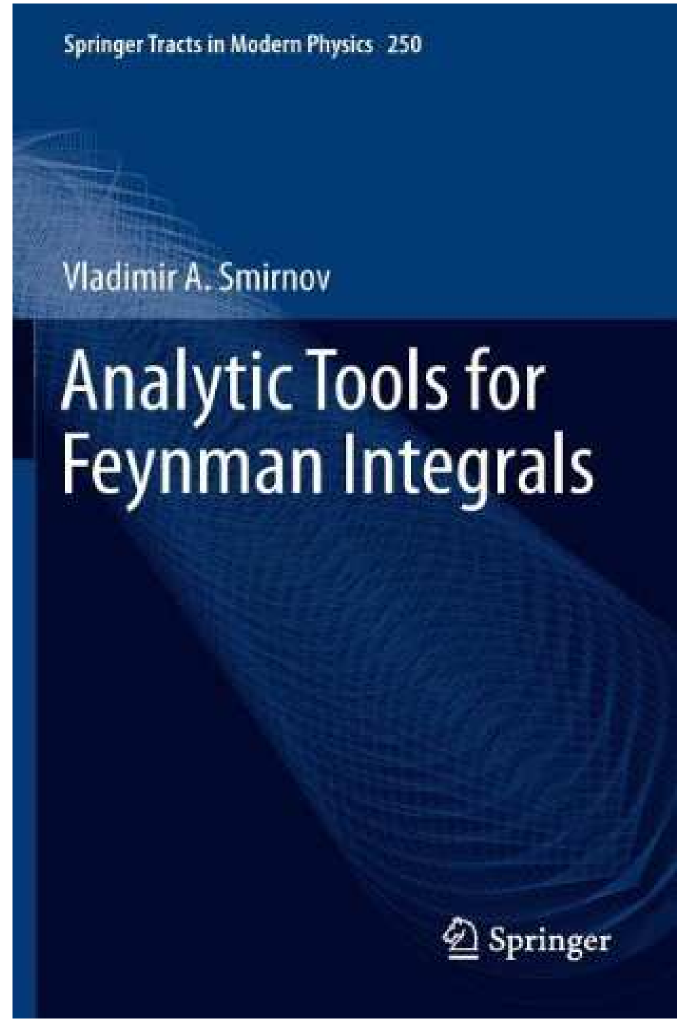
Now: Evaluation of FI = established, fastly developing part of math. physics





Now:
Expansion of FI=established, fastly developing part of math. Physics





This talk:

I will concentrate on the "most mutiloop" branch of multiloop QCD: the evaluation of RG-functions (that is β-functions and anomalous dimensions) based on **massless** propagators.

Note that the same methods are equally applicable to a tightly related problem of computing massless current-current QCD correlators. The latter are connected to many interesting physical observables:

total cross-section of e^+e^- annihilation to hadrons (R-ratio)

Z, τ-lepton and Higgs decays rates to hadrons

quark masses (via QCD sum rules)

running of α_{EM} due to strong interaction in the SM

...

...

An alternative approach, based on completely **massive vacuum tadpoles** will be discussed today by A. Bednyakov

INTRO: Z-factors and R-operation

Consider a simple theory, the ϕ^4-model with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4$$

and let

$$\Gamma[\mathcal{L}, \phi]$$

is the the generating functional of all 1PI Green functions corresponding to the Lagrangian $\mathcal{L}$ (some UV regularization is assumed). Renormalizability of the model means that there exits such a choice of the *renormalization constants* Z_2, Z_m and Z_4 (in the form of the *formal* series in the cc g, starting from 1) that the *renormalized* generating functional

$$\Gamma_r[\phi] \equiv \Gamma[\mathcal{L}^c, \phi]$$

with

$$\mathcal{L}^c = \frac{1}{2}Z_2(\partial\phi)^2 - \frac{1}{2}Z_m m^2\phi^2 - \frac{g}{4!}Z_4\,\phi^4$$

produces finite (after regularization is removed) Green functions in every order of PT in the coupling g.

All information on RG-functions (that is β-functions and anomalous dimesnions) of a theory is siiting in Z-factors (renormalization constants).

To prove the renormalzability Bogolyubov and Parasyuk invented the R-operation. Let'remind some definitions.

Let $< \Gamma >$ be a Feynman integral (FI) corresponding to a diagram Γ, then R-operation is defined as

$$R < \Gamma >= \sum_{\gamma_1,\dots\gamma_j} \prod_i \Delta(\gamma_i) < \Gamma >= R' < \Gamma > + \Delta(\Gamma) < \Gamma >$$

1. sum goes over all sets $\{\gamma_1, \dots \gamma_j\}$ of (pairwise) disjoint 1PI subgraphs, with $\Delta(\emptyset) = 1$

2. $\Delta(\gamma)$ is a counterterm (c-) operation which acts as follows:

$$\Delta(\gamma) < \Gamma >= P_\gamma * < \Gamma/\gamma >$$

3. $P_\gamma = \Delta < \gamma >$ is a polynomial in external momenta (mandatory) and masses (desirable) of FI $< \gamma >$ which is inserted into the vertex v_γ inside of the reduced graph Γ/γ

4. a specific choice of the c-operation $\longleftrightarrow$ choice of a renormalization scheme

Main (analytical) theorem of the R-operation:

If FI $<\Gamma>$ does not contain IR divergences (which is certanly true if all lines are massive or external momenta are off-shell), then the $renormalized$ **FI** $R <\Gamma>$ **can be made finite in the limit of removed** $UV\ regularization$ **with a proper choice of the c-operation**

In terms of the R-operation the generating functional of the renormalized Green function is written as:

$$\Gamma_r[\phi] = R\,\Gamma[\mathcal{L}, \phi]$$

Connection to the multiplicative regularization and the Lagrangian with counter-terms is given by the following

Main (combinatorial) theorem of the R-operation:

$$\mathcal{L}^c \equiv \Delta\,\Gamma[\mathcal{L}, \phi]$$

The theorem provides us with a very convenient and flexible way of computing of contributions to Z-factors from *separate diagrams*.

Dimensional Regularization (DR) and related MS-scheme /t' Hooft and M. Veltman, (1972-1973)/ are most usefull for RG calculations.

The R-operation for MS-scheme is fixed by defining the c-operation as follows

$$\Delta(\gamma) < \gamma >= -K\, R' < \gamma >$$

where K picks up the pole part in $\epsilon = 2 - D/2$.

The most crucial for all the RG-business property of such the MS-R-operation is its commutativity with differentiations wrt masses and external momenta. This commutativity naturally leads to the following remarkable statement

Theorem 1. (J. Collins, 75) *Any UV counterterm for any Feynman integral and, consequently, any RG function in arbitrary minimally renormalized model is a polynomial in momenta and masses.*

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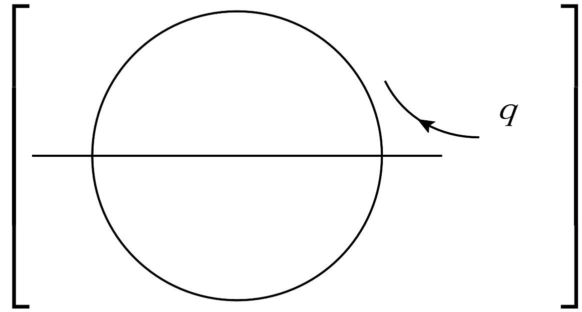
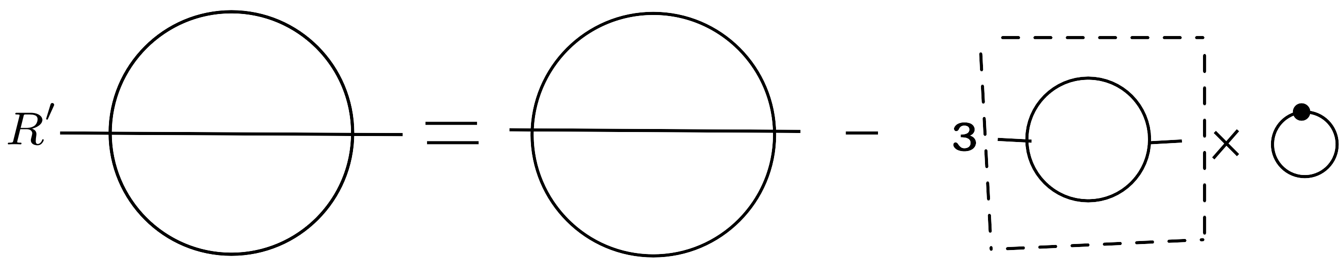
* Mathematically correct definition of DR requires the use of α-parameters and has been done for massive case by Breitelohner and Maison in 1977 and by V. Smirnov and K.Ch. in 1984 for a general case of FI with UV **and** IR divergences

Infrared Rearrangement Method /A. Vladimirov (1978)/

Suppose we want to compute contributions to Z_2 and Z_m from the 2-loop propagator of the ϕ^4-theory

$$q^2 \delta Z_2 + m^2 \delta Z_m = K R'$$





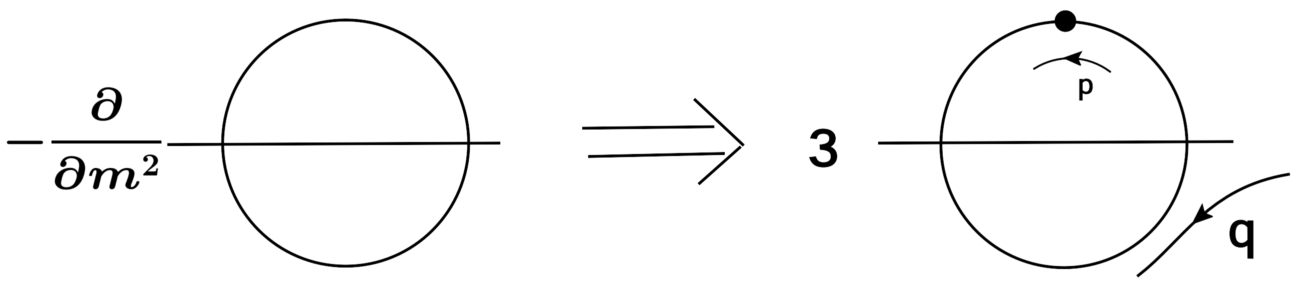
δZ_2 is simple: one just set $m = 0$ and the result directly comes from double application of the textbook 1-loop formula for *massless propagators* (called *p-integrals* later on):

$$\frac{1}{i}\int \frac{d^D l}{(-l^{2\alpha})(-(q-l)^{2\beta})} = \pi^{D/2}(-q^2)^{2-\epsilon-\alpha-\beta} G(\alpha,\beta)$$

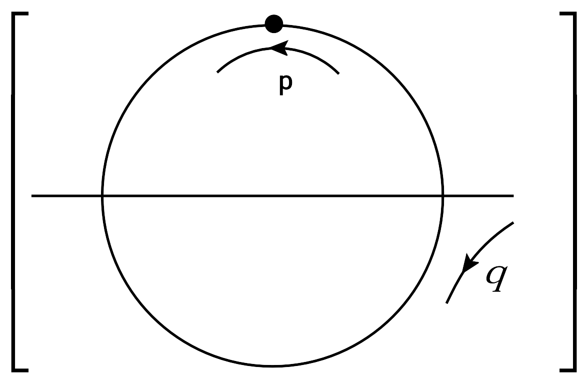
with $G(\alpha,\beta)$ being just a simple combination of 6 Γ-functions.

To compute δZ_m one could, of course, set $q = 0$ but resulting 2-loop massive vacuum graph is certainly more complicated than 1-loop p-integral. Anticipating even more complicated cases in future, let us try to stay with massless integrals . . . .

Let us first perform a derivative wrt m^2 of the initial integral:



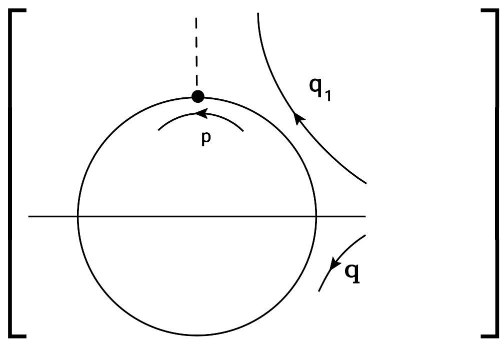
$$-\frac{1}{3}\,\delta Z_m = KR'$$



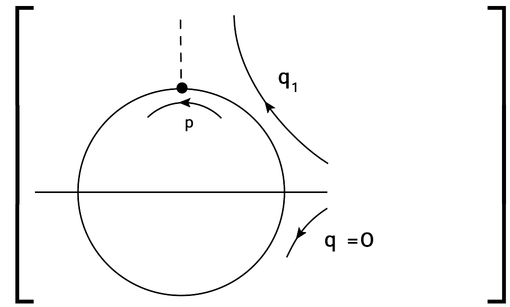
Now would be nice to set m=0 but we can not as the dotted line corresponds to $\frac{1}{p^4}$ and leads to an IR divergency! (which certainly spoil the result for δZ_m)

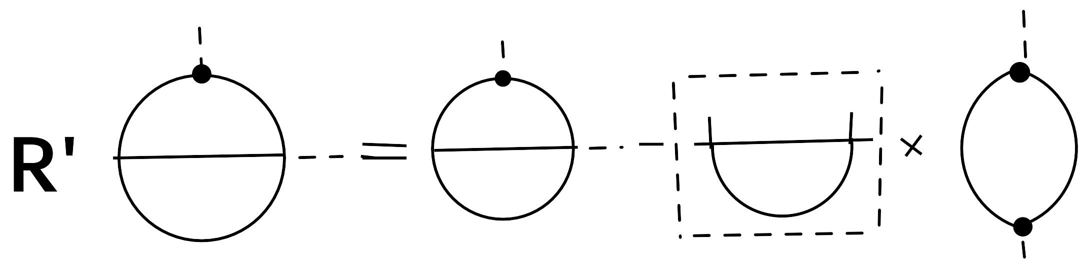
But as we are dealing with log-divergent integral, its UV counter-term is just a pole without *any* dependence on external momenta. So one could freely *change* external momenta without touching δZ_m!

$$-\frac{1}{3}\delta Z_m = KR'$$



$$= KR'$$



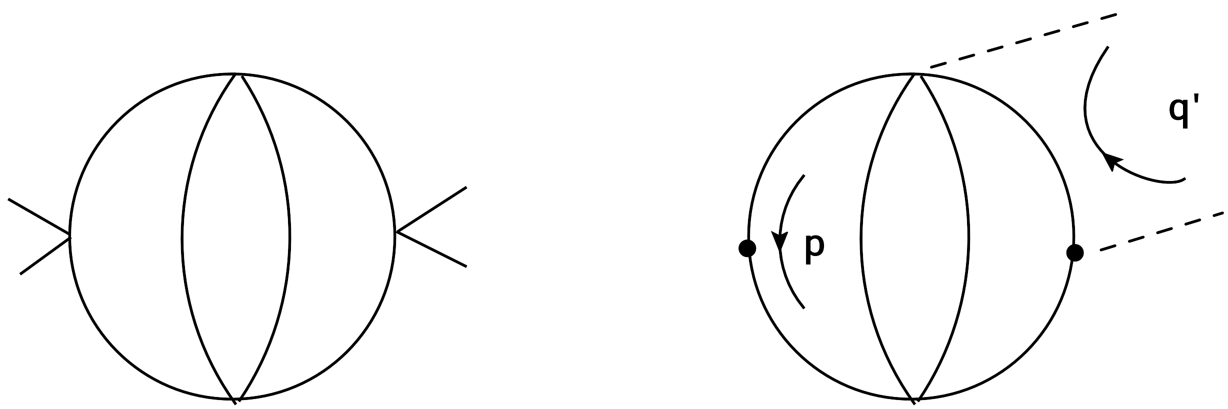


With $q = m == 0$ and $q' \neq 0$ there is no IR divergences and we could easily perform integrations using massless formulas.

The IRR in many cases is able to reduce the problem of evaluation of (L+1)-loop UV countertem to evaluation of some L-loop p-integrals (the latter is necessary to know up to and including the constant ϵ^0 part in the corresponding ϵ-expansion).

But *there are cases when it does not work*: no **simple** (read: flowing through exactly one line) choice of the external momentum in a massless FI can kill all IR problems:

An example:



Here the IR divergency in p-integration makes problems. One, of course, could regulate it with a small "auxiliary" mass:

$$\frac{1}{p^4} \to \frac{1}{(p^2+m^2)^2}$$

but that will complicates integration, leading to a 2-scale integral.

The idea how to overcome the problem (in fact, it came from the Bogolyubov's distributional approach to QFT) is very simple: to subtract the unwanted IR divergency with the help of an IR counterterm but now local in *position space*:

$$\frac{1}{p^4} \rightarrow \frac{1}{(p^4)} - \frac{c}{\epsilon}\delta^D(p)$$

with the constant c choosen such that there would be no IR poles coming from the integration region of small momentum p.

After such a replacement no IR poles survive and integrations are made easily.

This idea eventually led to the so-called R^*-operation[1]:

a generalization of the R-operation which recursively subtracts all UV and IR divergences from any (Euclidean!) Feynman integral.

It was also a starting point of the so-called $extension$ **principle**[2]

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[1] **K.Ch., F. Tkachov (1982); K.Ch., V. Smirnov (1984 – . . .)**

[2] **F. Tkachov (1983); F. Tkachov, G. Pivovarov (1983 –. . .)**

The main use of the R^* -operation is in proof of the following statement

Theorem 2. *Any (L+1)-loop UV counterterm for any Feynman integral may be expressed in terms of pole and finite parts of some appropriately constructed L-loop p-integrals.*

Theorem 2 is a key tool for multiloop RG calculations as it reduces the general task of evaluation of (L+1)-loop UV counterterms to a well-defined and clearly posed purely mathematical problem: the calculation of L-loop p-integrals (that is massless propagator-type FI's).

In the following we shall refer to the latter as the L-loop Problem.

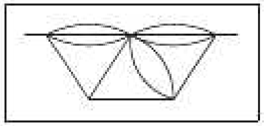
1. 1-loop Problem is trivial

2. the 2-loop Problem was solved after inventing and developing the Gegenbauer polynomial technique in x-space (GPTX) (K.Ch.,F. Tkachov (1980); further important developments in works by D. Broadhurst and A. Kotikov).

In principle GTPX is applicable to analytically compute some quite non-trivial three and even higher loop p-integrals. However, in practice calculations quickly get clumsy, especially for diagrams with numerators. . Nevertheless, it proved to be very usefull in cases of scalar diagrams with many multilinear vertexes /appear frequently in supersymmetric theories/

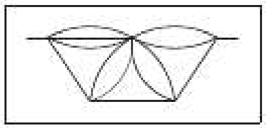
An impressive example of GPTX in action (8 loops!!)

$$\bar{I}_{8d} =$$



$$= \frac{1}{(4\pi)^8}\left(-\frac{1}{2048\,\varepsilon^4}+\frac{1}{192\,\varepsilon^3}-\frac{1}{64\,\varepsilon^2}-\frac{11}{192\,\varepsilon}\right),$$

$$\bar{I}_{8e} =$$



$$= \frac{1}{(4\pi)^8}\left(-\frac{1}{6144\,\varepsilon^4}+\frac{1}{256\,\varepsilon^3}-\frac{19}{384\,\varepsilon^2}+\frac{5}{16\,\varepsilon}\right).$$

For their evaluation, we used the *Gegenbauer polynomials x-space technique* (GPXT). We briefly review this technique in Appendix C.

+ many more similar integrals

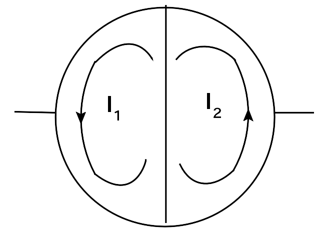
copied from the recent work (January of 2013):

“The Leading Order Dressing Phase in ABJM Theory”

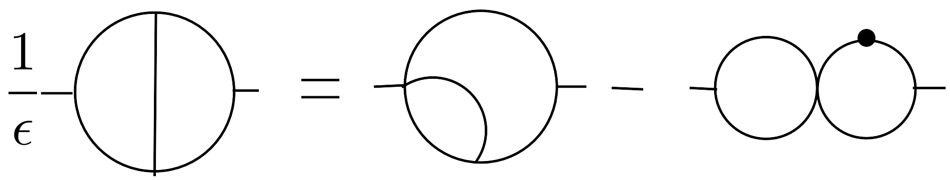
Andrea Mauri, Alberto Santambrogio, Stefano Scoleri, arXiv:1301.7732 [hep-th]

The main breakthrough at the three loop level happened with elaborating the method of integration by parts (IBP) of DR integrals. An example:

$$\mathbf{0} = \int d^D\ell_1 \, d^D\ell_2 \, \frac{\partial}{\partial \ell_1^\alpha} (\ell_1 - \ell_2)_\alpha$$



which is equivalent to the **exact** D-dimensional equality (first derived with GPTX in 1979):



Historical references:

At one loop, IBP (for DR integrals) was used in [*], a crucial step — an appropriate modification of the *integrand* before differentiation was undertaken in [**] (in momentum space, 2 and 3 loops) and in [***] (in position space, 2 loops)

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[*] G. 't Hooft and M. Veltman (1979)
[***] A. Vasiliev, Yu. Pis'mak and Yu. Khonkonen (1981)
[**] F. Tkachov (1981); K. Ch. and F. Tkachov (1981)

COMMENTS

- **IBP identities are *exact ones* valid* for general Feynman amplitudes**

- **IBP identities relate complicated topologies to simpler ones**

- **For a given class of FI's there exist only *finite* very limited (empirical, well established fact) number of (further irreducible) so-called *master* integrals which provide *basis* to express all other members of the class**

- **IBP relations also play the crucial role in many (but not all) powerful approaches to compute the very master FI's; see below**

- **As a result: ". . . IBP relations evolved into a fantastically universal and efficient method for reducing all integrals of a given topology to a few *master integrals*"****

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* to the best of my knowledge a rigorous proof is available for the euclidean case K.Ch., V. smirnov (1984)/; practical calculations find no selfconsitency in on-shell situation too . . .

** A. Grozin, from Int.J.Mod.Phys. A27 (2012) 1230018

With the use of IBP identities the 3-loop Problem was completely solved and corresponding (manually constructed) algorithm was effectively implemented first in SCHOONSCHIP CAS (Gorishny, Larin, Surguladze, and Tkachov) and then with FORM (Vermaseren, Larin, Tkachov, /1991/ ... Vermaseren 2000–2012).

Note that all (notrivial) masters for MINCER were provided by GPTX.

This achievement resulted to a host of various important 3- and 4-four loop calculations performed by different teams during 80-th and 90-th.

Note that the 4-loop correction to the QCD β function was done only as late as in 1996 and using "massive" way /van Ritbergen, Vermaseren,and Larin/; the reason was too complicated combinatorics of the IR reduction

4 ways to reduce a Feynman integral to Masters

- Empiric /sit and think/ way, basically limited to 3 loops (/Mincer,Matad/);

- Arithmetic way: direct solution of /thousands or even millions!/ IBP eqs. /Laporta, Remiddi (96); Gehrmann, Schröder, Anastasiou, Czakon, Sturm , Marquard. . . , A. Smirnov, Manteuffel and Studerus, (2011)

- New Representation for CF's /Baikov (96), Steinhauser, Smirnov . . . /

  $\Downarrow$

  - $1/D$ expansion of CF's /Baikov (2001-09) /

- Very New: a Mathematica assisted AI method: a heuristic search of *algebraic* reduction rules (see the talk by R. Lee)

Feynman parameters:

$$\frac{1}{m^2 - p^2} \approx \int d\ \alpha\ \mathrm{e}^{i\alpha(m^2 - p^2)}$$

New parameters:

$$\frac{1}{m^2 - p^2} \approx \int\ \frac{d\ x}{x}\ \delta(x - (m^2 - p^2))$$

Now for a given topology one can make loop integrations once and forever with the result:

Baikov's Representation:

$$F(\underline{n}) \quad \sim \quad \int \dots \int \frac{\mathbf{d}x_1 \dots \mathbf{d}x_N}{x_1^{n_1} \dots x_N^{n_N}} [P(\underline{x})]^{(D-h-1)/2} \ ,$$

where $P(\underline{x})$ is a polynomial on $x_1, \dots, x_N$ (and masses and external momenta)

New representation obviously meets the same set IBP'id as the original integral but it has much more flexibility! (Due to choice of the integration contours)

MAIN IDEA: to use (1) as a "template" for the very CF's!

reduction to Masters: $1/D$ expansion[1]

- coefficient functions in front of *master integrals* depend on D in simple way:

$$C^{\alpha}(D) = \frac{P^n(D)}{Q^m(D)} \underset{D\to\infty}{=\!=\!=} \sum_k C_k^{\alpha} \quad (1/D)^k$$

- The terms in the $1/D$ expansion expressible (with the use of the Baikov's representation) through simple Gaussian integrals

- sufficiently many terms in $1/D$ and $C_k^{\alpha} \longrightarrow C^{\alpha}(D)$

- computing time and required resources: could be huge

- the only method currently able to reduce 4-loop p-integrals with many (up to 3) dots and complicated numerators

- interestingly enough, unlike every other method no(!) IBP relations are directly involved

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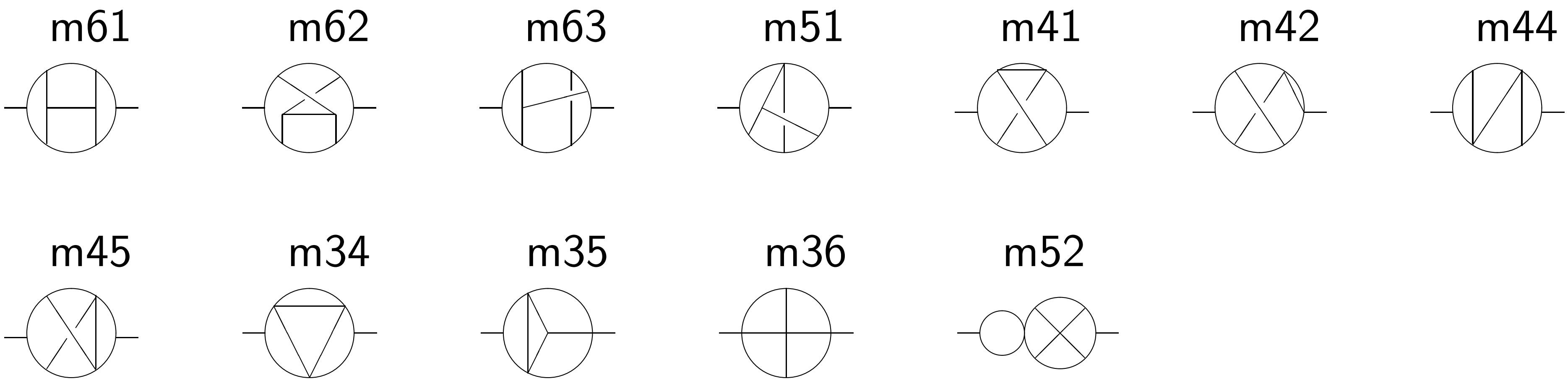
[1] Baikov, Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003

(incomplete) **Tool-Box for Evaluation of masters**

- High-Precision or even analytical: Differential Equations /Kotikov, 1991, . . . ; Remiddi, 2000, . . . / Difference Equations /Laporta, 2001, . . . /
- **very new and powerfull:**
  Dimensional recurrence relations (DRR) /R. Lee, 2010, R.Lee, V.Smirnov, A. Smirnov (2011)/
- "Mellin-Barnes"
  /V.A. Smirnov (1999), Tausk, (1999) . . . Czakon, Anastasiou, . . . /
  and, of course, all kinds of (irregular) tricks with dispersion relations, Feynman parameters, heavy mass expansions, etc.
  /Usually works for not-too-complicated cases/
- Direct numerical evaluation via sector decomposition
  (in general less precise) /Binoth and Heinrich (2000), . . . ;
  Bogner and Weinzierl (2008); A.V. Smirnov and Tentyukov (2008)/ See, also, talks by G. Heinrich and M. Kompaniets on Friday
- Glue-and-Cut symmetry (K.Ch. and Tkachov /1981/) + Reduction to Master FI's: works only for p-integrals but for all loops (if Reduction to Master FI's is available)

Important Case for RG-claculations:

for all non-trivial 4-loop massless master propagators



about 10 terms of ϵ expansion (up to the transcendentality level 12!) have been computed first numerically (around **500 significant** digits!) and then their full analytical structure has been reconstructed (V. Smirnov & R. Lee, 2011).

Tools: IBP + recurrence relations in the space-time dimension D (O. Tarasov,1996) + a lot of ingenuity

Note: up to the transcendentality level 7 /enough for 5-lop RG/ the masters were first evaluated with the Glue-and-Cut method +IBP /P. Baikov and K.Ch. 2004 – 2010/

4-loop Problem has been under study in the Karlsruhe-Moscow group (P. Baikov, K.Ch., J. Kühn . . .) since late 90th. It is essentially solved by now with the help of $1/D$ expansion /reduction to masters, implemented as a FORM program **BAICER**/ and Glue-and-Cut symmetry (analytical evaluation of all necessary masters)

As a result during last 10 years in our group the the results for $R^{VV}(s)$ and a closely related quantity – Z-decay rate into hadrons have been extended by one more loop (that is to order α_s^4).

These results +some others related to 5 and 4-loop correlators (Higgs decays into hadrons, etc.) can be found in:

Phys.Rev.Lett. 88 (2002) 01200
Phys.Rev.Lett. 95 (2005) 012003
Phys.Rev.Lett. 96 (2006) 012003
Phys.Rev.Lett. 97 (2006) 061803
Phys.Rev.Lett.101:012002,2008
Phys.Rev.Lett. 102 (2009) 212002
Phys.Rev.Lett.104:132004,2010
Phys.Rev.Lett. 108 (2012) 222003
JHEP 1207 (2012) 017
Phys.Lett. B714 (2012) 62-65

IMPORTANT

21 century $\mathcal{O}(\alpha_s^4)$ calculations would hardly be feasible without excellent possibilities for dealing with gigantic data streams offered by FORM 3 &4 and, especially, such its versions as

ParFORM and T-FORM:

M. Tentyukov et al. "ParFORM: Parallel Version of the Symbolic Manipulation Program", PoS ACAT2010 (2010) 072

M. Tentyukov, H. M. Staudenmaier, and J. A. M. Vermaseren. "ParFORM: Recent development". *Nucl. Instrum. Meth.*, A559:224–228, 2006.

M. Tentyukov and J. A. M. Vermaseren. "The multithreaded version of FORM", hep-ph/0702279"

⋮

J.A.M. Vermaseren , "Potential of FORM 4.0", PoS LL2012 (2012) 031

see, also the today's talk "Recent developments on FORM" by Takahiro Ueda

Example of Phenomenological Relevance

- With previous α_s^3 calculation* of Γ_Z^h, the theoretical errors were comparable with the experimental ones and, in despair, everybody was using the famous Kataev&Starshenko /1993/ estimation of the α_s^4 term **which (incidentally?) has happened to be quite close to the true number!**

- After our calculations the situation has become significantly better, especially for Γ_Z^h, where the the theoretical error was reduced by a factor of four!

- α_s^4 correction to the τ decay rate has decreased the theoretical error and improved stability wrt the scale variation

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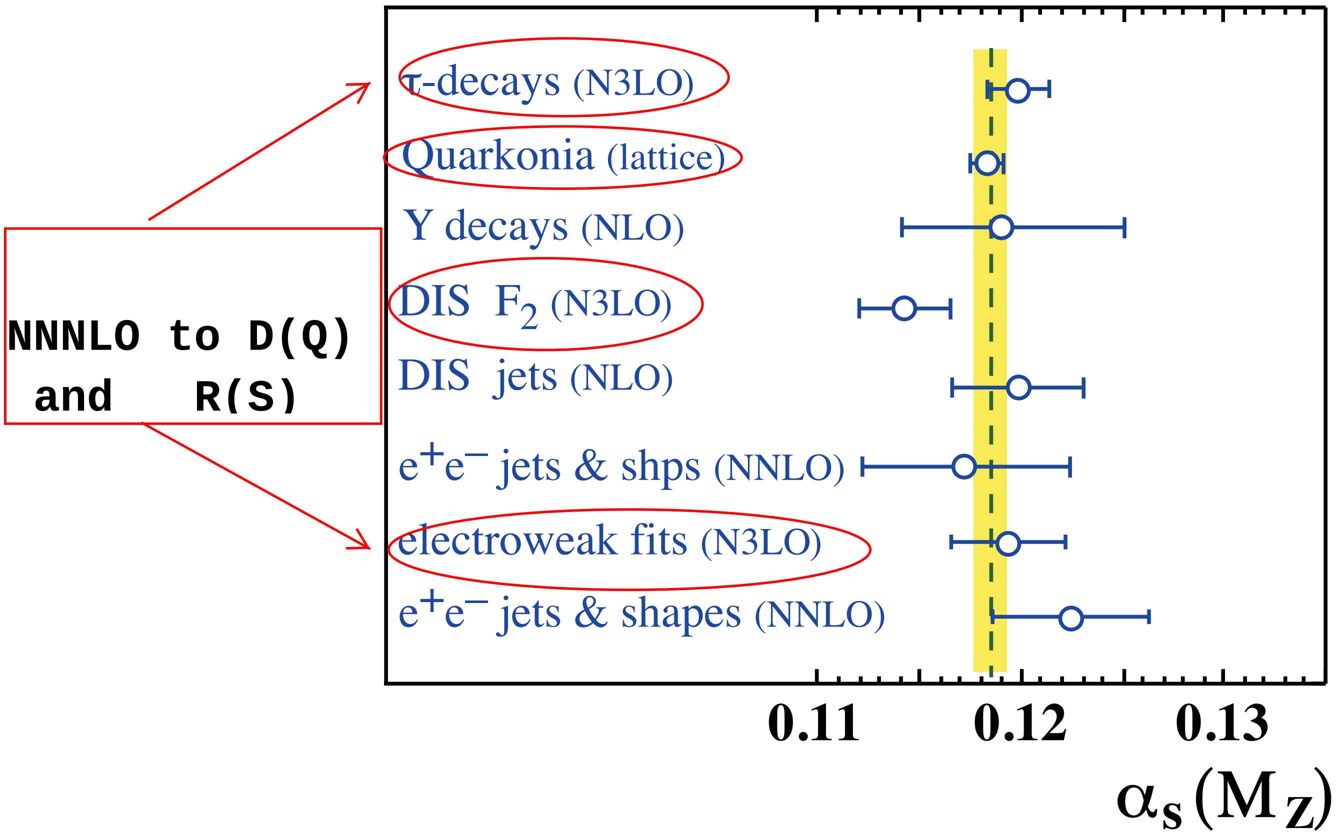
* Gorishnii, Kataev, Larin, (1991); Surguladze, Samuel, (1991); (both used Feynman gauge); K. Ch. (1997) (in general covariant gauge)

World Summary of α_s 2009:



–> $\alpha_s(M_Z) = 0.1184 \pm 0.0007$

(Bethke, arXiv:0908.1135)

Next aim of the K-M group: to compute analytically the QCD beta-function and quark mass anom. dim. at 5-loops!

Phenomenogically result for $\beta(\alpha_s)$ would be of some importance for the analysis of the τ-decay rate within so-called CIPT and for various QCD "optimization" schemes like PMS and PMC (the Principles of Maximal Sensitivity P. Stevenson, 1981) and of Maximal Conformality (S. Brodsky, X. G. Wu,L. Di Giustino,M. Mojaza, 2012).

We have started from the quark mass anom. dim. $\gamma_m(\alpha_s)$

Quark Mass Anomalous Dimension $\gamma_m = -\sum_{i\geq 0} \gamma_i \, a_s^i$: history

3-loops: /O, Tarasov (82, with IRR reduced to 2-loop p-integrals);

3-loops: /S. Larin/ (92; direct evaluation of 3-loop p-integrals with MINCER)

4-loops: /K. Chetyrkin/ (97; with R^*-operation all FI's were reduced to 3-loop p-integrals; the latter were performed with MINCER)

4-loops: /J.A.M. Vermaseren, S.A. Larin, T. van Ritbergen/ (97; direct evaluation of the **completely massive 4-loop tadpoles** /via a kind of Laporta machine (?)/)

$$\gamma_0 = 1 \quad \gamma_1 = \frac{1}{16}\left\{ \frac{202}{3} + n_f \left[-\frac{20}{9} \right] \right\}, \gamma_2 = \frac{1}{64}\left\{ 1249 + n_f \left[-\frac{2216}{27} - \frac{160}{3}\zeta(3) \right] + n_f^2 \left[-\frac{140}{81} \right] \right\}$$

$$\begin{aligned} \gamma_3 = \frac{1}{256} \Bigg\{ & \frac{4603055}{162} + \frac{135680}{27}\zeta(3) - 8800\,\zeta(5) \\ + \quad & n_f \left[-\frac{91723}{27} - \frac{34192}{9}\zeta(3) + 880\,\zeta(4) + \frac{18400}{9}\zeta(5) \right] \\ + \quad & n_f^2 \left[\frac{5242}{243} + \frac{800}{9}\zeta(3) - \frac{160}{3}\zeta(4) \right] + n_f^3 \left[-\frac{332}{243} + \frac{64}{27}\zeta(3) \right] \Bigg\}. \end{aligned}$$

Quark Mass Anomalous Dimension $\gamma_m = \sum_{-i\geq 0} \gamma_i \, a_s^i$: today

New result (preliminary) /P. Baikov, J. Künh, K. Ch./ (2013; with R^*-operation all was reduced to 4-loop p-integrals; the latter were performed with BAICER)

$$
\begin{aligned}
\gamma_4 =& \frac{-1}{4^5}\Bigg\{ -\frac{99512327}{162} - \frac{46402466}{243}\zeta_3 - 96800\,\zeta_3^2 + \frac{698126}{9}\zeta_4 \\
& \qquad + \frac{231757160}{243}\zeta_5 - 242000\,\zeta_6 - 412720\,\zeta_7 \\
+ & \quad n_f \left[\frac{150736283}{1458} + \frac{12538016}{81}\zeta_3 + \frac{75680}{9}\zeta_3^2 - \frac{2038742}{27}\zeta_4 \right. \\
& \qquad \left. - \frac{49876180}{243}\zeta_5 + \frac{638000}{9}\zeta_6 + \frac{1820000}{27}\zeta_7 \right] \\
+ & \quad n_f^2 \left[-\frac{1320742}{729} - \frac{2010824}{243}\zeta_3 - \frac{46400}{27}\zeta_3^2 + \frac{166300}{27}\zeta_4 + \frac{264040}{81}\zeta_5 - \frac{92000}{27}\zeta_6 \right] \\
+ & \quad n_f^3 \left[-\frac{91865}{1458} - \frac{12848}{81}\zeta_3 - \frac{448}{9}\zeta_4 + \frac{5120}{27}\zeta_5 \right] + n_f^4 \left[\frac{260}{243} + \frac{320}{243}\zeta_3 - \frac{64}{27}\zeta_4 \right] \Bigg\}
\end{aligned}
$$

Quark Mass Anomalous Dimension $\gamma_m = -\sum_{i\geq 0} \gamma_i\, a_s^i$: con-ed

For $n_f = 3$ the result reads:

$$\gamma_4^{n_f=3} = -\frac{156509815}{497664} + \frac{23663747}{124416}\,\zeta_3 - 85\zeta_3^2 - \frac{23765}{256}\,\zeta_4 + \frac{22625465}{62208}\,\zeta_5 - \frac{1875}{32}\,\zeta_6 - \frac{118405}{576}\,\zeta_7$$

Numerically:

$$\gamma_m^{n_f=3} = -\left\{a_s + 3.792\, a_s^2 + 12.420\, a_s^3 + 44.263\, a_s^4 + 198.906\, a_s^5\right\}$$

To construct scale-inavariant mass (or to run the quark mass) one needs also β-function at 5-loop (not yet available)

$$\beta(n_f = 3) = -\left(\beta_0 = \frac{4}{9}\right) \cdot \left\{a_s + 1.777\, a_s^2 + 4.4711\, a_s^3 + 20.990\, a_s^4 + \overline{\beta}_4\, a_s^5\right\}$$

It is natural to estimate $\overline{\beta}_4$ as sitting in the interval 50 –100

The mass evolution is described by equation $\frac{m(\mu)}{m(\mu_0)} = \frac{c(a_s(\mu))}{c(a_s(\mu_0))}$ where

$$
\begin{aligned}
c(x) &= \exp\left\{\int \frac{dx'}{x'}\frac{\gamma_m(x')}{\beta(x')}\right\}(x)^{\bar{\gamma}_0}\{1 + (\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)x \\
&+ \frac{1}{2}\left[(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)^2 + \bar{\gamma}_2 + \bar{\beta}_1{}^2\bar{\gamma}_0 - \bar{\beta}_1\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_0\right]x^2 \\
&+ \left[\frac{1}{6}(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)^3 + \frac{1}{2}(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)(\bar{\gamma}_2 + \bar{\beta}_1{}^2\bar{\gamma}_0 - \bar{\beta}_1\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_0)\right. \\
&\left. + \frac{1}{3}\left(\bar{\gamma}_3 - \bar{\beta}_1{}^3\bar{\gamma}_0 + 2\bar{\beta}_1\bar{\beta}_2\bar{\gamma}_0 - \bar{\beta}_3\bar{\gamma}_0 + \bar{\beta}_1\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_2\right)\right]x^3 + \mathcal{O}(x^4)\}
\end{aligned}
$$

$\bar{\gamma}_i = \gamma_i/\beta_0$, $\bar{\beta}_i = \beta_i/\beta_0$, (i=1,2,3) and β_i are the coefficients of the QCD beta-function

Running (strange quark) mass from the RGI mass $\hat{m} \equiv m(\mu_0)/c(a_s(\mu_0))$:

$$m_s(\mu) = c(a_s(\mu))\hat{m}_s$$

with ($c_s(x) \equiv c(x)$ in QCD with $n_f = 3$)

$$c_s(x) = x^{4/9}(1 + 0.895062\,x + 1.37143\,x^2 + 1.95168\,x^3 + (15.6982 - 0.1111\overline{\beta}_4)\,x^4)$$

The function $c(x)$ is used, e.g, by the **ALPHA** lattice collaboration to find the ($\overline{\text{MS}}$) mass of the strange quark at a lower scale from the RGI mass determined from lattice simulations

Example (setting $a_s(\mu = 2\,\text{GeV}) = \frac{\alpha_s(\mu)}{\pi} = .1$; h counts loops)

$$m_s(2\,\text{GeV}) = \hat{m}_s \cdot (a_s(2\,\text{GeV}))^{\frac{4}{9}} \cdot$$

$$\left(1 + 0.0895\,h^2 + 0.0137\,h^3 + 0.00195\,h^4 + (0.00157 - .000011\,\overline{\beta}_4)\,h^5\right)$$

Note that for any reasonable value of $\overline{\beta}_4$ (positive and ≤ 100) the (apparent) convergency of the above series is quite good even at rather small energy scale

β-function in 6 loop in ϕ^4-model

First 5-loop RG calculation in a 4-dim model (ϕ^4-model) was done *long before* the 4-loop Problem was solved:

/K.Ch, Gorishnii, Larin, and Tkachov, Phys.Lett. B132 (1983) 351/; D.I. Kazakov,Phys.Lett. B133 (1983) 406; Kleinert, Neu, Schulte-Frohlinde , K.Ch., and Larin, Phys.Lett. B272 (1991) 39/

The reason: relative simplicity of the corresponding Feynman amplitudes (only limited number of topologies, no numerators).

What about 6 loops? (Would be of some use for the the statistical physics /critical indexes/)

Analytically: no hope at present (imho)

Numerically: no hope at present (imho)

Mixed way: YES! (imho)

WHY?

Lets neglect all IR singularities (having in mind the possibility of their complete removal with $\tilde{R}$-operation). Then the IR reduction by one loop could be easily understood as follows (for a log-divergent (L+1)loop FI $<\Gamma>$):

1. set zero all (except for one) massess and all external momenta
2. cut the massive line with internal momentum ℓ, then there is an obvious formal representation:

$$<\Gamma>=\int <\Gamma'>(\ell)\,\frac{d\ell}{m^2+\ell^2}$$

Now in order to find the UV div. of $<\Gamma>$ one should, obviously, compute the p-integral $<\Gamma'>(\ell)$ including its constant (ϵ^0 part) + some some "easy" FI's with less # of loops then L. (Note that since $<\Gamma'>(\ell)\approx 1/\ell^{2\alpha}$ the last integral over $d\ell$ is trivial!) That is esentially the statement of Theorem 2!

3. for the ϕ^4 model in many cases the FI $<\Gamma'>(\ell)$ could be chosen to be a product of 2 FI's with loop numbers less then 5 $\rightarrow$ representable in terms of 4-loop p-ntegrals

4. All *primitive* (that is without UV subdivergences) 6-loop contributions to the ϕ^4 β-function are known with high accuracy since long (via GPTx, D. Broadhurst, D. Kreimer (1995))

5. the rest (difficult) diagrams amount to comprise not more then 2% from all and can be eavluated numerically (see M. Kompaniets tomorrow)

Concluding Notes I:

- IRR based on R^* operation significantly simplifies RG calculations. It reduces (L+1)-loop RG function in any model to a combination of properly constructed p-integrals; the latter include not only standard UV- but also **IR** subtractions. It is always possible to do at the level of *separate* diagrams. IR counterterms are expressible diagramwise through UV-ones.
- IRR + R^* + Baikov Algorithm to reduce 4-loop p-integrals + paralell Form (J. Vermaseren, M. Tentyukov + ...) + known 4-loop masters (P. Baikov, K.Ch.) $\Longrightarrow$ the 5-loop RG functions are *in principle* doable in *any* model.
- But: global representation of neccessary IR subtractions (that is on the level of Green functions) strongly depends on the problem and not always easy.

Concluding Notes II:

- The 5-loop quark anomalous dimension γ_m is done QCD. The phenomenological implications seem to be not very dramatic.

- The 5-loop QCD β-function is significantly more complicated; first results are expected in a year or so. (The full QED β function in 5-loops is available since recently: P. Baikov, K. Ch., JH. Kühn, J. Rittinger, JHEP 1207:017,2012.).

- Truly remarkable fact: N=4 SYM theory seems to be simpler than QCD: "Konishi" (anomalous dimension of a specific operator in N=4 SYM) in 5-loop has been recenltly computed with a via IRR + p-intergrals + Laporta machine + a lot of ingenuity; the result confirms the prediction from non-perturbative methods (" Five-loop Konishi in N=4 SYM", B. Eden, P. Heslop, G. Korchemsky, V. Smirnov, E. Sokatchev, arXiv:1202.5733)

Concluding Notes III:

- There are some theoretical problems requiring analytical evaluation of 6-loop anomalous dimensions: e.g. "Konishi" (anomalous dimension of a specific operator in N=4 SYM) in 6-loop is already available from non-perturbative methods:

  Six and seven loop Konishi from Luscher corrections. Z. Bajnok, R. Janik e-Print: arXiv:1209.0791

  Here the main problem is the very reduction to masters (the way to compute the resulting masters is known /K.Ch. and Baikov, 2010). BUT: shear # of contributing diagrams in "normal" gauge theories would presumably be prohibitively large for, say, QCD 6-loop β-function. Probably the situation should be better for N=4 SYM and such quantities as $R(s)$ and DIS sum rules (here the next loop number is 5 *not* 6!)

- The 6-loop β-function in the ϕ^4-model is certainly doable in the "mixed" anaytical-numerical way (see the talk by M. Kompaniets tomorrow). But a diagram-wise computer algebra implementation of the R^* operation is requied; it is certainly doable /the scalar theories are much simpler to deal with than the gauge ones, but not completely trivial/.