Ratios of Bc meson decay constants in $\mathrm{N}^3\mathrm{LO}$ QCD

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Based on
Wei Tao, Zhen-Jun Xiao, Rui-Lin Zhu,
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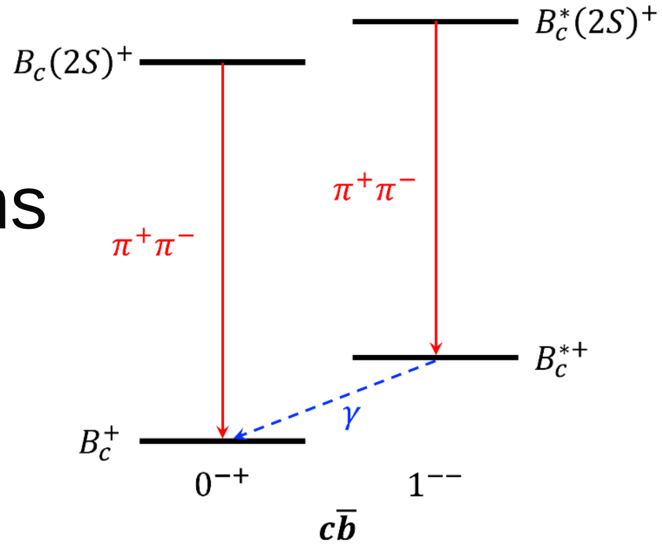
Outline

1.1

Motivation



- ➢ To help discover Bc*(1S) and P-wave $c\bar{b}$ mesons
  - 1998, Bc(1S)
  - 2014, Bc(2S) or Bc*(2S) ?
  - 2019, Bc(2S) & Bc*(2S)

[CDF, PRL(1998)]
[ATLAS, PRL(2014)]
[CMS, PRL(2019)]
[LHCb, PRL(2019)]

- ➢ To understand similarities and differences in $c\bar{b}$ meson leptonic decays via various currents
- ➢ To study new physics by calculating $c\bar{b}$ decay constants involving (axial-)tensor currents
- ➢ To test perturbative expansion convergence in Non-Relativistic Quantum Chromodynamics (NRQCD) effective theory

1.2 Review NRQCD multi-loop calculation for $c\bar{b}$ decay constants

- One-loop for Bc [E.Braaten,S.Fleming,PRD(1995)]
- One-loop for Bc* [D.S.Hwang,S.Kim,PRD(1999)]
- One-loop with relativistic corrections for Bc & Bc* [J.Lee,W.Sang,S.Kim,JHEP(2011)]
- Approximate two-loop for Bc [A.I.Onishchenko,O.L.Veretin,EPJC(2007)]
- Full analytical two-loop for Bc [L.B.Chen,C.F.Qiao,PLB(2015)]
- Three-loop for Bc [F.Feng,et al.,2208.04302]
- Three-loop for Bc* [W.Sang,H.F.Zhang,M.Z.Zhou,PLB(2023)]

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1. Introduction
2. Calculation Procedure
3. Results
4. Summary

2.1 Definition

➢ QCD currents

$$j_s = \bar{\psi}_b \psi_c,$$

$$j_p = \bar{\psi}_b \gamma_5 \psi_c,$$

$$j_v^\mu = \bar{\psi}_b \gamma^\mu \psi_c,$$

$$j_a^\mu = \bar{\psi}_b \gamma^\mu \gamma_5 \psi_c,$$

$$j_t^{\mu\nu} = \bar{\psi}_b \sigma^{\mu\nu} \psi_c,$$

$$j_{t5}^{\mu\nu} = \bar{\psi}_b \sigma^{\mu\nu} \gamma_5 \psi_c,$$

$$\{B_c(^1S_0), B_c^*(^3S_1), B_{c0}^*(^3P_0), B_{c1}(^3P_1)\}$$

➢ QCD definition for decay constants

$$\langle 0|j_p|B_c(q)\rangle \doteq f_{B_c}^p m_{B_c},$$

$$\langle 0|j_a^\mu|B_c(q)\rangle \doteq f_{B_c}^{a,0} q^\mu,$$

$$\langle 0|j_v^\mu|B_c^*(q,\varepsilon)\rangle \doteq f_{B_c^*}^{v,i} m_{B_c^*} \varepsilon^\mu,$$

$$\langle 0|j_t^{\mu\nu}|B_c^*(q,\varepsilon)\rangle \doteq f_{B_c^*}^{t,i0} (q^\mu \varepsilon^\nu - q^\nu \varepsilon^\mu),$$

$$\langle 0|j_{t5}^{\mu\nu}|B_c^*(q,\varepsilon)\rangle \doteq f_{B_c^*}^{t5,ij} \epsilon^{\mu\nu\alpha\beta} q_\alpha \varepsilon_\beta,$$

$$\langle 0|j_s|B_{c0}^*(q)\rangle \doteq f_{B_{c0}^*}^s m_{B_{c0}^*},$$

$$\langle 0|j_v^\mu|B_{c0}^*(q)\rangle \doteq f_{B_{c0}^*}^{v,0} q^\mu,$$

$$\langle 0|j_a^\mu|B_{c1}(q,\varepsilon)\rangle \doteq f_{B_{c1}}^{a,i} m_{B_{c1}} \varepsilon^\mu,$$

$$\langle 0|j_t^{\mu\nu}|B_{c1}(q,\varepsilon)\rangle \doteq f_{B_{c1}}^{t,ij} \epsilon^{\mu\nu\alpha\beta} q_\alpha \varepsilon_\beta,$$

$$\langle 0|j_{t5}^{\mu\nu}|B_{c1}(q,\varepsilon)\rangle \doteq f_{B_{c1}}^{t5,i0} (q^\mu \varepsilon^\nu - q^\nu \varepsilon^\mu),$$

NRQCD factorization

- NRQCD expansion

$$\tilde{j}_p = -\varphi_b^\dagger \chi_c,$$
$$\tilde{j}_a^0 = -\tilde{j}_p,$$
$$\tilde{j}_v^i = \varphi_b^\dagger \sigma^i \chi_c,$$
$$\tilde{j}_t^{i0} = \mathrm{i}\,\tilde{j}_v^i,$$
$$\tilde{j}_{t5}^{ij} = -\epsilon^{ijk}\,\tilde{j}_v^k,$$

$$\tilde{j}_s = -\varphi_b^\dagger \left(\frac{1}{2m_c} + \frac{1}{2m_b}\right) \vec{k}\cdot\vec{\sigma}\chi_c,$$
$$\tilde{j}_v^0 = -\frac{m_b - m_c}{m_b + m_c}\tilde{j}_s,$$
$$\tilde{j}_a^i = \varphi_b^\dagger \left[\left(\frac{1}{2m_c} - \frac{1}{2m_b}\right)\vec{k} + \mathrm{i}\left(\frac{1}{2m_c} + \frac{1}{2m_b}\right)\vec{k}\times\vec{\sigma}\right]^i \chi_c,$$
$$\tilde{j}_{t5}^{i0} = \mathrm{i}\,\varphi_b^\dagger \left[\left(\frac{1}{2m_c} + \frac{1}{2m_b}\right)\vec{k} + \mathrm{i}\left(\frac{1}{2m_c} - \frac{1}{2m_b}\right)\vec{k}\times\vec{\sigma}\right]^i \chi_c,$$
$$\tilde{j}_t^{ij} = \mathrm{i}\,\epsilon^{ijk}\,\tilde{j}_{t5}^{k0},$$

- NRQCD factorization of decay constants

[G.T.Bodwin,E.Braaten,G.P.Lepage,PRD(1995)]

$$f_{B_c}^p = \sqrt{\frac{2}{m_{B_c}}} C_p(m_b, m_c, \mu_f, \mu) \left\langle 0 \left| \chi_b^\dagger \psi_c \right| B_c(\mathbf{P}) \right\rangle (\mu_f) + O(v^2)$$

mathing coefficients — NRQCD LDMEs

$$f_{B_c^*}^{v,i} = \sqrt{\frac{2}{m_{B_c^*}}} C_{v,i}(m_b, m_c, \mu_f, \mu) \left\langle 0 \left| \chi_b^\dagger \sigma\cdot\varepsilon \psi_c \right| B_c^*(\mathbf{P}) \right\rangle (\mu_f) + O(v^2)$$

Matching formula

➢ QCD-NRQCD matching formula

$$\sqrt{Z_{2,b}^{\mathrm{OS}} Z_{2,c}^{\mathrm{OS}}}\, Z_J^{\mathrm{OS}} \left(Z_J^{\overline{\mathrm{MS}}}\right) \Gamma_J = \mathcal{C}_J \left(\overline{\mathcal{C}}_J\right) \sqrt{\widetilde{Z}_{2,b}^{\mathrm{OS}} \widetilde{Z}_{2,c}^{\mathrm{OS}}}\, \widetilde{Z}_J^{-1} \widetilde{\Gamma}_J$$

- $Z_J(\tilde{Z}_J)$: QCD(NRQCD) current renormalization constant
- $Z_2(\tilde{Z}_2)$: QCD(NRQCD) heavy quark field renormalization constant
- $\Gamma_J(\tilde{\Gamma}_J)$: QCD(NRQCD) current vertex function

➢ Relations between matching coefficients

$$\frac{f_{B_c}^{p}}{f_{B_c}^{a,0}} = \frac{\mathcal{C}_p}{\mathcal{C}_{a,0}} = \frac{\widetilde{Z}_p}{\widetilde{Z}_{a,0}} = 1$$

$$\frac{f_{B_c^*}^{v,i}}{f_{B_c^*}^{t,i0}} = \frac{\mathcal{C}_{v,i}}{\mathcal{C}_{t,i0}} = \frac{\widetilde{Z}_{v,i}}{\widetilde{Z}_{t,i0}} = 1 \qquad \widetilde{Z}_s = \widetilde{Z}_{v,0}$$

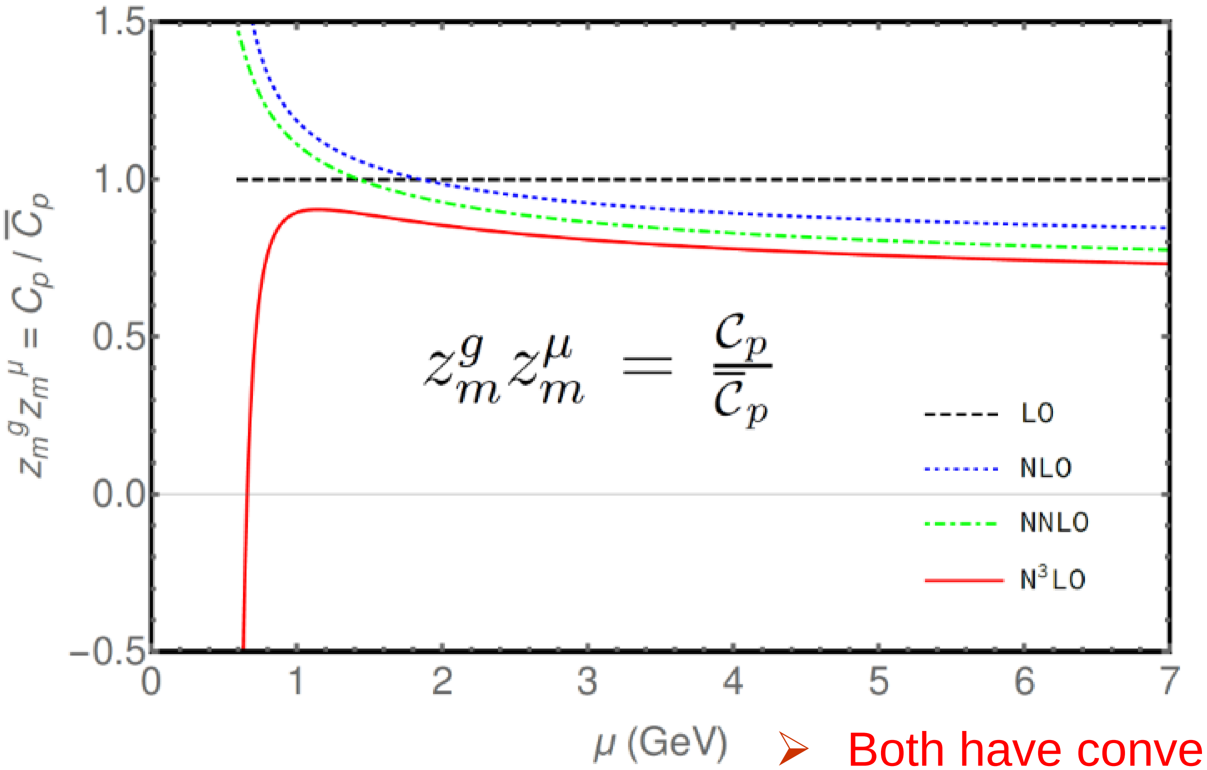
$$\frac{f_{B_{c0}^*}^{s}}{f_{B_{c0}^*}^{v,0}} = \frac{\mathcal{C}_s}{\mathcal{C}_{v,0}} = \frac{m_b Z_{m,b}^{\mathrm{OS}} + m_c Z_{m,c}^{\mathrm{OS}}}{m_b Z_{m,b}^{\mathrm{OS}} - m_c Z_{m,c}^{\mathrm{OS}}} \frac{m_b - m_c}{m_b + m_c}$$

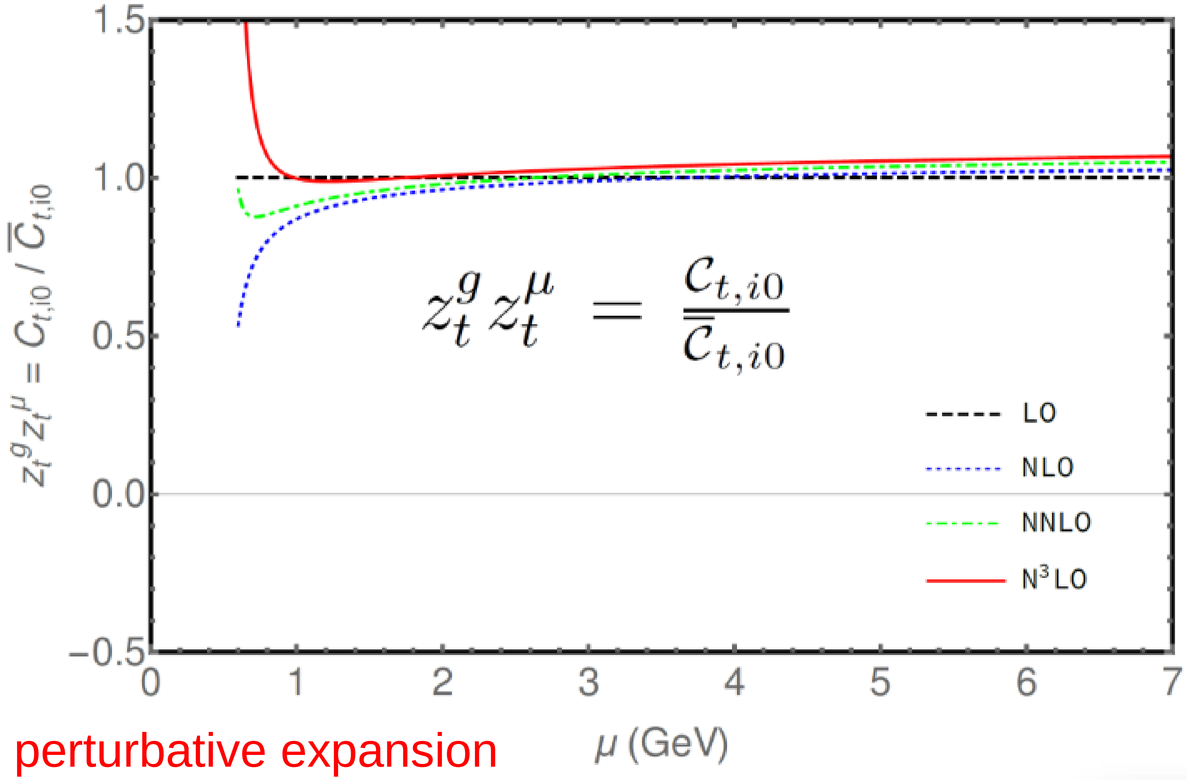
Solve for Z_J^{OS}

- Ratio of QCD current OS to $\overline{\mathrm{MS}}$ renormalization constants

$$\frac{\mathcal{C}_J}{\overline{\mathcal{C}}_J} = \frac{Z_J^{\mathrm{OS}}}{Z_J^{\overline{\mathrm{MS}}}} = z_J^g z_J^\mu + \mathcal{O}(\epsilon),$$

- Verify the correctness of $z_t^g z_t^\mu = \frac{\mathcal{C}_{v,i}}{\overline{\mathcal{C}}_{t,i0}}$



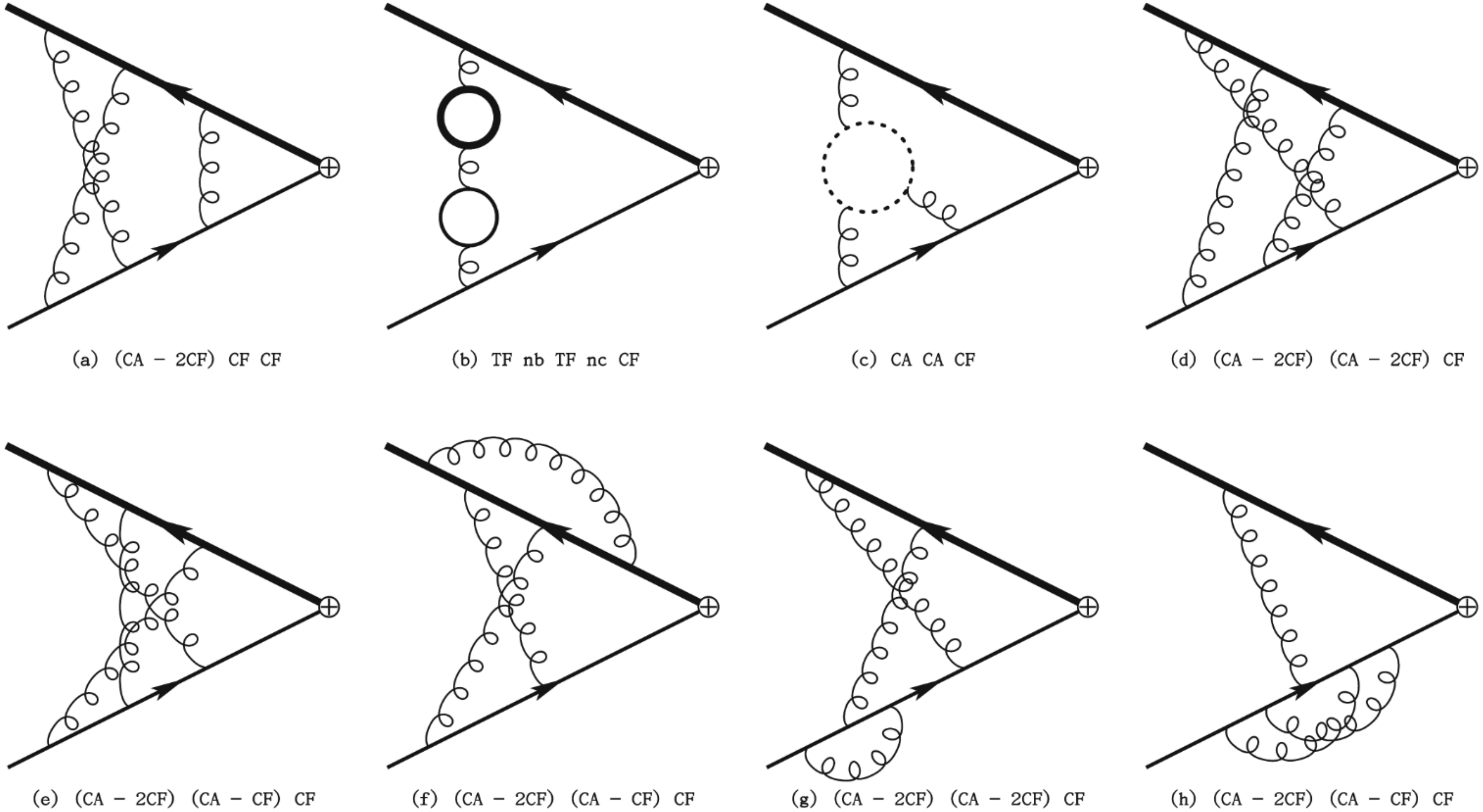


$\mu_f \in [0.4, 1.2, 7]\mathrm{GeV}$

- Both have convergent perturbative expansion
- Both have weak dependence on renormalization scale
- Both are independent of factorization scale

Feynman diagrammatic renormalization at $\mathcal{O}(\alpha_s^3)$

- Tree diagram inserted with one $\mathcal{O}(\alpha_s^3)$ counter-term vertex
- One-loop diagram inserted with one $\mathcal{O}(\alpha_s^2)$ counter-term vertex
- One-loop diagram inserted with two $\mathcal{O}(\alpha_s)$ counter-term vertexes
- Two-loop diagrams inserted with one $\mathcal{O}(\alpha_s)$ counter-term vertex
- Three-loop diagrams



Multi-loop calculation process

- FeynCalc: [V.Shtabovenko, R.Mertig, F.Orellana, CPC(2025)]
  - Generate & filter Feynman diagrams & amplitudes;
  - Express amplitudes in terms of Feynman integrals;
  - Minimize the total number of families of Feynman integrals.
- FIRE / Kira / FiniteFlow: [A.V.Smirnov, F.S.Chukharev, CPC(2020)] [J.Klappert, et al., CPC(2021)] [T.Peraro, JHEP(2019)]
  - Reduce Feynman integrals to master integrals.
- Kira + FIRE + Mathematica code:
  - Minimize the total number of families of master integrals.
- AMFlow + FiniteFlow / Kira: [X.Liu, Y.Q.Ma, CPC(2023)]
  - Calculate master integrals.
- Renormalization.

268 three-loop diagrams
$\rightarrow \mathcal{O}(100)$ families of Feynman integrals
$\rightarrow$ 26 families of master integrals

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3.1 Expression of C_J

$$x \equiv \frac{m_c}{m_b},$$

$$L_\mu \equiv \ln \frac{\mu^2}{m_b m_c},$$

$$L_{\mu_f} \equiv \ln \frac{\mu_f^2}{m_b m_c}.$$

$$\begin{aligned}
\mathcal{C}_J(\mu_f,\mu,m_b,m_c) &= 1 + \frac{\alpha_s^{(n_l)}(\mu)}{\pi}\mathcal{C}_J^{(1)}(x) \\
&+ \left(\frac{\alpha_s^{(n_l)}(\mu)}{\pi}\right)^2 \left[\frac{\mathcal{C}_J^{(1)}(x)}{4}\beta_0^{(n_l)} L_\mu + \frac{\tilde{\gamma}_J^{(2)}(x)}{2} L_{\mu_f} + \mathcal{C}_J^{(2)}(x)\right] \\
&+ \left(\frac{\alpha_s^{(n_l)}(\mu)}{\pi}\right)^3 \left\{\frac{\mathcal{C}_J^{(1)}(x)}{16}\beta_0^{(n_l)2} L_\mu^2 + \left[\frac{\mathcal{C}_J^{(1)}(x)}{16}\beta_1^{(n_l)} + \frac{\mathcal{C}_J^{(2)}(x)}{2}\beta_0^{(n_l)}\right] L_\mu \right. \\
&+ \frac{\tilde{\gamma}_J^{(2)}(x)}{4}\beta_0^{(n_l)} L_\mu L_{\mu_f} + \left[\frac{\partial \tilde{\gamma}_J^{(3)}\left(L_{\mu_f};x\right)}{4\partial L_{\mu_f}} - \frac{\tilde{\gamma}_J^{(2)}(x)}{8}\beta_0^{(n_l)}\right] L_{\mu_f}^2 \\
&\left. + \frac{1}{2}\left[\mathcal{C}_J^{(1)}(x)\tilde{\gamma}_J^{(2)}(x) + \tilde{\gamma}_J^{(3)}\left(L_{\mu_f}=0;x\right)\right] L_{\mu_f} + \mathcal{C}_J^{(3)}(x)\right\} + \mathcal{O}\left(\alpha_s^4\right),
\end{aligned}$$

- One-loop

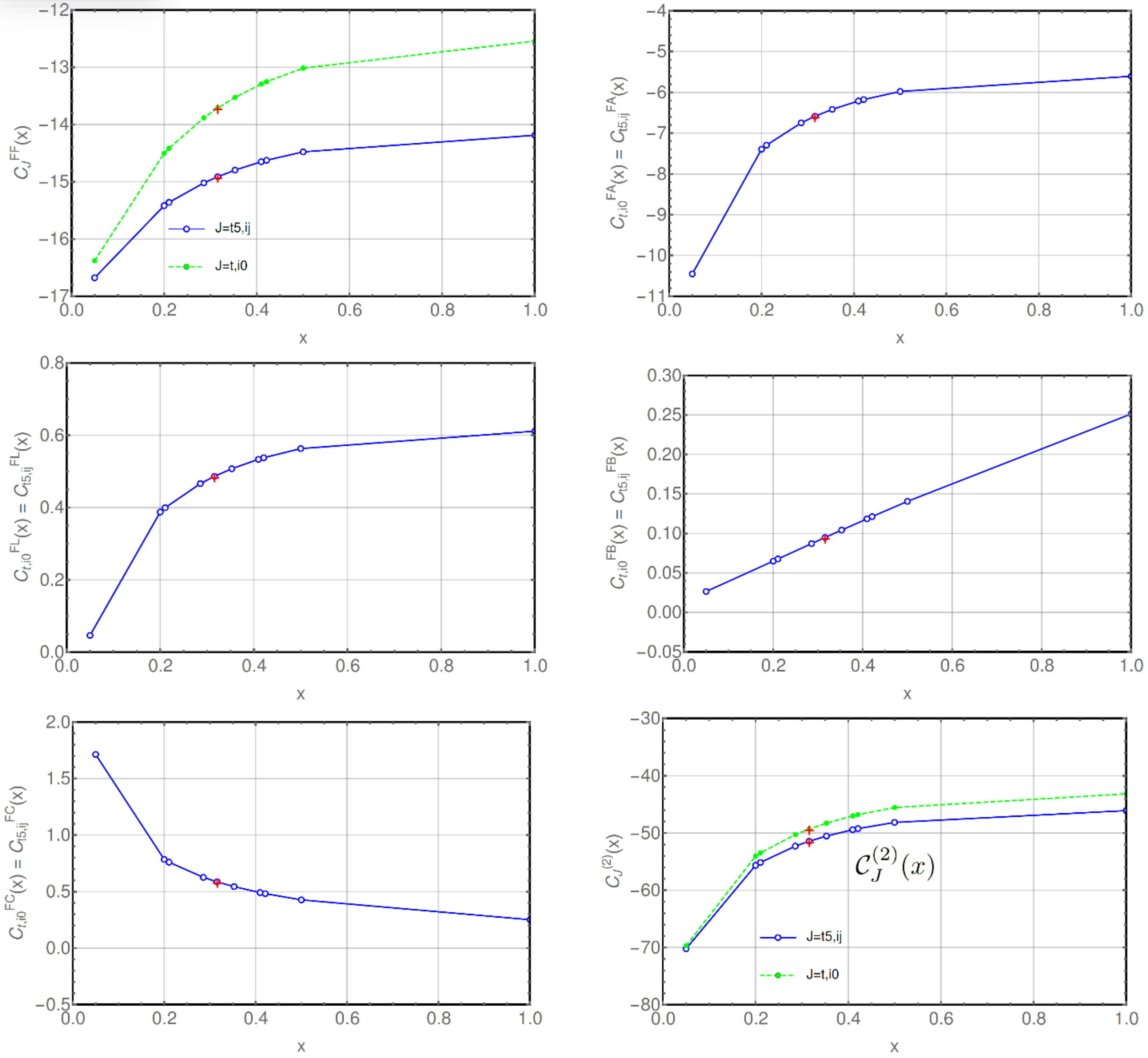
$$\mathcal{C}_p^{(1)}(x) = \mathcal{C}_{a,0}^{(1)}(x) = \frac{3}{4}C_F\left(\frac{x-1}{x+1}\ln x - 2\right);$$

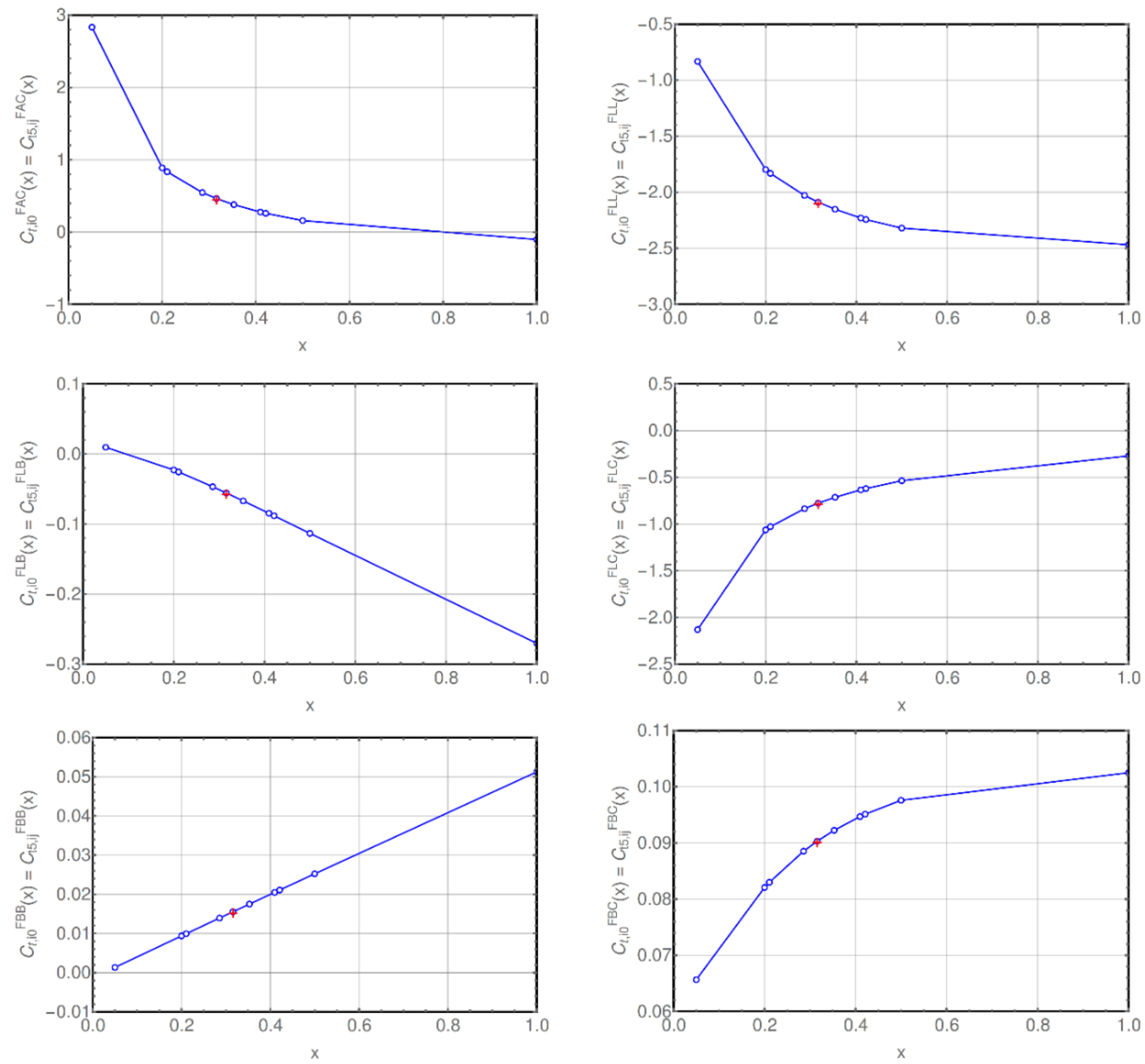
$$\mathcal{C}_{v,i}^{(1)}(x) = \mathcal{C}_{t,i0}^{(1)}(x) = \mathcal{C}_{t5,ij}^{(1)}(x) = \frac{3}{4}C_F\left(\frac{x-1}{x+1}\ln x - \frac{8}{3}\right);$$

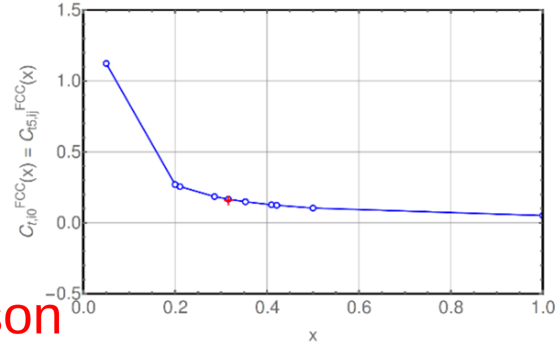
$$\mathcal{C}_s^{(1)}(x) = \frac{3}{4}C_F\left(\frac{x-1}{x+1}\ln x - \frac{2}{3}\right), \mathcal{C}_{v,0}^{(1)}(x) = \frac{3}{4}C_F\left(\frac{x+1}{x-1}\ln x - \frac{2}{3}\right);$$

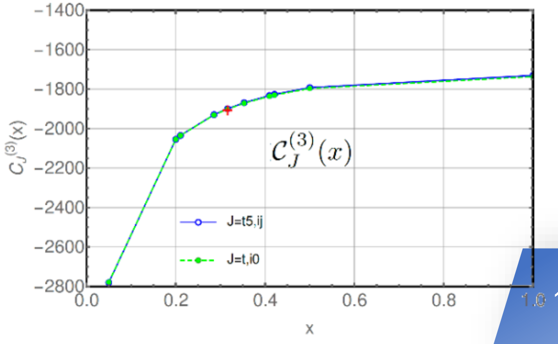
$$\mathcal{C}_{a,i}^{(1)}(x) = \mathcal{C}_{t,ij}^{(1)}(x) = \mathcal{C}_{t5,i0}^{(1)}(x) = \frac{3}{4}C_F\left(\frac{x-1}{x+1}\ln x - \frac{4}{3}\right).$$

3.2 Mass dependence of C_J



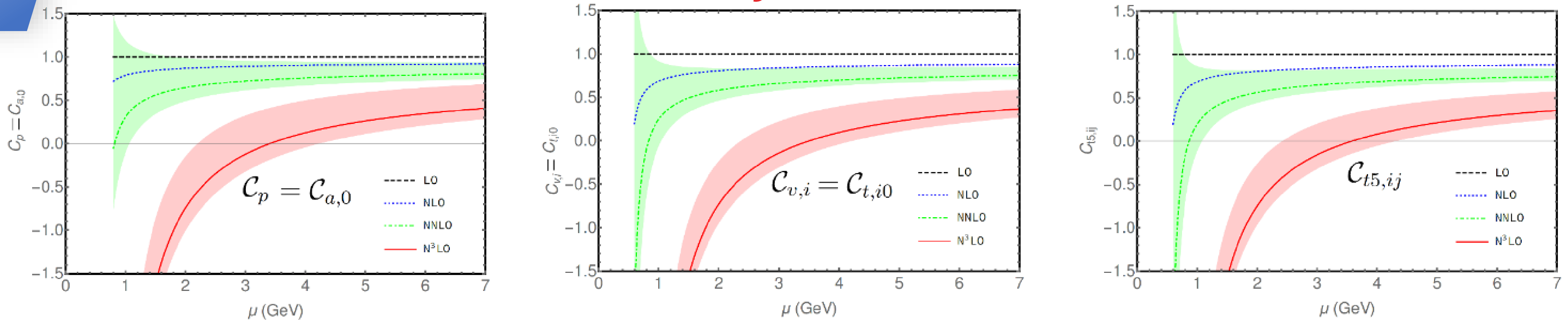




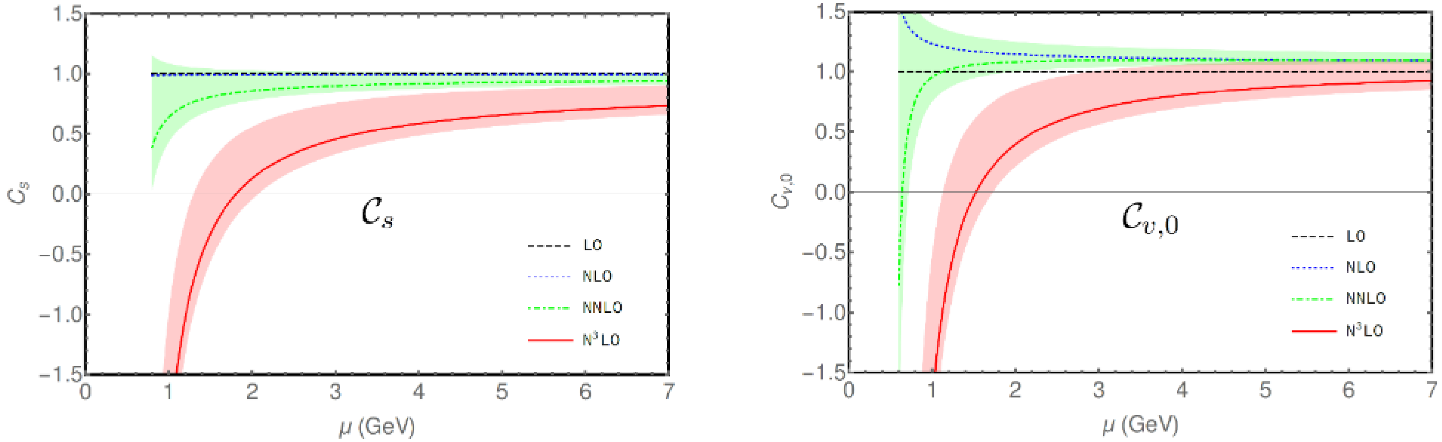


Weak dependence on heavy quark mass ratio x ⇒
$c\bar{b}$: both a heavy-heavy meson and a heavy-light meson

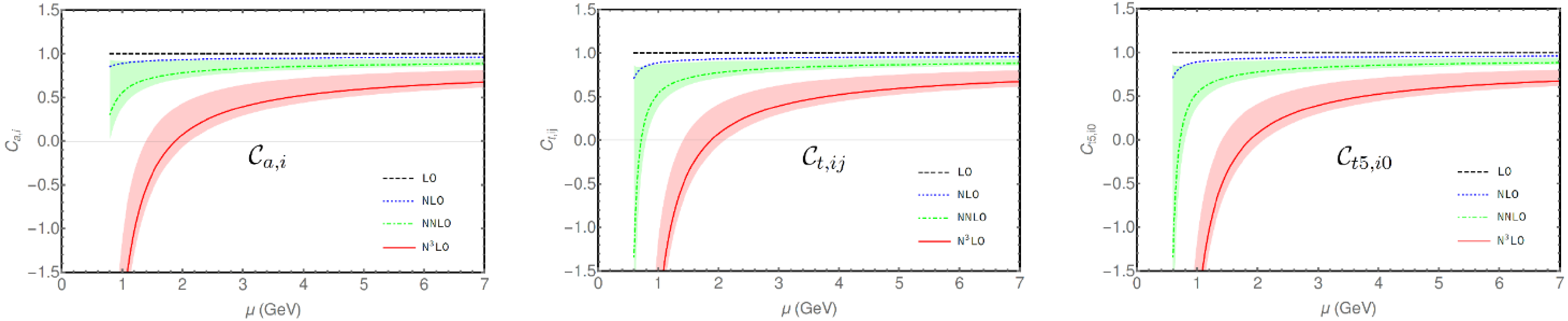
Scale dependence of C_J



$\boldsymbol{\mu_f \in [0.4, 1.2, 2]}$**GeV**



- Substantial nonconvergence at $\mathcal{O}(\alpha_s^3)$
- Strong dependence on renormalization (factorization) scale

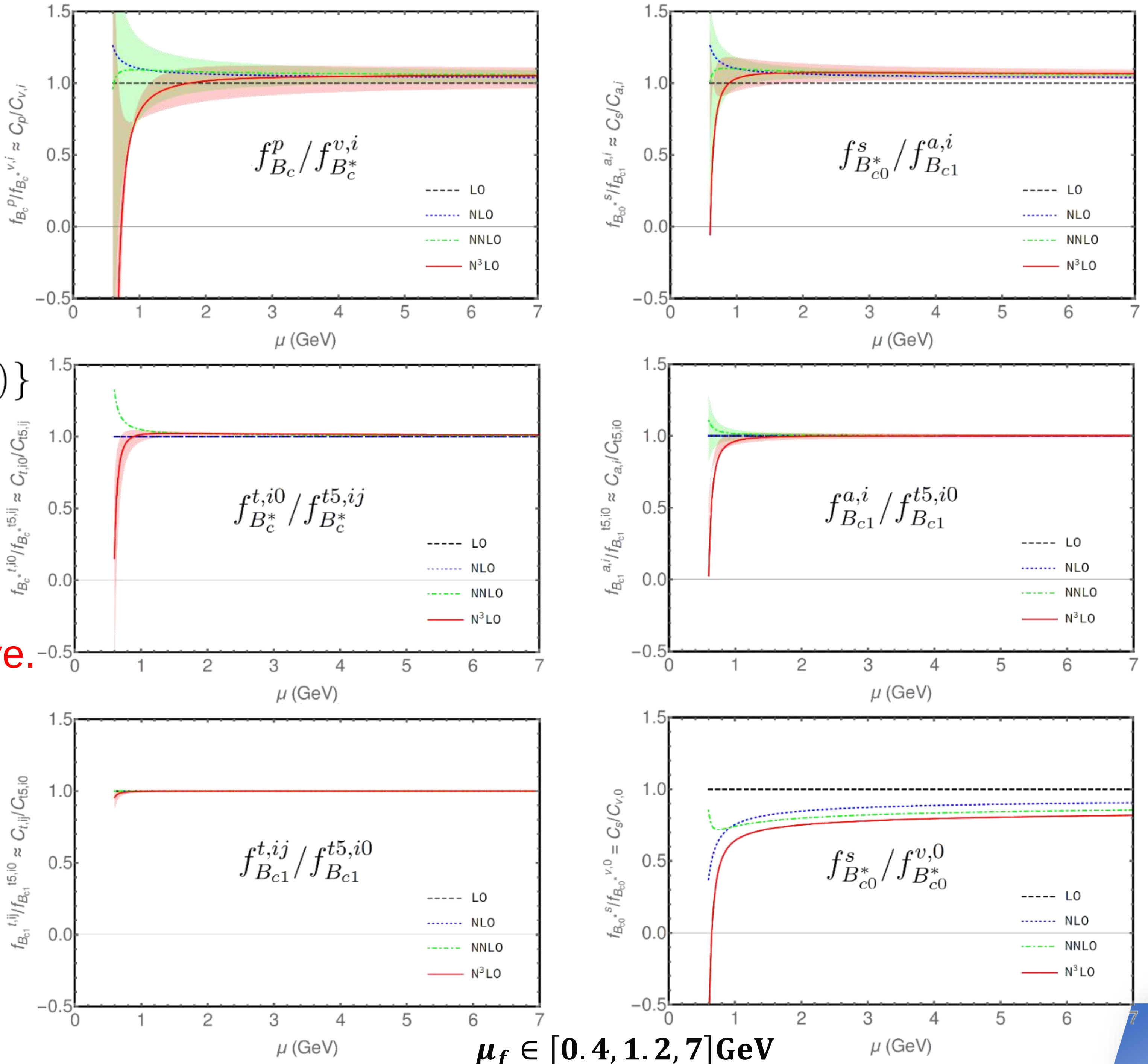


3.4 Ratios of decay constants

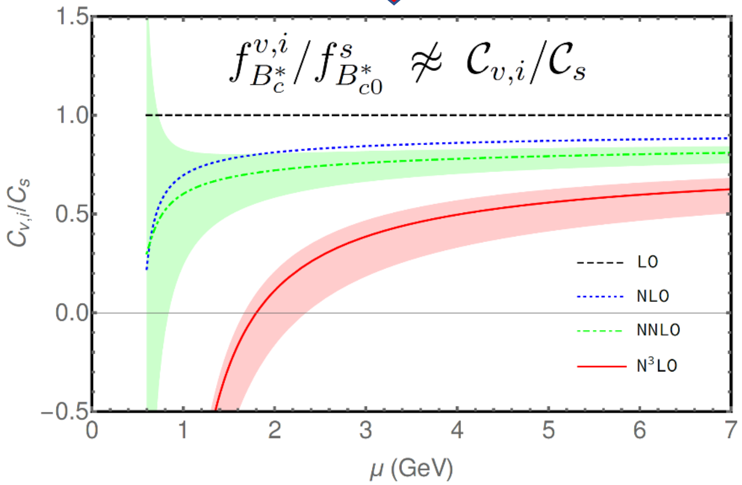
$$\frac{f_{X_1}^{J_1}}{f_{X_2}^{J_2}} \approx \frac{\mathcal{C}_{J_1}\,\tilde{f}_{X_1}^{J_1}}{\mathcal{C}_{J_2}\,\tilde{f}_{X_2}^{J_2}} \approx \frac{\mathcal{C}_{J_1}}{\mathcal{C}_{J_2}}$$

$$X \in \{B_c(^1S_0), B_c^*(^3S_1), B_{c0}^*(^3P_0), B_{c1}(^3P_1)\}$$

Convergent $\mathbf{N^3LO}$ correction with weak scale-dependence.



Cannot apply heavy-quark spin symmetry between S-wave and P-wave.



$\boldsymbol{\mu_f \in [0.4, 1.2, 7]}$**GeV**

Perturbative QCD predictions up to $\mathbf{N^3LO}$

	LO	NLO	NNLO	N^{3}LO
$\frac{\mathcal{C}_{v,i}}{\mathcal{C}_s}$	1	$0.84400^{-0+0.03953+0+0}_{+0-0.06519-0-0}$	$0.75905^{-0.09541+0.05127+0.00091+0.02692}_{+0.05944-0.07471-0.00114-0.04304}$	$0.38591^{-0.20176+0.23967+0.00030+0.05279}_{+0.08251-0.65866-0.00062-0.09918}$
$\frac{f^p_{B_c}}{f^{v,i}_{B_c^*}}$	1	$1.05200^{-0-0.01318+0+0}_{+0+0.02173-0-0}$	$1.06946^{-0.09157-0.01228+0.00153+0.00593}_{+0.05704+0.01393-0.00203-0.00473}\pm 0.01746$	$1.03816^{-0.12195+0.01312+0.00417-0.00050}_{+0.08313-0.06443-0.00519+0.00294}\pm 0.04966$
$\frac{f^s_{B^*_{c0}}}{f^{a,i}_{B_{c1}}}$	1	$1.05200^{-0-0.01318+0+0}_{+0+0.02173-0-0}$	$1.07143^{-0.03434-0.01315+0.00021-0.00007}_{+0.02139+0.01592-0.00037+0.00122}\pm 0.01943$	$1.07225^{-0.05482-0.00602+0.00094-0.00131}_{+0.04668-0.00573-0.00135+0.00411}\pm 0.02126$
$\frac{f^{t,i0}_{B_c^*}}{f^{t5,ij}_{B_c^*}}$	1	1	$1.01298^{-0-0.00575-0.00068+0.00186}_{+0+0.01312+0.00074-0.00276}\pm 0.01298$	$1.01822^{-0.00670-0.00559-0.00111+0.00143}_{+0.00417+0.00481+0.00124-0.00267}\pm 0.01822$
$\frac{f^{a,i}_{B_{c1}}}{f^{t5,i0}_{B_{c1}}}$	1	1	$1.00430^{-0.01145-0.00190+0.00005+0.00178}_{+0.00713+0.00435-0.00010-0.00207}\pm 0.00430$	$1.00045^{-0.01767+0.00141+0.00030+0.00208}_{+0.01638-0.00875-0.00041-0.00229}\pm 0.00816$
$\frac{f^{t,ij}_{B_{c1}}}{f^{t5,i0}_{B_{c1}}}$	1	1	1	$0.99964^{-0.00067+0.00021+0.00001+0.00011}_{+0.00042-0.00067-0.00001-0.00009}\pm 0.00036$
$\frac{f^s_{B^*_{c0}}}{f^{v,0}_{B^*_{c0}}}$	1	$0.87385^{-0+0.03197+0.00455-0.01197}_{+0-0.05271-0.00485+0.01906}$	$0.82199^{-0+0.03401+0.00886-0.01078}_{+0-0.04341-0.00974+0.02050}$	$0.78047^{-0+0.03860+0.01240-0.00839}_{+0-0.05744-0.01390+0.01920}$

$\mu_f = 1.2^{+5.8}_{-0.8}\,\text{GeV},\ \mu = 3^{+4.0}_{-1.5}\,\text{GeV},\ m_b = 4.75 \pm 0.5\,\text{GeV}, m_c = 1.5 \pm 0.5\,\text{GeV},$

- ➢ Uncertainty ordering:

$$\tilde{f}^{J_1}_{X_1}/\tilde{f}^{J_2}_{X_2} = 1 \pm \left(|(\mathcal{C}_{J_1}/\mathcal{C}_{J_2})_{\alpha_s^2}| + |(\mathcal{C}_{J_1}/\mathcal{C}_{J_2})_{\alpha_s^3}| \right)$$

$$\Delta_{\mu_f} \gtrsim \Delta_{\text{LDME}} \gtrsim \Delta_{\mu} \gg \Delta_{m_c} \gtrsim \Delta_{m_b}$$

- ➢ Hierarchical relationship among $c\bar{b}$ meson decay constants

$$f^p_{B_c} = f^{a,0}_{B_c} > f^{v,i}_{B_c^*} = f^{t,i0}_{B_c^*} > f^{t5,ij}_{B_c^*} > f^{v,0}_{B^*_{c0}} > f^s_{B^*_{c0}} > f^{a,i}_{B_{c1}} \gtrsim f^{t5,i0}_{B_{c1}} \gtrsim f^{t,ij}_{B_{c1}}$$

Summary

- At three-loop level, obtain QCD on-shell renormalization constant and matching coefficients for tensor currents.
- In $\mathbf{N^3LO}$ QCD, obtain convergent Bc meson decay constant ratios involving ten currents.

Thank you!