

Inclusive hadron production at low energy e^+e^-

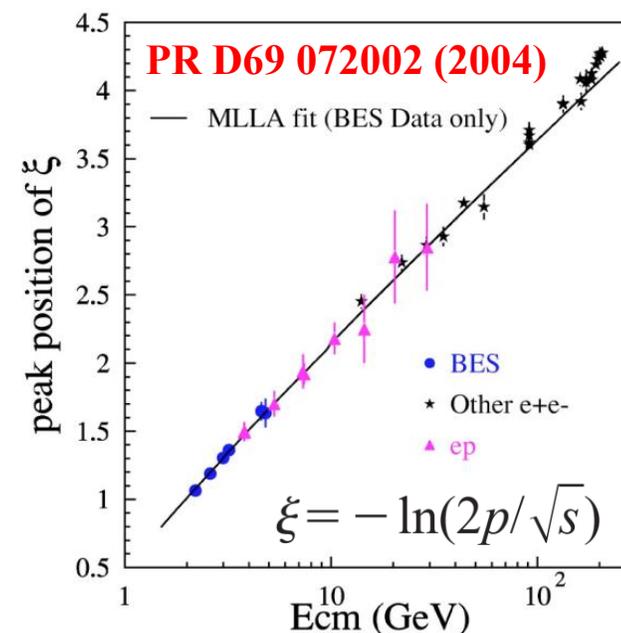
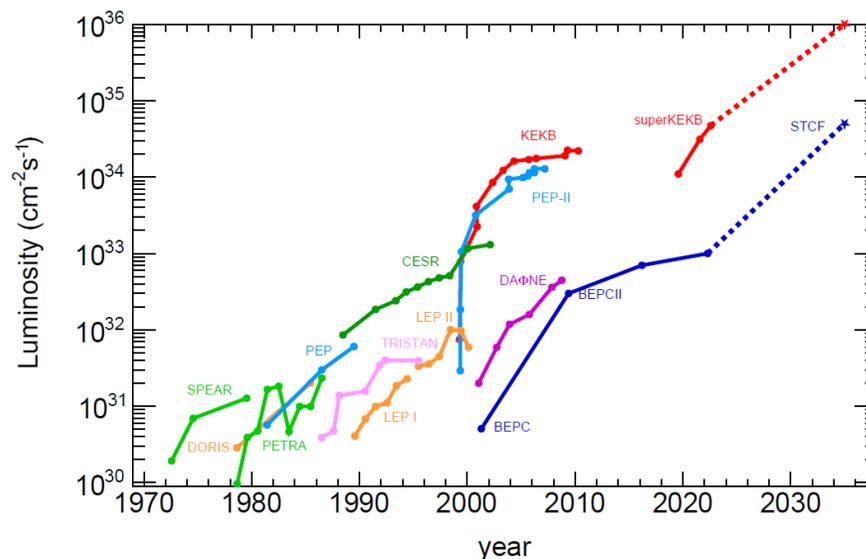
鄢文标(中国科学技术大学)



Leading Quark TMDFFs $\bigcirc \rightarrow$ Hadron Spin $\bigcirc \rightarrow$ Quark Spin

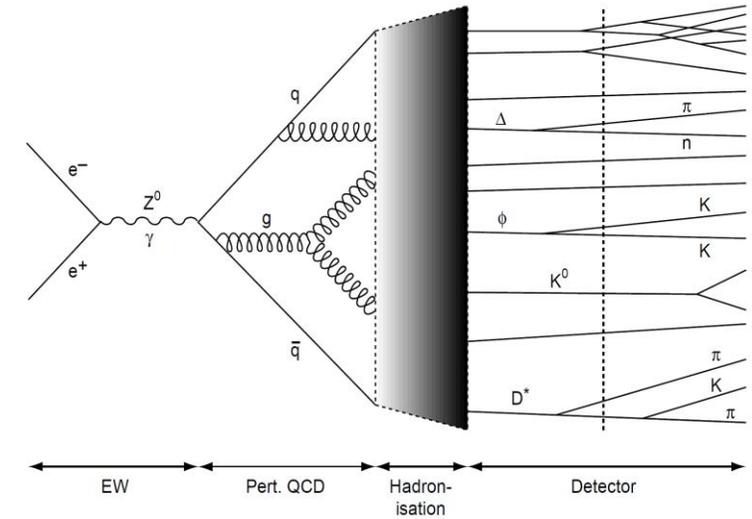
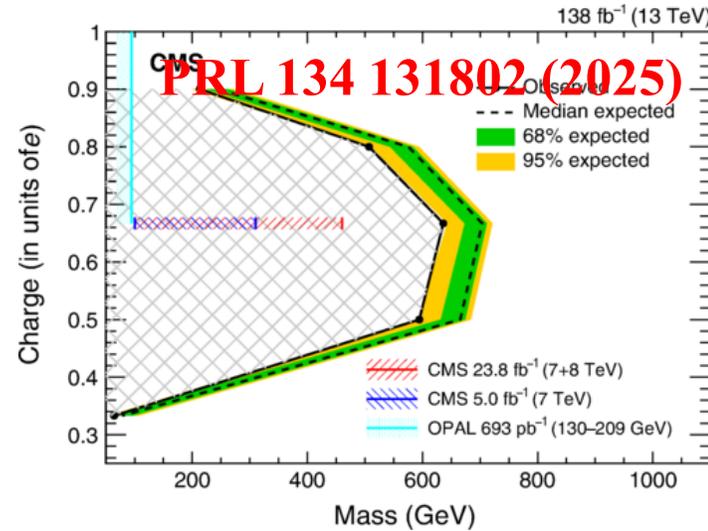
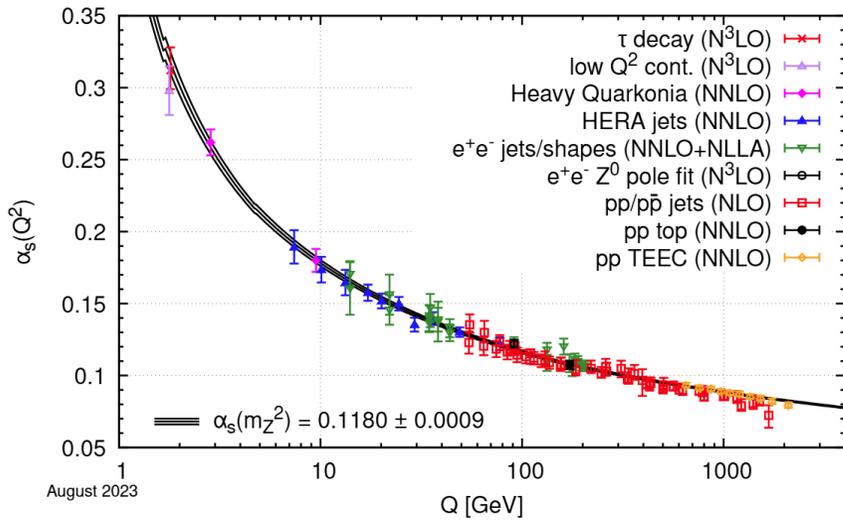
		Quark Polarization		
		Un-Polarized (U)	Longitudinally Polarized (L)	Transversely Polarized (T)
Unpolarized (or Spin 0) Hadrons		$D_1 = \bigcirc$ Unpolarized		$H_1^\perp = \bigcirc - \bigcirc$ Collins
	L		$G_1 = \bigcirc - \bigcirc$ Helicity	$H_{1L}^\perp = \bigcirc - \bigcirc$
Polarized Hadrons	T	$D_{1T}^\perp = \bigcirc - \bigcirc$ Polarizing FF	$G_{1T}^\perp = \bigcirc - \bigcirc$	$H_1 = \bigcirc - \bigcirc$ Transversity

arXiv:2304.03302

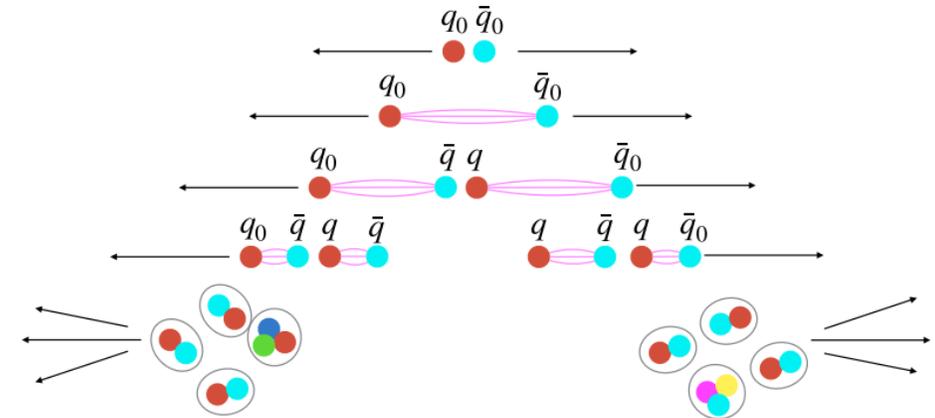


BESIII实验物理讨论会, 2026.02.07, 昌平

QCD: Asymptotic freedom & Confinement



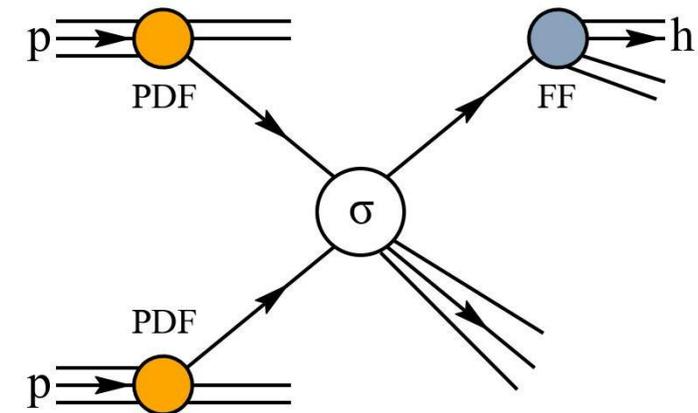
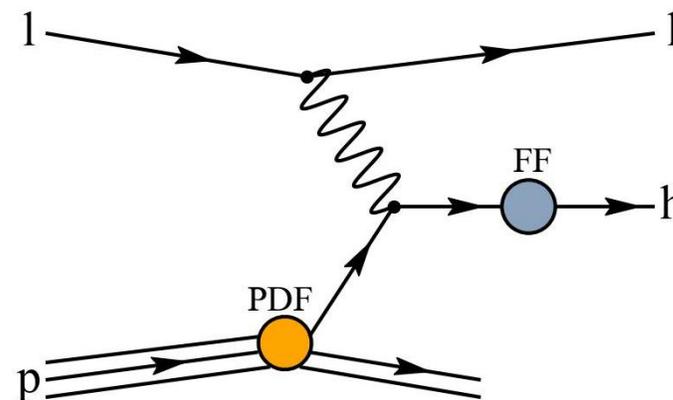
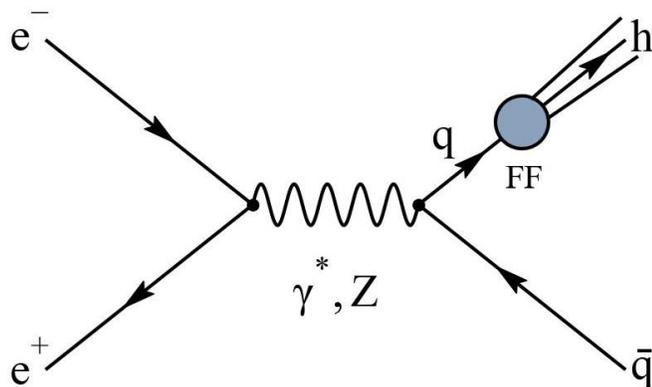
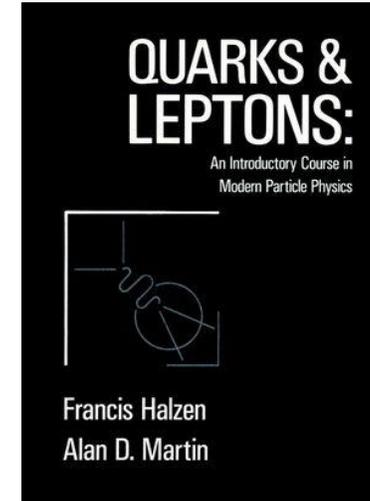
- QCD coupling constant $\alpha_s(Q)$
 - ✓ high Q : asymptotic freedom, perturbative QCD
 - ✓ low Q : **non-perturbative QCD**
- Confinement: partons do not exist as free particles, but are always confined in hadron
- Essence of confinement? Why & how?
 - ✓ **hadronization models** & **Fragmentation function**
 - ✓ **LPHD**: Local Parton Hadron Duality hypothesis
 - ✓ **machine learning** for hadronization



Eur. Phys. J. C85 16(2025)

Fragmentation function: integrated $D_1^h(z)$

- Fragmentation function $D_q^h(z)$: probability that hadron h is found in the debris of a parton carrying a fraction z of parton's energy
- **Consequence of confinement**
- **FF: QCD first principle (NOT YET)**
 - ✓ FF evolution function: DGLAP
 - ✓ fitting: parametrization & experimental data
 - ✓ universality: e^+e^- , DIS, pp , $p\bar{p}$ data
- FFs contribute to **virtually all processes**



FFs with quark/hadron polarization

Hadron polarization	Quark polarization @ PPNP 91 136 (2016)		
	Unpolarized	Longitudinally	Transversely
Unpolarized	D_1^h		$H_1^{\perp h}$
Longitudinally		G_1^h	$H_{1L}^{\perp h}$
Transversely	$D_{1T}^{\perp h}$	G_{1T}^h	$H_1^h \quad H_{1T}^{\perp h}$

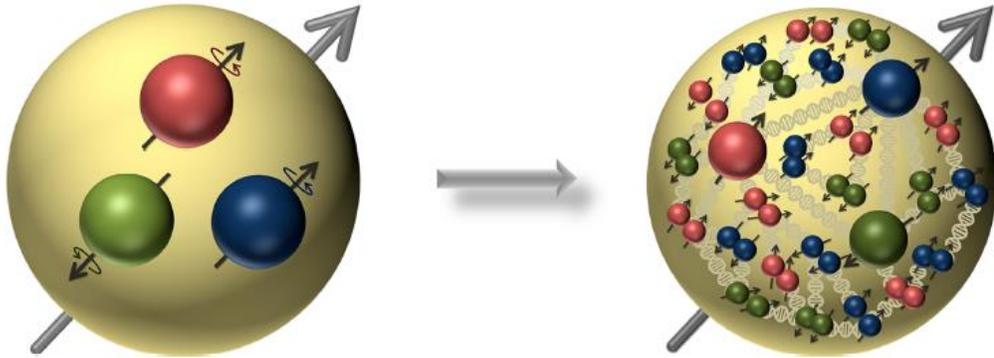
PLB396 (1993) 161



$$D_{hq^\uparrow}(z, P_{h\perp}) = D_1^q(z, P_{h\perp}^2) + H_1^{\perp q}(z, P_{h\perp}^2) \frac{(\hat{\mathbf{k}} \times \mathbf{P}_{h\perp}) \cdot \mathbf{S}_q}{zM_h}$$

- Theoretically many more, in particular with **polarized hadrons** in the final state and **transverse momentum dependence (TMD)**

FFs for EIC & EicC

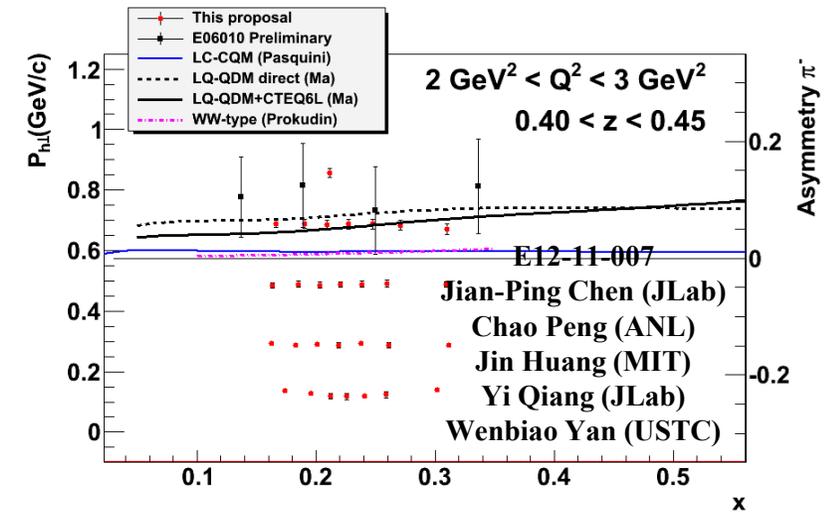
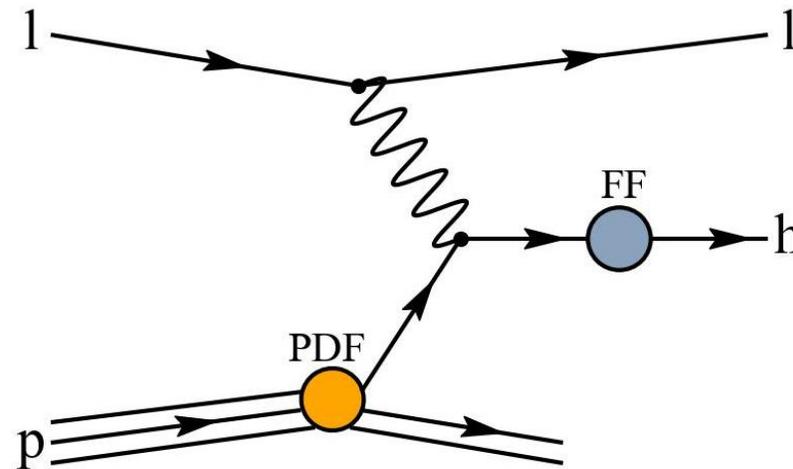
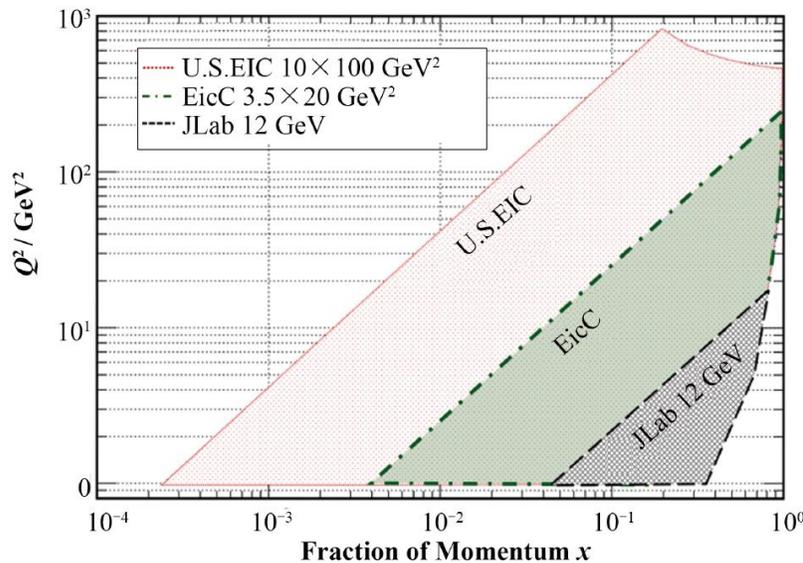


Preprints: JLAB-THY-23-3780, LA-UR-21-20798, MIT-CTP/5386

arXiv:2304.03302

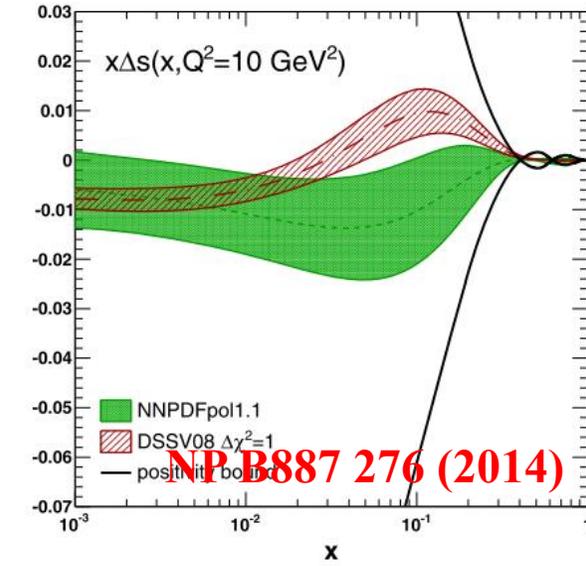
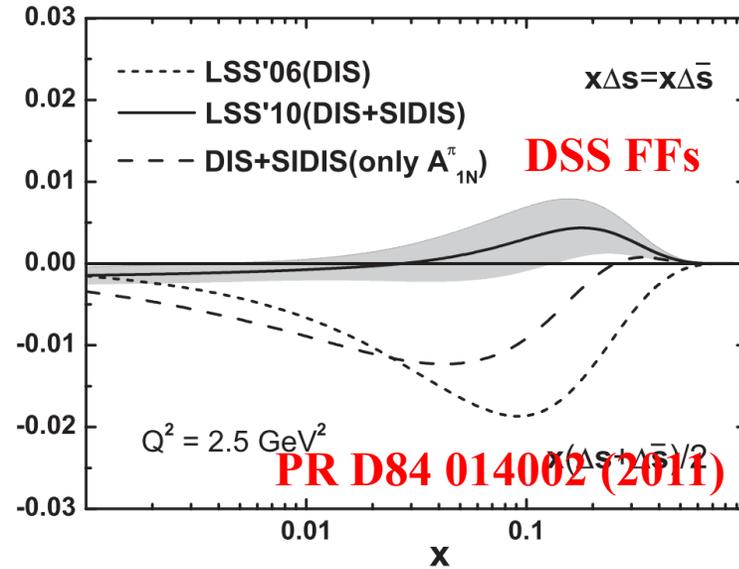
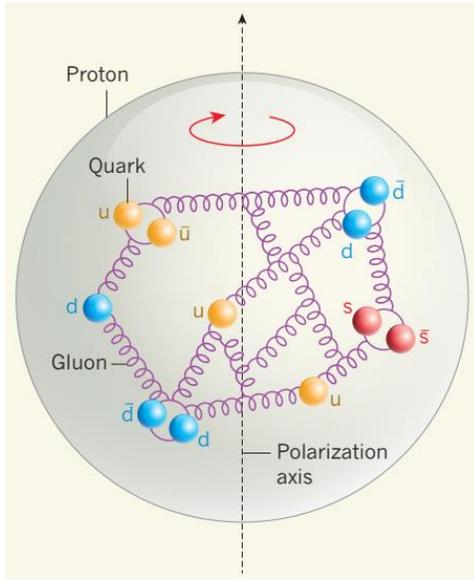
TMD Handbook

Renaud Boussarie¹, Matthias Burkardt², Martha Constantinou³, William Detmold⁴, Markus Ebert^{4,5}, Michael Engelhardt², Sean Fleming⁶, Leonard Gamberg⁷, Xiangdong Ji⁸, Zhong-Bo Kang⁹, Christopher Lee¹⁰, Keh-Fei Liu¹¹, Simonetta Liuti¹², Thomas Mehen¹³, Andreas Metz³, John Negele⁴, Daniel Pitonyak¹⁴, Alexei Prokudin^{7,16}, Jian-Wei Qiu^{16,17}, Abha Rajan^{12,18}, Marc Schlegel^{2,19}, Phiala Shanahan⁴, Peter Schweitzer²⁰, Iain W. Stewart⁴, Andrey Tarasov^{21,22}, Raju Venugopalan¹⁸, Ivan Vitev¹⁰, Feng Yuan²³, Yong Zhao^{24,4,18}

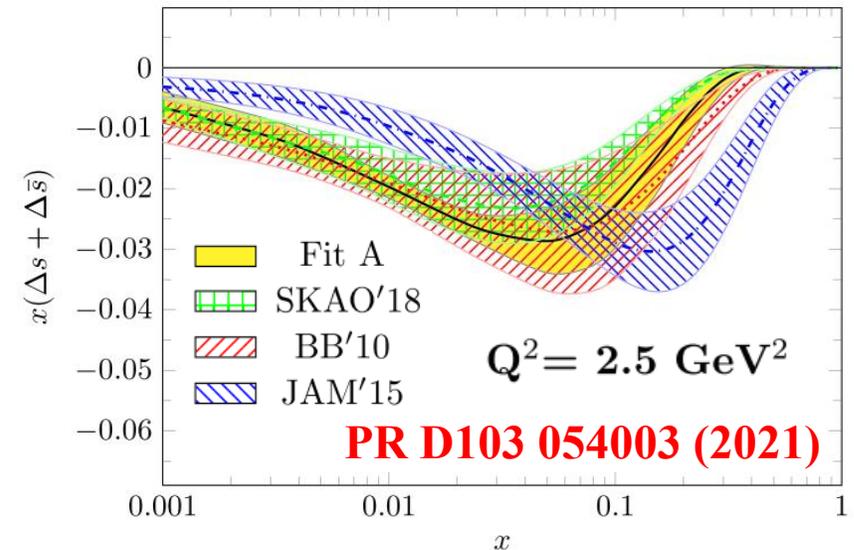


Precise knowledge of FFs will be crucial

Strange quark polarization puzzle

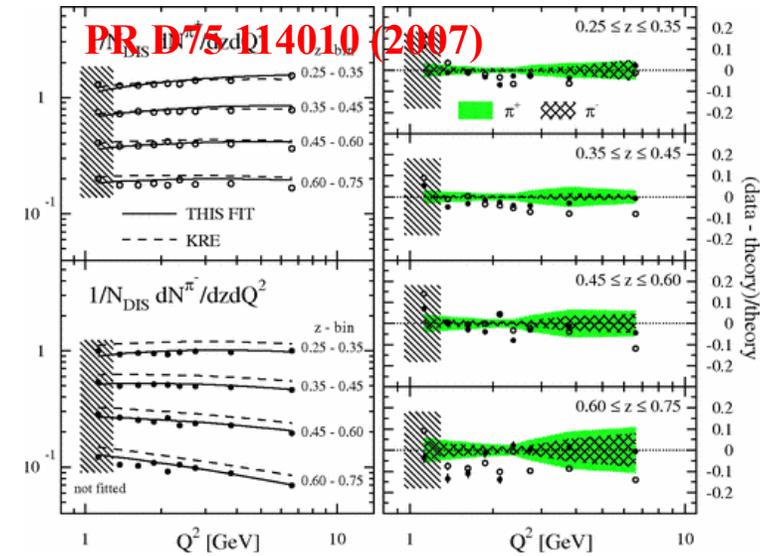
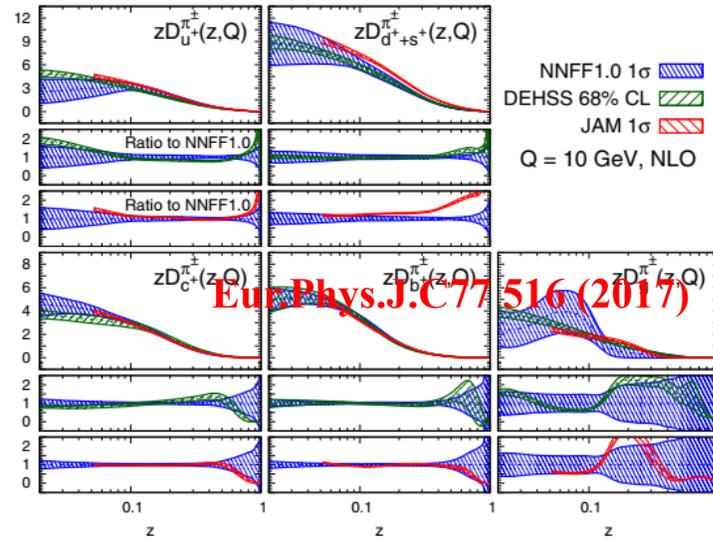
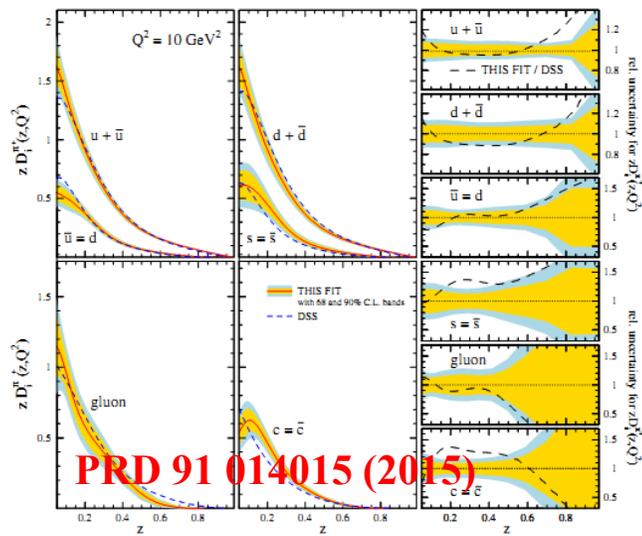


- Strange quark density function: $\Delta s(x) + \Delta \bar{s}(x)$
 - ✓ inclusive DIS: only proton PDF
 - a. **negative** for all values of x
 - ✓ semi-inclusive DIS: proton PDF & kaon FF
 - a. DSS FFs: **positive** for most of measured x
 - b. NNPDF FF: **negative**
 - c. JAM FFs: **negative**

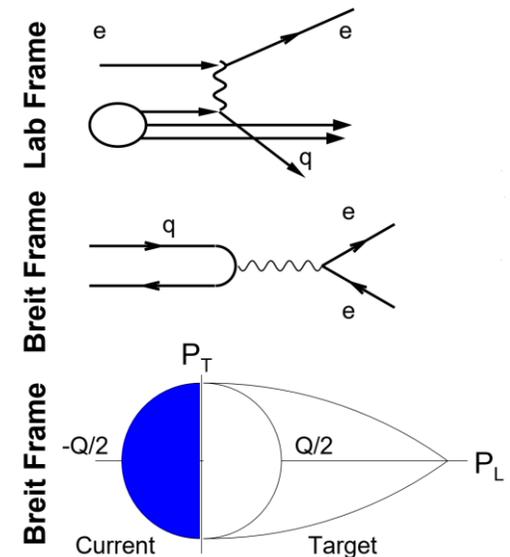


- Reliable FFs knowledge ? important !!!

Pion FF: Best known FF

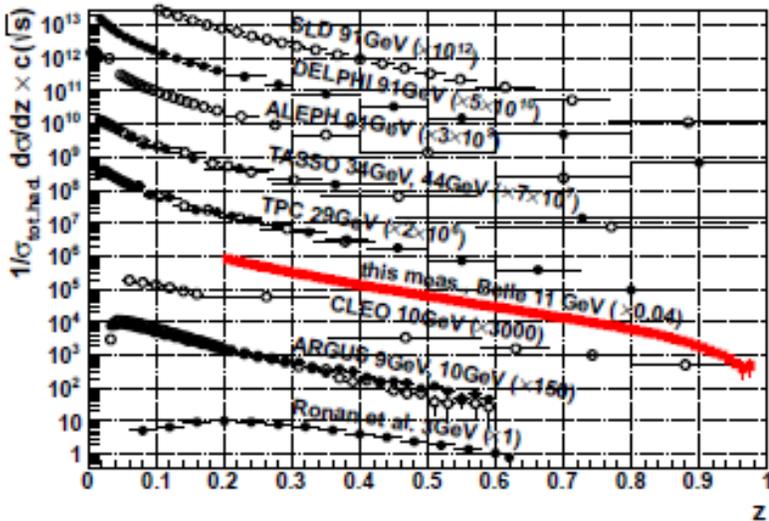


- For $z \geq 0.8$, uncertainty rapidly increase because of lack of experimental data
- Breit frame @ DIS: incoming quark scatters off photon and returns along same axis
 - ✓ current region is analogous to half of e^+e^- collision
 - ✓ Born level: DIS $Q = e^+e^- \sqrt{s}$
- DSS FFs: HERMES ep pion data at **10% level**
 - ✓ describe BESIII data at **?% level**

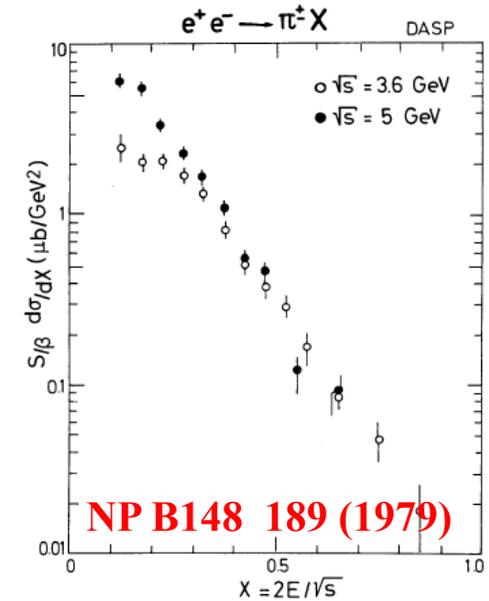
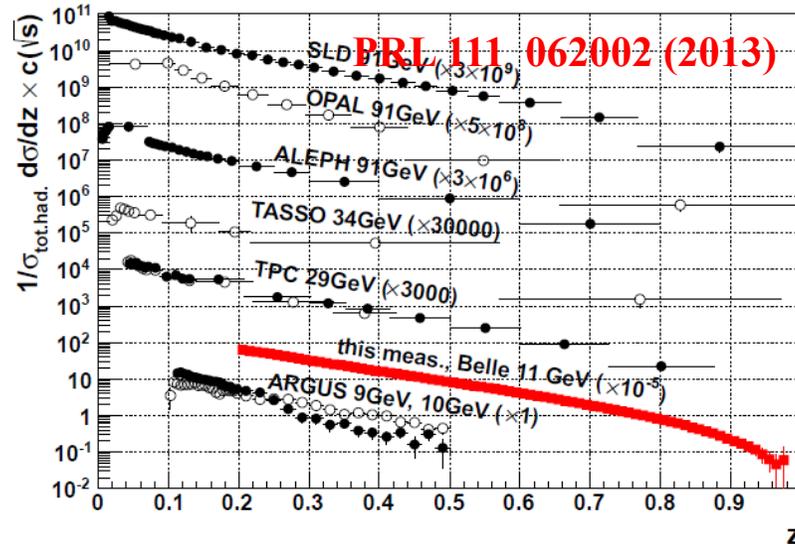


World π & K data @ e^+e^- collision

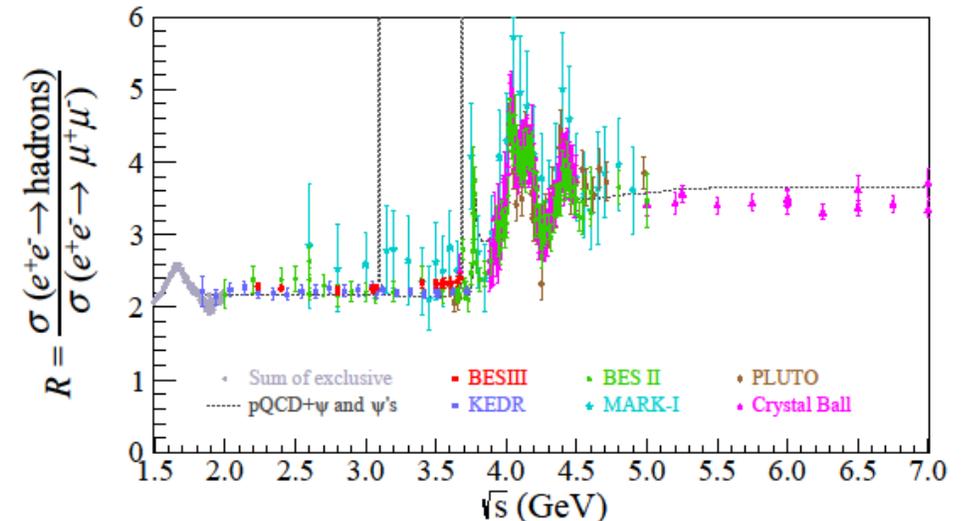
World Data (Sel.) for $e^+e^- \rightarrow \pi^+X$ Production



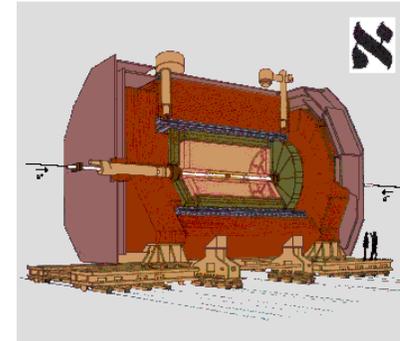
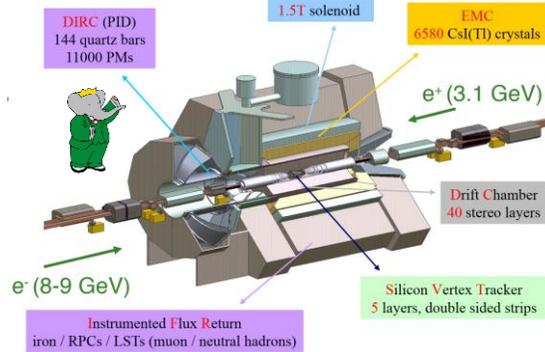
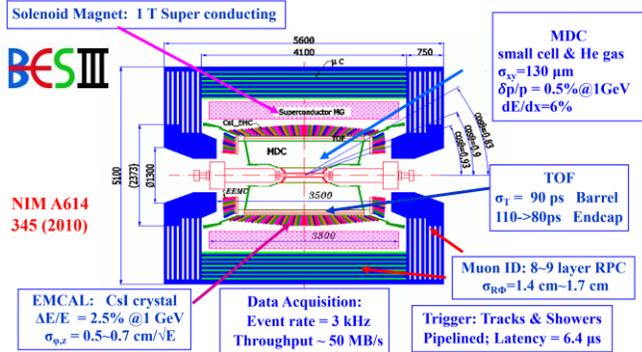
World Data (Sel.) for $e^+e^- \rightarrow K^+X$ Production



- Data sets at high z ?
- Data sets at $\sqrt{s} < 10$ GeV e^+e^- collision ?
- Charged π data by DASP
 - ✓ stat. uncertainty: **18%**
- BESIII & STCF: **opportunities & challenges**
 - ✓ e^+e^- : the cleanest process for FF study

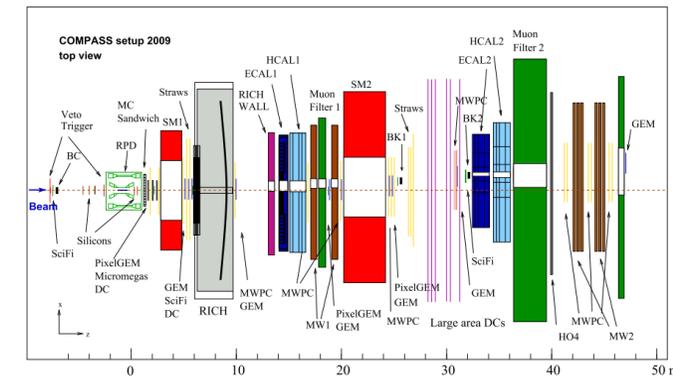
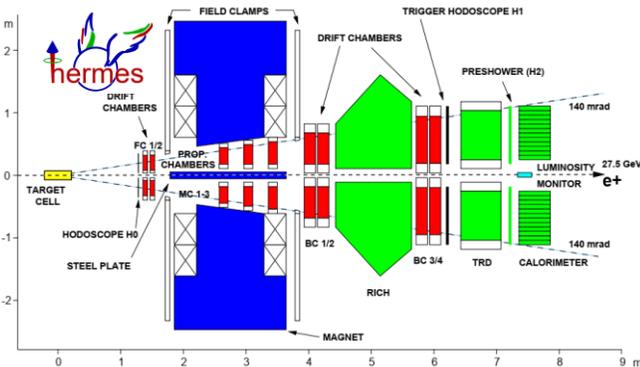
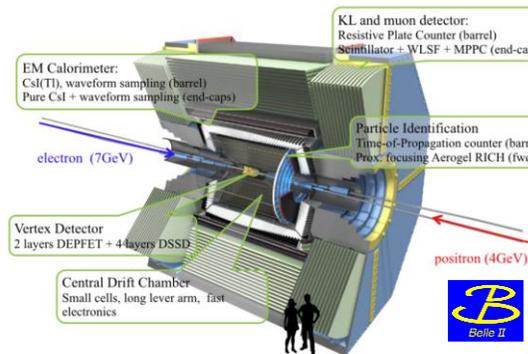
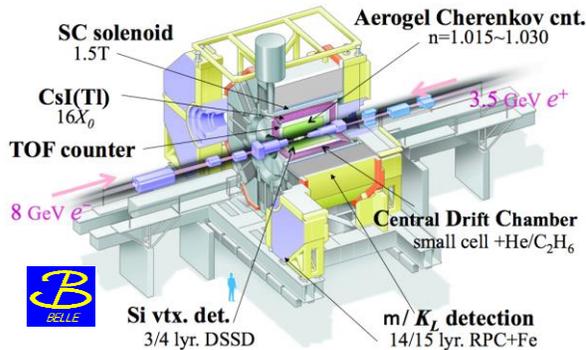
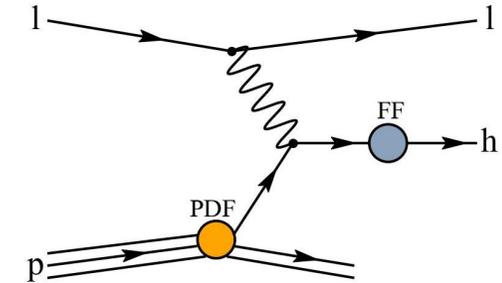
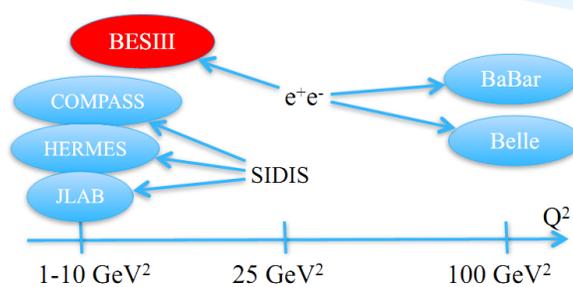
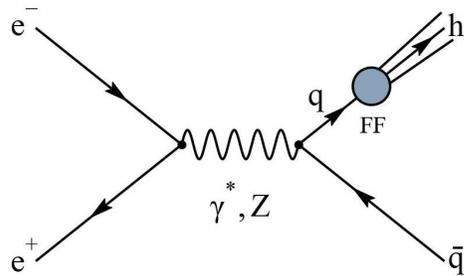
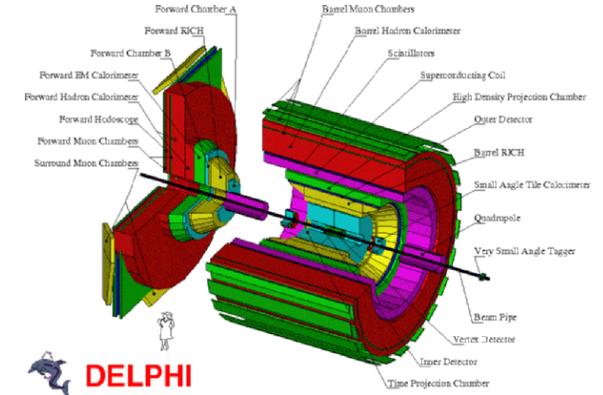


Hadron production with QCD @ detectors



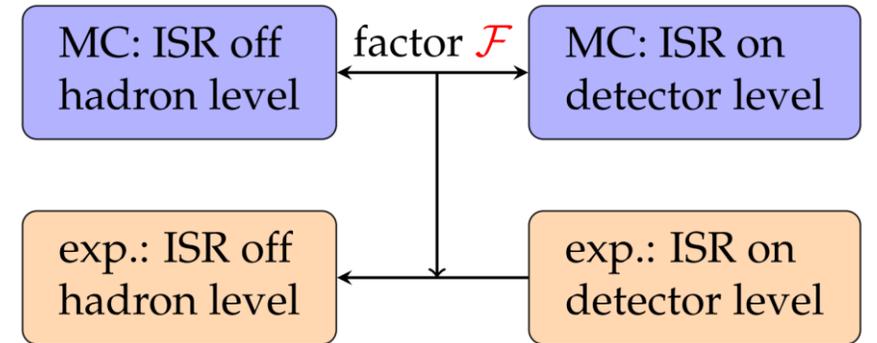
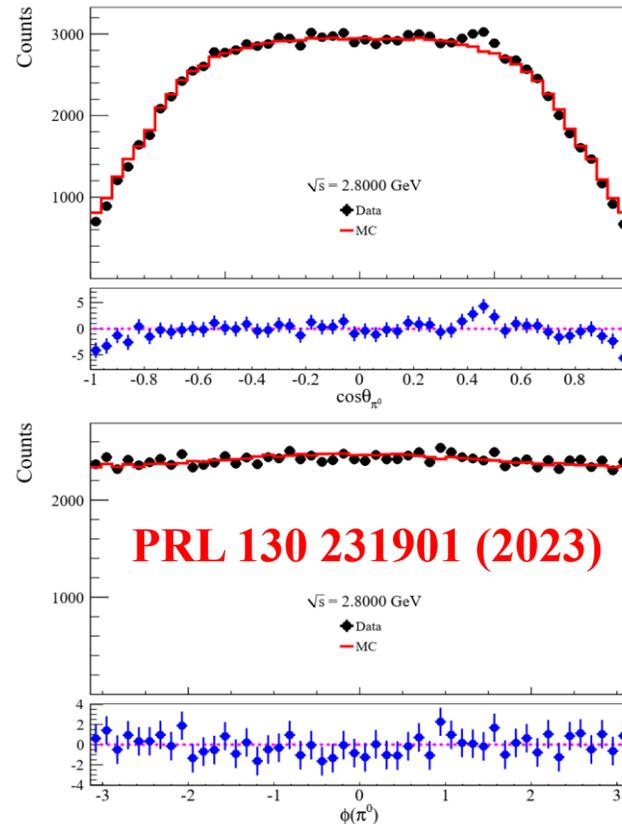
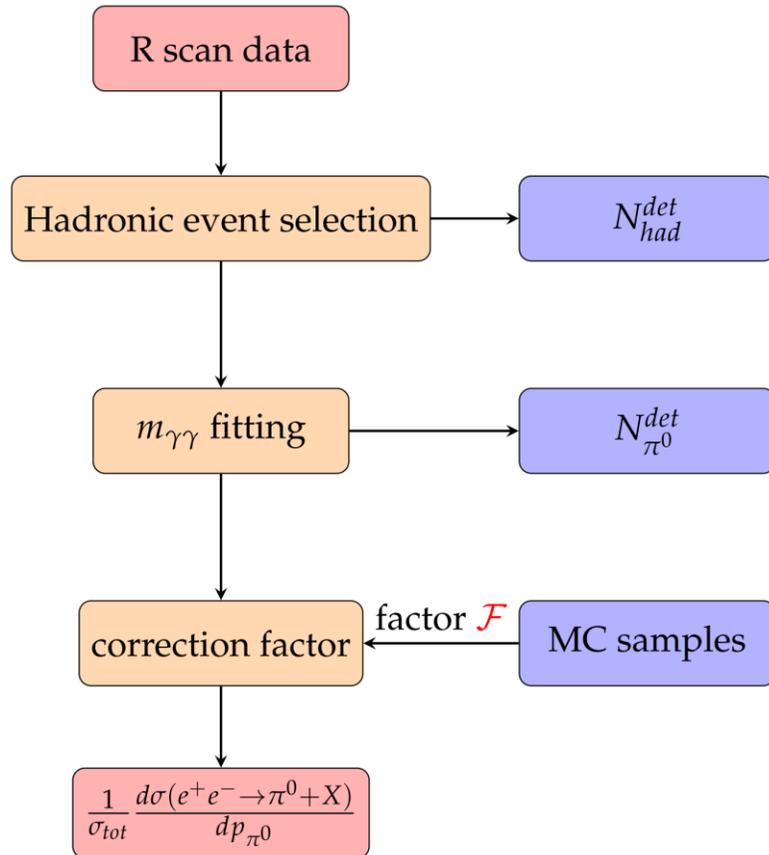
The ALEPH Detector

- Vertex Detector
- Inner Tracking Chamber
- Time Projection Chamber
- Electromagnetic Calorimeter
- Superconducting Magnet Coil
- Hadron Calorimeter
- Muon Chambers
- Luminosity Monitors



Measurement of $e^+e^- \rightarrow \pi^0 + X$ @ BESIII

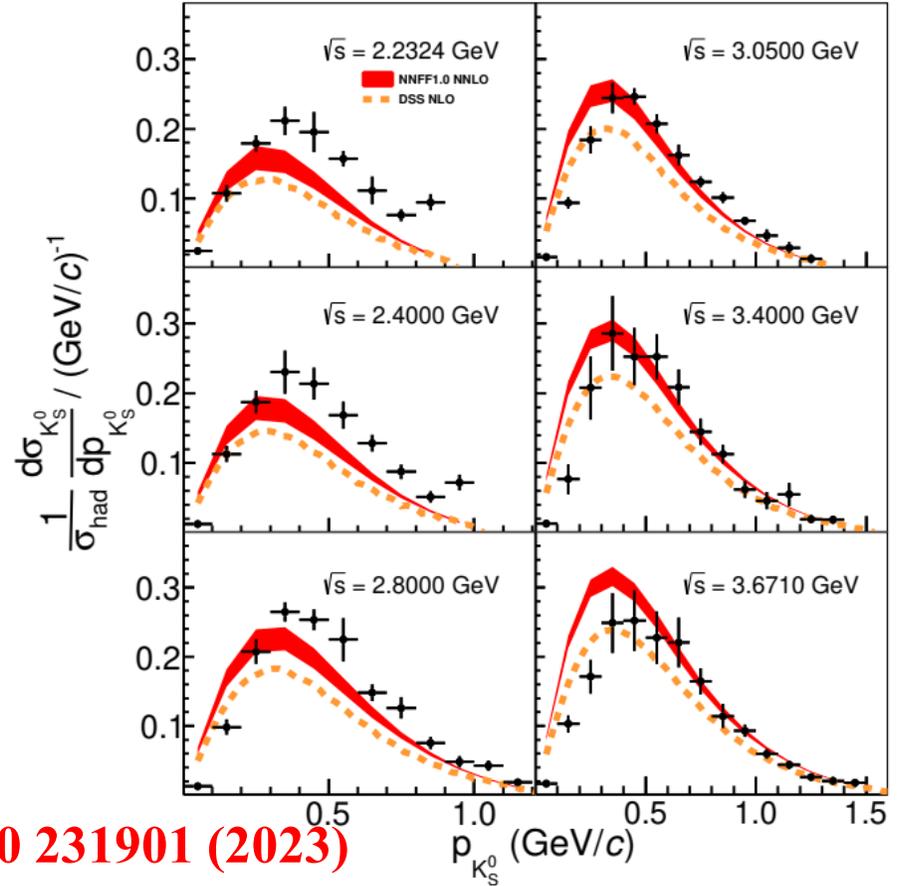
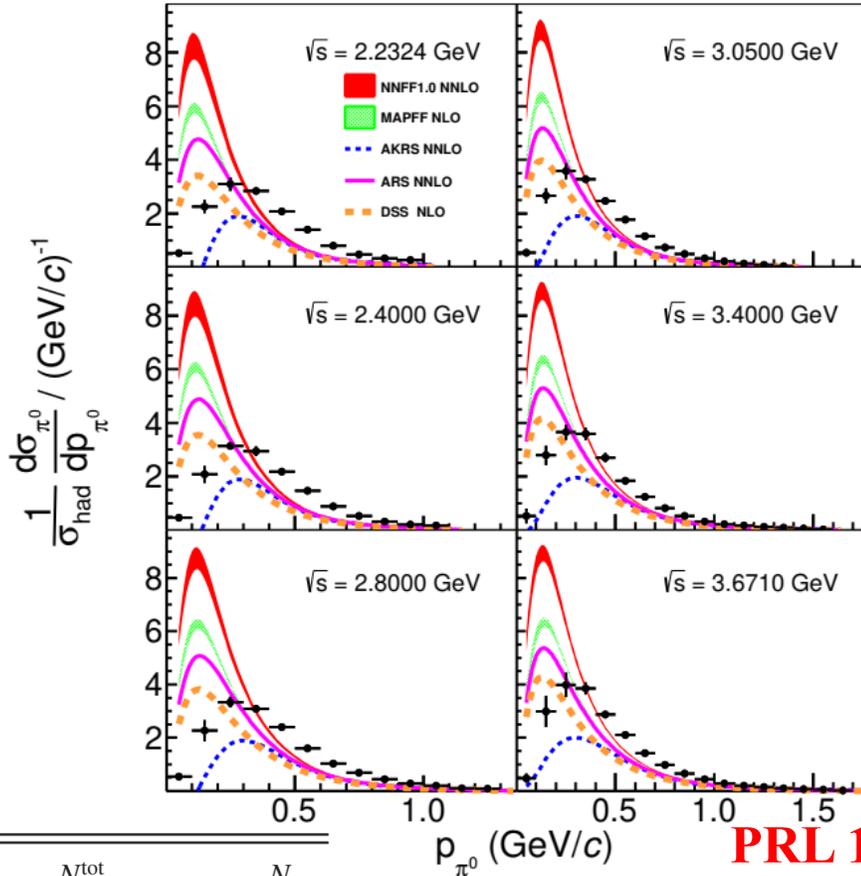
$$\frac{1}{\sigma_{total}} \frac{d\sigma(e^+e^- \rightarrow \pi^0 + X)}{dp_{\pi^0}} \Rightarrow \frac{1}{N_{had}} \frac{N_{\pi^0}}{\Delta p_{\pi^0}} = \mathcal{F} \frac{1}{N_{had}^{det}} \frac{N_{\pi^0}^{det}}{\Delta p_{\pi^0}}$$



$$\mathcal{F} = \frac{N_{\pi^0}^{true}(off)}{N_{had}^{true}(off)} / \frac{N_{\pi^0}^{det}(on)}{N_{had}^{det}(on)}$$

- **Bin-by-bin unfolding method**
- **Correction factor \mathcal{F}**
 - ✓ event selection efficiency
 - ✓ radiative correction

$e^+e^- \rightarrow \pi^0/K_S^0 + X @ \text{BESIII}$

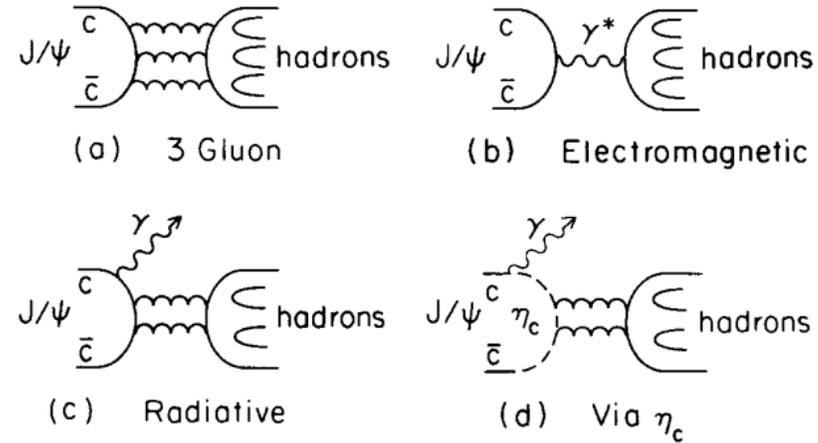
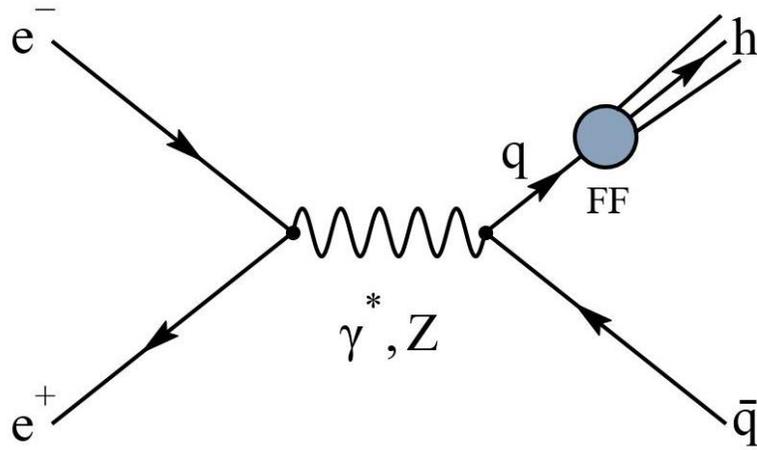


PRL 130 231901 (2023)

\sqrt{s} (GeV)	\mathcal{L} (pb ⁻¹)	$N_{\text{had}}^{\text{tot}}$	N_{bkg}
2.2324	2.645	83227	2041
2.4000	3.415	96627	2331
2.8000	3.753	83802	2075
3.0500	14.89	283822	7719
3.4000	1.733	32202	843
3.6710	4.628	75253	6461

- Dominated uncertainty: MC generator
- Inclusive π^0 production: **surprise , disagreement !!!**
- Inclusive K_S^0 production: **not so bad**

$e^+e^- \rightarrow \pi^0/K_S^0 + X @ \text{BESIII}$

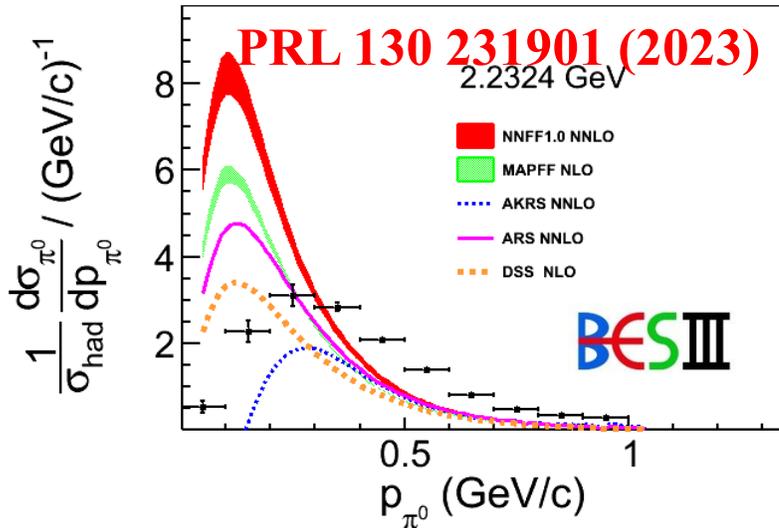


- From theory side:

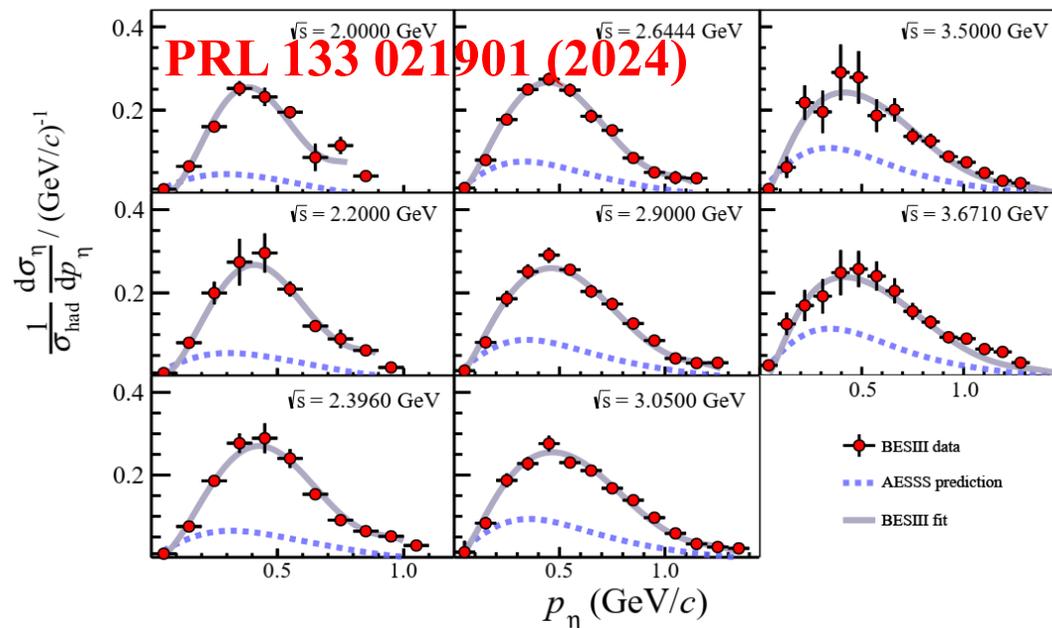
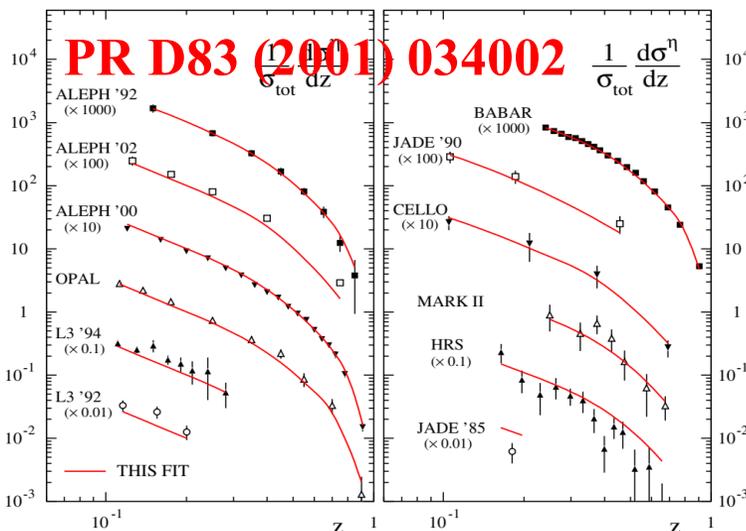
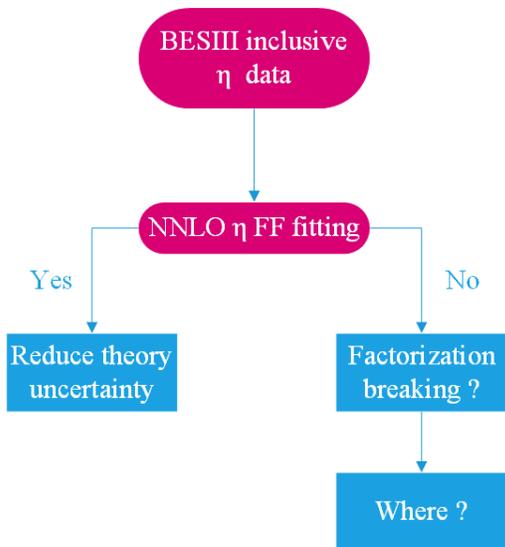
- ✓ fit with BESIII data, hadron mass effect, higher twist contribution, and so on
- ✓ factorization breaking

- From experimental side

- ✓ primary hadron vs from resonance decay
 $\Rightarrow e^+e^- \rightarrow \rho(\omega, \phi) + X$, and so on
- ✓ contribution of vector states ρ^*, ω^* and ϕ^*
 $\Rightarrow e^+e^- \rightarrow \rho^*/\omega^*/\phi^* \rightarrow h + X$



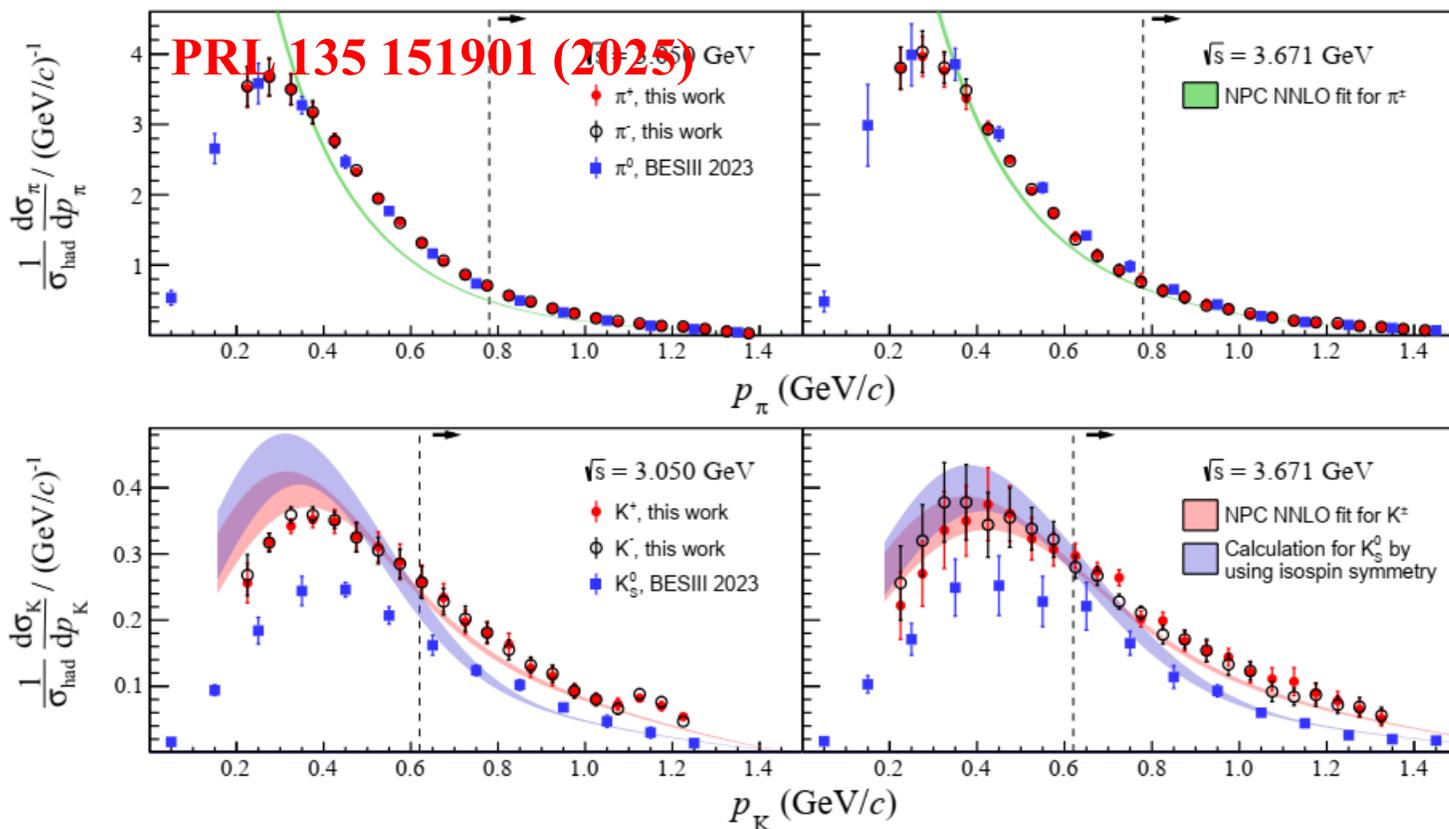
$e^+e^- \rightarrow \eta + X @ \text{BESIII}$



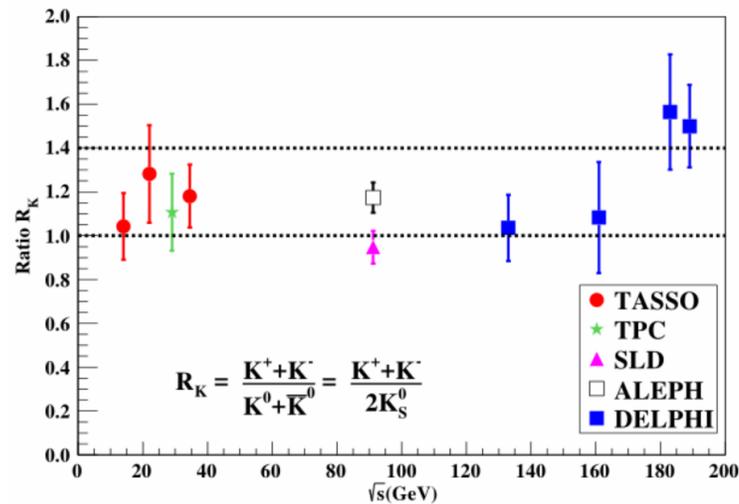
- η FF @ NLO: data at $\sqrt{s} > 10 \text{ GeV } e^+e^-$
 - ✓ missing theory uncertainty
- PRD83 (2001) 034002 vs. BESIII data
 - ✓ **underestimate**
- BESIII fit: PR D111 034030 (2025)
 - ✓ NNLO accuracy, hadron mass & higher twist
 - ✓ $\sqrt{s} > 10 \text{ GeV } e^+e^-$ data + **BESIII data**

\sqrt{s} (GeV)	\mathcal{L} (pb ⁻¹)	$N_{\text{had}}^{\text{tot}}$	N_{bkg}
2.0000	10.074	350 298 ± 591	8722 ± 93
2.2000	13.699	445 019 ± 666	10 737 ± 103
2.3960	66.869	1 869 906 ± 1365	47 550 ± 218
2.6444	33.722	817 528 ± 902	21 042 ± 145
2.9000	105.253	2 197 328 ± 1478	56 841 ± 238
3.0500	14.893	283 822 ± 531	7719 ± 87
3.5000	3.633	62 670 ± 249	1691 ± 41
3.6710	4.628	75 253 ± 273	6461 ± 80

Inclusive π^\pm & K^\pm vs. pQCD



- Inclusive pion cross section
 - ✓ $\pi^+ \approx \pi^- \approx \pi^0$
 - ✓ Isospin symmetry
- Inclusive kaon cross section
 - ✓ $K^+ \approx K^- \approx 1.4 K_S$
 - ✓ **Large isospin violation ?**



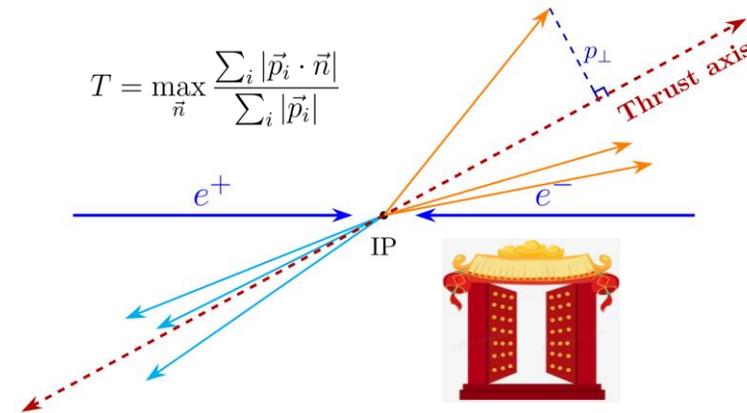
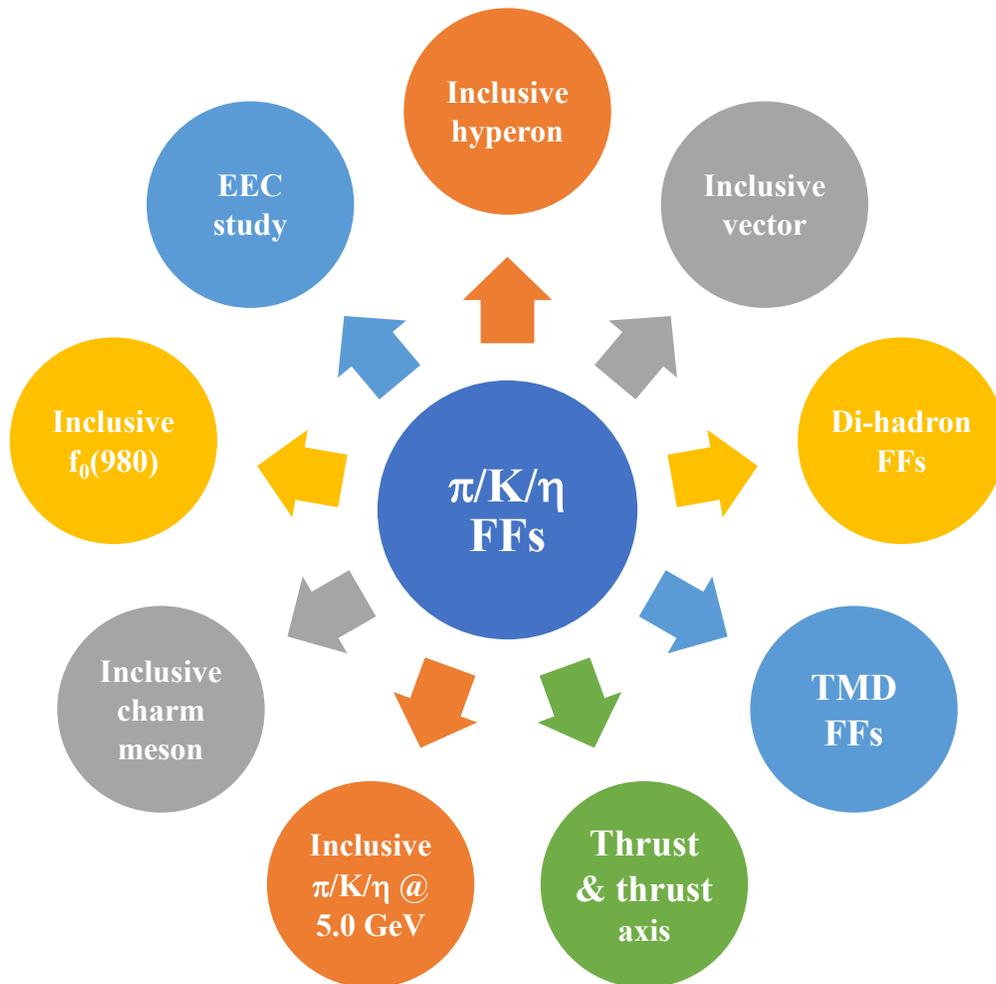
- NPC NNLO: $\sqrt{s} > 3.0$ GeV & $E_h > 0.8$ GeV @ BESIII
- **PRL 135 041902 (2025):** Validity of QCD factorization and pQCD calculation at energy scales down to 3 GeV
- Test of isospin symmetry with FF fitting

$$D_i^{\pi^0}(z, Q) = \frac{1}{2} D_j^{\pi^\pm}(z, Q) \quad @\text{NPB 803 42 (2008)}$$

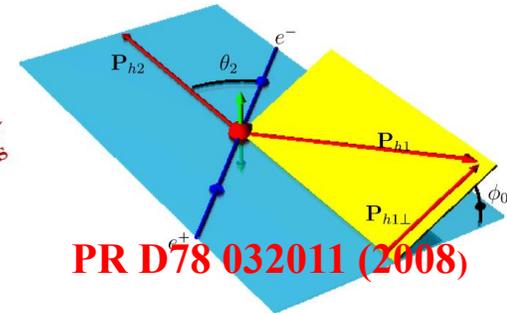
$$D_i^{K_S^0}(z, Q) = \frac{1}{2} D_j^{K^\pm}(z, Q) \quad \begin{cases} \text{if } i = d(u), j = u(d) \\ \text{otherwise } i = j \end{cases}$$

To do list @ BESIII

Theoretically many more, in particular with **polarized hadrons** in the final state and **transverse momentum dependence (TMD)**



$$T = \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}$$



PR D96 032005 (2017)

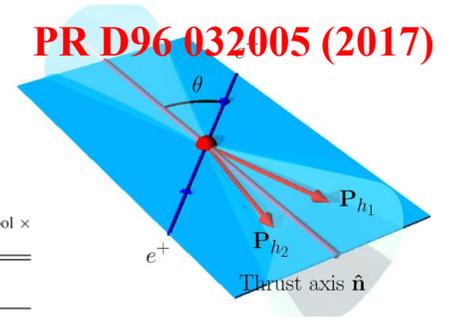


TABLE II. The 18 leading twist components of the FFs for quark fragments to spin-1 hadrons. The symbol \times means that the corresponding FF disappears after the integration over transverse momentum.

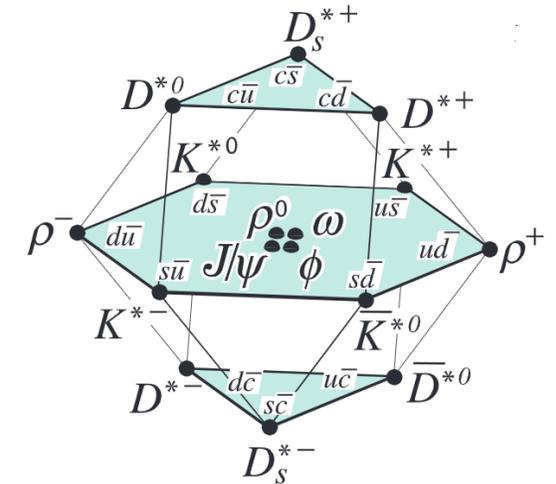
Quark polarization	Hadron polarization	TMD FFs	Integrated over $\vec{k}_{F\perp}$	Name	
U	U	$D_1(z, k_{F\perp})$	$D_1(z)$	Number density	
	T	$D_{1T}^+(z, k_{F\perp})$	\times		
	LL	$D_{1LL}^+(z, k_{F\perp})$	$D_{1LL}(z)$	Spin alignment	
	LT	$D_{1LT}^+(z, k_{F\perp})$	\times		
	TT	$D_{1TT}^+(z, k_{F\perp})$	\times		
	L	L	$G_{1L}(z, k_{F\perp})$	$G_{1L}(z)$	Spin transfer (longitudinal)
T		$G_{1T}^+(z, k_{F\perp})$	\times		
LT		$G_{1LT}^+(z, k_{F\perp})$	\times		
TT		$G_{1TT}^+(z, k_{F\perp})$	\times		
T		U	$H_1^+(z, k_{F\perp})$	\times	Collins function
		$T(\parallel)$	$H_{1T}(z, k_{F\perp})$	$H_{1T}(z)$	Spin transfer (transverse)
	$T(\perp)$	$H_{1T}^+(z, k_{F\perp})$	\times		
	L	$H_{1L}^+(z, k_{F\perp})$	\times		
	LL	$H_{1LL}^+(z, k_{F\perp})$	\times		
	TT	$H_{1TT}^+(z, k_{F\perp}), H_{1TT}^-(z, k_{F\perp})$	$H_{1TT}(z)$		

PR D94 034003 (2016)



Inclusive hadron: beyond pion/kaon/eta

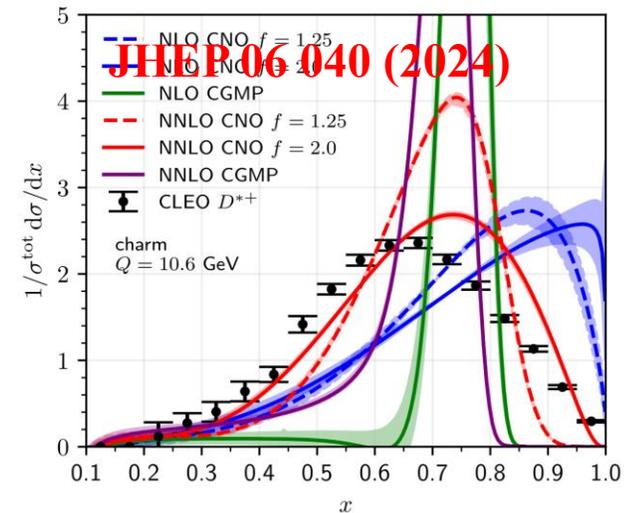
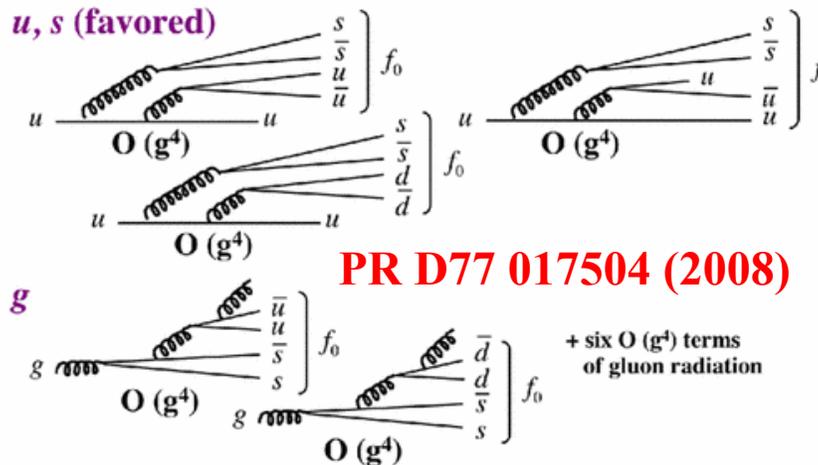
collaboration	NNFF	JAM	DSS+	BDSSV	MAP	NPC
SIA (ee)	✓	✓	✓	✓	✓	✓
SIDIS (ep)	✗	✓	✓	✓	✓	✓
pp incl. hadron	✗	✗	✓	✗	✗	✓
hadron in jet	✗	✗	✗	✗	✗	✓
FFs (charged h)	π^\pm, K^\pm, p	π^\pm, K^\pm	π^\pm, K^\pm, p	π^\pm	π^\pm, K^\pm	π^\pm, K^\pm, p
FFs (neutral h)			η			K^0, η, Λ
pQCD order	NNLO	NLO	NLO	appr. NNLO	appr. NNLO	NLO



- Existing hadron FFs
- Inclusive $f_0(980)$ production
 - ✓ nature of $f_0(980)$
- **FFs of charm quark**
 - ✓ inclusive D^0 production
 - ✓ **inclusive π/K with D-tag**

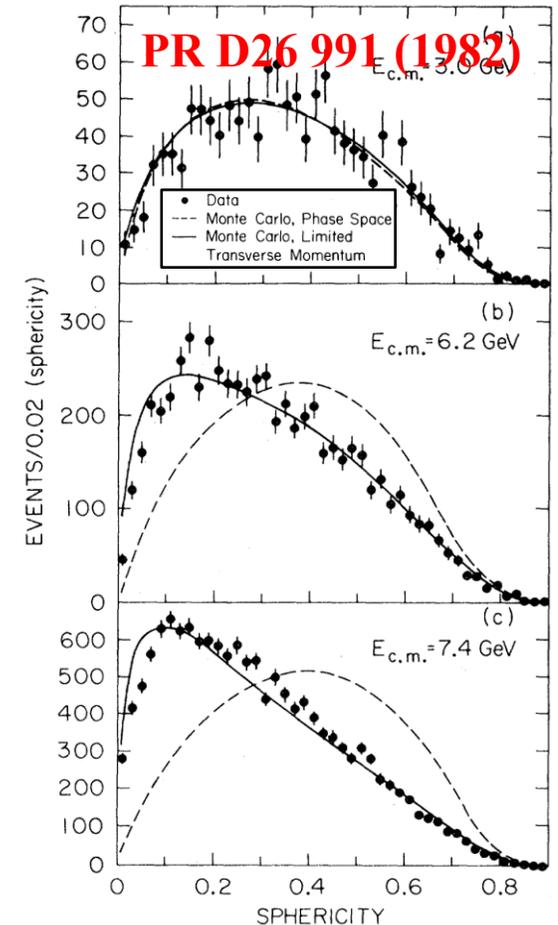
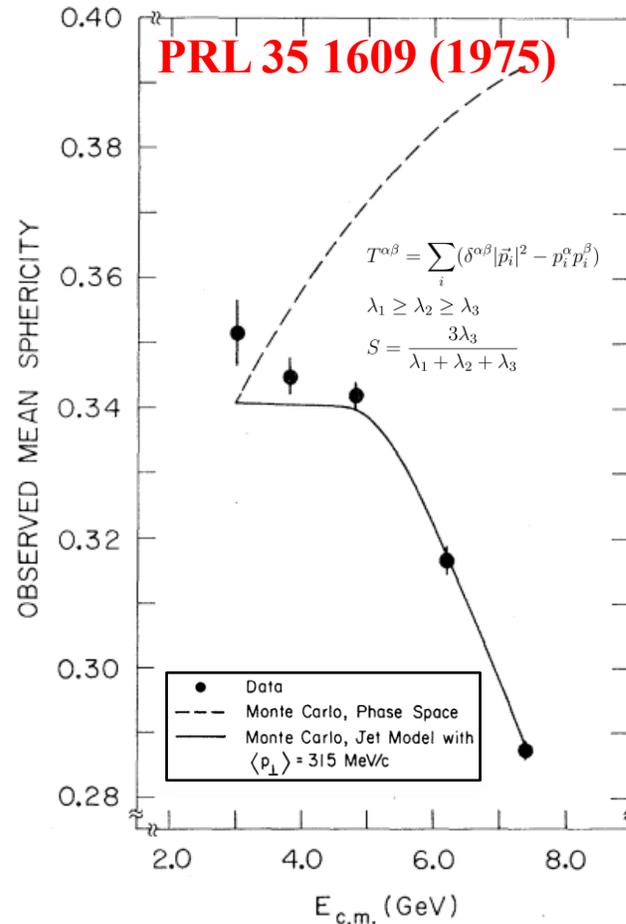
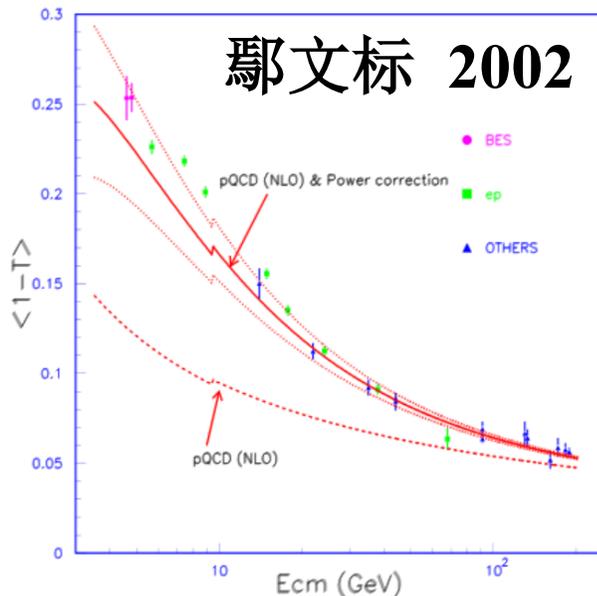
Tetraquark picture for $f_0(980)$

$$f_0 = (u\bar{u}s\bar{s} + d\bar{d}s\bar{s}) / \sqrt{2}$$



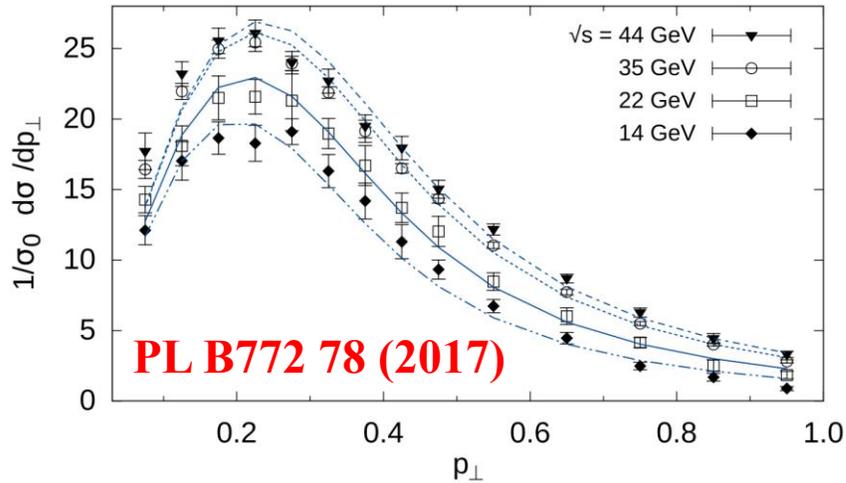
Thrust axis: more is different

- BESIII & STCF
 - ✓ correctly thrust axis ?
- Phase space model vs. jet model
 - ✓ reliable @ $\sqrt{s} \geq 5 \text{ GeV}$?
- Usage of thrust axis
 - ✓ two hadron ϵ/\notin of a quark
 - ✓ transverse momentum



Edward Farhi @ PRL 39 1587 (1977): Hadronic jets have been observed in electron positron annihilation experiments at energies above 5 or 6 GeV

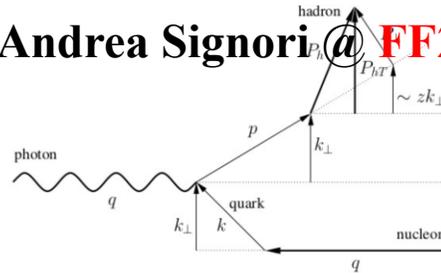
TMD FFs: $D_1^h(\mathbf{z}) \Rightarrow D_1^h(\mathbf{z}, p_T)$



Extractions – SIDIS

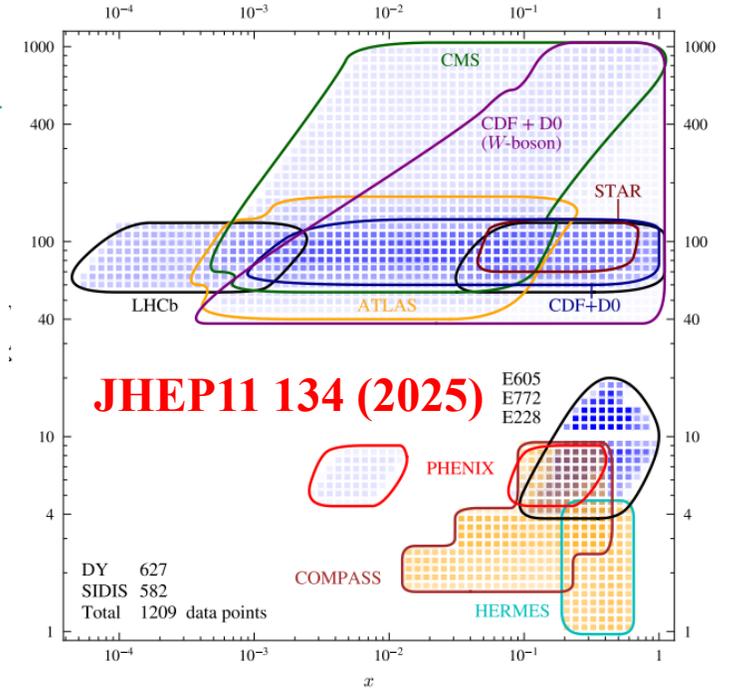
The **only available fits** are from **SIDIS** data, no e+e- data yet

Andrea Signori @ FF2019

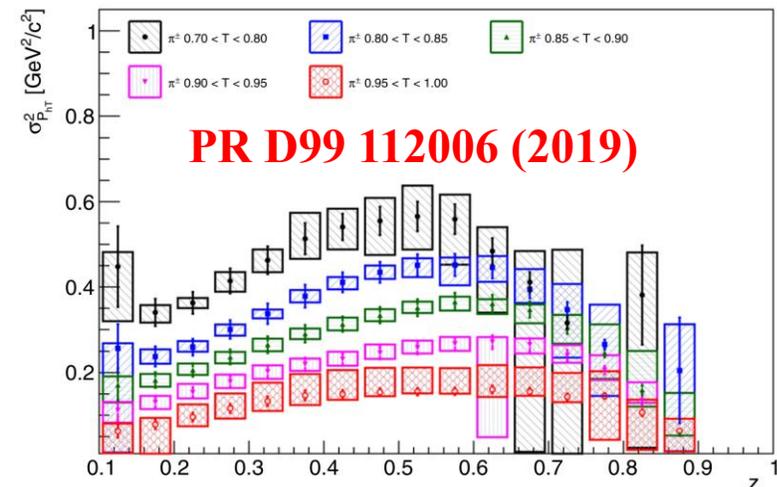


The **priority** is to get **independent information from e+e-**

In SIDIS the **information on FFs** is always **convoluted** with (TMD) PDFs



- **Collinear factorization vs. TMD factorization**
- **Published $d\sigma/dp_T$ of e^+e^- collision**
 - ✓ **stricter** constraint $p_T/(zQ)$ @ SIDIS TMD
- **Belle: Gaussian shape @ low p_T**
 - ✓ similar results @ HERMES & COMPAS
- **SIDIS & Drell-Yan: proton PDF**
- **e^+e^- : cleanest process**
 - ✓ TMD evolution effect & universality



Di-hadron FFs

PR D96 032005 (2017)

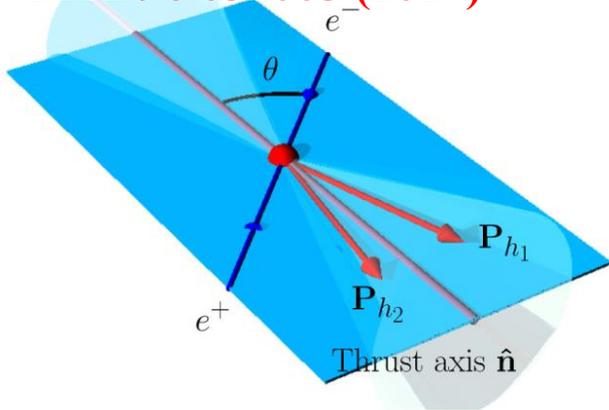


TABLE I. Data from experiment and LQCD used in this analysis for the DiFFs D_1 and H_1^\leftarrow and the transversity PDF h_1 . PR D109 034024 (2024)

Collaboration	References	Observable	Process	Nonperturbative function(s)
Belle	[64]	$d\sigma/dz dM_h$	$e^+e^- \rightarrow (\pi^+\pi^-)X$	D_1
Belle	[112]	$A^{e^+e^-}$	$e^+e^- \rightarrow (\pi^+\pi^-)(\pi^+\pi^-)X$	D_1, H_1^\leftarrow
HERMES	[118]	A_{UT}^{SIDIS}	$ep^\uparrow \rightarrow e'(\pi^+\pi^-)X$	D_1, H_1^\leftarrow, h_1
COMPASS	[117]	A_{UT}^{SIDIS}	$\mu\{p, D\}^\uparrow \rightarrow \mu'(\pi^+\pi^-)X$	D_1, H_1^\leftarrow, h_1
STAR	[97,121]	A_{UT}^{pp}	$p^\uparrow p \rightarrow (\pi^+\pi^-)X$	D_1, H_1^\leftarrow, h_1
ETMC	[77]	$\delta u, \delta d$	LQCD	h_1
PNDME	[71]	$\delta u, \delta d$	LQCD	h_1

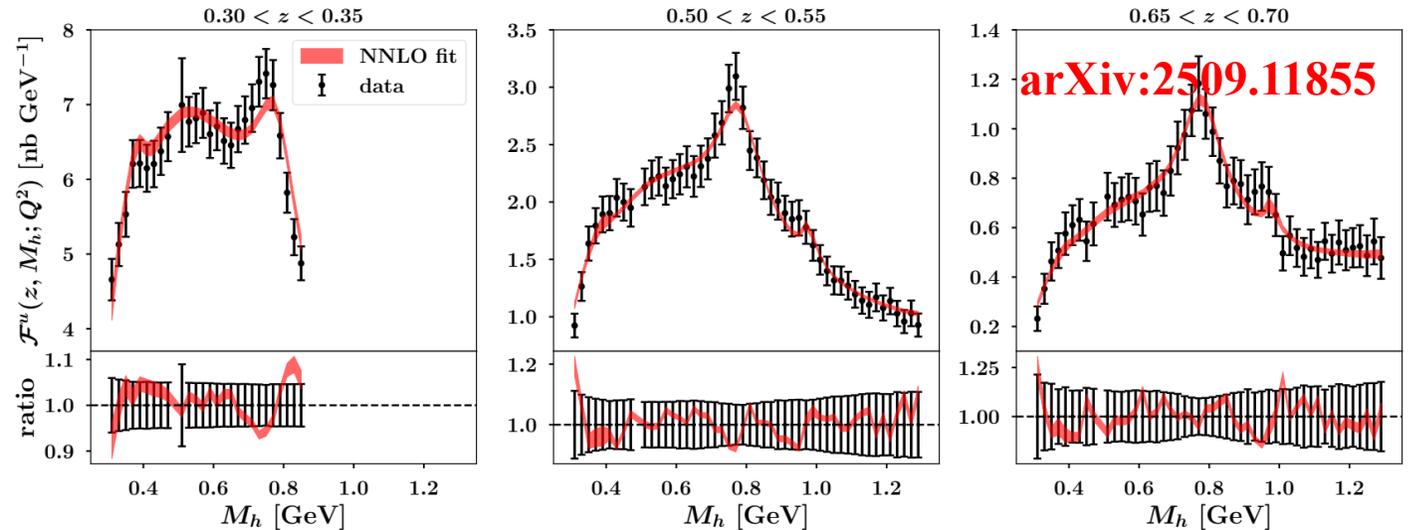
- e^+e^- vs. SIDIS

- ✓ di-hadron FFs evolution
- ✓ universality @ e^+e^- & SIDIS

- MAP: di-hadron FFs at NNLO

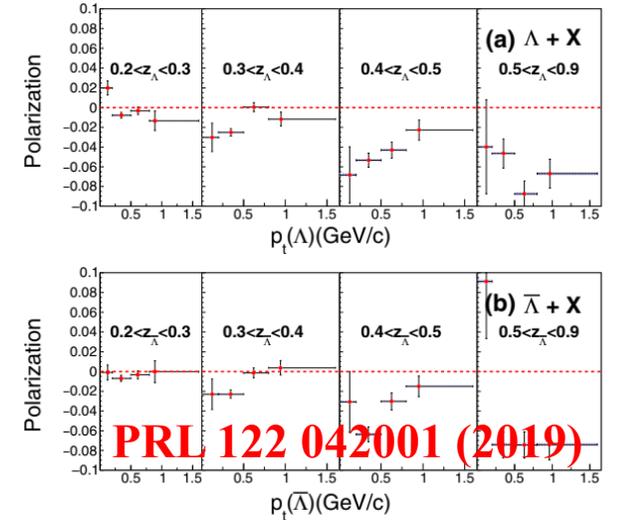
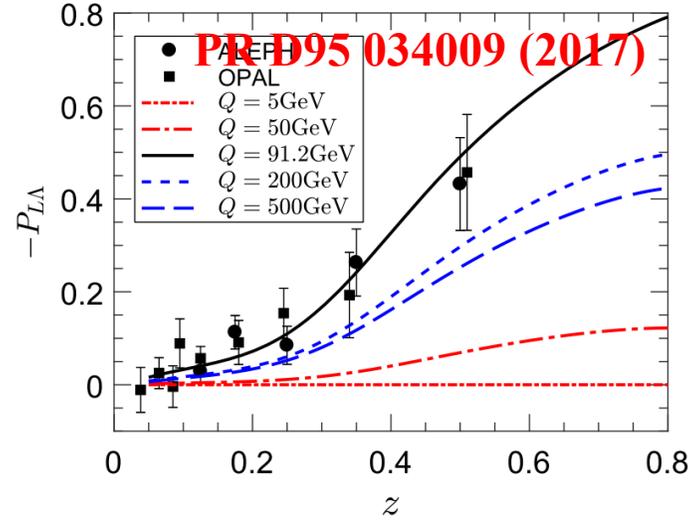
- Truth distribution @ BESIII

- ✓ with thrust axis: possible
- ✓ without thrust axis: ratio ?



Spin @ hadronization: Λ polarization

Group	Reaction	\sqrt{s}/GeV	Measurements	Result
BES	$e^+ + e^- \rightarrow \Lambda + X$	~ 5	$P_T^\Lambda \sim D_T^\Lambda$	Furture
Belle [44]	$e^+ + e^- \rightarrow \Lambda/\bar{\Lambda} + X$	10.58	$P_T^\Lambda \sim D_{1T}^{\Lambda/\bar{\Lambda}}$	~ -0.06
	$e^+ + e^- \rightarrow \Lambda/\bar{\Lambda} + h^\pm + X$		$P_T^\Lambda \sim D_1^h D_{1T}^{\Lambda/\bar{\Lambda}}$	~ 0.1
BelleII [45]	$e^+ + e^- \rightarrow \Lambda_c + X$	10.58	Polarization	Furture
	$\Lambda_c \rightarrow \Lambda + e^+ + \nu$		Polarization	-0.86 ± 0.04
	$\Lambda_c \rightarrow \Lambda + \pi^+$		Polarization	-0.91 ± 0.15
OPAL [46]	$e^+ + e^- \rightarrow \Lambda/\bar{\Lambda} + X$	91.2	$P_T^\Lambda \sim D_{1T}^{\Lambda/\bar{\Lambda}}$	~ 0
	$e^+ + e^- \rightarrow \Lambda/\bar{\Lambda} + X$		$P_L^\Lambda \sim G_{1L}^\Lambda$	-0.329
ALEPH [47]	$e^+ + e^- \rightarrow \Lambda/\bar{\Lambda} + X$	91.2	$P_T^\Lambda \sim D_{1T}^{\Lambda/\bar{\Lambda}}$	~ 0
	$e^+ + e^- \rightarrow \Lambda/\bar{\Lambda} + X$		$P_L^\Lambda \sim G_{1L}^\Lambda$	-0.32
TASSO [49]	$e^+ + e^- \rightarrow \Lambda/\bar{\Lambda} + X$	14, 22, 34	$P_T^\Lambda \sim D_{1T}^{\Lambda/\bar{\Lambda}}$	~ 0
	$e^+ + e^- \rightarrow \Lambda/\bar{\Lambda} + X$		$P_L^\Lambda \sim G_{1L}^\Lambda$	~ 0



- **Longitudinal Λ polarization**

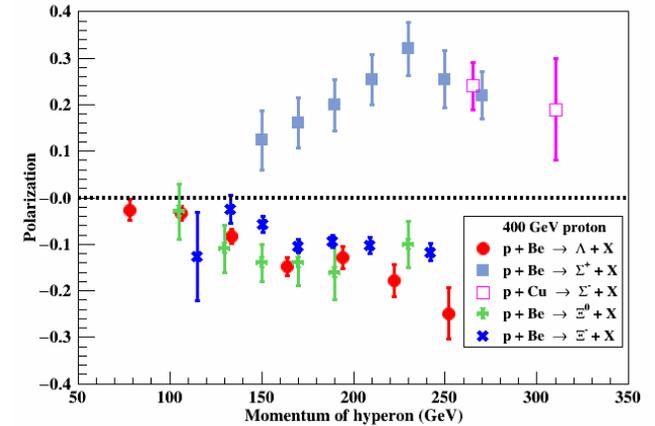
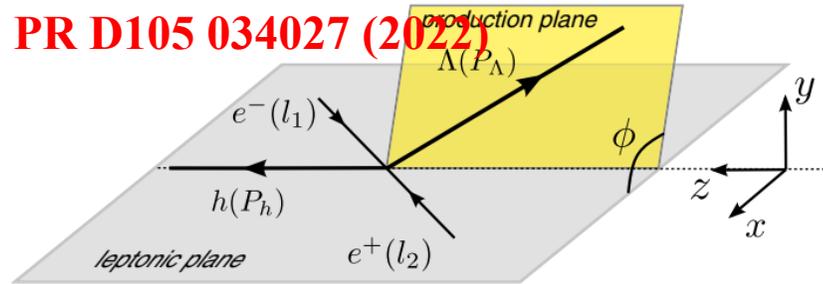
- **Transverse Λ polarization**

- ✓ increases with increasing z_Λ
- ✓ due to hadronization effect

- **Hyperons, beyond Λ ?**

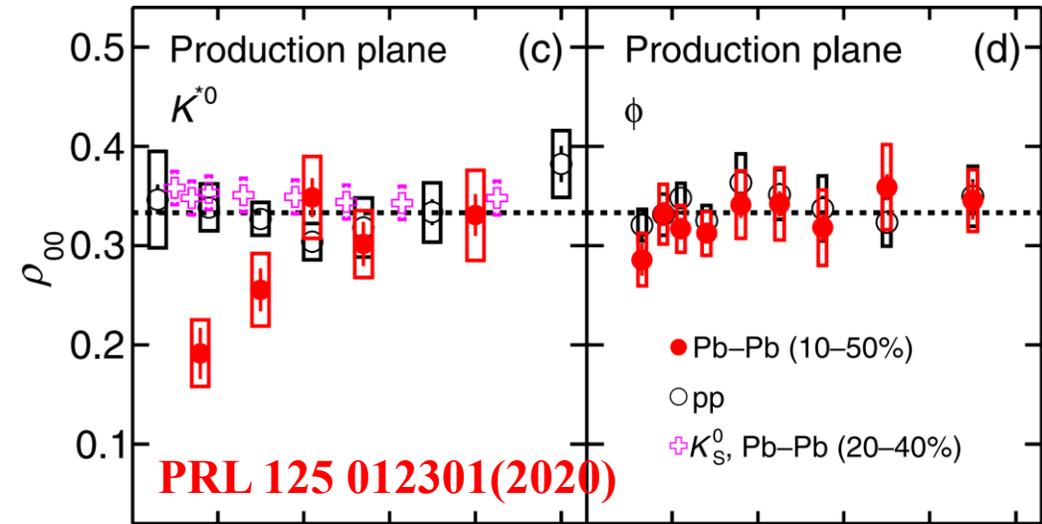
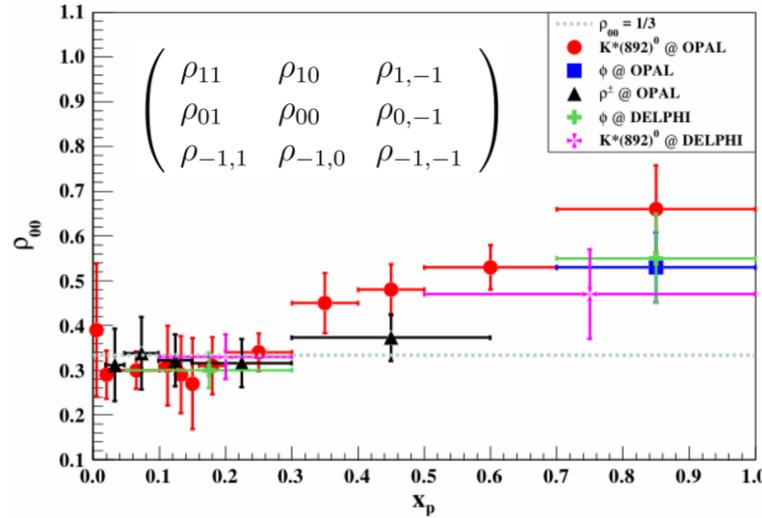
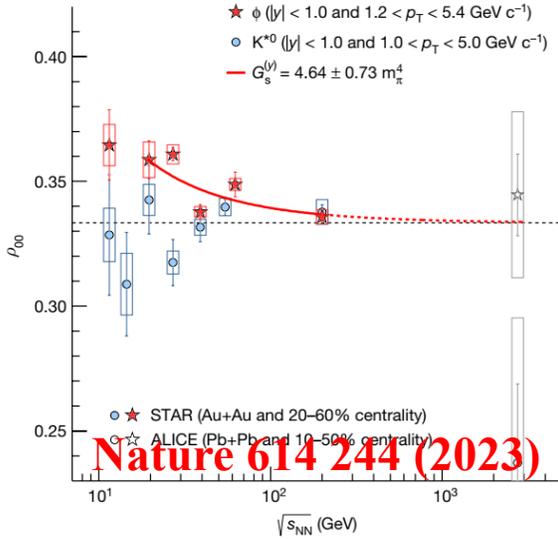
- ✓ negative polarization for Λ , Ξ^0 and Ξ^-
- ✓ positive polarization for Σ^+ and Σ^-
- ✓ Σ^0 vs Σ^+ and Σ^- , Σ^0 vs Λ (uds)

PR D105 034027 (2022)

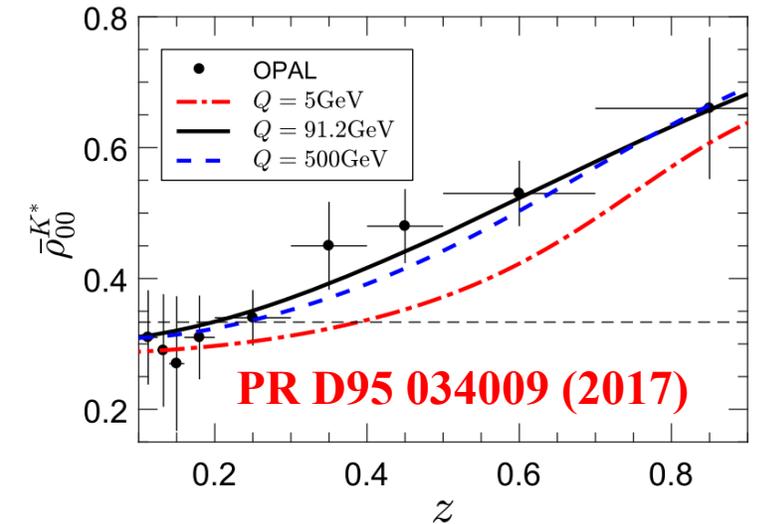


$$\frac{d\sigma^{e^+e^-}}{dz_h dz_\Lambda dy d^2P_{\perp\Lambda}} = \frac{2\pi N_c \alpha_{em}^2}{Q^2} \{ \mathcal{A}(y) F_{UU}^T + \mathcal{B}(y) F_{UU}^L + \mathcal{C}(y) \cos \phi F_{UU}^{\cos \phi} + \mathcal{B}(y) \cos 2\phi F_{UU}^{\cos 2\phi} + \lambda_\Lambda [\mathcal{C}(y) \sin \phi F_{UL}^{\sin \phi} + \mathcal{B}(y) \sin 2\phi F_{UL}^{\sin 2\phi}] + S_{\Lambda T} [\mathcal{C}(y) \sin \phi F_{UT}^{\sin \phi} + \mathcal{B}(y) \sin 2\phi F_{UT}^{\sin 2\phi}] + S_{\Lambda N} [\mathcal{A}(y) F_{UT}^T + \mathcal{B}(y) F_{UT}^L + \mathcal{C}(y) \cos \phi F_{UT}^{\cos \phi} + \mathcal{B}(y) \cos 2\phi F_{UT}^{\cos 2\phi}] \},$$

Spin @ hadronization: spin alignment



- Heavy ion collision: QGP & spin-orbital coupling
 - ✓ STAR: **unexpectedly large** than $1/3$ @ ϕ meson
- LEP: e^+e^- collision, fragmentation, Z^0 dominant
 - ✓ quark/antiquark: **longitudinally polarized**
 - ✓ $x_p < 0.3$, $\rho_{00} \approx 1/3$; $x_p > 0.3$, $\rho_{00} > 1/3$
- pp collision: PDF & fragmentation
- BESIII/Belle II: e^+e^- collision, fragmentation, γ^* dominant
 - ✓ **energy dependence, any surprise ?**



Energy-energy correlator (EEC) @ e^+e^-

- **Not a new idea @ theory**

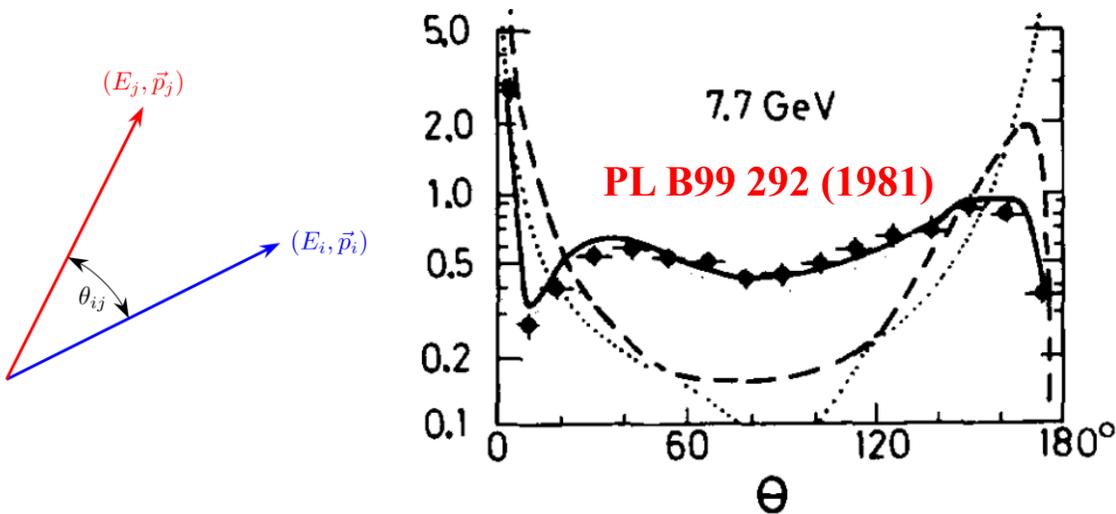
- ✓ infrared & collinear safe event shapes

$$EEC(\theta_L) \equiv \frac{1}{\sigma_{total}} \frac{d\Sigma(\theta_L)}{d\theta_L} = \frac{1}{\sigma_{total}} \int \sum_{ij} \frac{E_i E_j}{(\sqrt{s}/2)^2} d\sigma_{e^+e^- \rightarrow i+j+X} \delta(\theta_L - \theta_{ij})$$

- **Not a new topic @ e^+e^-**

- ✓ widely measured at e^+e^-

- ✓ extract α_s at NNLO+NNLL @ $\sim 2\%$



Energy Correlations in Electron-Positron Annihilation: Testing Quantum Chromodynamics

C. Louis Basham, Lowell S. Brown, Stephen D. Ellis, and Sherwin T. Love
 Department of Physics, University of Washington, Seattle, Washington 98195
 (Received 21 August 1978)

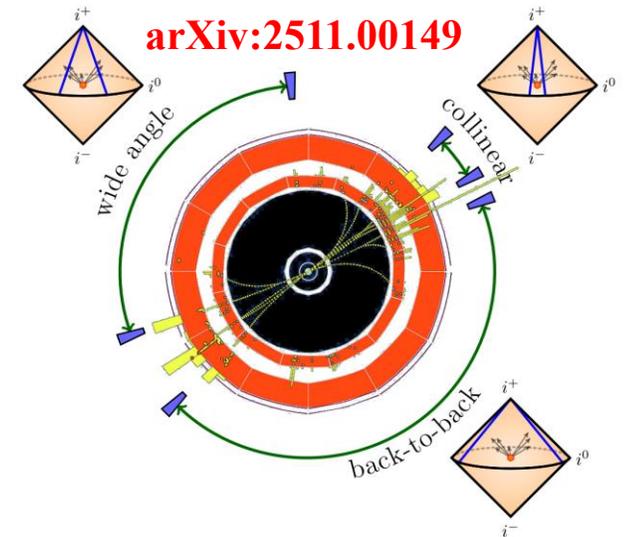
PRL 41 1585 (1978)

An experimental measure is presented for a precise test of quantum chromodynamics. This measure involves the asymmetry in the energy-weighted opening angles of the jets of hadrons produced in the process $e^+e^- \rightarrow \text{hadrons}$ at energy W . It is special for several reasons: It is reliably calculable in asymptotically free perturbation theory; it has rapidly vanishing (order $1/W^2$) corrections due to nonperturbative confinement effects; and it is straightforward to determine experimentally.

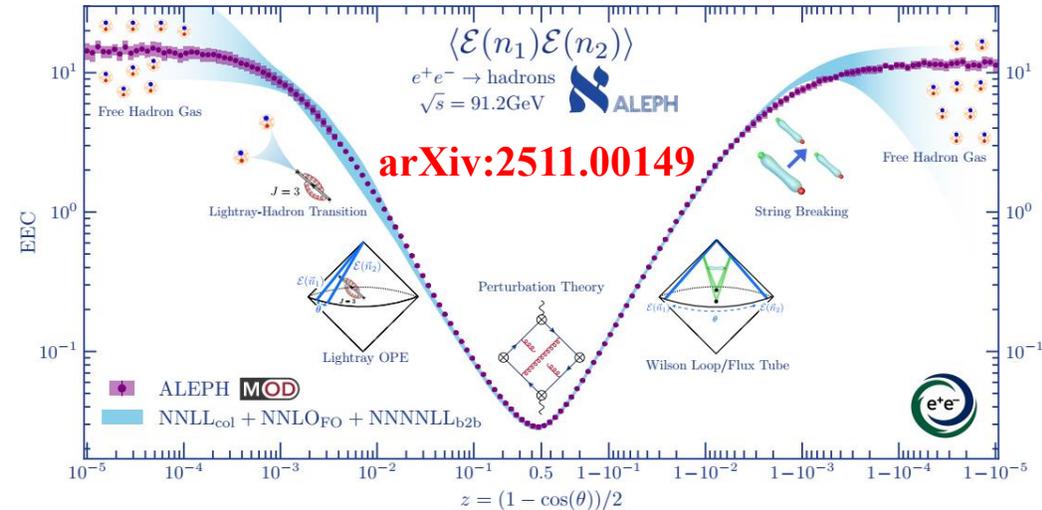
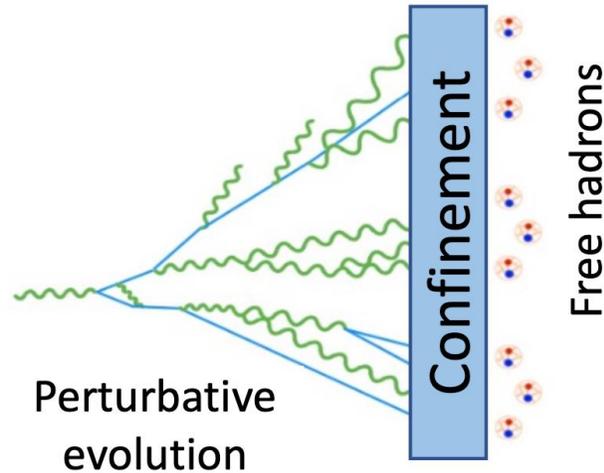
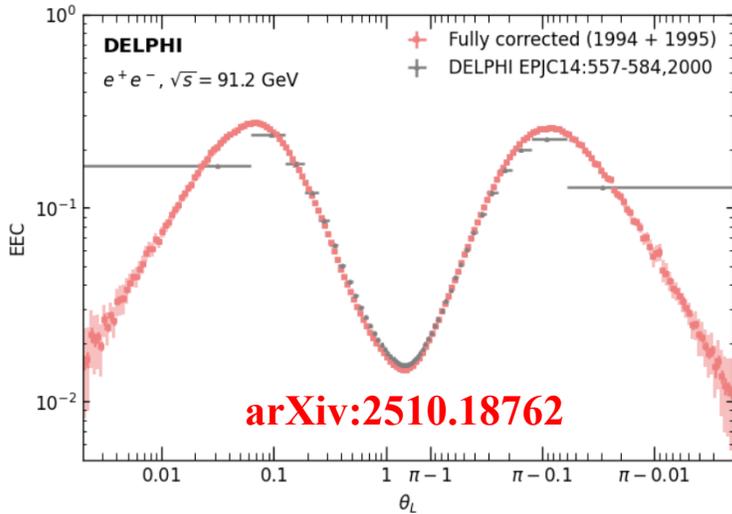
$$EEC(\theta_L) = \frac{1}{N} \sum_{events} \frac{1}{\Delta\theta_L} \int_{\theta_L - \Delta\theta_L/2}^{\theta_L + \Delta\theta_L/2} \sum_{ij} \frac{E_i E_j}{(\sqrt{s}/2)^2} \delta(\theta_L - \theta_{ij}) d\theta$$

Experiment	\sqrt{s} , GeV, data	\sqrt{s} , GeV, MC	Events
SLD [47]	91.2 (91.2)	91.2	60,000
OPAL [50]	91.2 (91.2)	91.2	336,247
OPAL [51]	91.2 (91.3)	91.2	128,032
L3 [48]	91.2 (91.2)	91.2	169,700
DELPHI [49]	91.2 (91.2)	91.2	120,600
TOPAZ [52]	59.0–60.0 (59.5)	59.5	540
TOPAZ [52]	52.0–55.0 (53.3)	53.3	745
TASSO [53]	38.4–46.8 (43.5)	43.5	6434
TASSO [53]	32.0–35.2 (34.0)	34.0	52,118
PLUTO [58]	34.6 (34.6)	34.0	6964
JADE [54]	29.0–36.0 (34.0)	34.0	12,719
CELLO [57]	34.0 (34.0)	34.0	2600
MARKII [56]	29.0 (29.0)	29.0	5024
MARKII [56]	29.0 (29.0)	29.0	13,829
MAC [55]	29.0 (29.0)	29.0	65,000
TASSO [53]	21.0–23.0 (22.0)	22.0	1913
JADE [54]	22.0 (22.0)	22.0	1399
CELLO [57]	22.0 (22.0)	22.0	2000
TASSO [53]	12.4–14.4 (14.0)	14.0	2704
JADE [54]	14.0 (14.0)	14.0	2112

Eur. Phys. J. C78 498 (2018)

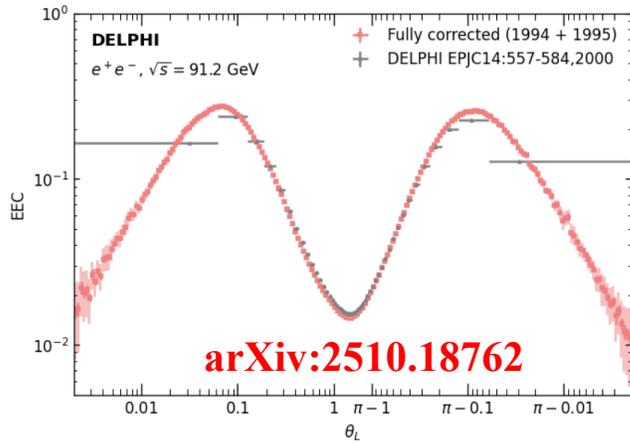


Revisit LEP data

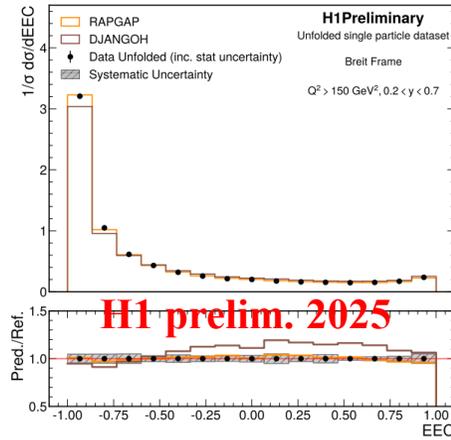


- The 2-point energy correlator (E2C)
 - ✓ limited binning @ small & large θ_L
- Left peak: collinear, E2C inside jets
 - ✓ large θ_L : partonic, perturbative
 - ✓ small θ_L : hadronic phase
 - ✓ middle θ_L : confinement transition
- Mapping time evolution of jet formation onto EEC angular scaling
- Right peak: back-to-back, TMD physics
 - ✓ peak and transition between Sudakov limit and parton shower
 - ✓ a unique opportunity in e^+e^-
- State-of-the-art prediction vs. BESIII/Belle II
 - ✓ physics at collinear & back-to-back limit
- Only pairs of charged particle

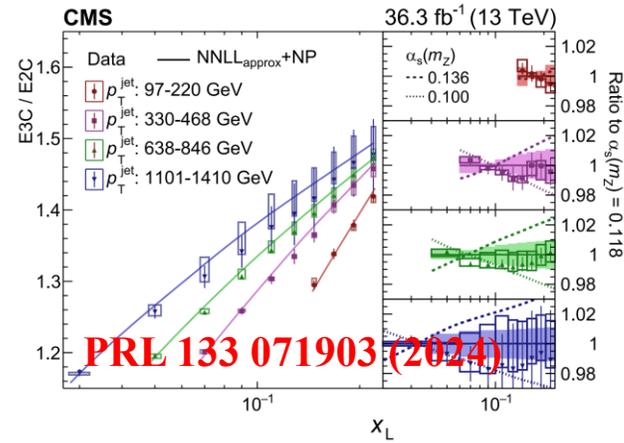
Study of EEC: active area of research !!!



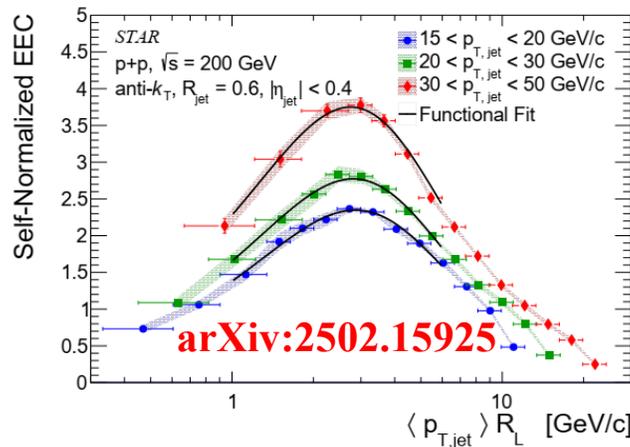
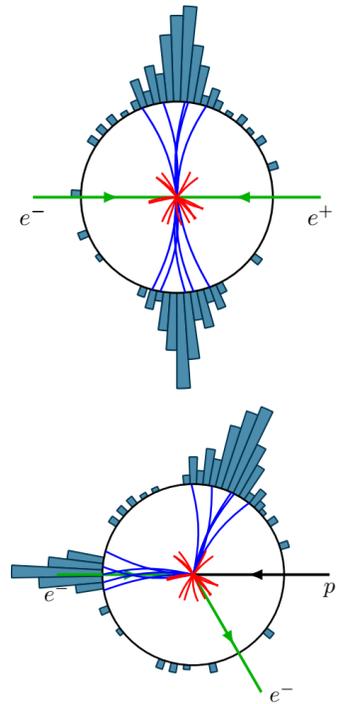
arXiv:2510.18762



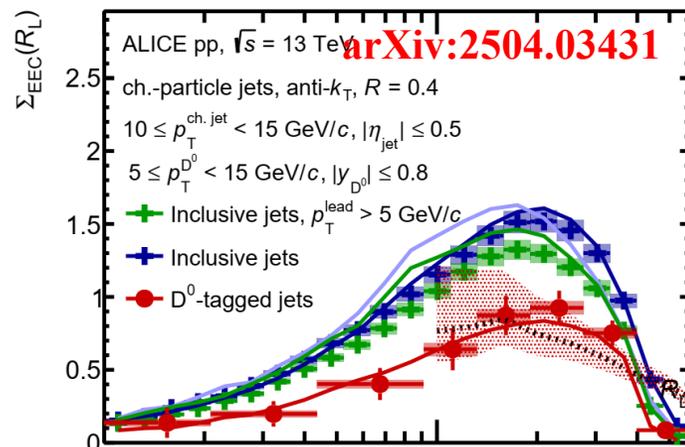
H1 prelim. 2025



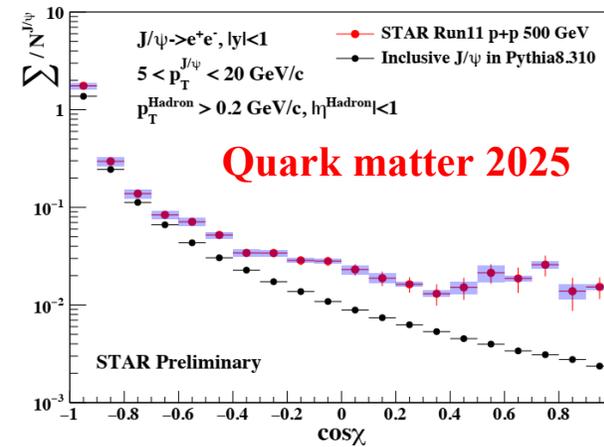
PRL 133 071903 (2024)



arXiv:2502.15925

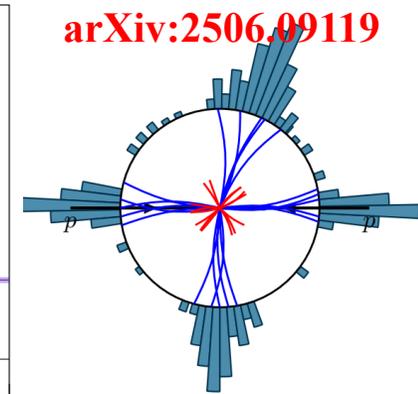


arXiv:2504.03431

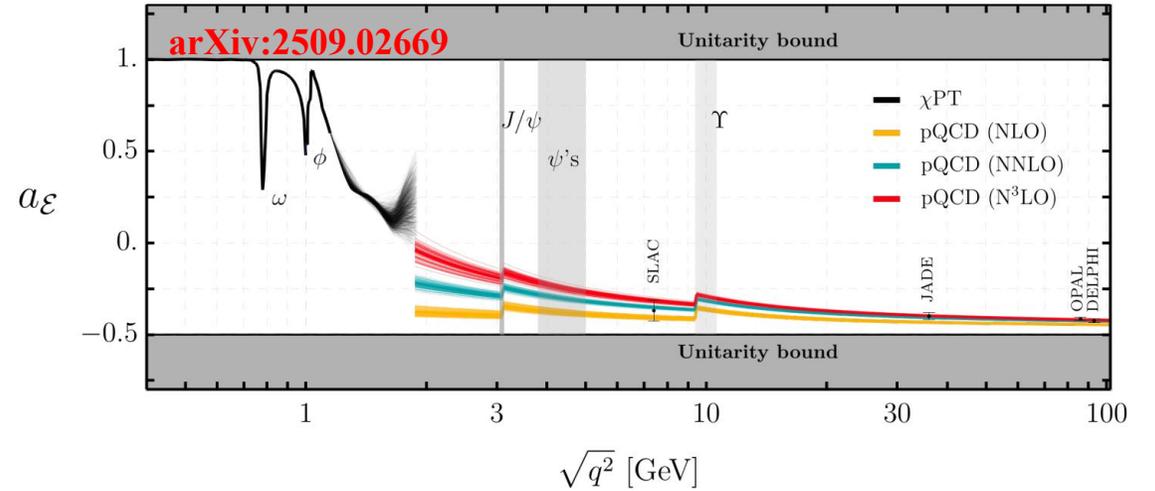
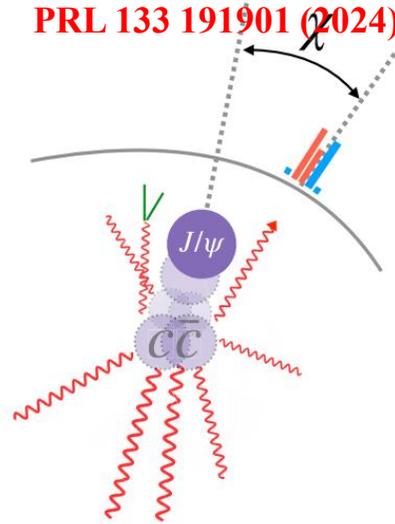
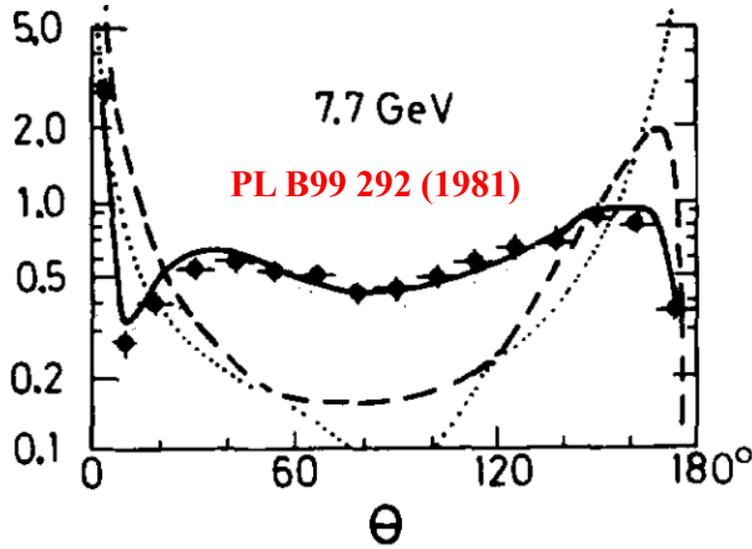


Quark matter 2025

arXiv:2506.09119



EEC @ BESIII



- Energy dependence of EEC

- ✓ results with $\sqrt{s} < 7$ GeV: not yet

- Quarkonium energy correlator

- ✓ inclusive $\rho/\omega/\phi/K^*(892)$?

- One-point energy correlators

- ✓ zero @ \sqrt{s} ? flavor dependence ?

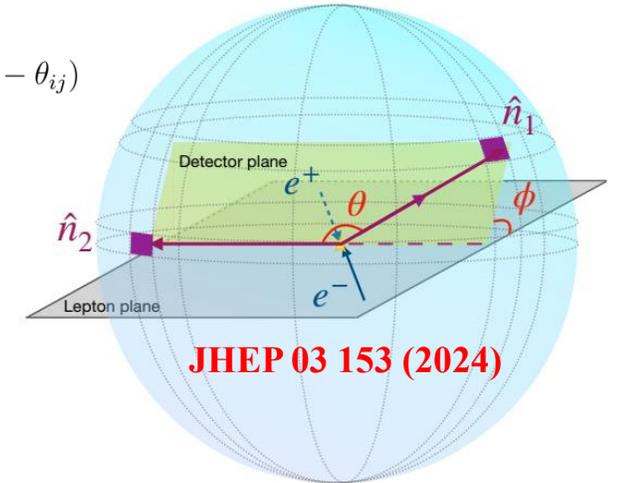
- Azimuthal dependent EEC

$$EEC(\theta_L) \equiv \frac{1}{\sigma_{total}} \frac{d\Sigma(\theta_L)}{d\theta_L} = \frac{1}{\sigma_{total}} \int \sum_{ij} \frac{E_i E_j}{(\sqrt{s}/2)^2} d\sigma_{e^+e^- \rightarrow i+j+X} \delta(\theta_L - \theta_{ij})$$

$$\Sigma(\cos \chi) = \int d\sigma \sum_i \frac{E_i}{M} \delta(\cos \chi - \cos \theta_i)$$

$$\langle \mathcal{E}_n \rangle = \frac{\langle \mathcal{E} \rangle}{4\pi} \left[1 + a_{\mathcal{E}} \left(\frac{3}{2} \sin^2 \theta - 1 \right) \right]$$

$$EEC_{e^+e^-}(\tau, \phi) \equiv \frac{d\Sigma_{e^+e^-}}{d\tau d\phi} = \frac{1}{2} \sum_{1,2} \int d\sigma_{z_1 z_2} \delta\left(\tau - \frac{1 + \cos \theta_{12}}{2}\right) \delta(\phi - \phi_{12})$$



Summary & Outlook

- The FF play a crucial role in describing hadronization process, and understanding of non-perturbative QCD dynamics.
- For FFs @ BESIII
 - ✓ Inclusive $\pi/K/\eta$ @ BESIII: **validity of QCD factorization** and NNLO calculation at energy scales down to 3 GeV, importance of **hadron mass & higher twist**
 - ✓ FFs with hadron: **beyond pion/kaon/eta**
 - ✓ **TMD FFs & Di-hadrons FFs**: possible
 - ✓ **Spin effect** at hadronization: **surprises** @ low energy ?
 - ✓ **Thrust & EEC**: **old & new topics**



TMD FFs: $H_1^{\perp h}(z, p_T)$

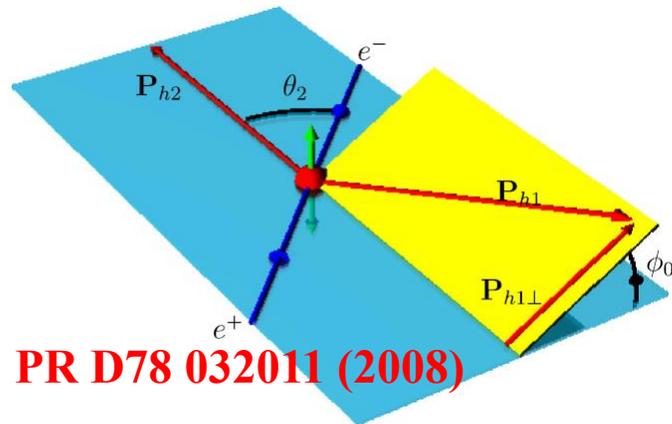
PL B396 (1993) 161



- **Collins Function** H_1 : a transversely polarized quark into a spinless hadron h

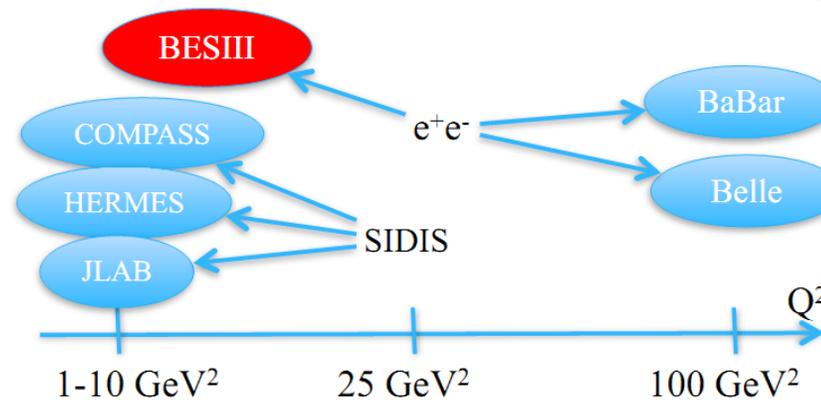
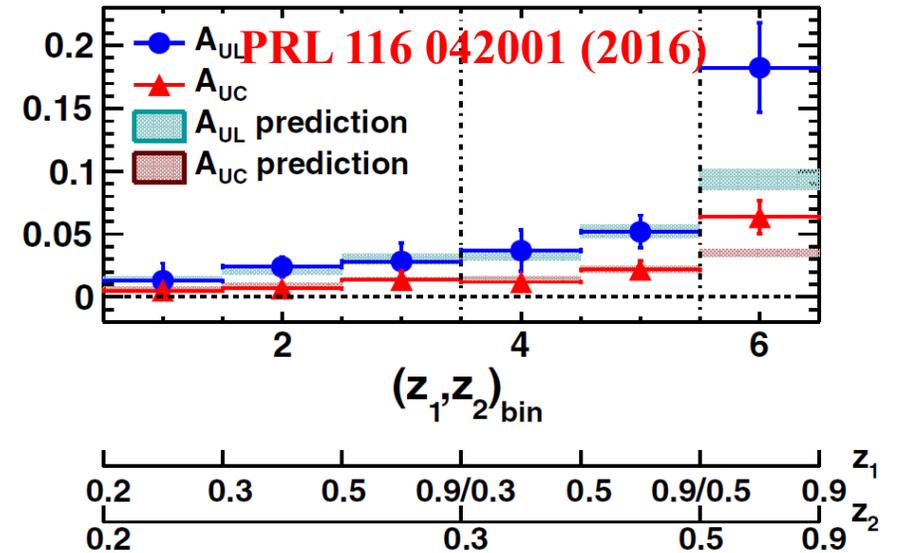
- ✓ Save as “quark polarimeter”
- ✓ Use **back-to-back hadron events**

- TMD FFs evolution



PR D78 032011 (2008)

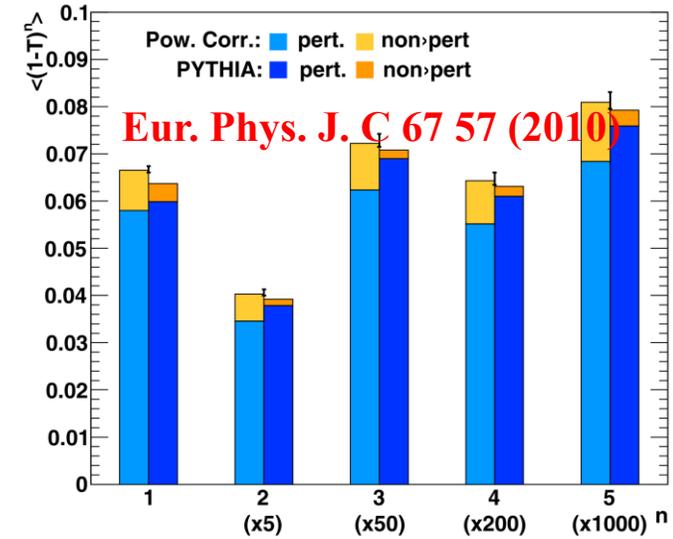
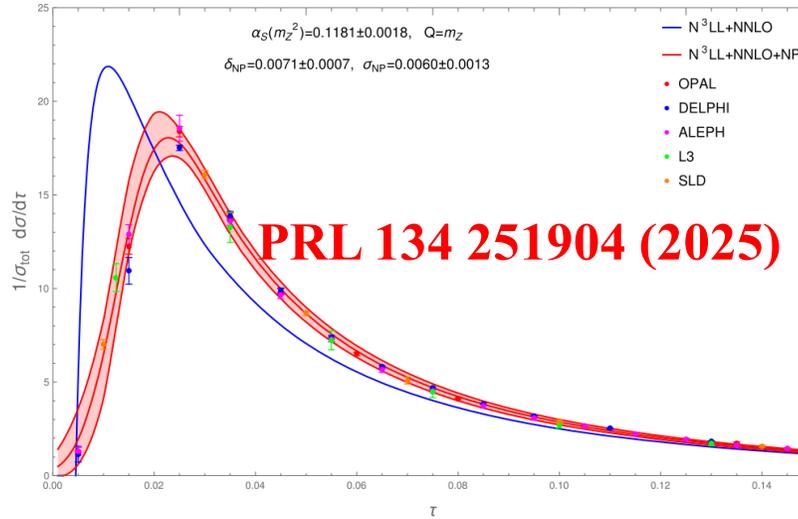
$$D_{hq^\uparrow}(z, P_{h\perp}) = D_1^q(z, P_{h\perp}^2) + H_1^{\perp q}(z, P_{h\perp}^2) \frac{(\hat{\mathbf{k}} \times \mathbf{P}_{h\perp}) \cdot \mathbf{S}_q}{zM_h}$$



Distribution of thrust $\tau \equiv 1 - T$

TABLE I. Partonic contributions to the thrust distribution in perturbative QCD. **PRL 99 132002 (2007)**

LO	$\gamma^* \rightarrow q\bar{q}g$	tree-level
NLO	$\gamma^* \rightarrow q\bar{q}g$	one-loop
	$\gamma^* \rightarrow q\bar{q}gg$	tree-level
NNLO	$\gamma^* \rightarrow q\bar{q}q\bar{q}$	tree-level
	$\gamma^* \rightarrow q\bar{q}g$	two-loop
	$\gamma^* \rightarrow q\bar{q}gg$	one-loop
	$\gamma^* \rightarrow q\bar{q}q\bar{q}$	one-loop
	$\gamma^* \rightarrow q\bar{q}q\bar{q}g$	tree-level
	$\gamma^* \rightarrow q\bar{q}ggg$	tree-level



- Fixed order calculation, **NNLO**
- Small τ ($\tau \ll 1$) due to soft & collinear gluons
 - ✓ re-summation of large logarithmic corrections $\ln(1/\tau)$ to all orders in α_s
 - ✓ **N⁴LL** in full QCD @ **PRL 134 251904 (2025)**
- NNLO + **N⁴LL** vs NNLO + **N³LL** @ **Z⁰**: few permille
 - ✓ low \sqrt{s} e^+e^- data: motivation for theory ???
- Perturbative & no-perturbative contribution
- Re-analysis LEP data with modern method

