LiteRed: a new powerful tool for the reduction of multiloop integrals

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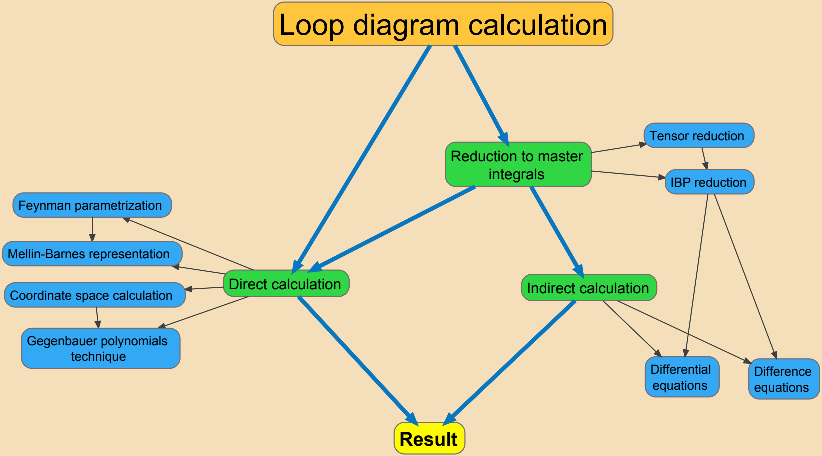
Outline

1. Introduction
   - Automatic loop calculations
   - Relations between integrals: symmetries, IBP, LI
   - Equations for the masters: differential and dimensional recurrences
2. LiteRed program
   - Description of LiteRed
   - Installation
   - Example
3. Summary

General path of the loop calculations



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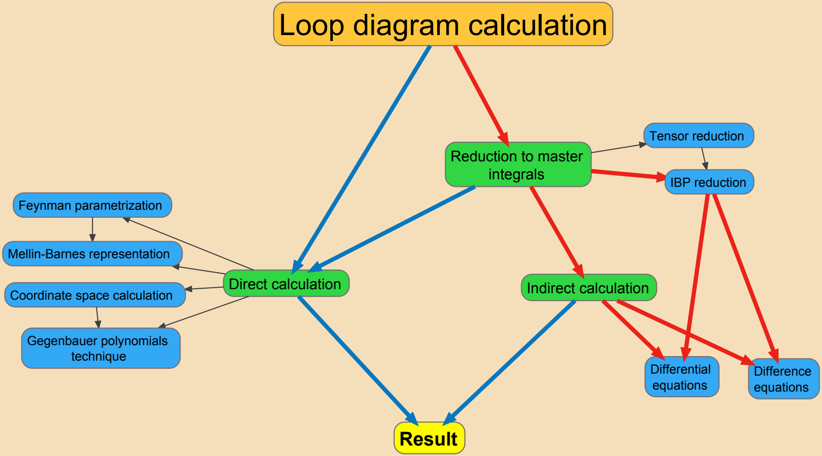
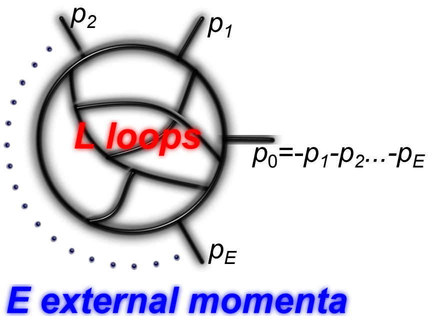


Loop Integral

L loop, $E+1$ legs



Loop integral

$$J(\mathbf{n}) = \int \frac{d^{\mathscr{D}} l_1}{\pi^{\mathscr{D}/2}} \cdots \frac{d^{\mathscr{D}} l_L}{\pi^{\mathscr{D}/2}} j(\mathbf{n})$$

$$= \int \frac{d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L}{\pi^{\frac{L\mathscr{D}}{2}} D_1^{n_1} \ldots D_N^{n_N}}$$

$D_1, \ldots, D_M$ — denominators of the diagram,
$D_{M+1}, \ldots, D_N$ conveniently chosen numerators.

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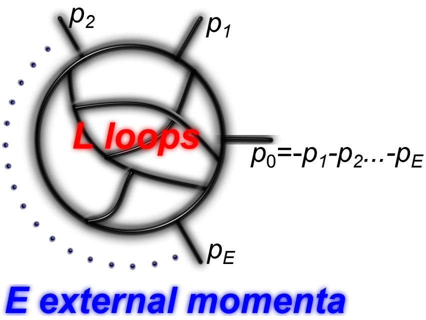

Loop integral

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$$= \int \frac{d^{\mathscr{D}} l_1 \dots d^{\mathscr{D}} l_L}{\pi^{\frac{L\mathscr{D}}{2}} D_1^{n_1} \dots D_N^{n_N}}$$

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Prerequisites

All D_k linearly depend on $s_{ij} = l_i \cdot q_j$, any s_{ij} can be expressed via D_k. $\Longrightarrow$ $N = \#s_{ij} = L(L+1)/2 + LE$

Notation

$$q_{1,..L} = l_{1,..L}$$
$$q_{L+1,..L+E} = p_{1,..E}$$

Operator representation

Operators $A_1, \ldots, A_N, B_1, \ldots, B_N$

In order to write identities between integrals with different indices, it is convenient to introduce the operators:

$$(A_\alpha f)(n_1, \ldots, n_N) = n_\alpha f(n_1, \ldots, n_\alpha + 1, \ldots, n_N),$$
$$(B_\alpha f)(n_1, \ldots, n_N) = f(n_1, \ldots, n_\alpha - 1, \ldots, n_N).$$

Commutator

$$[A_\alpha, B_\beta] = \delta_{\alpha\beta}$$

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Compact form of identities

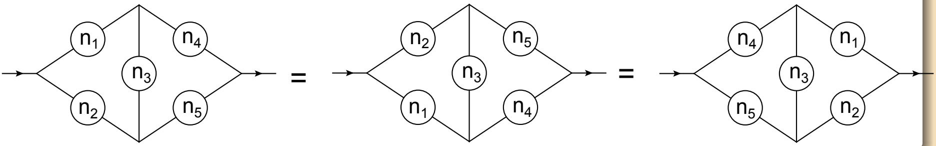
$$n_1 J(n_1 + 1, n_2) = J(n_1, n_2 - 1) + J(n_1, n_2) \implies A_1 J = B_2 J + J$$

Relations between the integrals

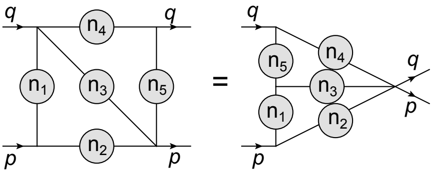
Symmetries

The **symmetry relations** arise from the shifts of loop momenta which map denominators to denominators.

Usually they correspond to the symmetries of the graph



But sometimes they don’t



Relations between the integrals

IBP&LI identities

The **integration-by-part identities** arise due to the fact, that, in dimensional regularization the integral of the total derivative is zero (Tkachov 1981, Chetyrkin and Tkachov 1981)

IBP identities

$$\int d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L \, O_{ij} j(\mathbf{n}) = 0 \qquad \text{(IBP)}$$

IBP operators

$O_{ij} = \frac{\partial}{\partial l_i} \cdot q_j$

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The **Lorentz-invariance identities** arise due to the fact that loop integrals are scalar functions of the external momenta (Gehrmann and Remiddi 2000).

LI identities

$$p_{1\mu} p_{2\nu} M^{\mu\nu} J = 0 \qquad \text{(LI)}$$

Lorentz generators

$M^{\mu\nu} = \sum_e p_e^{[\mu} \partial_e^{\nu]}$

Reduction and Calculation

Important fact!

Reduction not only reduces the number of integrals to be calculated. It also allows one to obtain for the master integrals the closed systems of equations: differential and/or difference. Solving these equations is often simpler then the direct integration.

Differential equations

Differentiating with respect to external parameter and performing IBP reduction of the result, we obtain **differential equation** for a given master integral(Kotikov 1991, Remiddi 1997).

Differential equation

$$\frac{\partial}{\partial a} J = f(a) J + h(a). \qquad \text{(DE)}$$

External parameter

$$s = \begin{cases} \text{mass} & \text{(Kotikov, 1991)} \\ \text{invariant of } p_e & \text{(Remiddi, 1997)} \end{cases}$$

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- ***n*-scale integrals ($n \geq 2$) can be investigated by the differential equation method.**
  Initial conditions for the differential equation are put in the point where the chosen parameter is expressed via the rest (or equal to $0, \infty$) $\Longrightarrow$ The problem is reduced to the calculation of integrals with $n-1$ scales.

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  Initial conditions for the differential equation are put in the point where the chosen parameter is expressed via the rest (or equal to $0, \infty$) $\Longrightarrow$ The problem is reduced to the calculation of integrals with $n-1$ scales.
- **One-scale integrals have obvious dependence on this scale. Differential equations cannot help.**

Dimensional recurrences

Dimesional recurrence relation (Tarasov 1996) relates integrals in d and $d+2$ dimensions. For the integral without numerators, representable by a graph, it reads:

Dimensional recurrence

$$J^{(d-2)}(\mathbf{n}) = \mu^L \sum_{\text{trees}} \left(\prod_{\text{chords}} A_k \right) J^{(d)}(\mathbf{n}) \,.$$

For automatic derivation of DRR it is convenient to use explicit formula without any reference to a graph (see below).

Equation for the masters

Reducing right-hand side, we obtain difference equations for the master.

Description of LiteRed

One more reduction package?

Many reduction packages on the market: FIRE, Reduze, etc., why creating another?

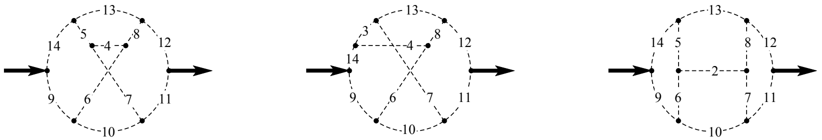
- The reduction is based on search of the universal rules, not on Laporta algorithm. The rules can be then used in LiteRed itself, or other programs.
- The search of symmetries works very fast. It determines not only graphic symmetries, but all shift symmetries.
- The convenient tools for the construction of differential equations and dimensional recurrence relations.

Installation

1. Download the archived package from it site: http://www.inp.nsk.su/∼lee/programs/LiteRed/
2. Unpack to your *UserBaseDirectory*/Applications, where *UserBaseDirectory* can be determined by evaluating `$UserbaseDirectory` in *Mathematica* session.

Example:Baikov&Chetyrkin integrals

Four-loop massless propagators (Baikov 2006, Baikov and Chetyrkin 2010)




```
SetDirectory[NotebookDirectory[]](*setting working directory*)
<<LiteRed` (*loading package*)
SetDim[d]; (*dimension variable*)
Declare[{l1, l2, l3, l4, q}, Vector];(*l1,l2,l3,l4 --- loop momenta*)
sp[q, q] = 1;(*q --- incoming momentum *)
NewBasis[p4, {l4-q,l4-l2,l2-l3+l4,l1-l2+l3-l4,l4-l1,l1-l2,l2-l3,
l3-l4,l1+q,l2+q,l3+q,l3,l4,l1},{l1,l2,l3,l4},Directory->"p4 dir"]
(*1st argument --- list of D_i,2nd argument --- list of loop mom.,
"p4 dir" --- directory to save rules and more*)
```

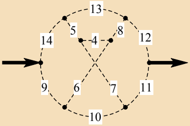
Example:Baikov&Chetyrkin integrals

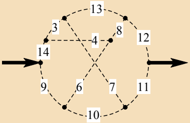
IBP generation

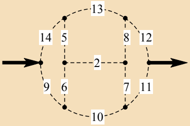
```
In[]:=GenerateIBP[p4];(*generating ibp identities*)
Integration-By-Part&Lorentz-Invariance identities are generated.
    IBP[p4]  --- integration-by-part identities,
    LI[p4] --- Lorentz invariance identities.
In[]:=IBP[p4][n1,n2,n3,n4,n5,n6,n7,n8,n9,n10,n11,n12,n13,n14]
(*print IBP relations*)
Out[]:=A very large output was generated. Here is a sample of it:
{n6 j[p4,n1,-1+n2,n3,n4,n5,1+n6,n7,n8,n9,n10,n11,n12,n13,n14]
+n6 j[p4,n1,n2,-1+n3,n4,n5,1+n6,n7,n8,n9,n10,n11,n12,n13,n14]
+<<17>>
-n4 j[p4,n1,n2,n3,1+n4,n5,n6,n7,n8,n9,n10,n11,n12,n13,-1+n14],
<<19>>}
```

Example:Baikov&Chetyrkin integrals

Zero sectors search



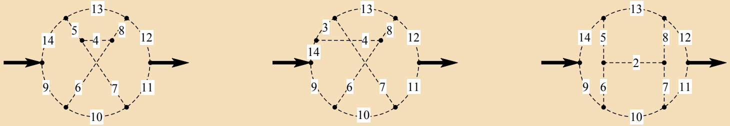





```
In[]:=Timing[AnalyzeSectors[p4,{0,0,0,__}|{0,0,_,_,0,__}|{0,_,0,0,__}];]
Found 2882 zero sectors out of 4096.
    ZeroSectors[p4] --- zero sectors,
    NonZeroSectors[p4] --- nonzero sectors,
    SimpleSectors[p4] --- simple sectors (no nonzero subsectors),
    BasisSectors[p4] --- basis sectors (at least one immediate subsect
    ZerojRule[p4] --- a rule to nullify all zero j[p4,...].
Out[]:={136, Null}
```

Example:Baikov&Chetyrkin integrals

Zero sectors search

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In[]:=Timing[AnalyzeSectors[p4,{0,0,0,__}|{0,0,_,_,0,__}|{0,_,0,0,__}];]
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    ZerojRule[p4] --- a rule to nullify all zero j[p4,...].
Out[]:={136, Null}
```

Algorithm

In version 1.3 the search is based on Feynman parameterization and works flawlessly for all cases. In particular, the sectors containing massless onshell propagators are now detected as zero.

Example:Baikov&Chetyrkin integrals

Symmetries search

In[]:=Timing[`FindSymmetries`[p4];]

```
Found 1110 mapped sectors and 104 unique sectors.
    UniqueSectors[p4] --- unique sectors.
    MappedSectors[p4] --- mapped sectors.
    SR[p4][...] --- symmetry relations for j[p4,...] from UniqueSector
    jSymmetries[p4,...] --- symmetry rules for the sector js[p4,...] i
    jRules[p4,...] --- reduction rules for j[p4,...] from MappedSector
Out[]:={120, Null}
```

Algorithm

1. The equivalent simple sectors are determined from FP (Very similar to A. Pak's `TSort` (Pak 2012))
2. The symmetries for simple sectors are determined from momenta shifts
3. The symmetries for higher sectors are picked up from those of simple sectors.

Example:Baikov&Chetyrkin integrals

Heuristic search for the reduction rules

```
In[]:=Timing[SolvejSector[UniqueSectors[p4],DiskSave->True]]
Sector js[p4,0,0,0,0,0,1,1,1,1,0,0,0,1,0]
    Master integrals found: j[p4, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0,
    jRules[p4, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 1, 0] --- reduction
    MIs[p4] --- updated list of the masters.
...
Sector js[p4,0,1,0,0,1,1,1,1,1,1,1,1,1,1]
    Master integrals found: j[p4, 0, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1,
    jRules[p4, 0, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] --- reduction
    MIs[p4] --- updated list of the masters.
Out[]:={~35hours,{1,1,1,1,0,0,0,1,...0,1,1,1}}
(*output lists  the number of masters in each sector*)
```

Example:Baikov&Chetyrkin integrals

Form and application of the found rules

```
In[]:=jRules[p4, 0, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
{j[p4,n1_?NonPositive,n2_?NonPositive,n3_?NonPositive,n4_?Positive,
n5_?Positive,n6_?Positive,n7_?Positive,n8_?Positive,n9_?Positive,n10_?Positive,
n11_?Positive,n12_?Positive,n13_?Positive,n14_?Positive]/;!(n1==0||n13==1)->
<<48>>+<<1>>/<<1>>+((-1-n1) j[p4,<<14>>])/(-1+n13),<<127>>,
j[p4,n1_?NonPositive,<<11>>,n13_?Positive,n14_?Positive]/;<<1>>-><<1>>}
In[]:=Timing[IBPReduce[j[p4, -1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]]]
{~20min, <<1>>}
(**)
```

Example:Baikov&Chetyrkin integrals

Aditional tools:differentiation

```
In[]:=Dinv[2*j[p4, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1],sp[q,q]]/.sp[q,q]-
>1(*derivative ∂/∂q^2*)
Out[]:=
j[p4,0,-1,0,1,1,1,1,1,1,2,1,1,1,1]+j[p4,0,0,-1,1,1,1,1,1,1,2,1,1,1,1]
-j[p4,0,0,0,1,1,1,0,1,1,2,1,1,1,1]-j[p4,0,0,0,1,1,1,1,0,1,2,1,1,1,1]
-3j[p4,0,0,0,1,1,1,1,1,1,1,1,1,1,1]+j[p4,0,0,0,1,1,1,1,1,1,2,0,1,1]
-j[p4,0,0,0,1,1,1,1,1,1,1,2,1,1,1]+j[p4,0,0,0,1,1,1,1,1,2,1,0,1,1]
-j[p4,0,0,0,1,1,1,1,1,1,2,1,1,0,1]-j[p4,0,0,0,1,1,1,1,1,1,2,1,1,1,1]
+j[p4,0,0,0,1,1,1,1,1,2,1,1,1,1,0]-j[p4,0,0,0,1,1,1,1,1,2,1,1,1,1,1]
In[]:=IBPReduce[%]
Out[]:=(4d-22)*j[p4, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
(**)
```

Example:Baikov&Chetyrkin integrals

The algorithm for dimensional recurrence relations

LiteRed uses

Lowering&Raising DRR from Baikov's formula (Lee 2010)

$$J^{(d+2)}(\mathbf{n}) = \frac{(2\mu)^L \left[V(p_1,\ldots,p_E)\right]^{-1}}{(\mathscr{D}-E-L+1)_L} P(B_1,\ldots,B_N)\, J^{(d)}(\mathbf{n})\,. \qquad \text{(LDRR)}$$

$$J^{(d-2)}(\mathbf{n}) = \mu^L \det\left[\sum_k \frac{\partial D_k}{\partial s_{ij}} A_k \bigg|_{i,j=1,\ldots L}\right] J^{(d)}(\mathbf{n})\,. \qquad \text{(RDRR)}$$

Work fine for the integrals with numerators, for non-standard denominators, and for non-graphical integrals.

Example:Baikov&Chetyrkin integrals

Aditional tools:dimensional recurrences

```
In[]:=RaisingDRR[p4, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
(*right-hand side of the raising DRR*)
Out[]:=
j[p4,0,0,0,1,1,1,1,1,1,2,1,2,2,2]+j[p4,0,0,0,1,1,1,1,1,1,2,2,1,2,2]
+j[p4,0,0,0,1,1,1,1,1,2,2,1,2,2,1]+j[p4,0,0,0,1,1,1,1,1,2,2,2,1,2,1]
+<<161>>+j[p4,0,0,0,2,2,2,1,1,1,1,2,1,1,1]
+j[p4,0,0,0,2,2,2,1,1,1,2,1,1,1,1]+j[p4,0,0,0,2,2,2,1,1,2,1,1,1,1,1]
```

Example:Baikov&Chetyrkin integrals

Aditional tools:dimensional recurrences

```
In[]:=RaisingDRR[p4, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
(*right-hand side of the raising DRR*)
Out[]:=
j[p4,0,0,0,1,1,1,1,1,1,2,1,2,2,2]+j[p4,0,0,0,1,1,1,1,1,1,2,2,1,2,2]
+j[p4,0,0,0,1,1,1,1,1,2,2,1,2,2,1]+j[p4,0,0,0,1,1,1,1,1,2,2,2,1,2,1]
+<<161>>+j[p4,0,0,0,2,2,2,1,1,1,1,2,1,1,1]
+j[p4,0,0,0,2,2,2,1,1,1,2,1,1,1,1]+j[p4,0,0,0,2,2,2,1,1,2,1,1,1,1,1]
In[]:=Timing[IBPReduce[%]]
Out[]:={~2.5hours, <<1>>}
(**)
```

Summary

- LiteRed is publicly available from http://www.inp.nsk.su/∼lee/programs/LiteRed/
- LiteRed implements a new approach to the reduction: heuristic search+application.
- The rules found can be reused and shared (see ready-to-use bases on package web site).
- The results obtain in LiteRed can be used in other programs to dramatically extend their limits (import already implemented in FIRE by A. Smirnov and V. Smirnov).
- LiteRed also contains some convenience tools, like constrution of the differential and DRR equations.
- Outlook: FP-based search of the symmetries
- Outlook: major improvements in the heuristic search

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