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“FAPT”: a Mathematica package

for calculations in

QCD Fractional Analytic Perturbation Theory

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Motivation

Analytic Perturbation Theory **(APT)** [Shirkov, Solovtsov (1996,1997)]

Fractional Analytic Perturbation Theory **(FAPT)** [Bakulev, Mikhailov, Stefanis (2005-2010)], [Bakulev, Karanikas, Stefanis (2007)]

APT and FAPT:

- Closed theoretical scheme without singularities and additional parameters;
- RG-invariance, Q^2-analyticity;
- Power PT set in $\overline{MS}$-scheme $\{\bar{\alpha}_{\rm s}^k(Q^2)\}$ $\Rightarrow$ a non-power APT expansion set $\bar{\mathcal{A}}_k(Q^2)$ **in Euclidian domain** and $\bar{\mathfrak{A}}_k(s)$ **in Minkowskian domain** with both $\bar{\mathcal{A}}_k(Q^2)$ and $\bar{\mathfrak{A}}_k(s)$ regular in the IR region.

$$
\begin{array}{rclr}
\bar{\alpha}_{\rm s}^k & \to & \bar{\mathcal{A}}_k, & \bar{\mathfrak{A}}_k \\
\sum d_k \bar{\alpha}_{\rm s}^k & \to & \sum d_k \bar{\mathcal{A}}_k, & \sum d_k \bar{\mathfrak{A}}_k \\
d_k & - & \text{numbers} &
\end{array}
$$

Functions $\bar{\mathcal{A}}_k(Q^2)$ and $\bar{\mathfrak{A}}_k(s)$ cannot be trivial calculated in higher orders.
The main our goal is to simplify the calculations in framework of APT&FAPT.

Introduction

This talk is based on recent publication A.P. Bakulev and V.L. Khandramai *Comp. Phys. Comm.* **184** (2013) 183-193. All relevant formulas which are necessary for the running of $\bar{\mathcal{A}}_\nu[L]$, $L = \ln(Q^2/\Lambda^2)$ and $\bar{\mathfrak{A}}_\nu[L_s]$, $L_s = \ln(s/\Lambda^2)$ in Minkowskian domain in framework of FAPT are collected.

- We provide here easy-to-use `Mathematica` system procedures collected in the package **"FAPT"**
- Our package is organized as well-known package **RunDec** [Chetyrkin, Kühn, Steinhauser (2000)]
- This task has been partially realized for APT as the `Maple` package **QCDMAPT** and as the `Fortran` package **QCDMAPT_F** [Nesterenko, Simolo (2010)]

Outline:

1. Theoretical framework: *from standard PT to Analytic Perturbation Theory and its generalization – Fractional APT*;
2. APT/FAPT Applications: Bjorken sum rule and RG-evolution;
3. Package ``FAPT'': *description of procedures and examples of usage.*

Theoretical Framework

Running coupling

The QCD running of the coupling constant $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s[L]$ is defined through RG equation

$$\frac{d\, a_s[L]}{d\, L} = -a_s^2 - c_1\, a_s^3 - c_2\, a_s^4 - c_1\, a_s^3 - \ldots, \quad L = \ln\left(\frac{\mu^2}{\Lambda^2}\right), \quad c_k(n_f) \equiv \frac{b_k(n_f)}{b_0(n_f)^{k+1}}\,.$$

The solutions of RG equation in $\overline{MS}$-scheme can be obtained exactly at LO and NLO

$$a_s^{(1)}[L] = \frac{1}{L}\,; \quad \textbf{(LO)}$$

$$a_s^{(2)}[L; n_f] = \frac{-c_1^{-1}(n_f)}{1 + W_{-1}\left(z_W[L]\right)} \quad \text{with} \quad z_W[L] = -c_1^{-1}(n_f)\, e^{-1-L/c_1(n_f)}\,. \quad \textbf{(NLO)}$$

The higher-order solutions $a_s^{(\ell)}[L; n_f]$ can be expanded in powers of the two-loop one, $a_s^{(2)}[L; n_f]$, it has been suggested in [Kourashev, Magradze, (1999-2003)]:

$$a_s^{(\ell)}[L; n_f] = \sum_{n\geq 1} C_n^{(\ell)} \left(a_s^{(2)}[L; n_f]\right)^n.$$

Heavy quark mass thresholds

For realization of so-called "global scheme" it needs to match the running coupling values in Euclidian domain at Q corresponding to quark masses:

$$\begin{aligned}
\alpha_s^{(\ell)}\left[L_4(\Lambda_3);3\right] &= \alpha_s^{(\ell)}\left[L_4(\Lambda_3)+\lambda_4;4\right]\,;\\
\alpha_s^{(\ell)}\left[L_5(\Lambda_3)+\lambda_4;4\right] &= \alpha_s^{(\ell)}\left[L_5(\Lambda_3)+\lambda_5;5\right]\,;\\
\alpha_s^{(\ell)}\left[L_6(\Lambda_3)+\lambda_5;5\right] &= \alpha_s^{(\ell)}\left[L_6(\Lambda_3)+\lambda_6;6\right]\,;
\end{aligned}$$

where $L_k(\Lambda_3) \equiv \ln\left(M_k^2/\Lambda_3^2\right)$ at the thresholds M_k ($M_4 = m_c$, $M_5 = m_b$ and $M_6 = m_t$). We use all logarithms L with depending of three-flavor scale QCD Λ_3:

$$\begin{aligned}
\alpha_s^{\text{glob};(\ell)}(Q^2,\Lambda_3) &= \alpha_s^{(\ell)}\left[L(Q^2);3\right]\theta\left(Q^2<M_4^2\right)\\
&+ \alpha_s^{(\ell)}\left[L(Q^2)+\lambda_4^{(\ell)}(\Lambda_3);4\right]\theta\left(M_4^2\leq Q^2<M_5^2\right)\\
&+ \alpha_s^{(\ell)}\left[L(Q^2)+\lambda_5^{(\ell)}(\Lambda_3);5\right]\theta\left(M_5^2\leq Q^2<M_6^2\right)\\
&+ \alpha_s^{(\ell)}\left[L(Q^2)+\lambda_6^{(\ell)}(\Lambda_3);6\right]\theta\left(M_6^2\leq Q^2\right),
\end{aligned}$$

and recalculate $\alpha_s^{(\ell)}$ to all other scales with the help of finite additions $\lambda_k \equiv \ln\left(\Lambda_3^2/\Lambda_k^2\right)$.

Why we need APT and FAPT?

For standard QCD PT in Euclidian domain the expressions for hadronic quantities $F\left(Q^2=\mu^2,\,\bar{a}_s\right)$ are based on expansions in a series over the powers of running coupling

$$F[L] = 1 + f_1\,\bar{a}_s[L] + f_2\,\bar{a}_s^2[L] + f_3\,\bar{a}_s^3[L] + f_4\,\bar{a}_s^4[L] + \ldots .$$

Hadronic quantities calculated in terms of a power-series are not everywhere well defined because of unphysical **coupling singularities**: $\bar{a}_s^{(1)}[L] = 1/L$, $\bar{a}_s^{(2)}[L] \sim 1/\sqrt{L + c_1 \ln c_1}$,...

In standard QCD PT we have not only power series **but also:**

- RG-improvement to account for higher-orders $\to$

$$Z[L] = \mathbf{exp}\left\{\int \frac{\gamma(a)}{\beta(a)}\,da\right\} \xrightarrow{\text{1-loop}} \left[\bar{a}_s[L]\right]^{\gamma_0/(2\beta_0)}\,;$$

- Factorization $\to (\bar{a}_s[L])^n\,L^m$;
- Two-loop case $\to (\bar{a}_s)^{\nu}\,\ln(\bar{a}_s)$.

Basics of APT

The analytic images of the strong coupling powers:

$$\bar{a}^n \xrightarrow{Euclidian} \bar{\mathcal{A}}_n^{(\ell)}[L;n_f] = \int_0^\infty \frac{\bar{\rho}_\nu^{(\ell)}(\sigma;n_f)}{\sigma+Q^2}\,d\sigma\,, \quad \bar{a}^n \xrightarrow{Minkowskian} \bar{\mathfrak{A}}_n^{(\ell)}[L_s;n_f] = \int_s^\infty \frac{\bar{\rho}_n^{(\ell)}(\sigma;n_f)}{\sigma}\,d\sigma$$

are defined through the spectral density

$$\bar{\rho}_n^{(\ell)}[L;n_f] = \frac{1}{\pi}\,\mathbf{Im}\left(\bar{\alpha}_{\rm s}^{(\ell)}\left[L-i\pi;n_f\right]\right)^n\,.$$

Leading order (integer powers):

$$\bar{\rho}_1^{(1)}(\sigma) = \frac{4}{b_0}\,\mathbf{Im}\,\frac{1}{L_\sigma - i\pi} = \frac{4\pi}{b_0}\,\frac{1}{L_\sigma^2+\pi^2}\,.$$

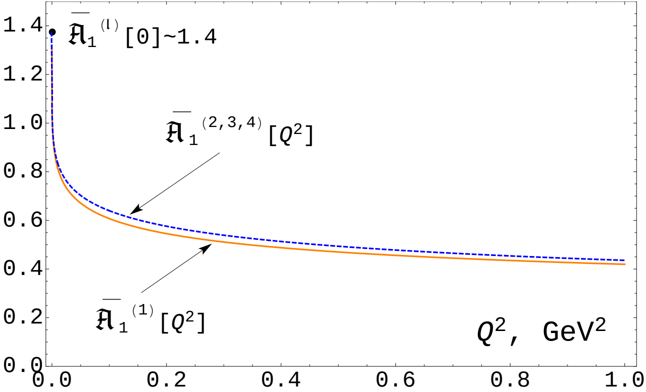
$\mathcal{A}_1^{(1)}$ [Shirkov, Solovtsov (1996, 1997)] and $\mathfrak{A}_1^{(1)}$ [Jones, Solovtsov (1995); Jones, Solovtsov, Solovtsova (1995); Milton, Solovtsov (1996)]:

$$\begin{aligned} \bar{\mathcal{A}}_1^{(1)}[L] &= \frac{4\pi}{b_0}\left(\frac{1}{L} - \frac{1}{e^L-1}\right)\,, \quad L = \ln\left(Q^2/\Lambda^2\right)\,; \\ \bar{\mathfrak{A}}_1^{(1)}[L_s] &= \frac{4}{b_0}\,\arccos\left(\frac{L_s}{\sqrt{L_s^2+\pi^2}}\right)\,, \quad L_s = \ln\left(s/\Lambda^2\right)\,. \end{aligned}$$

APT Running Coupling

All APT couplings have the following properties:

- Universal finite IR values: $\bar{\mathcal{A}}(0) = \bar{\mathfrak{A}}(0) = 4\pi/b_0$;
- Week dependence on the number of loops;
- Correspondence with perturbative $\bar{\alpha}_s(Q^2)$ at $Q^2 \gg 1$ GeV^2.



FAPT: one-loop Euclidian $\bar{\mathcal{A}}_\nu[L]$

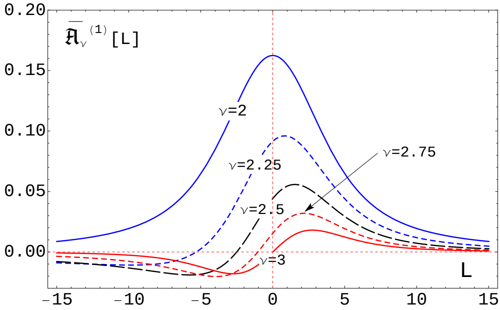
Euclidian coupling:

$$\bar{\mathcal{A}}_\nu^{(1)}[L] = \frac{4\pi}{b_0}\left[\frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}\right], \quad L = \ln\left(\frac{Q^2}{\Lambda^2}\right).$$

$F(z,\nu)$ is Lerch transcendent function.

Properties:

- $\bar{\mathcal{A}}_0[L] = 1$;
- $\bar{\mathcal{A}}_{-m}[L] = L^m$ for $m \in \mathbb{N}$;
- $\bar{\mathcal{A}}_m[L] = (-1)^m \bar{\mathcal{A}}_m[-L]$ for $m \geq 2\,, \; m \in \mathbb{N}$;
- $\bar{\mathcal{A}}_m[\pm\infty] = 0$ for $m \geq 2\,, \; m \in \mathbb{N}$.



FAPT: one-loop Minkowskian $\bar{\mathfrak{A}}_\nu[L]$

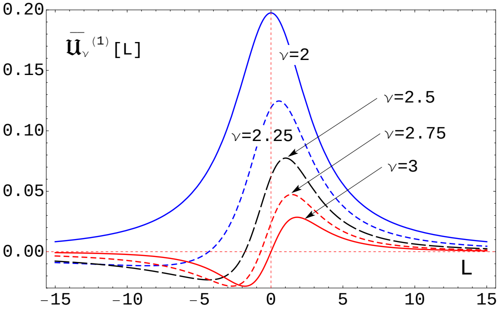
Minkowskian coupling:

$$\bar{\mathfrak{A}}_\nu^{(1)}[L] = \frac{4}{b_0}\frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}, \quad L_s = \ln\left(\frac{s}{\Lambda^2}\right).$$

Here we need the elementary functions only.

Properties:

- $\bar{\mathfrak{A}}_0[L] = 1; \quad \bar{\mathfrak{A}}_{-1}[L] = L;$
- $\bar{\mathfrak{A}}_{-2}[L] = L^2 - \frac{\pi^2}{3}, \quad \bar{\mathfrak{A}}_{-3}[L] = L\left(L^2 - \pi^2\right), \ \ldots;$
- $\bar{\mathfrak{A}}_m[L] = (-1)^m \bar{\mathfrak{A}}_m[-L]$ for $m \geq 2, \ m \in \mathbb{N};$
- $\bar{\mathfrak{A}}_m[\pm\infty] = 0$ for $m \geq 2, \ m \in \mathbb{N}.$



APT/FAPT Applications

Non-power APT expansions

The polarized Bjorken Sum Rule is given by a sum of two series in powers of α_s and OPE higher twists corrections μ_{2i}^{p-n}:

$$\begin{aligned} \Gamma_1^{p-n}(Q^2) &= \frac{|g_A|}{6} C_{\mathrm{Bj}} + \sum_{i=2}^{\infty} \frac{\mu_{2i}^{p-n}}{Q^{2i-2}}\,, \\ C_{\mathrm{Bj}}(Q^2) &\equiv 1 - \Delta_{\mathrm{Bj}}(Q^2)\,,\ |g_A| = 1.2701 \pm 0.0025\,. \end{aligned}$$

Perturbative power-correction:

[Baikov, Chetyrkin, Kühn (2010)]

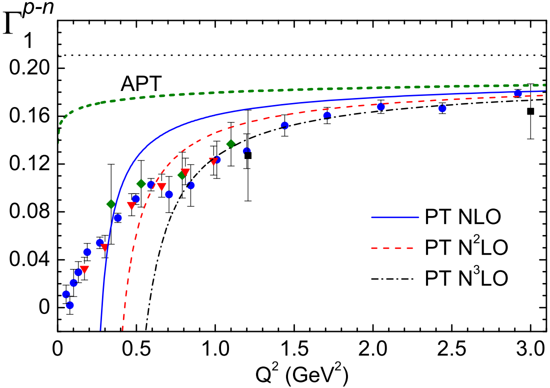
$$\Delta_{\mathrm{Bj}}^{\mathrm{PT}}(Q^2) = 0.318\,\bar{\alpha}_s(Q^2) + 0.363\,\bar{\alpha}_s^2(Q^2) + 0.652\,\bar{\alpha}_s^3(Q^2) + 1.804\,\bar{\alpha}_s^4(Q^2) + \ldots$$

Instead of universal power-in-α_s expansion in APT one should use non-power functional expansions:

$$\Delta_{\mathrm{Bj}}^{\mathrm{APT}}(Q^2) = 0.318\bar{\mathcal{A}}_1(Q^2) + 0.363\bar{\mathcal{A}}_2(Q^2) + 0.652\bar{\mathcal{A}}_3(Q^2) + 1.804\bar{\mathcal{A}}_4(Q^2) + \ldots$$

Loop stabilization

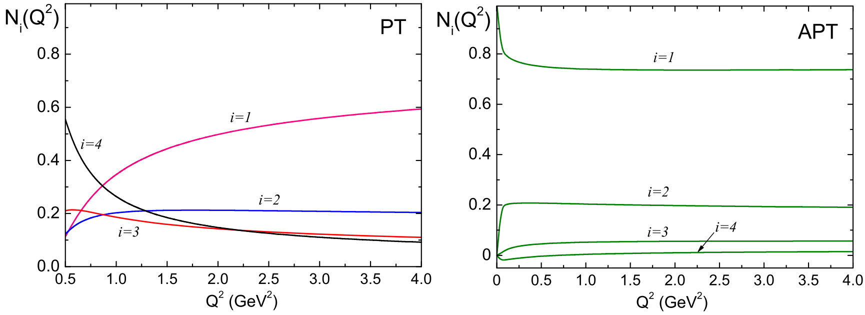
The $\Gamma_1^{p-n}(Q^2)$ (**with HT=0**) in both the standard **PT** and **APT** approaches with the combined set of the Jefferson Lab and SLAC data.



- The behavior of the **PT** curves is changed order by order.
- The **APT** curves in all three orders practically coincide with each other.
- The deviation of the APT curves from the data shows for necessity of the higher-twist contribution.

Convergence

The relative contributions of separate terms in **PT** and **APT** expansions for $\Delta_{\mathrm{Bj}}(Q^2)$, $\mathrm{N}_i(Q^2) = \delta_i(Q^2)/\Delta_{\mathrm{Bj}}(Q^2)$, as a functions of Q^2 from [Khandramai *et. al* (2012)]



- Asymptotic structure manifestation: in the region $Q < 1$ GeV the dominant contribution to the **PT** correction comes from the α_s^4-term; its relative contribution increases with decreasing Q.
- Good loop convergence in **APT**: the higher order contributions are stable at all Q^2 values; the 3^{rd} and 4^{th} terms contribute less than 5% and 1% respectively.

The RG evolution of the higher-twist $\mu_4^{p-n}(Q^2)$

See [Pasechnik *et al.* (PRD,2010)]

The expression for the evolution of the higher twist μ_4^{p-n} in the Bjorken sum rule is known:

$$\mu_{4,\mathrm{PT}}^{p-n}(Q^2) = \mu_{4,\mathrm{PT}}^{p-n}(Q_0^2)\left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right]^{\nu}, \quad \nu = \frac{\gamma_0}{8\pi\beta_0}, \quad \gamma_0 = \frac{16}{3}C_F\,, \quad C_F = \frac{4}{3}.$$

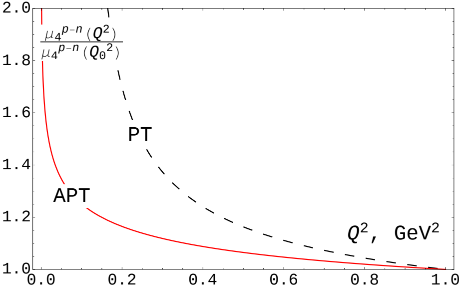
For the RG-evolution of the μ_4^{p-n} in the **Fractional APT** we need change the fractional powers of the QCD running coupling constant by the corresponding analytic images:

$$\mu_{4,\mathrm{APT}}^{p-n}(Q^2) = \mu_{4,\mathrm{APT}}^{p-n}(Q_0^2)\,\frac{\mathcal{A}_{\nu}^{(1)}(Q^2)}{\mathcal{A}_{\nu}^{(1)}(Q_0^2)}\,.$$

Note, at the same time the evolution of the higher twist μ_6^{p-n}, μ_8^{p-n}, ... is still unknown.

The RG evolution of the higher-twist $\mu_4^{p-n}(Q^2)$

The Q^2-dependence of evolution factors in the standard **PT** and **Fractional APT**.



- The influence of non-physical singularities complicate the analysis of the higher-twist evolution in the low-energy region of the standard **PT**.
- The evolution from 1 GeV^2 to Λ^2_{QCD} in the **APT** increases the absolute value of μ_4^{p-n} (normalized at $Q_0^2 = 1$ GeV^2) by about 20 %.

The RG evolution of the higher-twist $\mu_4^{p-n}(Q^2)$

Using the above-mentioned total APT expressions for Bjorken sum rule fitted to experimental data we extract the coefficients of the higher twist μ_4 OPE corrections without and with their RG-evolution.

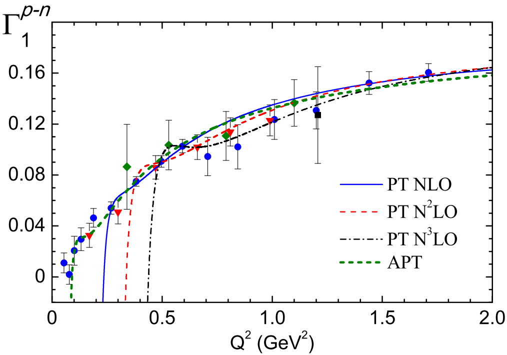
Method	Q^2_{min} GeV^2	μ_4/M^2	μ_6/M^4	μ_8/M^6
NNLO APT no evolution	0.47	-0.055(3)	-	-
	0.17	-0.062(4)	0.008(2)	-
	0.10	-0.068(4)	0.010(3)	-0.0007(3)
NNLO APT with evolution	0.47	-0.051(3)	-	-
	0.17	-0.056(4)	0.0087(4)	-
	0.10	-0.058(4)	0.0114(6)	-0.0005(8)

Note, we do not study account RG-evolution for the PT calculations since the only effect of that would be the enhancement of the Landau singularities by extra divergences at Λ_{QCD}, whereas at higher Q^2 the evolution is negligible.

Extracted values of μ_4^{p-n} in **APT** become more stable with respect to Q_{min} variations.

Higher twist analysis

The three-parametric $(\mu_4 + \mu_6 + \mu_8)$-fits of the Bjorken SR data in various orders of the **PT** and the all-order **APT**.



The APT application leads to accurate data description down to $Q^2 \sim 0.1$ GeV^2 always at the two-loop APT level.

Package ''FAPT''

''FAPT'' package review

Title of program: `FAPT`

Available from:

`http://theor.jinr.ru/~bakulev/fapt.mat/FAPT.m`

`http://theor.jinr.ru/~bakulev/fapt.mat/FAPT_Interp.m`

Computer: Any work-station or PC where `Mathematica` is running.

Operating system or monitor under which the program has been tested: Mathematica (versions 5,7,8).

''FAPT'' package contains the calculations of the required objects:

1. $\bar{\alpha}_{\mathrm{s}}^{(\ell)}[L, n_f]$, $\bar{\alpha}_{\mathrm{s}}^{(\ell);\mathrm{glob}}$
2. $\bar{\rho}^{(\ell)}[L_\sigma, n_f, \nu]$, $\rho^{(\ell);\mathrm{glob}}[L_\sigma, \nu, \Lambda_{n_f=3}]$
3. $\bar{\mathcal{A}}_{\nu}^{(\ell)}[L, n_f]$, $\mathcal{A}_{\nu}^{(\ell);\mathrm{glob}}[L, \nu, \Lambda_{n_f=3}]$
4. $\bar{\mathfrak{A}}_{\nu}^{(\ell)}[L, n_f]$, $\mathcal{A}_{\nu}^{(\ell);\mathrm{glob}}[L, \nu, \Lambda_{n_f=3}]$

Numerical parameters

The pole masses of heavy quarks and Z-boson, collected in the set NumDefFAPT (all mass variables and parameters are measured in GeVs):

$$\begin{array}{llll} \texttt{MQ4}: & M_c = 1.65 \text{ GeV}\,, & \texttt{MQ5}: & M_b = 4.75 \text{ GeV}\,; \\ \texttt{MQ6}: & M_t = 172.5 \text{ GeV}\,, & \texttt{MZboson}: & M_Z = 91.19 \text{ GeV}\,. \end{array}$$

** The package RunDec is using the set NumDef with slightly different values of these parameters ($M_c = 1.6$ GeV, $M_b = 4.7$ GeV, $M_t = 175$ GeV, $M_Z = 91.18$ GeV).*

Collection in the set setbetaFAPT the following rules of substitutions $b_i \to b_i(n_f)$

$$\texttt{b0}: b_0 \ \to\ 11 - \frac{2}{3}\, n_f\,, \quad \texttt{b1}, \quad \texttt{b2}, \quad \texttt{b3}.$$

** Here we follow the same substitution strategy as in RunDec, but our b_i differ from b_i^{RunDec} by factors 4^{i+1}: $b_i = 4^{i+1}\, b_i^{RunDec}$.*

α_s calculations

\[CapitalLambda]$\ell[\Lambda_3, n_f]$ returns ℓ-loop QCD scales with n_f active quarks by using matching conditions and three-flavors QCD scale Λ_3:

$$\texttt{\textbackslash[CapitalLambda]}\ell[\Lambda, k] = \Lambda\ell[\Lambda_3, n_f = k] = \Lambda_k^{(\ell)}(\Lambda_3)\,,\ (\ell = 1 \div 4\,;\ k = 2, 4 \div 6)\,,$$

\[Alpha]Bar$\ell[Q^2, n_f, \Lambda]$ returns ℓ-loop running QCD couplings at scale Q^2 with fixed n_f and QCD scale Λ:

$$\texttt{\textbackslash[Alpha]}\mathrm{Bar}\ell[Q^2, n_f, \Lambda] = \alpha\mathrm{Bar}\ell[Q^2, n_f, \Lambda] = \bar{\alpha}_s^{(\ell)}[\ln(Q^2/\Lambda^2); n_f],\ (\ell = 1 \div 4)$$

\[Alpha]Glob$\ell[Q^2, \Lambda_3]$ returns global ℓ-loop QCD coupling at scale Q^2 and QCD scale Λ_3, corresponds to $n_f = 3$:

$$\texttt{\textbackslash[Alpha]}\mathrm{Glob}\ell[Q^2, \Lambda_3] = \alpha\mathrm{Glob}\ell[Q^2, \Lambda_3] = \alpha_s^{\mathrm{glob};(\ell)}(Q^2, \Lambda_3)\,,\ (\ell = 1 \div 4)$$

Example 1 (for Standard PT)

We assume that the three-loop QCD scale is fixed at the value $\Lambda_3 = 0.387$ GeV. We want to evaluate the corresponding values of $\alpha_s^{glob;(\ell)}(Q^2, \Lambda)$ at the scale $Q = 4$ GeV.

```
In[1]:= SetDirectory[NotebookDirectory[]];
<< FAPT.m
<< RunDEC.m

In[2]:=  L33=0.387;
In[3]:=  Q1=4;
In[4]:=  A=\[Alpha]Glob3[Mb^2,L33]
Out[4]=  0.2383

————TEST with RunDEC————
WE USE OBTAINED RESULT AS INITIAL VALUE FOR AsRunDec
In[5]:=  Q2=10;

In[6]:=  {\[Alpha]Glob3[Q2^2,L33], AsRunDec[A,Q1,Q2,3]}
Out[6]=  {0.1837, 0.1846} —>>> ERROR LESS THEN 0.5%
DUE TO SMALL DIFFERENCE IN POLE QUARK MASSES
```

ρ_ν calculations

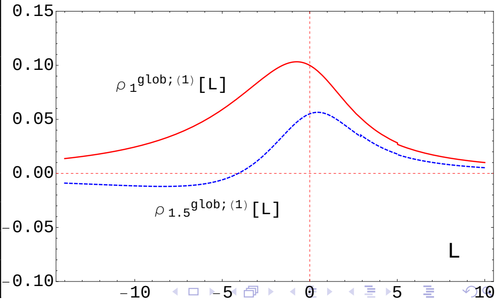
RhoBar$\ell[L, n_f, \nu]$ returns ℓ-loop spectral density $\bar{\rho}_\nu^{(\ell)}$ of fractional-power ν at $L = \ln(Q^2/\Lambda^2)$ and at fixed number of active quark flavors n_f:

$$\texttt{RhoBar}\ell[L, k, \nu] = \bar{\rho}_\nu^{(\ell)}[L; n_f = k]\,, \quad (\ell = 1 \div 4)$$

RhoGlob$\ell[L, \nu, \Lambda_3]$ returns the global ℓ-loop spectral density $\bar{\rho}_\nu^{(\ell);\text{glob}}[L; \Lambda_3]$ of fractional-power ν at $L = \ln(Q^2/\Lambda_3^2)$ and with Λ_3 corresponds to $n_f = 3$:

$$\texttt{RhoGlob}\ell[L, \nu, \Lambda_3] = \bar{\rho}_\nu^{(\ell);\text{glob}}[L; \Lambda_3]\,, \quad (\ell = 1 \div 4)$$

```
In[7]:= Show[
Plot[RhoGlob3[L, 1, 0.387],
{L, -14, 10},
PlotStyle -> {Thick, Red}],
Plot[RhoBar3[L, 3, 1],
{L, -14, 10},
PlotStyle -> {Thick,Dashed,
Blue}]]

Out[7]= see in Fig. right
```



$\bar{\mathcal{A}}_\nu$ and $\bar{\mathfrak{A}}_\nu$ calculations

AcalBar$\ell[L, n_f, \nu]$ returns ℓ-loop analytic image of fractional-power ν coupling $\bar{\mathcal{A}}_\nu^{(\ell)}[L; n_f]$ in Euclidian domain,

$$\texttt{AcalBar}\ell[L, k, \nu] = \bar{\mathcal{A}}_\nu^{(\ell)}[L; n_f = k]\,, \quad (\ell = 1 \div 4)$$

AcalGlob$\ell[L, \nu, \Lambda_3]$ returns ℓ-loop analytic image of fractional-power ν coupling $\mathcal{A}_\nu^{(\ell);glob}[L, \Lambda_3]$ in Euclidean domain

$$\texttt{AcalGlob}\ell[L, \nu, \Lambda_3] = \mathcal{A}_\nu^{(\ell);\text{glob}}[L, \Lambda_3]\,, \quad (\ell = 1 \div 4)$$

UcalBar$\ell[L, n_f, \nu]$ returns ℓ-loop analytic image of fractional-power ν coupling $\bar{\mathfrak{A}}_\nu^{(\ell)}[L, n_f]$ in Minkowskian domain

$$\texttt{UcalBar}\ell[L, k, \nu] = \bar{\mathfrak{A}}_\nu^{(\ell)}[L; n_f = k]\,, \quad (\ell = 1 \div 4)$$

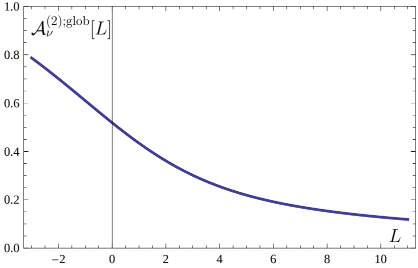
UcalGlob$\ell[L, \nu, \Lambda_3]$ returns ℓ-loop analytic image of fractional-power ν coupling $\mathfrak{A}_\nu^{(\ell);glob}[L, \Lambda_3]$ in Minkowskian domain

$$\texttt{UcalGlob}\ell[L, \nu, \Lambda_3] = \mathfrak{A}_\nu^{(\ell);glob}[L, \Lambda_3]\,, \quad (\ell = 1 \div 4)$$

Example 2 (for FAPT)

Construction of a two plot of $\mathcal{A}_{\nu}^{(2);\text{glob}}[L, \text{L23APT}]$ and $\mathfrak{A}_{\nu}^{(2);\text{glob}}[L, \text{L23APT}]$ for $L \in [-3, 11]$ with indication of the needed time (in seconds):

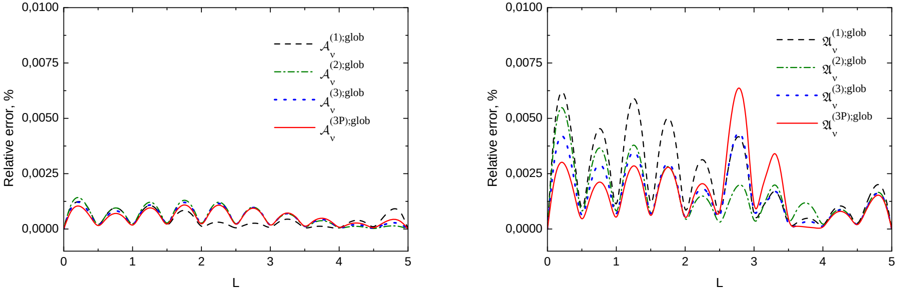
```
In[8]:= Plot[AcalGlob2[L,1,L23APT],{L,-3,11}]//Timing
Out[8]= {19.843, Graphics
(see in the left panel of Fig. below)}

In[9]:= Plot[UcalGlob2[L,1,L23APT],{L,-3,11}]//Timing
Out[9]= {14.656, Graphics
(see in the right panel of Fig. below)}
```



Interpolation

To obtain the fast results one can use the package ‘‘`FAPT_Interp`’’ which consists of procedures AcalGlobℓi$[L,\nu,\Lambda_3]$ and UcalGlobℓi$[L,\nu,\Lambda_3]$, which are based on interpolation using the basis of the precalculated data.

Relative error of the interpolation procedure for $\mathcal{A}^{glob}_{\nu=1.1}$ and $\mathfrak{A}^{glob}_{\nu=1.1}$.



The relative error for $\mathcal{A}_\nu^{(1);glob}$ is less than for $\mathfrak{A}_\nu^{(1);glob}$. In any case, using the same pre-computed data provides an error less than 0.01 %.

Summary

APT provides natural way for coupling constant and related quantities with

- Universal (loop independent) IR limit;
- Weak dependence on the number loops.

Fractional APT provides effective tool to apply Analytic approach for RG improved perturbative amplitudes.

This approaches are used in several applications:

- Higgs boson decay [Bakulev, Mikhailov, Stefanis (2007)];
- calculation of binding energies and masses of quarkonia [Ayala, Cvetič (2013)];
- analysis of the structure function $F_2(x)$ behavior at small values of x [Kotikov, Krivokhizhin, Shaikhatdenov (2012)];
- resummation approach [Bakulev, Potapova (2011)].

We collected in "FAPT" package all the procedures in APT and FAPT which are needed to compute the analytic images of the $\bar{\alpha}_s$ powers up to N^{3}LO based on the system Mathematica.

In memory of Alexander Bakulev

25.06.1956 – 28.09.2012

Professor
Alexander Bakulev

Leading Researcher, Bogolyubov Lab, JINR, Dubna

About 70 papers (*average citations per paper = 20*), including popular papers on **Fractional APT**:

A. Bakulev, S. Mikhailov, N. Stefanis, Phys.Rev. D72 (2005) **[CI=60]**

A. Bakulev, S. Mikhailov, N. Stefanis, Phys.Rev. D75 (2007) **[CI=47]**

A. Bakulev, Phys.Part.Nucl. 40 (2009) **[CI=30]**

Thanks for your attention!

And I thank especially organizers of ACAT2013
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