Theory of phase transitions and critical phenomena: new approach to numerical calculation of anomalous dimensions.

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Outline

- Field theoretic renormalization group in critical behaviour theory.
- New approach to numerical calculation of anomalous dimensions.
  Normalization point scheme. Current results
- Sector decomposition and graph symmetries

φ^4 model

$$S(\varphi) = -\int d\mathbf{x} \left(\frac{1}{2}\tau\varphi(\mathbf{x})^2 + \frac{1}{2}(\vec{\bigtriangledown}\varphi(\mathbf{x}))^2 + \frac{1}{24}g(\varphi(\mathbf{x})^2)^2 \right)$$

$O(n)$-symmetric φ^4 model in statistical physics describes second order phase transition in:

- $n = 1$
  - liquid-gas system
  - critical mixing point in binary mixtures
- $n = 2$
  - planar Heisenberg magnet
  - transition to the superfluid phase of liquid ^4He
- $n = 3$
  - isotropic Heisenberg magnet

φ^4 model. Experimentally observables quantities

$$S(\varphi) = -\int d\mathbf{x} \left(\frac{1}{2}\tau\varphi(\mathbf{x})^2 + \frac{1}{2}(\overrightarrow{\nabla}\varphi(\mathbf{x}))^2 + \frac{1}{24}g(\varphi(\mathbf{x})^2)^2 \right)$$

For liquid-gas system near T_c the following quantities can be compared with experiment:

- critical exponents ν, γ, α
- amplitude ratio B_+/B_-

Correlation length r_c and some thermodynamic quantities show singular behavior in the limit $T \to T_c$

$$r_c \cong r_0\tau^{-\nu}, \qquad \tau = (T - T_c)/T_c, \qquad \nu \approx 0.63\ .$$

$$C_P \simeq A_\pm|\tau|^{-\gamma}, \qquad \gamma \approx 1.24, \qquad C_V \simeq B_\pm|\tau|^{-\alpha}, \qquad \alpha \approx 0.1\ .$$

φ^4 model discribes only equal-time correlation functions and thermodinamical quantities

Critical dynamics

Models of critical dynamics are designed to describe critical slowing down effect and critical behaviour of

- speed of sound
- viscosity
- thermal conductivity coefficient
- diffusion coefficient

The behaviour of these values near critical point can be measured experimentally, for example in experiments on light scattering, sound propagation and so on

Critical dynamics

Models of critical dynamics are constructed on basis of models like φ^4, but there are much more models because of

- conservation or non-conservation of order parameter
- different mode couplings

Action for model H of critical dynamics
(liquid-gas critical point)

$$S(\psi, \psi', \vec{v}, \vec{v}') = -\lambda\psi'\partial^2\psi' + \psi'\left[-\partial_t\psi - \lambda\partial^2(\partial^2\psi - \tau\psi - g_1\psi^3/6) - \right.$$

$$\left. -v_i\partial_i\psi\right] - \lambda^{-1}g_2^{-1}v_i'\partial^2 v_i' + v_i'\left[-a\partial_t v_i + \lambda^{-1}g_2^{-1}\partial^2 v_i + \psi\,\partial_i(\partial^2\psi)\right],$$

ψ – order parameter (density fluctuations), coupling with velocity field fluctuations $\vec{v}$,
ψ' and $\vec{v}\,'$– auxiliary fields, by which the transition from the stochastic equations to QF model is made

Critical dynamics

Recent review:
Critical dynamics: a field-theoretical approach,
Reinhard Folk (Linz U.) , **Hans-Guenther Moser** (Salzburg U.)
J.Phys. A39 (2006) R207-R313

model	system	loops	fields
A	Relaxational dynamics	3	2s
B/D	Diffusive dynamics	$\equiv \phi^4$	2s / 4s
C/C'	Relaxational dyn. + energy cons.	2	4s / 6s
E/E'	Planar ferromagnet $h_z = 0$	2	2s 2v / 2s 4v
F	Superfluid transition ^4He Planar ferromagnet $h_z \neq 0$	2	4s
F'	Superfluid transition ^3He - ^4He mixtures	2	6s
G	Heisenberg antiferromagnet	2	2s 2v
H	Liquid-Gas transition	2	2s 2v
H'	Liquid-Gas and Liquid-Liquid transition in binary mixtures	1	4s 2v
J	Heisenberg ferromagnet	1	2s

Renormalization group

Most successfull method applied to this models is Renormalization group method[1]

Model φ^4

- $d = 4 - 2\epsilon \rightarrow$ 5-loop order (analytical)[2]
- $d = 2 \rightarrow$ 5-loop order (numerical)[3]
- $d = 3 \rightarrow$ 6-loop order (numerical)[4]

Critical dynamics:
Most models are calculated only in 2-loop order.
Simplest model "A" is calculated in 3-loop order.

[1] was firstly applied to statictical physics problems by **Kennet Wilson** in 1971
[2] **Chetyrkin K.G, Kataev A.L., Tkachev F.V.**
[3] **Orlov E. V., Sokolov A. I.**
[4] **Nickel B., Meiron D., Baker G.**

Why more loops are needed?

- for more precise Borel resummation (all series expansions for critical exponents are asymptotic)
- to distinguish concurrent asymptotic regimes
- to clarify inconsistency between RG-results ($d = 4 - \epsilon$ and $d = 2$) and exact solution for 2D Ising model

Normalization Point Scheme

Normalization point scheme. Motivation

Our approach has the following advantages:

- anomalous dimensions (γ_i) and β-function are expressed **directly from diagrams of 1-PI renormalized Green functions**, without calculation of renormalization constants
- for renormalized diagrams we use representation where there is **no pole terms at all**. (in contrast to MS scheme where pole terms cancel each other)

Minimal Subtraction (MS) scheme

$$S = -\frac{1}{2}(m^2 Z_1 + k^2 Z_2 + \delta m^2)\varphi^2 - \frac{1}{24} g\mu^\varepsilon Z_3 \varphi^4 .$$

$d = 4 - \epsilon$, Euclidean space

$$Z_1 = Z_{m^2} Z_\varphi^2, \quad Z_2 = Z_\varphi^2, \quad Z_3 = Z_g Z_\varphi^4.$$

$$\Gamma_i^R = R\Gamma_i = (1-K)R'\Gamma_i$$

R' - incomplete R-operation (eliminating divergences in subgraphs)

K - subtraction operation

Renomalization group equations (MS scheme)

$$(\mu\partial_\mu + \beta\partial_g - \gamma_{m^2} m^2 \partial_{m^2})\Gamma_n^R = n\gamma_\varphi \Gamma_n^R .$$

$$\beta = \frac{-\varepsilon\, g}{1 + g\partial_g \ln Z_g}, \qquad \gamma_i = \frac{-\varepsilon\, g\partial_g \ln Z_i}{1 + g\partial_g \ln Z_g} .$$

At fixed point g_* RG-equations turn into equations of critical scaling with exponents $\gamma_i(g_*)$

Normalization point (NP) scheme

We use renormalization scheme where RG-equations keep the simple form (like in MS scheme)

$$(\mu\partial_\mu + \beta\partial_g - \gamma_{m^2} m^2 \partial_{m^2})\Gamma_n^R = n\gamma_\varphi \Gamma_n^R \,.$$

RG-functions depend only on coupling constant g (as in MS-scheme)

In NP scheme anomalous dimensions can be expressed through renormalized 1-PI Green functions

$$\gamma_i = \frac{2f_i}{1+f_2}, \qquad i = 2,4$$

where

$$f_i = -R\; m^2 \partial_{m^2} \bar{\Gamma}_i|_{p=0,\mu=m}, \qquad \bar{\Gamma}_2 = -\partial_{p^2}\Gamma_2, \quad \bar{\Gamma}_4 = -\frac{\Gamma_4}{g\mu^\varepsilon}.$$

Subtraction operation for this scheme is defined by following normalization conditions

$\bar{\Gamma}_2^R|_{p=0,\mu=m} = 1, \quad \bar{\Gamma}_2^R|_{p=0,m=0} = 0, \quad \bar{\Gamma}_4^R|_{p=0,\mu=m} = 1$

Normalization point (NP) scheme

This scheme is in some sense intermediate between minimal subtraction(MS) scheme and subtraction at zero momenta (ZM)

- RG-functions do not depend on mass, this results in simple RG-equations as in MS scheme (RG-equations in ZM scheme have more complex form)
- Each diagram of renormalized Green function can be expressed as uniform integral without cancellation of pole terms and large – large subtractions (difficult in MS scheme)

Subtractions in ZM and NP scheme can be expressed as remainder of Taylor expansion
$(1-K)F(k) = F(k) - \sum_{m=0}^{n} \frac{k^m}{m!} F^{(m)}|_{k=0} = \frac{1}{n!}\int_0^1 da(1-a)^n \partial_a^{n+1} F(ak)$.

This leads to the following representation for R-operation

$$R\Gamma = \prod_i \frac{1}{n_i!} \int_0^1 da_i (1-a_i)^{n_i} \partial_{a_i}^{n_i+1} \Gamma(\{a\}),$$

Normalization point scheme. Summary

$$\gamma_i = \frac{2f_i}{1+f_2}, \qquad i = 2,4$$

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$$R\Gamma = \prod_i \frac{1}{n_i!}\int_0^1 da_i (1-a_i)^{n_i}\partial_{a_i}^{n_i+1}\Gamma(\{a\}),$$

- RG-function can be calculated directly from diagrams without divergences
- there is no need for pole extraction.
- ϵ-expansion for these diagrams can be obtained by simple Taylor expansion of integrand
- this representation can also be used for renormalization group in "real space"($d = 2,\ d = 3$)

We developed **fully automated** software that allows to calculate anomalous dimensions in wide range of models (e.g. ϕ^3, ϕ^4 models and models of critical dynamics)

This software includes

- graph generation
- generic algorithm for finding UV and IR subgraphs
- generation of integrand in momentum/Feynman representation
- sector decomposition (strategy S with symmetries)
- numerical evaluation of integrals (using external programs)
- combining final result for RG-function

(NEW!) Using this approach we calculated RG-functions for ϕ^3 model up to 4-loop order [5]

These calculations can be performed without any tricks directly in momentum representation or using Feynman parameters.

[5] Adzhemyan L. Ts., Kompaniets M. V., Theor. and Math. Phys., 169(1): 1450–1459 (2011)

For 5-loops and higher Sector Decomposition is required. But in our case Sector decomposition is much more simplier than that for MS scheme, because we don't need to extract pole residues.

- $O(N)$-symmetric ϕ^4 model ($d = 4 - \epsilon$) in 5-loop order
  Results agree within 10^{-6} with analitical results[6]
- $O(N)$-symmetric ϕ^4 model ($d = 2$) in 5-loop order
  Results agree within 10^{-6} with results obtained by Orlov and Sokolov.[7]
- NEW! Preliminary results for $O(N)$-symmetric ϕ^4 model ($d = 2$) in 6-loop order
- NEW! Preliminary result for model A of critical dynamics in 4-loop order

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[6] Chetyrkin K.G, Kataev A.L., Tkachev F.V., Phys.Lett., B99, 147 (1981); B101,457(E) (1981) Kleinert H., Neu J., Shulte-Frohlinde V., Chetyrkin K.G., Larin S.A., Phys.Lett., B272,39 (1991); Erratum: B319, 545 (1993)

[7] Orlov E. V., Sokolov A. I., Physics of the Solid State vol. 42 issue 11 November 2000. p. 2151-2158

Remarks on ϕ^4 model in $d = 4 - \epsilon$

Each diagram calculated in Normalization Point scheme can be simply recalculated to that in Minimal Subtraction scheme.

We also perform diagram by diagram verification of analitical results in 5-loop order.[8]

[8] This is first fully independent crosscheck of these results

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IN PROGRESS: 6-loop order (in collaboration with K.G.Chetyrkin)

loops	total	factorized	primitive	4-loop reducible	4-loop irreducible
Γ_2	50	0	0	48	2
Γ_4	627	124	10	481	12

[8] This is first fully independent crosscheck of these results

Sector decomposition and graph symmetries

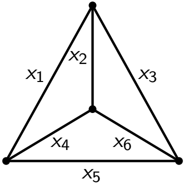
Sector decomposition and graph symmetries

If we are going to apply Sector Decomposition to 6-7 loop integrals one of the problems that arises is a huge[9] number of sectors

- new strategies?
- any additional information about integrand structure?

[9] for 6-loop graphs with strategy S, number of sectors $> 10^5$

Integrand symmetries



Graph is fully symmetric

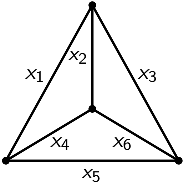
$$I = \int_0^1 dx_1 \ldots \int_0^1 dx_6\ f(x_1, x_2, x_3, x_4, x_5, x_6)$$

where

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = \delta(1 - \textstyle\sum_{i=1}^{6} x_i)(x_1x_2x_4 + x_1x_2x_5 + x_1x_2x_6 + x_1x_3x_4 + x_1x_3x_5 + x_1x_3x_6 + x_1x_4x_6 + x_1x_5x_6)^{(-2+\epsilon)}$$

Integrand has the same symmetries as graph e.g. $(x2, x4) \to (x3, x5)$

Integrand symmetries



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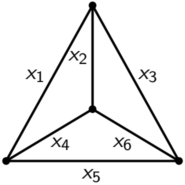
We can use graph symmetries to find integrand symmetries

Integrand symmetries



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We can use graph symmetries to find integrand symmetries

Integrand symmetries allows us to find equal sectors

Primary sectors, msn-notation

Strategy S

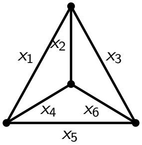
6 primary sectors: (def: $x_1 > x_2, x_3 \equiv \theta(x_1 - x_2)\theta(x_1 - x_3)$)

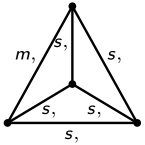
$x_1 > x_2, x_3, x_4, x_5, x_6$; $x_2 > x_1, x_3, x_4, x_5, x_6$; $x_3 > x_1, x_2, x_4, x_5, x_6$;

$x_4 > x_1, x_2, x_3, x_5, x_6$; $x_5 > x_1, x_2, x_3, x_4, x_5$; $x_6 > x_1, x_2, x_3, x_4, x_5$;

Lets mark graph lines with indexes **m**, **s** and **n**[10]

for sector $x_1 > x_2, x_3, x_4, x_5, x_6$ $x_1 \to$ **m**, $x_2, x_3, x_4, x_5, x_6 \to$ **s**





In further decompositions each sector will be associated with graph with such multi-indexes on lines

[10] indexes correspond to role of variable in current decomposition:
- **m** - main variable
- **s** – secondary variable
- **n** – neutral (variable are not included in current decomposition subspace)

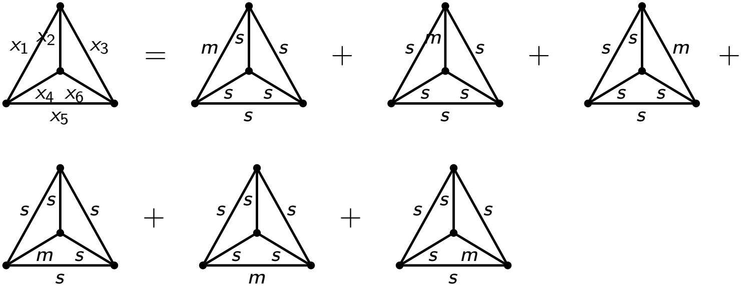
Primary Sectors

strategy S

6 primary sectors:

$x_1 > x_2, x_3, x_4, x_5, x_6;$ $\quad x_2 > x_1, x_3, x_4, x_5, x_6;$ $\quad x_3 > x_1, x_2, x_4, x_5, x_6;$

$x_4 > x_1, x_2, x_3, x_5, x_6;$ $\quad x_5 > x_1, x_2, x_3, x_4, x_5;$ $\quad x_6 > x_1, x_2, x_3, x_4, x_5;$



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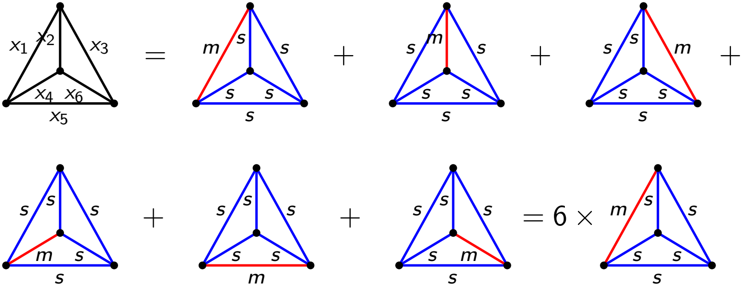
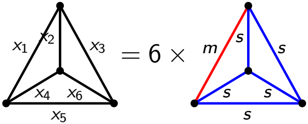
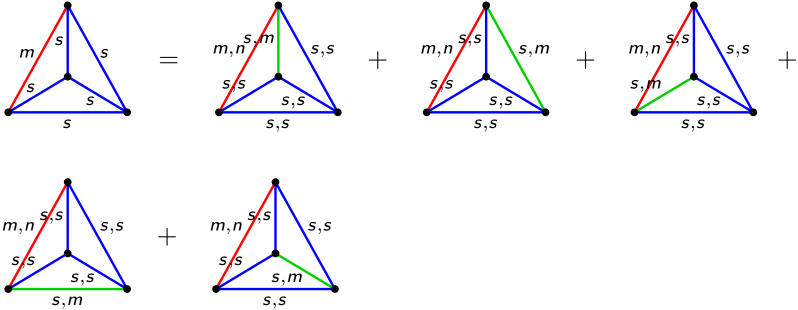


Second decomposition



Decomposition space is $(x_2, x_3, x_4, x_5, x_6)$



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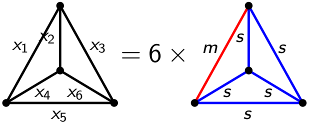

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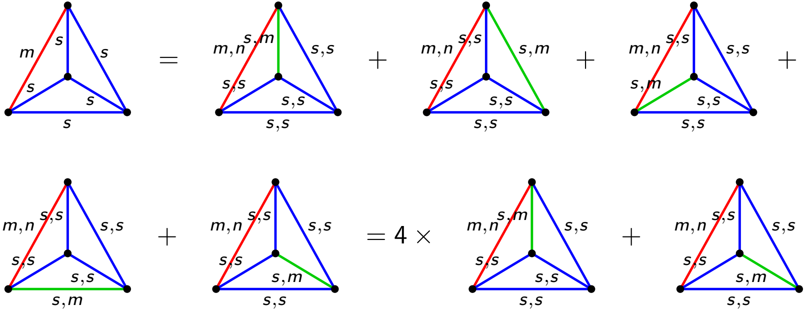

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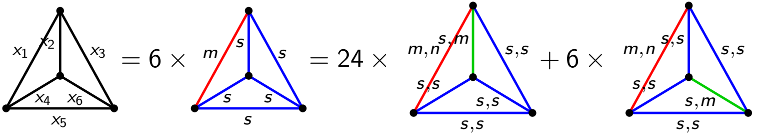

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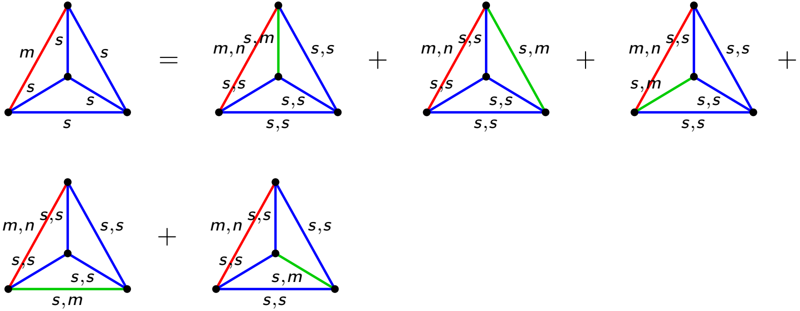
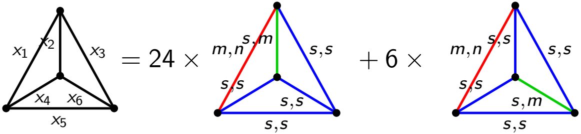
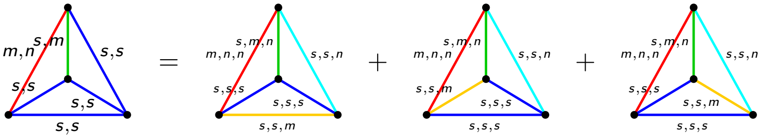


Third decomposition



1. Decomposition space (x_4, x_5, x_6)



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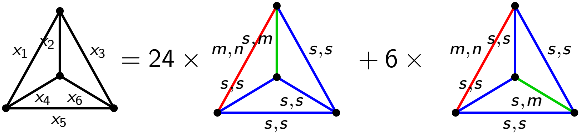

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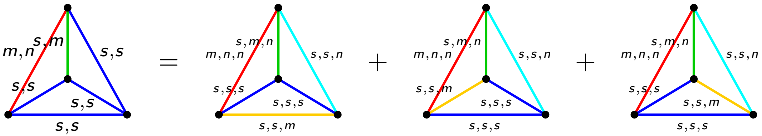
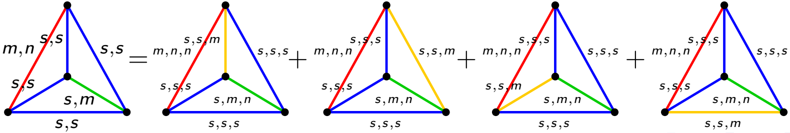
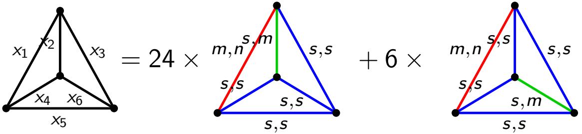


2. Decomposition space (x_2, x_3, x_4, x_5)



Third decomposition



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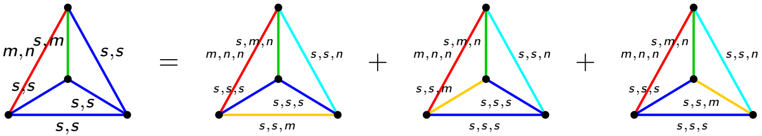

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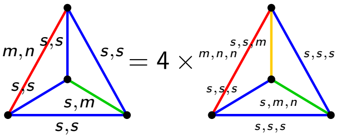
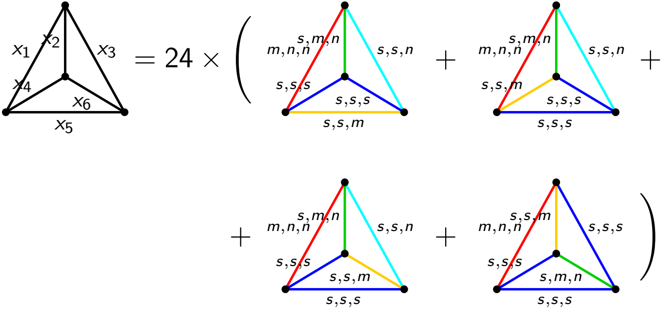


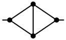
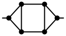
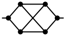
Decomposition result



- we need to calculate only 4 sectors
  - original strategy: 96 sectors
  - primary sectors equivalence[11]: 16 sectors
- Sectors we need to calculate can be easily reconstructed from msn-notation

[11] SecDec, FIESTA

More examples

								
A[1]	orig.	215	8851	fail	78	7529	8854	7126
	uniq.	12	2208		28	3215	2603	4264
B/C[1]	orig.	96	1080	14520	48	1080	1080	960
	uniq.	4	135	1452	12	270	135	480
X[1]	orig.	96	1170	15350	48	1304	1114	986
	uniq.	8	287	2182	12	324	215	584
S[2]	orig.	96	1080	14520	16	180	216	210
	uniq.	4	135	1452	4	45	27	105

[1] modified sector_decomposition code from C.Bogner and S. Weinzierl
[2] original implementation

How to find isomorphic graphs? graph_state library

- original algorithm[12] used by B. Nickel et al. for graph identification
- generalized algorithm
  - directed and undirected graphs
  - different types of edges
  - multi-index labels
- can be used for
  - graph identification
  - calculating symmetry factors
  - finding graph automorphisms
- complexity of algorithm is close to $O(N)$ (N – number of verticies)
- written on Python (Win/Lin/Mac)
- available at http://code.google.com/p/rg-graph/downloads

[12] **Nickel B., Meiron D., Baker G.** - University of Guelf Report,1977
J.F.Nagle, J. Math. Phys. 7, 1588 (1966)

Thank you for attention!