New Strucutes in Feynman Integrals

Xing Wang (王 星)

The Chinese University of Hong Kong, Shenzhen

Based on 2506.09124 (PRL accepted), 2507.23594 (JHEP), 2511.15381 with ε-collaboration,

and 2603.18576, 2605.xxxxx, with Yefan Wang and Jian Wang

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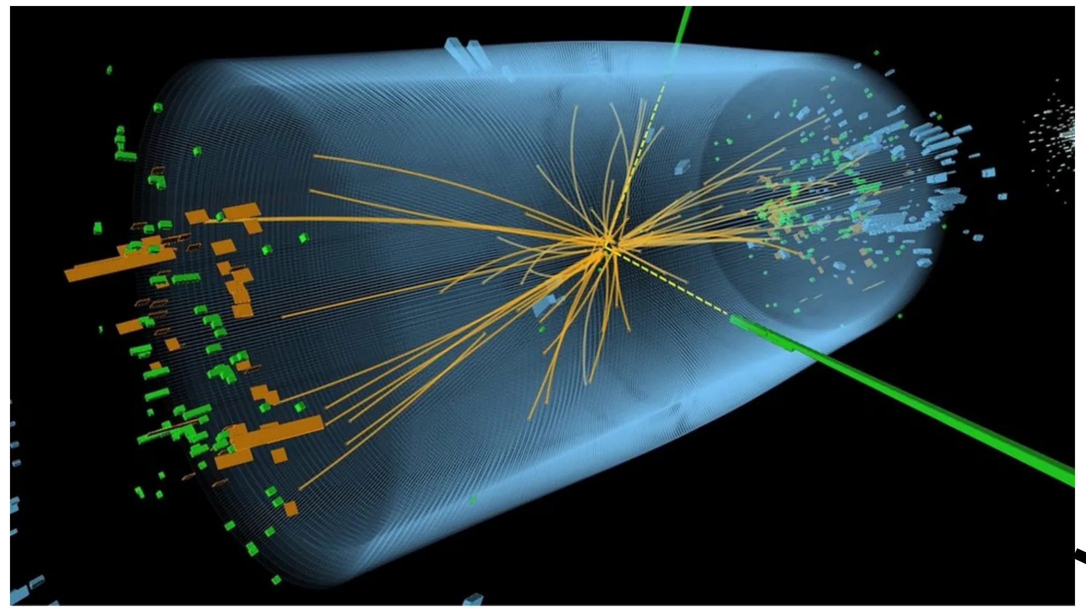
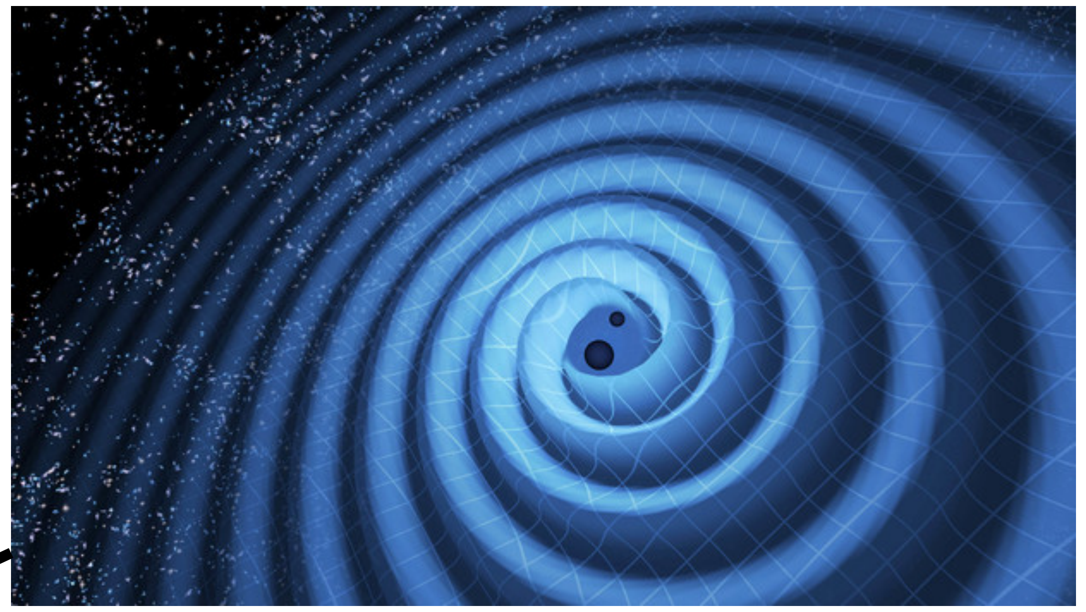
Motivation of Feynman Integrals

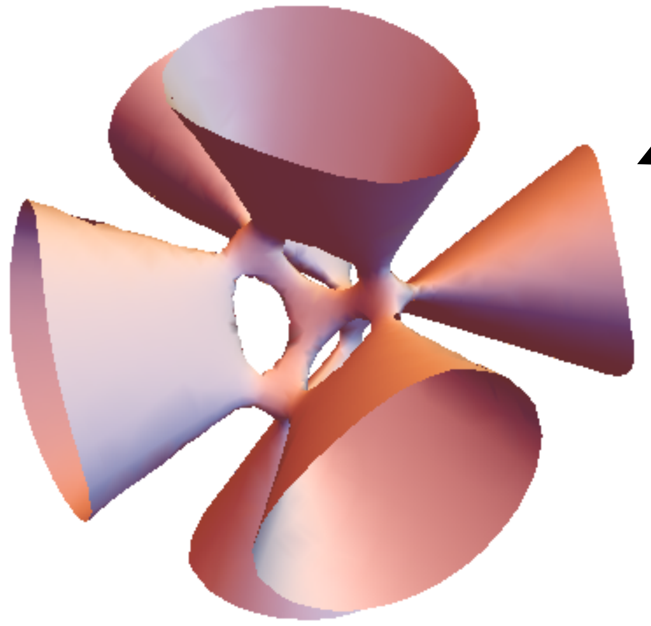


precision prediction with QFTs



$$O\left(\alpha, \{x\}\right) = \sum_A c_A F_A$$

F_A: **Feynman Integrals**



mathematics

$\mathcal{N} = 4$ sYM, string…

formal theory

General Aspects of Feynman Integrals

IBP coeffs.

$$F_{\nu_1\nu_2\cdots\nu_N}(\varepsilon,\{x\}) = \sum_{k=1}^{N_F} c_k \cdot I_k$$

master integral

[Kotikov '91; Remiddi '97;

Gehrmann, Remiddi '00]

+ lots of analytic applications

+ AMFlow (numerics rev.)

IBP

bottlenecks

MIs by DEs

LiteRed, FIRE, Reduze, Kira, NeatIBP, Blade, Finiteflow…

See talks by **Feng, Zhang, Wu for recent progress**.

$$\mathrm{d}\begin{pmatrix} I_1 \\ \vdots \\ I_{N_F} \end{pmatrix} = A_{N_F\times N_F}(\varepsilon,\{x\})\begin{pmatrix} I_1 \\ \vdots \\ I_{N_F} \end{pmatrix}$$

IBP complexity explodes fast!

Solving this linearly coupled PDE system is non-trivial!

Outline and Takehome Message

It is well-known that FIs have **block structures**, specified by sectors.

They also have a **multi-layered (Hodge-like) structure** in a sector.

☑ Improve IBP (with the Laporta algorithm), in general.

☑ An algorithm to derive ε-factorizing (canonical) DEs of MIs.

The structure is **universal**, as are two aspects of the method.

A Toy Model of IBP

$$
\begin{aligned}
14732\, I_1 - 2514\, I_2 - 5\, I_3 - 7\, I_4 &= 0, \\
9872\, I_1 - 17294\, I_2 + 3\, I_3 - 11\, I_4 &= 0, \\
5068\, I_1 - 49336\, I_2 + 18\, I_3 - 22\, I_4 &= 0.
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= \frac{1237}{3025750} I_3 + \frac{1229}{3025750} I_4, \\
I_2 &= \frac{1231}{3025750} I_3 - \frac{1223}{3025750} I_4
\end{aligned}
\qquad
\begin{aligned}
I_3 &= 1223 I_1 + 1229 I_2 \equiv 1223 J_1 + 1229 J_2, \\
I_4 &= 1231 I_1 - 1237 I_2 \equiv 1231 J_1 - 1237 J_2
\end{aligned}
$$

This shows that I_1, I_2 as MIs are simpler than I_3, I_4, because no large denominators are involved.

A Practical Example of IBP

$$F_{\nu_1\nu_2\nu_3\nu_4} = \int \frac{\mathrm{d}^D q_1}{i\pi^{D/2}} \frac{\mathrm{d}^D q_2}{i\pi^{D/2}} \frac{1}{\left[(q_1+p-zk)^2-m^2\right]^{\nu_1}\left[q_1^2-m^2\right]^{\nu_2}\left[q_2^2-m^2\right]^{\nu_3}\left[(q_1+q_2+\bar{z}k-p)^2-m^2\right]^{\nu_4}}$$

$$F_{1311} = -\frac{3\varepsilon(3\varepsilon-1)(z\varepsilon+z-2\varepsilon-1)}{4(z-1)(2(z-2)\varepsilon-1)} \cdot F_{1111} + z\frac{-8z^2\varepsilon(\varepsilon+1)+2z\left(8\varepsilon^2+6\varepsilon-1\right)-2\varepsilon+1}{4(z-1)^2(2(z-2)\varepsilon-1)} \cdot F_{2111}$$

$$F_{1311} = -\frac{3\varepsilon(\varepsilon+1)}{2(1+2\varepsilon)} \cdot \underbrace{F_{2111}}_{J_1} + \frac{2z\epsilon+2z-1}{2(2z-1)(1+2\varepsilon)} \cdot \underbrace{\left(-2\varepsilon \cdot F_{1311} - 2z \cdot F_{2211} + (z+\varepsilon)\cdot F_{1212}\right)}_{J_2} .$$

This shows that we should choose $\{J_1, J_2\}$ as MIs. The new structure can find such a basis systematically. This pattern is inherited by DEs.

ε-factorization (canonicalization)

ε-factorization:

Via **rotations** of basis (i.e., $\vec{I} \to \vec{K}$) and **variable change**, *ε-dependence factorizes in the connection matrix, with suitable boundary conditions.* [Henn '13]

$$\mathrm{d}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} = \varepsilon\, B_{N_F \times N_F}(\{q(x)\}) \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} \Longrightarrow$$

$$\vec{K} = \sum \varepsilon^n \vec{K}^{(n)} \longrightarrow \vec{K}^{(n+1)} = \int B_{N_F \times N_F} \vec{K}^{(n)} + \text{boundary}$$

$$\hookrightarrow \quad \vec{K} = \mathbf{P}\exp\left(\varepsilon \int B_{N_F \times N_F}\right) \vec{K}_{\text{boundary}}$$

MIs can be written as Chen's iterated integrals [Chen '77].

Once the ε-factorized (canonical) form is derived, FIs (MIs) are viewed as solved.

Very Hard! Lots of progress recently, e.g., Baune, Bönisch, Broedel, ε-collaboration, Dlapa, Duhr, Frellesvig, Görges, Henn, Klement, Jiang, Maggio, Nega, Porkert, Sauer, Sohnle, Stawinski, Tancredi, Wager, Wilhelm, Yan, Yang, Zhang, Zhu + many more…

Sketch of the Algorithm: $\vec{I} \to \vec{J} \to \vec{K}$

Step I: Given any pre-basis $\vec{I}$, derive a good (pre-) basis $\vec{J}$, compatible with filtrations (滤过), borrowed from Hodge theory.

$$\mathrm{d}\vec{J} = \left[\frac{1}{\varepsilon^{N_V}}\mathbf{B}^{(-N_V)}(x) + \frac{1}{\varepsilon^{N_V-1}}\mathbf{B}^{(-N_V+1)}(x) + \cdots + \mathbf{B}^{(0)}(x) + \varepsilon\,\mathbf{B}^{(1)}(x)\right]\vec{J}, \quad \vec{J} = \mathbf{R}_1 \cdot \vec{I}$$

$\mathbf{B}^{(-N_V)}(x), \cdots, \mathbf{B}^{(0)}(x)$ are in a good block lower-triangular form. Both $\mathbf{B}^{(i)}(x), \mathbf{R}_1(x)$ rational!

Step II: $\vec{J}$ may be canonical. If not, one can always rotate it to be algorithmically.

$$\mathbf{R}_2^{(-N_V)}\mathbf{R}_2^{(-N_V+1)}\cdots\mathbf{R}_2^{(0)}\,\vec{K} = \vec{J}$$

$\mathbf{R}_2^{(i)}(x)$ may be transcendental! The algorithm applies to FIs related to different geometries.

Multi-layered Structures of Feynman Integrals

Baikov Rep. : Integrals Hard, but Integrands Simple

$\mathrm{d}^D l \to \mathrm{d}P_1(l)\mathrm{d}P_2(l)\cdots$, i.e., propagators, P_i's, as integration variables: $z_i = P_i$

↪ non-trivial "Jacobian", **twist** $u(z) = \prod_{j\in\text{all}} \left[p_j(z)\right]^{\frac{1}{2}(a_i+b_i\varepsilon)}$, $a_i = 0,1$; $b_i \in \mathbb{Z}$:

$$F_i = C_{\text{Baikov}} \int_{\text{MC}} \underbrace{u\left(z\right) \frac{q_i(z)}{\prod_{j\in\text{all}} \left[p_j(z)\right]^{\mu_j}} \mathrm{d}z_{N_V} \wedge \cdots \wedge \mathrm{d}z_1}_{\hat{\phi}}, \qquad \mu_j \in \mathbb{Z}$$

Packages: [*Baikovletter*, Jiang, Yang; *BaikovPackage*, Frellesvig; *SOFIA*, Correia, Giroux, Mizera]

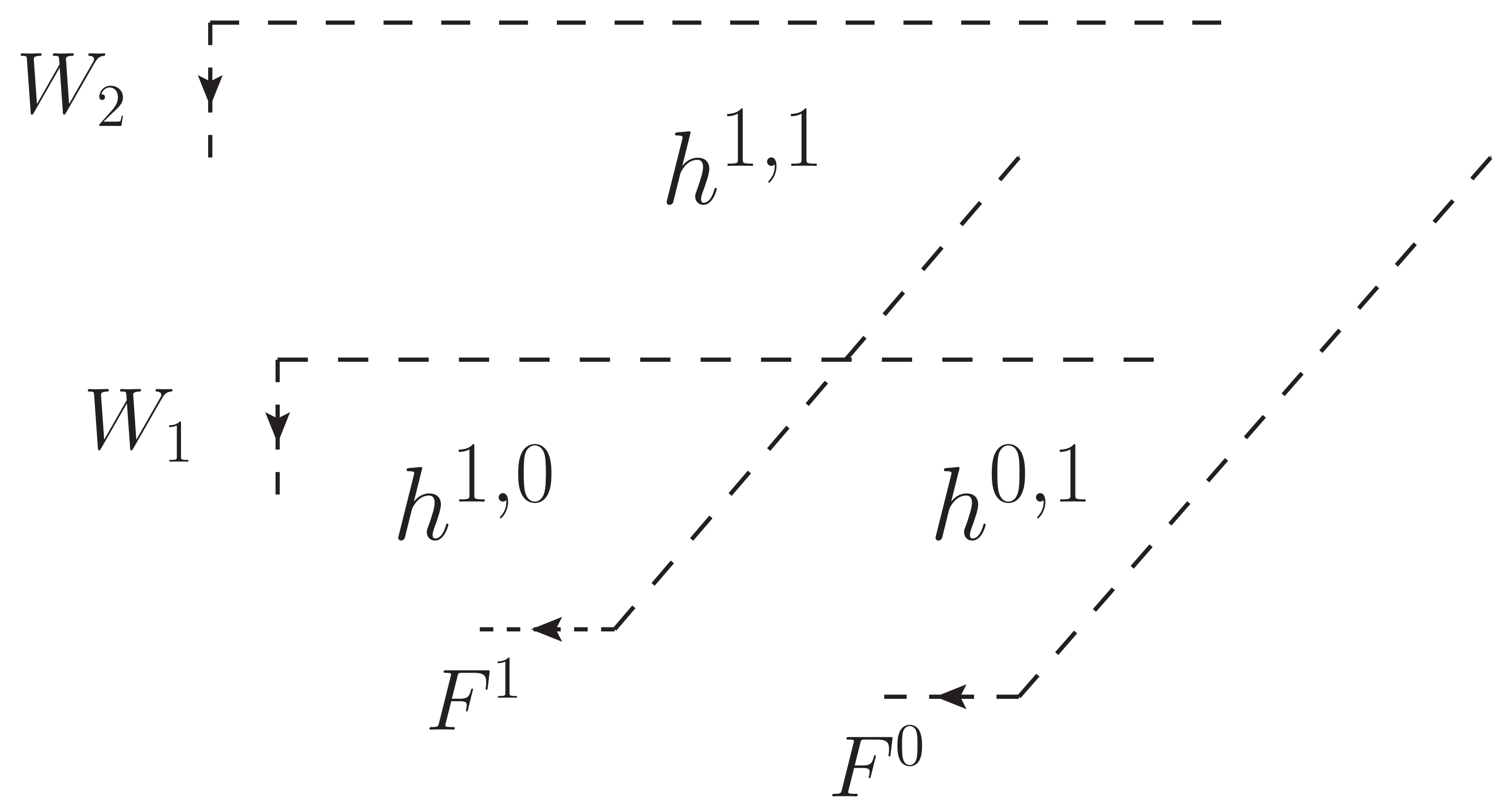
- Not for calculating FIs, but rather for studying the structures!
- It translates FIs to twisted cohomology language: $u\hat{\phi} \sim_u u\left(\hat{\phi} + \nabla_u \omega\right)$;
- Perfect for (onshell) propagator cuts: $\text{cut}_i = \text{res}_{z_i=0}$.

Filtrations

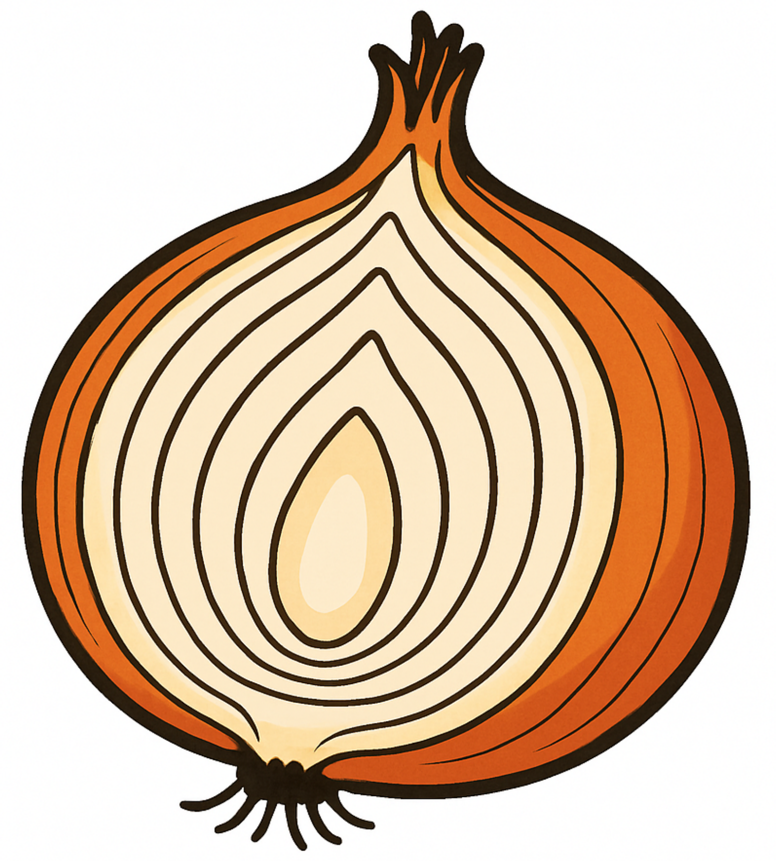
"Layered" decomposition (**filtration, 滤**) of $H_\omega^{N_V} = \{\hat{\phi}\}/\sim_u$ into subspaces! （剥洋葱）

$$\cdots \subseteq F^{p+1} H_\omega^{N_V} \subseteq F^p H_\omega^{N_V} \subseteq \cdots$$

$$\cdots \supseteq W_w H_\omega^{N_V} \supseteq W_{w-1} H_\omega^{N_V} \supseteq \cdots$$



two filtrations cut an "onion" into 3 subspaces in this example

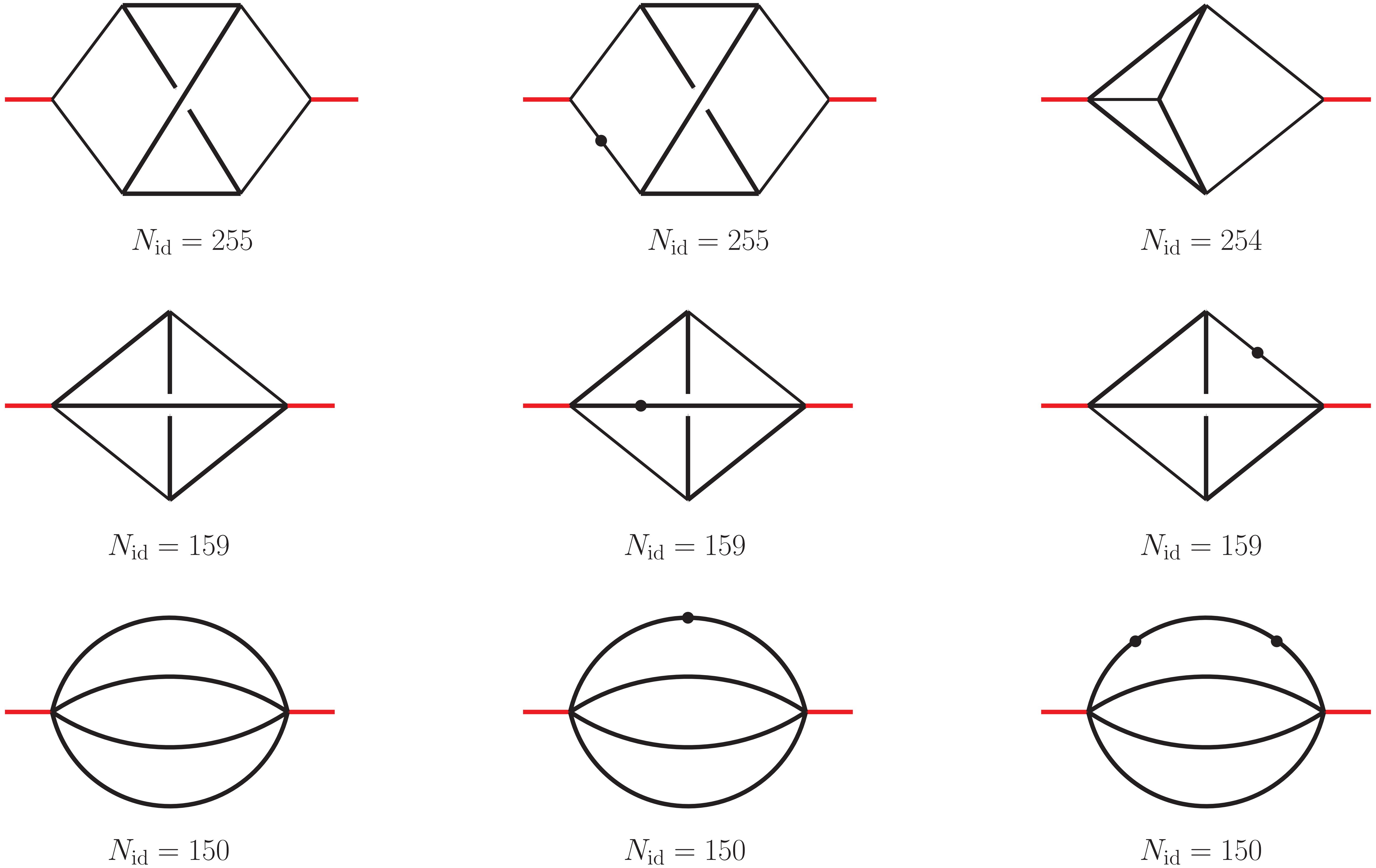


one filtration of onion

Given $\Psi_{\mu_0\cdots\mu_{N_D}}[Q]$, two more ordering criteria (two **filtrations**): pole order o, and the number of non-zero residues r . *These two integers indicate "layer" numbers (两种剥洋葱的指标).*

$$p = N_V - o + r; \qquad q = o; \qquad w = p + q = N_V + r\,.$$

An Example



They contribute (with four massive cuts) to $\Gamma_{H\to b\bar{b}}$, see the talk by Yefan Wang.

An Example: Layered Structure

$$I_i^{(255)} = C_{\text{Baikov}}^{(255)} \cdot \int_{\mathscr{C}_{255}} u(z_1)\, \hat{\phi}_i^{(255)}(z_1)\,; \quad u(z_1) = \left[z_1\left(z_1+t\right)\left(z_1\left(z_1+t\right)-4t\right)\right]^{-\frac{1}{2}-\varepsilon}$$

$\downarrow$ $\downarrow$

$C_{\text{Baikov}}^{(255)}$ defines required pre-factor $C_{\text{abs}}^{(255)}$; $U^{(255)}(z_0,z_1) = P_0^{2\varepsilon} \cdot P_1^{-1/2-\varepsilon} \cdot P_2^{-1/2-\varepsilon} \cdot P_3^{-1/2-\varepsilon}, \ (P_0 = z_0)\,.$

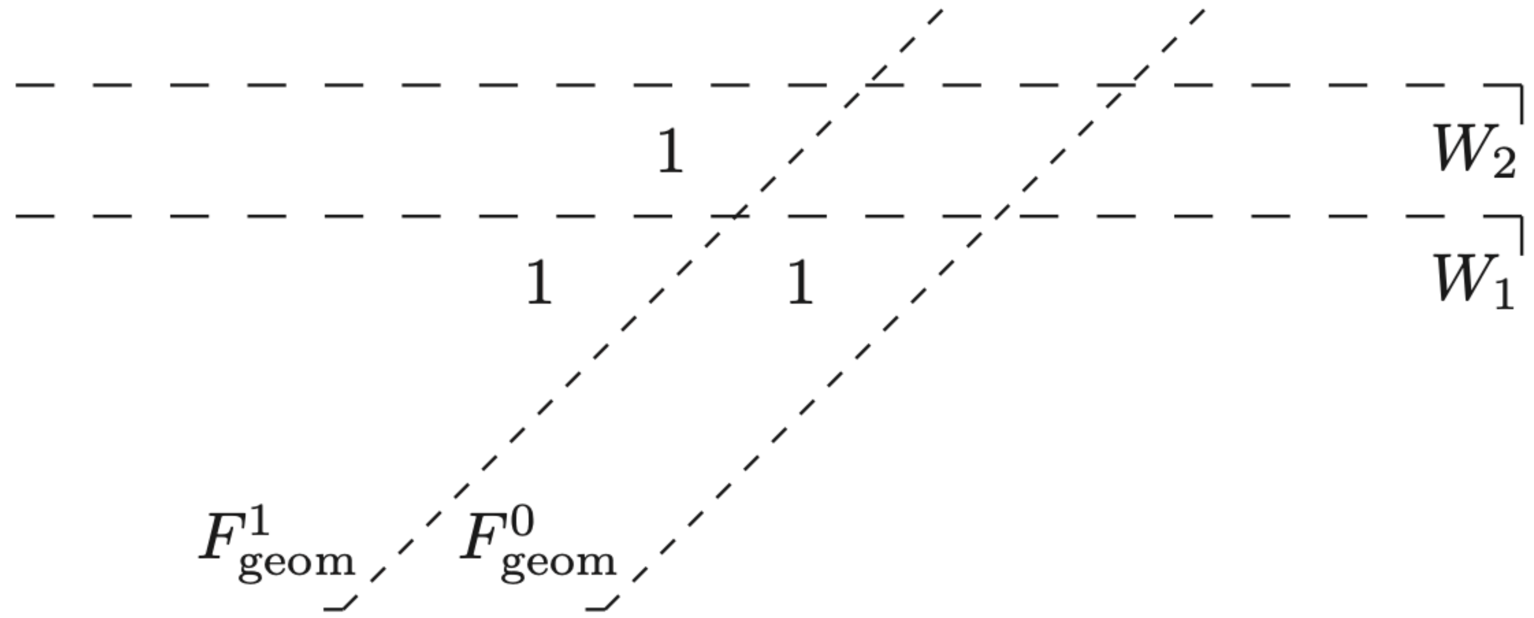
- $w = N_V + r = 1+1$: only at $P_0 = z_0 = 0 \to \hat{\phi}_3^{(255)} = C_{\text{abs}}^{(255)} \cdot 2 \cdot \frac{z_1}{P_0}$;
- $w = N_V + r = 1+0$: two choices. $\hat{\phi}_1^{(255)} = C_{\text{abs}}^{(255)} \cdot 1,\ \hat{\phi}_2^{(255)} = C_{\text{abs}}^{(255)} \cdot \left(-\frac{1}{2}-\varepsilon\right) \cdot \varepsilon^{-1} \cdot \frac{z_0 z_1}{P_3}.$

An Example: Layered Structure

- $w = N_V + r = 1 + 1$: only at $P_0 = z_0 = 0 \ \to \ \hat{\phi}_3^{(255)} = C_{\text{abs}}^{(255)} \cdot 2 \cdot \dfrac{z_1}{P_0}$;
- $w = N_V + r = 1 + 0$: two choices: $\hat{\phi}_1^{(255)} = C_{\text{abs}}^{(255)} \cdot 1$, $\hat{\phi}_2^{(255)} = C_{\text{abs}}^{(255)} \cdot \left(-\dfrac{1}{2} - \varepsilon\right) \cdot \varepsilon^{-1} \cdot \dfrac{z_0 z_1}{P_3}$.

  $\hat{\phi}_1^{(255)}$ and $\hat{\phi}_2^{(255)}$ are specified by the other layer (Hodge filtration).

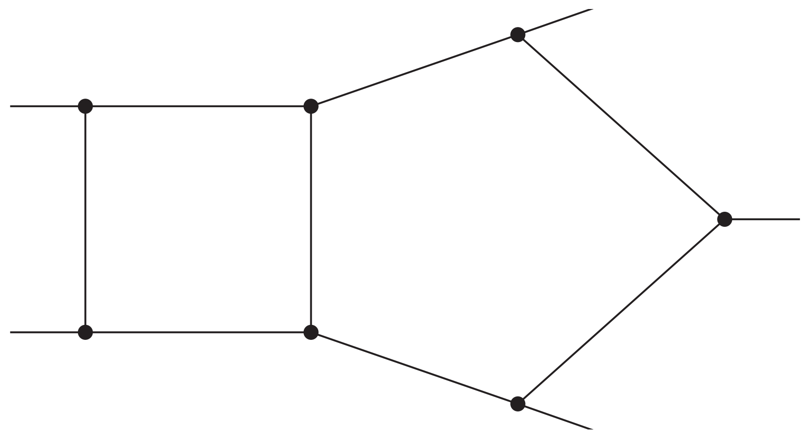
  - ☑ $\hat{\phi}_1^{(255)}$: holomorphic, $p = N_V - o + r = 1 - 0 + 0 = 1$;
  - ☑ $\hat{\phi}_2^{(255)}$: $o = \lfloor 3/2 \rfloor = 1$, no residue, $p = 1 - 1 + 0 = 0$.

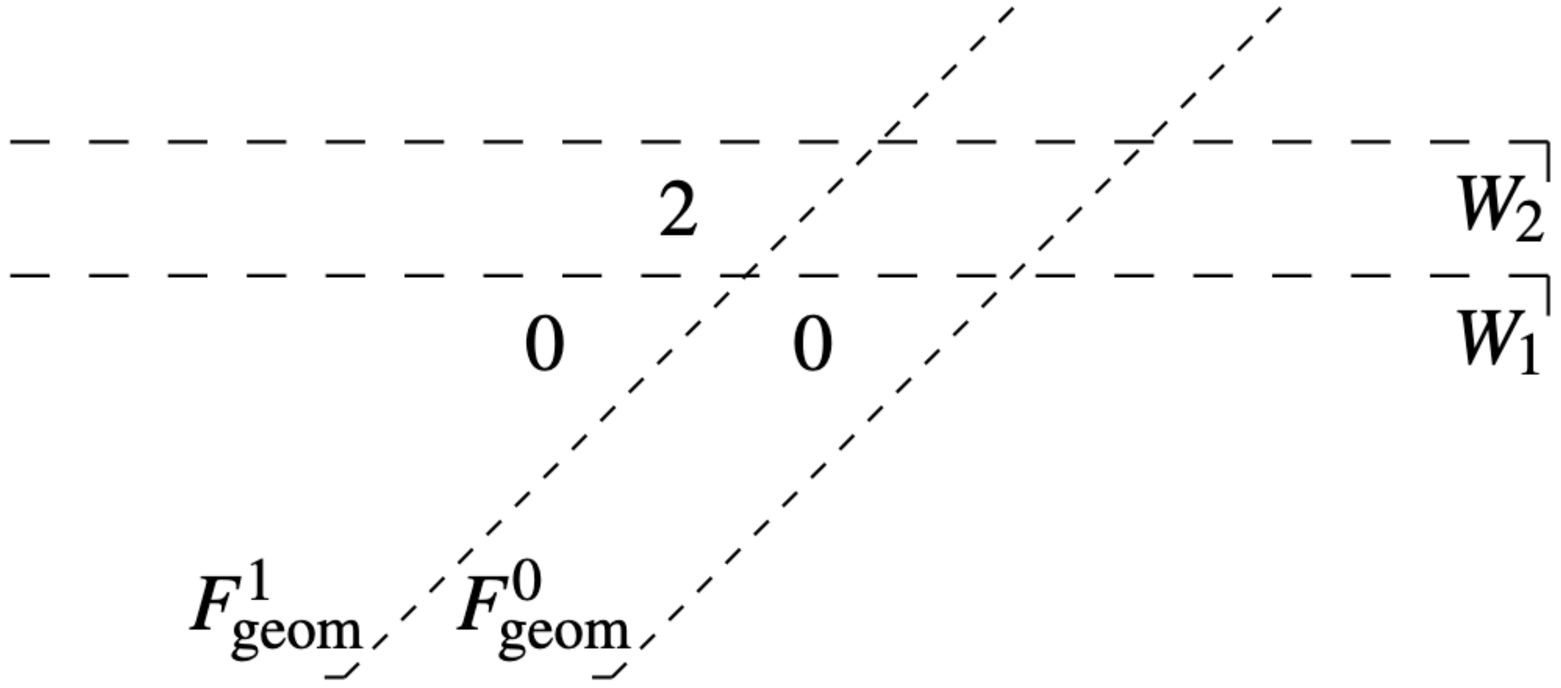


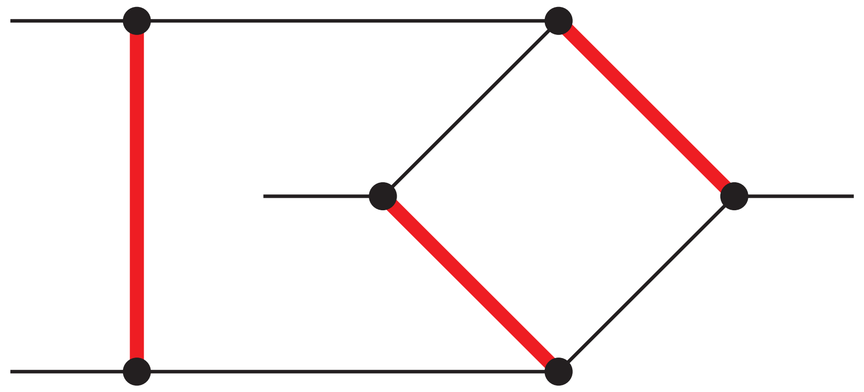
- We can't proceed, and have exhausted all possibilities.

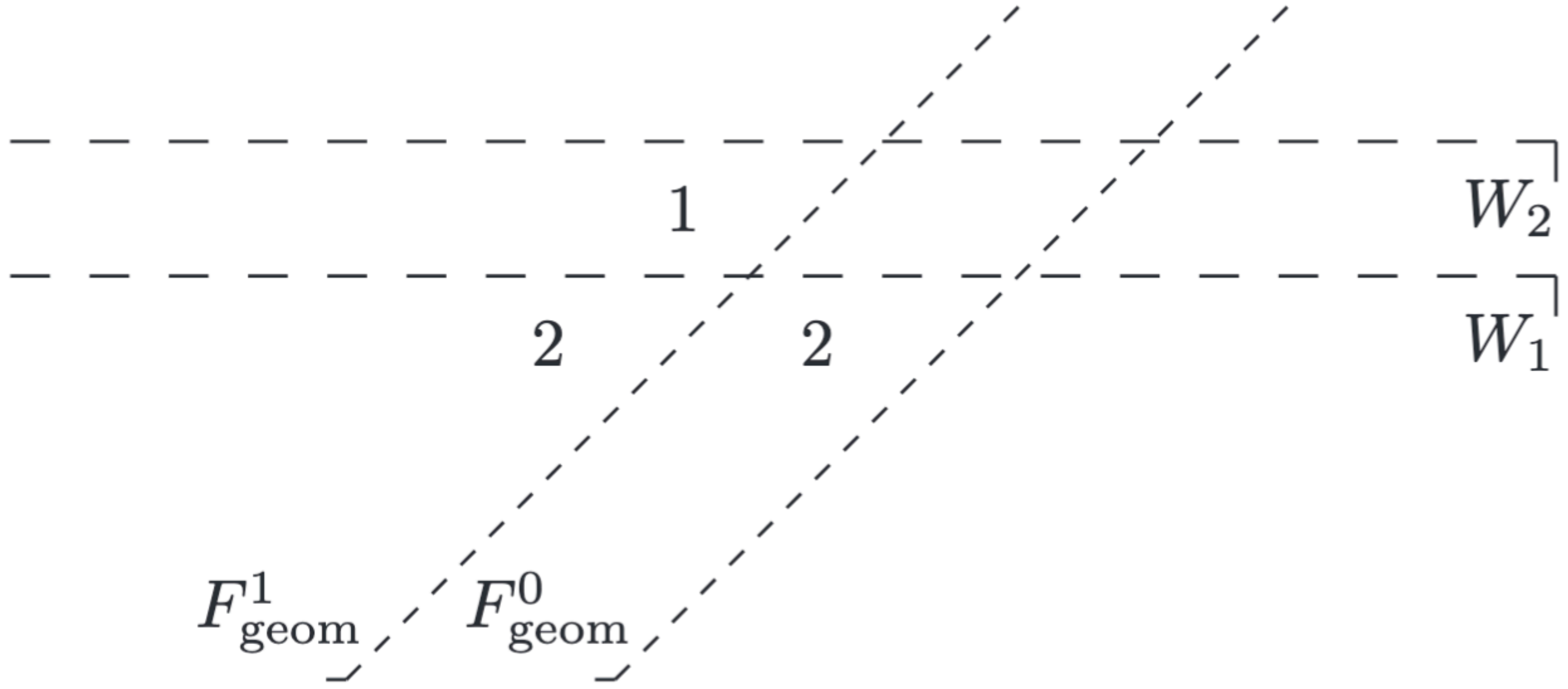
Hodge Diamond $\longrightarrow$ Feynamn Triangle

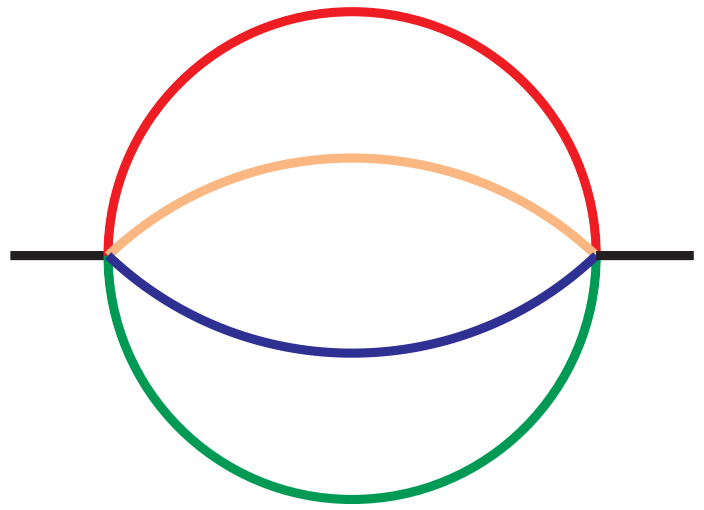
“Feynamn Triangle”: Some Other Exampes

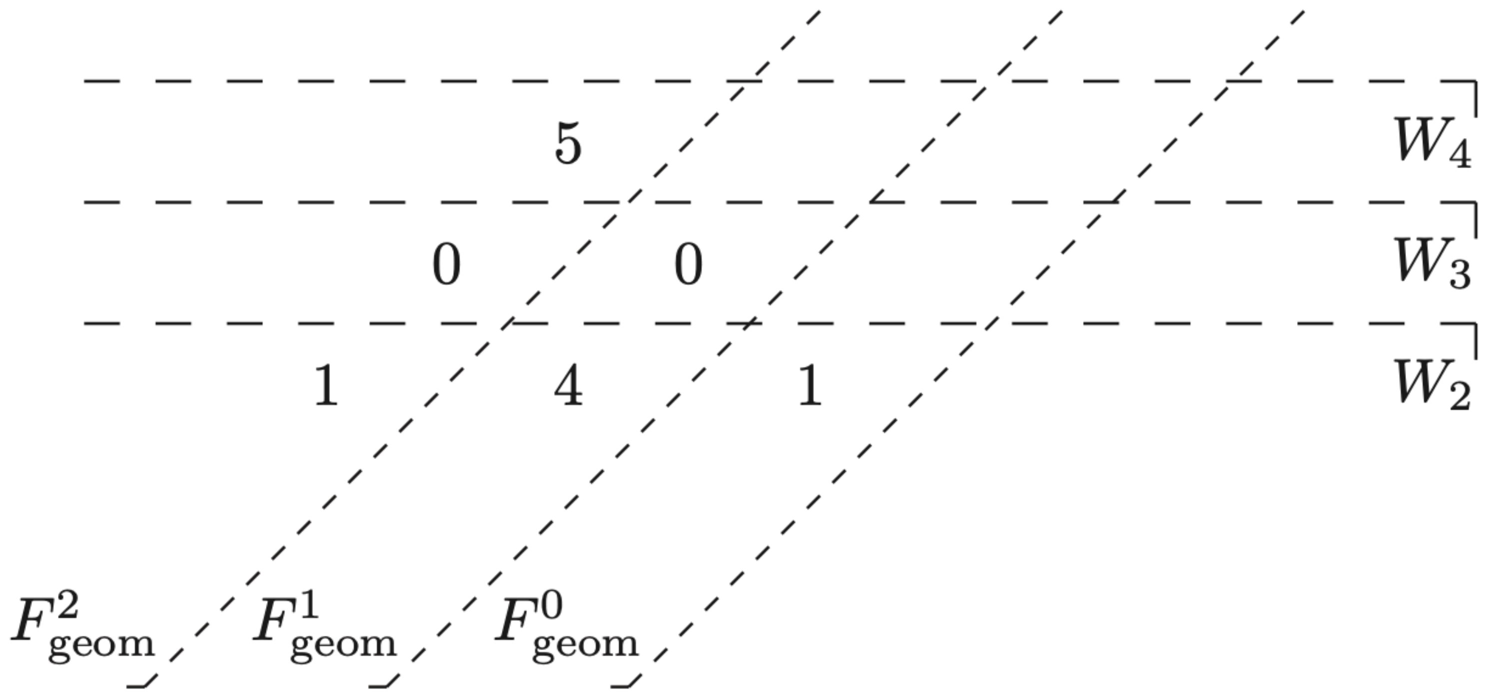












An Example: Layered Structure → Canolicalizing

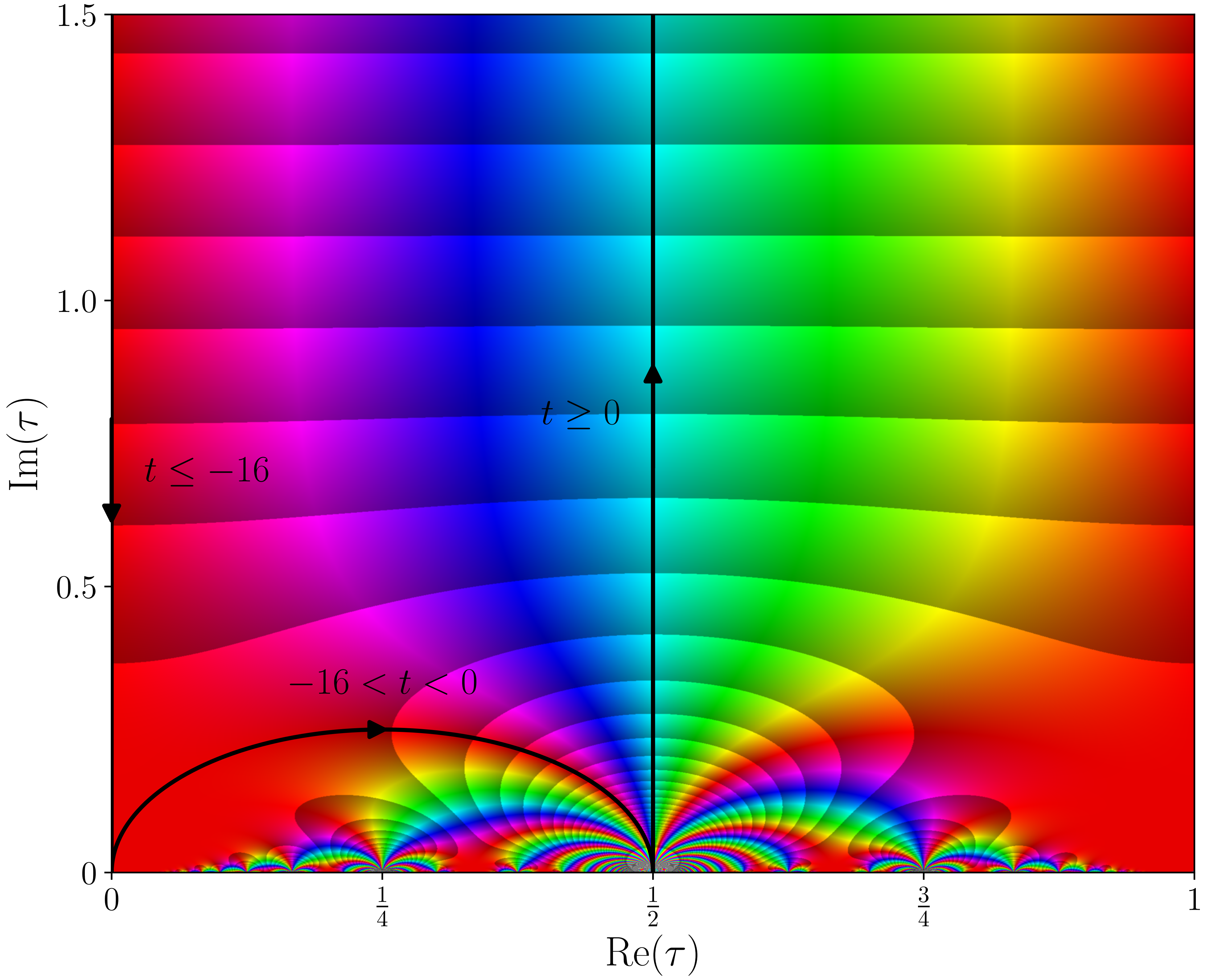
- Back to integrals: $J_1 = t(1-2\varepsilon)\varepsilon^5 \cdot F_{111111110}$, $J_2 = t(1-2\varepsilon)\varepsilon^4 \cdot F_{111111120}$;
- Its DE takes a nice form, which can be rotated to be canonical easily.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} J_1 \\ J_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{t} & 0 \\ \frac{1}{\varepsilon}\frac{1}{t(t+16)} & -\frac{1}{t+16} \end{bmatrix}\begin{bmatrix} J_1 \\ J_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{6}{t(t+16)} & 0 \end{bmatrix}\begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$$

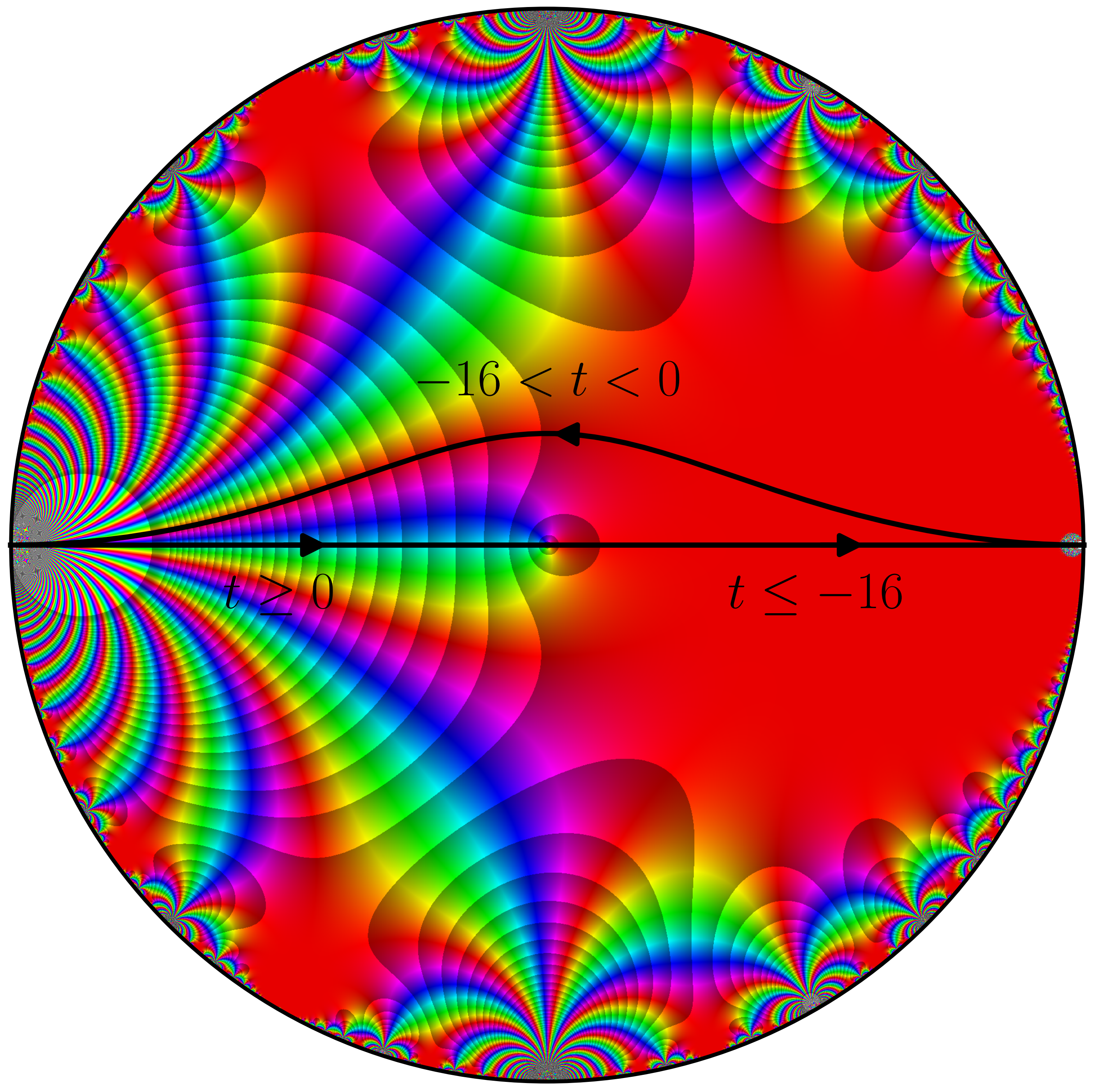
$$\mathbf{R}_{255}^{(-1)} \cdot \mathbf{R}_{255}^{(0)} \cdot \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}$$

- The integrals are elliptic. However, we do not need this knowledge in advance. But this knowledge can certainly help to identify the transcendental functions.

Some Extra Information



(a) τ plane



(b) q-disk

$$-\frac{1}{t} = y = \frac{\eta(\tau)^8\eta(4\tau)^{16}}{\eta(2\tau)^{24}} = \frac{1}{16}\lambda(2\tau)$$

Gemini 3.1 Pro generated the Matplotlib code!

Summary

- FIs not only have block structures, but also have **multi-layered structures**, specified by filtrations.
- After filtrations, FIs enjoy a Hodge-like triangle: "**Feynman triangle**".
- The filtered basis does not have spurious polynomials during IBP, and makes canonicalization always achievable.

The algorithm is universal and cracks the complexity to the minimum.

Thank you for listening.