Four-loop splitting functions in QCD: non-singlet case

杨通智（华南师范大学）

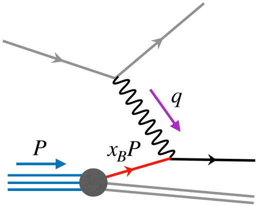
with Thomas Gehrmann, Andreas von Manteuffel and Vasily Sotnikov

Based on 2604.09534, JHEP 04 (2023) 041

2026 年 4 月 24-28 日，重庆大学, 山城国际会议中心

第八届全国重味物理与量子色动力学研讨会

Parton densities and splitting functions



- Bjorken variable

$$x_B = \frac{-q^2}{2P\cdot q}$$

- Factorization

$$\sigma \sim \sum_a f_{a|N}(x_B) \otimes \hat{\sigma}_a(x_B)$$

- Quark parton density in axial gauge

$$f_{q|N}(x_B) = \int \frac{dt}{2\pi} e^{-i\, x_B\, t\Delta\cdot p} \langle N(P)|\bar{\psi}(t\Delta)\frac{\not{\Delta}}{2}\psi(0)|N(P)\rangle \,,\ \Delta^2 = 0$$

- Splitting functions govern the DGLAP evolutions of PDFs

$$\frac{df_{i|N}}{d\ln\mu} = 2\sum_k P_{ik} \otimes f_{k|N}, \qquad P_{ik} = a_s P_{ik}^{(0)} + a_s^2 P_{ik}^{(1)} + a_s^3 P_{ik}^{(2)} + a_s^4 P_{ik}^{(3)}$$

Splitting functions & Anomalous dimensions (A.D.)

- Mellin transformation

$$f_q(n) = -\int_0^1 dx\, x^{n-1} f_q(x)\,, \quad \gamma_{ij}(n) = -\int_0^1 dx\, x^{n-1} P_{ij}(x)$$

- DGLAP evolution in n-space

$$\frac{d}{d\ln\mu} f_q(n,\mu^2) = -2\sum_j \gamma_{qj}(n)\, f_j(n,\mu^2)$$

- PDFs in n-space are hadronic operator matrix elements (OMEs)

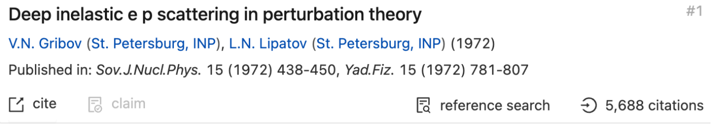
$$f_q(n) \sim \langle N(P)|\, \bar\psi \not{\Delta} (\Delta\cdot D)^{n-1} \psi\, |N(P)\rangle = \langle N(P)|\, O_q\, |N(P)\rangle$$

- Twist-two operators, twist = [mass] - spin

$$O_q = \mathcal{O}_q^{\mu_1\cdots\mu_n}\, J_{\mu_1\cdots\mu_n} = \left[\bar\psi\gamma^{\mu_1} D^{\mu_2}\cdots D^{\mu_n}\psi\right]\left[\Delta_{\mu_1}\cdots\Delta_{\mu_n}\right]$$

Formulation of DGLAP evolution equation

- Before Asymptotic Freedom



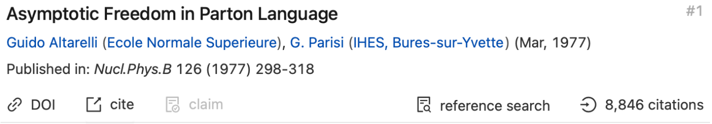
Deep inelastic e p scattering in perturbation theory #1

V.N. Gribov (St. Petersburg, INP), L.N. Lipatov (St. Petersburg, INP) (1972)

Published in: *Sov.J.Nucl.Phys.* 15 (1972) 438-450, *Yad.Fiz.* 15 (1972) 781-807

cite claim reference search 5,688 citations

- Preprint March 1977



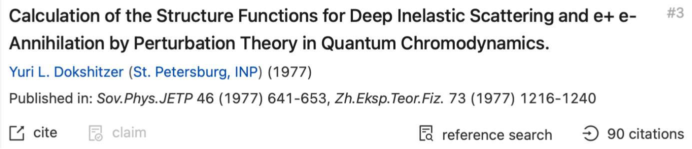
Asymptotic Freedom in Parton Language #1

Guido Altarelli (Ecole Normale Superieure), G. Parisi (IHES, Bures-sur-Yvette) (Mar, 1977)

Published in: *Nucl.Phys.B* 126 (1977) 298-318

DOI cite claim reference search 8,846 citations

- Submit to journal in April 1977



Calculation of the Structure Functions for Deep Inelastic Scattering and e+ e- Annihilation by Perturbation Theory in Quantum Chromodynamics. #3

Yuri L. Dokshitzer (St. Petersburg, INP) (1977)

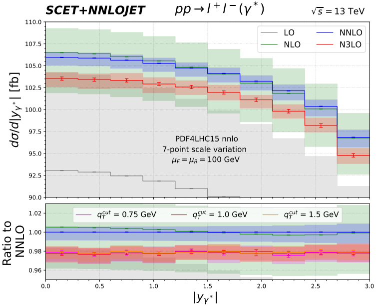
Published in: *Sov.Phys.JETP* 46 (1977) 641-653, *Zh.Eksp.Teor.Fiz.* 73 (1977) 1216-1240

cite claim reference search 90 citations

Motivation: required by high-precision physics

- (HL-)LHC and EIC will generate precise experimental data
- Several $\hat{\sigma}$ are available at N3LO → N3LO PDF ← $P_{ij}^{(3)}$

- For DY, scale uncertainty at N3LO using NNLO PDF remains at 1% [Chen,Gehrmann,Glover,Huss,TZY,Zhu 21']
- Fields in fitting N3LO PDFs are active
  - ▷ MSHT20 aN3LO[McGowan et al. 22']
  - ▷ NNPDF aN3LO[Ball et al. 24']
  - ▷ CTEQ is planning
- $P_{ij}^{(3)}$ crucial for N^{4}LL resummation: Q_t, threshold...



[Chen,Gehrmann,Glover,Huss,TZY,Zhu 21']

Motivation: theoretical interest

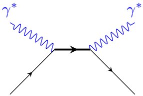
- Universal A.D. in $\mathcal{N}=4$ from QCD by principle of maximal transcendentality[Kotikov, Lipatov, Onishchenko, Velizhanin, 04']
- Reciprocity relation between space-like and time-like A.D.
  - In CFT: $2\gamma^{S}(n,\mu)=2\gamma^{T}(n+2\gamma^{S}(n,\mu),\mu)$[Basso, Korchemsky, 06']
  - Non-singlet in QCD $2\gamma^{S}(n,\mu)=2\gamma^{T}(n+2\gamma^{S}(n,\mu),\mu)$[Dokshitzer, Marchesini,Salam, 2006;Basso, Korchemsky, 06'; Mitov,Moch,Vogt,06']
  - Singlet in QCD: $2\gamma^{S}_{\pm}(n,\mu)=2\gamma^{T}_{\pm}(n+2\gamma^{S}_{\pm}(n,\mu),\mu)$[Chen, TZY, Zhu, Zhu, 20'] See also the talk by Yao Ji
- Simple mathematical structures
  - Only HPLs or Harmonic Sums (HSs) to three loops in QCD
  - Only HSs to 4-loop uni. A.D. in $\mathcal{N}=4$[B.A. Kniehl, V.N. Velizhanin, 21']
  - Strong hints: only HSs in four-loop QCD
  - This talk: Verified the above statement in non-singlet case

Status of four-loop splitting functions

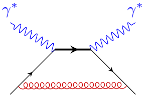
- Fixed moments
  - Non-singlet $\gamma_{\text{ns}}^{(3)}$ with $n \leq 16$[Moch,Ruijl,Ueda,Vermaseren,Vogt,17']
  - $\gamma_{\text{ps}}^{(3)}$, $\gamma_{qg}^{(3)}$, $\gamma_{gq}^{(3)}$ and $\gamma_{gg}^{(3)}$ with $n \leq 20$[Falcioni,Herzog,Moch,Pelloni,A. Vogt,23', 24',25']
  - $\gamma_{\text{ns}}^{(4)}$ with $n \leq 3$[Herzog,Moch,Ruijl,Ueda,Vermaseren,Vogt,18']
- Exact results with all-n dependence
  - Large-N_f limit to all-loop order[Gracey 94',96']
  - $\gamma_{\text{ns}}^{(3)}$ with leading color[Moch,Ruijl,Ueda,Vermaseren,Vogt,17']
  - N_f^2 term for $\gamma_{qg}^{(3)}$[Falcioni,Moch,Ruijl,Ueda,Vermaseren,Vogt,23']
  - N_f^2 for $\gamma_{\text{ps}}^{(3)}$, $N_f C_F^3$ for $\gamma_{\text{ns}}^{(3)}$ [Gehrmann,Manteuffel,Sotnikov,TZY,24']
  - N_f for $\gamma_{\text{ns}}^{(3)\pm}$ [Kniehl, Moch, Velizhanin, Vogt 25']
  - This talk: complete, analytic results for $\gamma_{\text{ns}}^{(3)}$

DIS method vs OME method

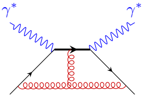
- Forward DIS (gauge invariant)



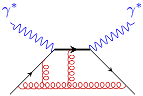
2 diagrams



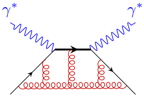
10 diagrams



143 diagrams

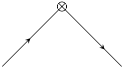


2922 diagrams

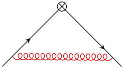


73634 diagrams

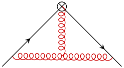
- Partonic off-shell OME (fewer diagrams, easier integrals)



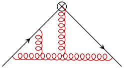
1 diagram



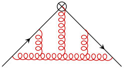
3 diagrams



37 diagrams



684 diagrams

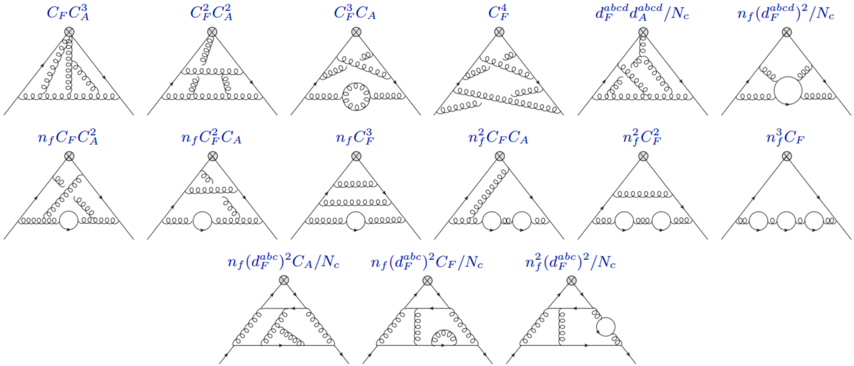


15901 diagrams

- Off-shell OMEs are not gauge invariant, physical operators mix with gauge-variant (GV) operators
- Non-singlet: problematic contributions cancel in $q - \bar{q}$
- Singlet: conceptually involved, need to find new, unknown GV contributions.

Full four-loop non-singlet splitting functions $\gamma_{\mathsf{ns}}^{(3)}$, $P_{\mathsf{ns}}^{(3)}$

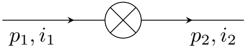
Sample Feynman diagrams



15 color structures, $\mathcal{O}(16k)$ diagrams by QGRAF

Tracing parameter to retain all-n dependence

- Non-standard terms appearing in the Feynman rules
- Example: Feynman rules for O_q at lowest order



$$\rightarrow \not{\Delta}(\Delta \cdot p_1)^{n-1}$$

- Retain all-n dependence?
  - Sum non-standard term into a linear propagator by a tracing parameter t[Ablinger,Bluemlein,Hasselhuhn,Klein,Schneiderm,Wissbrock,12']

$$(\Delta \cdot p)^{n-1} \rightarrow \sum_{n=1}^{\infty} t^n (\Delta \cdot p)^{n-1} = \frac{t}{1 - t\Delta \cdot p}$$

  - IBP reduction with respect to linear propagator
  - Parameter-t space → n-space by re-expanding t

$$H(1,1;t) = \sum_{n=1}^{\infty} t^n \left(-\frac{1}{n^2} + \frac{S(1,n)}{n} \right)$$

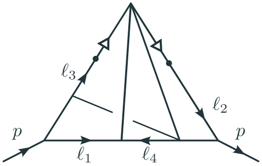
Technical aspects

- Raw integrand
  - Raw integrand generated by QGRAF,FORM
  - 549 top topologies with 13 denominators
  - 3GB (3M int.) & 200 MB (220K int.) before & after symmetry finding
- IBP reduction (major bottleneck)[Chetyrkin, Tkachov 81', Laporta 00'] See also the talk by Bo Feng and Longbin Chen
  - Finite field sampling + function reconstruction[Manteuffel, Schabinger 14'; Peraro 15']
  - Custom optimization of IBP equation generation/selection[Driesse, Jakobsen, Mogul, Plefka, Sauer, Usovitsch 24'; Guan, Liu, Ma, Wu 24'; Bern, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng 24'; von Hippel, Wilhelm 25'; Song, TZY, Cao, Luo, Zhu 25'; Lange, Usovitsch, Wu 25']
  - Divide big system into smaller systems by spanning cuts/sectors[Larsen, Zhang 15'; Guan, Liu, Ma, Wu,24']
- Differential equation
  - Derive DE[Kotikov 91'; Gehrmann,Remiddi,99'] with respect to t

Elliptic geometry from DE See also the talk by Xing Wang

- Analyze the leading singularity on maximal cut
- Many square roots
- Elliptic geometry, for example
  - A non-planar sector with 9 normal propagators and 2 linear propagators
  - 4 master integrals on maximal cut
  - ϵ-form achieved by ansatz method of [Bogner, Müller-Stach, Weinzierl,19']
  - Complete elliptic integrals

$$F(t) = \int_0^1 d\tau \frac{1}{\sqrt{(1-\tau^2)(1-t^2\tau^2)}}$$



Hints from four-loop $\mathcal{N}=4$ results: only HSs
Bypass the complexities: series expansion & reconstruct the A.D.

Master solution from series expansion of DE

- DE with respect to the tracing parameter t

$$d\vec{I}(t,\epsilon)/dt = A(t,\epsilon)\vec{I}(t,\epsilon)$$

- Largest DE from a top topology: $\mathcal{O}(2000)$ masters, $\mathcal{O}(700\text{MB})$ size
- Boundary condition: 28 four-loop self-energy master integrals in the $t \to 0$ limit[Baikov, Chetyrkin 10'; Lee, Smirnov, Smirnov 11']
  - Example: $\int \frac{1}{(1-t\Delta\cdot l_1)(1-t\Delta\cdot l_2)} \times \frac{1}{l_1^2(l_1-l_2)^2\cdots} \to \int \frac{1}{l_1^2(l_1-l_2)^2\cdots}$
- Derive recursion relations for each master

Ansatz: $I(t,\epsilon) = \sum_{m=m_{\text{min}}}^{m_{\text{max}}} \sum_{n=0}^{\infty} a_{m,n}\epsilon^m t^n$

Recursion: $a_{m,n} = (\#1)a_{m,n-1} + (\#2)a_{m-1,n} + \cdots +$ sub-sectors

- Compute each master to $\mathcal{O}(t^{4000})$ and enough ϵ order

Reconstruct A.D. from fixed moments

- Derive recursion relations for OME to obtain $\mathcal{O}(4000)$ moments
- Renormalization: $\langle q(p)|O_{\text{ns}}|q(p)\rangle^{\text{R}} = Z_\psi Z_{\text{ns}} \langle q(p)|O_{\text{ns}}|q(p)\rangle^{\text{B}}$
- $Z^{(4)}_{\text{ns}} = \frac{1}{4\epsilon}\gamma^{(3)}_{\text{ns}} + \frac{1}{\epsilon^2}\left(-\frac{1}{4}\beta_0\gamma^{(2)}_{\text{ns}} + \cdots\right) + \frac{1}{\epsilon^3}(\cdots) + \frac{1}{\epsilon^4}(\cdots)$
- Directly reconstruct $\gamma^{(3)}_{\text{ns}}$ by including lower-order counterterms
- Generic ansatz:

$$\gamma^{(3)}_{\text{ns}}(n) = \sum_a \sum_{k=0}^{7} \sum_{w=0}^{7-k} c_{akw} D^k_a S_w(n)\,, \qquad D^k_a = (n+a)^{-k}$$

$S_w(n)$ is HSs of weight w

We choose: $a = \{0,1\}$ for $\gamma^{(3)\pm}_{\text{ns}}$, $\{0,1,-1,2,-2\}$ for $\gamma^{(3)s}_{\text{ns}}$

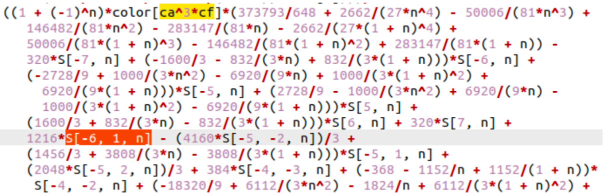
- Successfully reconstructed and verified $\gamma^{(3)}_{\text{ns}}$ using $\mathcal{O}(4000)$ moments

Analytic results for $\gamma_{\text{ns}}^{(3)}$

- Analytic results with all-n dependence via HSs to weight 7

$$\text{Example: } S_{-6,1}(n) = \sum_{\tau_2=1}^{n} \frac{(-1)^{\tau_2}}{\tau_2^6} \sum_{\tau_1=1}^{\tau_2} \frac{1}{\tau_1^1}$$

- Non-n_f terms for $\gamma_{\text{ns}}^{(3)\pm}$, n_f terms for $\gamma_{\text{ns}}^{(3)s}$ *New*



```
((1 + (-1)^n)*color[ca^3*cf]*(373793/648 + 2662/(27*n^4) - 50006/(81*n^3) +
   146482/(81*n^2) - 283147/(81*n) - 2662/(27*(1 + n)^4) +
   50006/(81*(1 + n)^3) - 146482/(81*(1 + n)^2) + 283147/(81*(1 + n)) -
   320*S[-7, n] + (-1600/3 - 832/(3*n) + 832/(3*(1 + n)))*S[-6, n] +
   (-2728/9 + 1000/(3*n^2) - 6920/(9*n) + 1000/(3*(1 + n)^2) +
     6920/(9*(1 + n)))*S[-5, n] + (2728/9 - 1000/(3*n^2) + 6920/(9*n) -
     1000/(3*(1 + n)^2) - 6920/(9*(1 + n)))*S[5, n] +
   (1600/3 + 832/(3*n) - 832/(3*(1 + n)))*S[6, n] + 320*S[7, n] +
   1216*S[-6, 1, n] - (4160*S[-5, -2, n])/3 +
   (1456/3 + 3808/(3*n) - 3808/(3*(1 + n)))*S[-5, 1, n] +
   (2048*S[-5, 2, n])/3 + 384*S[-4, -3, n] + (-368 - 1152/n + 1152/(1 + n))*
    S[-4, -2, n] + (-18320/9 + 6112/(3*n^2) - 1824/n + 6112/(3*(1 + n)^2) +
```

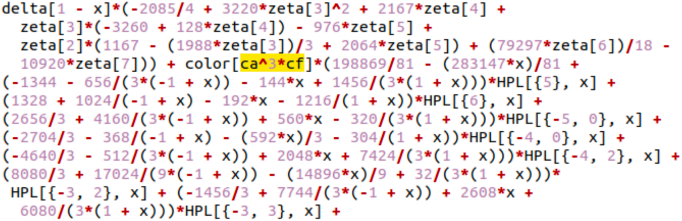
- Agree with partially known results
  - n_f terms for $\gamma_{\text{ns}}^{(3)\pm}$ [Gehrmann,Manteuffel,Sotnikov,TZY,24';Kniehl, Moch, Velizhanin, Vogt 25']
  - $n \leq 16$ for $\gamma_{\text{ns}}^{(3)}$ [Moch,Ruijl,Ueda,Vermaseren,Vogt,17']

Analytic results for $P_{\mathsf{ns}}^{(3)}$

- Splitting functions $P_{\mathsf{ns}}^{(3)}$ by inverse Mellin transformation of $\gamma_{\mathsf{ns}}^{(3)}$

$$P_{\mathsf{ns}}^{(3)}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-n}\,\gamma_{\mathsf{ns}}^{(3)}(n)\,dn$$

  - By `HarmonicSums`[Ablinger,09',12']
- $P_{\mathsf{ns}}^{(3)}$ via HPLs to weight 6, constants to weight 7 $(\zeta_7, \zeta_3\zeta_4, \zeta_2\zeta_5)$
- Example expressions for $C_A^3 C_F$ of $P_{\mathsf{ns}}^{(3)+}$



```
delta[1 - x]*(-2085/4 + 3220*zeta[3]^2 + 2167*zeta[4] +
  zeta[3]*(-3260 + 128*zeta[4]) - 976*zeta[5] +
  zeta[2]*(1167 - (1988*zeta[3])/3 + 2064*zeta[5]) + (79297*zeta[6])/18 -
  10920*zeta[7])) + color[ca^3*cf]*(198869/81 - (283147*x)/81 +
(-1344 - 656/(3*(-1 + x)) - 144*x + 1456/(3*(1 + x)))*HPL[{5}, x] +
(1328 + 1024/(-1 + x) - 192*x - 1216/(1 + x))*HPL[{6}, x] +
(2656/3 + 4160/(3*(-1 + x)) + 560*x - 320/(3*(1 + x)))*HPL[{-5, 0}, x] +
(-2704/3 - 368/(-1 + x) - (592*x)/3 - 304/(1 + x))*HPL[{-4, 0}, x] +
(-4640/3 - 512/(3*(-1 + x)) + 2048*x + 7424/(3*(1 + x)))*HPL[{-4, 2}, x] +
(8080/3 + 17024/(9*(-1 + x)) - (14896*x)/9 + 32/(3*(1 + x)))*
 HPL[{-3, 2}, x] + (-1456/3 + 7744/(3*(-1 + x)) + 2608*x +
  6080/(3*(1 + x)))*HPL[{-3, 3}, x] +
```

Results in $x \to 0$ and $x \to 1$ limit

- $x \to 0$ limit

$$P_{\text{ns}}^{(3)+} = \frac{C_F^4}{9}\log^6 x + C_F^3\left(\frac{22C_A}{9} - \frac{4n_f}{9} - \frac{4C_F}{3}\right)\log^5 x + \log^4 x\left(-48\zeta_2\frac{d_R^{abcd}d_A^{abcd}}{N_R} + \cdots\right)$$

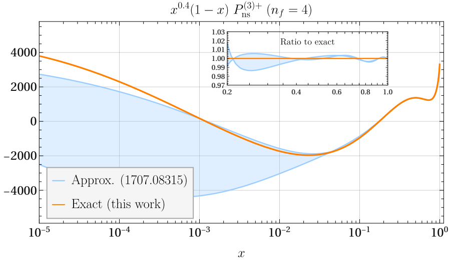
  - Quadratic color starts to contribute at $\log^4 x$, deviation at sub-LC with predictions of resummation[Davies,Kom,Moch,Vogt,22']

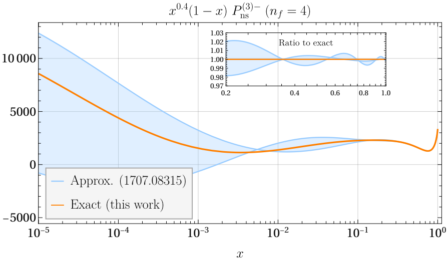
- $x \to 1$ limit

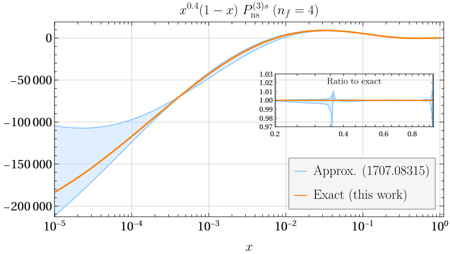
$$P_{\text{ns}}^{(3)\pm} = A_4\left[\frac{1}{1-x}\right]_+ + B_4\,\delta(1-x) + C_4\log(1-x) + D_4 - A_4 + \mathcal{O}(1-x)\,,$$

  - Confirm the four-loop cusp anomalous dimension A_4 [Henn,Korchemsky,Mistlberger 19';Manteuffel,Panzer,Schabinger 20']
  - Analytic virtual A.D. B_4, agree with numerical res.[Das,Mach,Vogt 19']
  - B_4 is the last missing piece to derive analytic rapidity A.D., agree well with numerical res.[Moult,Zhu,Zhu ' 21;Duhr,Mistlberger,Vita 21']
  - C_4 and D_4 confirm the conjecture to four-loop order $C = A^2, D = A\,(B + \beta/(2a_s))$[Dokshitzer,Marchesini,Salam 05']

Comparison with approximations







Approximations from[Moch,Ruijl,Ueda,Vermaseren,Vogt,17']

Good agreement except small-x region

Four-loop time-like non-singlet splitting functions

- Reciprocity relation for $\gamma_{\text{ns}}^{(3)}$ in QCD[Dokshitzer, Marchesini,Salam, 2006;Basso, Korchemsky, 06'; Mitov,Moch,Vogt,06']

$$2\gamma_{\text{ns}}^{S}(n,\mu) = 2\gamma_{\text{ns}}^{T}(n + 2\gamma_{\text{ns}}^{S}(n),\mu)$$

- Expanded form to four-loop order

$$\gamma_{\text{ns}}^{T(0)} - \gamma_{\text{ns}}^{S(0)} = 0,\ \gamma_{\text{ns}}^{T(1)} - \gamma_{\text{ns}}^{S(1)} = -2\gamma_{\text{ns}}^{S(0)}\frac{d\gamma_{\text{ns}}^{T(0)}}{dn},$$

$$\gamma_{\text{ns}}^{T(2)} - \gamma_{\text{ns}}^{S(2)} = -2\gamma_{\text{ns}}^{S(1)}\frac{d\gamma_{\text{ns}}^{T(0)}}{dn} - 2\gamma_{\text{ns}}^{S(0)}\frac{d\gamma_{\text{ns}}^{T(1)}}{dn} - 2\left(\gamma_{\text{ns}}^{S(0)}\right)^2\frac{d^2\gamma_{\text{ns}}^{T(0)}}{dn^2},$$

$$\gamma_{\text{ns}}^{T(3)} - \gamma_{\text{ns}}^{S(3)} = -2\gamma_{\text{ns}}^{S(2)}\frac{d\gamma_{\text{ns}}^{T(0)}}{dn} - 2\gamma_{\text{ns}}^{S(1)}\frac{d\gamma_{\text{ns}}^{T(1)}}{dn} - 2\gamma_{\text{ns}}^{S(0)}\frac{d\gamma_{\text{ns}}^{T(2)}}{dn}$$
$$- 4\gamma_{\text{ns}}^{S(0)}\gamma_{\text{ns}}^{S(1)}\frac{d^2\gamma_{\text{ns}}^{T(0)}}{dn^2} - 2\left(\gamma_{\text{ns}}^{S(0)}\right)^2\frac{d^2\gamma_{\text{ns}}^{T(1)}}{dn^2} - \frac{4}{3}\left(\gamma_{\text{ns}}^{S(0)}\right)^3\frac{d^3\gamma_{\text{ns}}^{T(0)}}{dn^3}$$

- $\gamma_{\text{ns}}^{T(3)} - \gamma_{\text{ns}}^{S(3)}$ only receive contributions from lower-loop order; Explicit res. given in[Moch,Ruijl,Ueda,Vermaseren,Vogt,17'] , we find full agreement

Towards four-loop singlet splitting functions

A new framework of deriving GV operators

- New framework

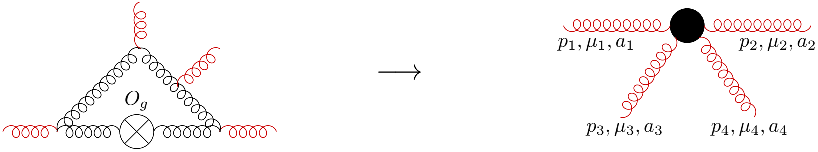
$$\begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^{\mathrm{R}} = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qA} \\ Z_{gq} & Z_{gg} & Z_{gA} \\ 0 & 0 & Z_{AA} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^{\mathrm{B}} + \begin{pmatrix} [ZO]_q^{\mathrm{GV}} \\ [ZO]_g^{\mathrm{GV}} \\ [ZO]_A^{\mathrm{GV}} \end{pmatrix}^{\mathrm{B}}$$

$$Z_{gA} = \mathcal{O}(a_s),\ Z_{qA} = \mathcal{O}(a_s^2),\ [ZO]_{g,q}^{\mathrm{GV}} = \sum_{l=2,3}^{\infty} a_s^l\ [ZO]_{g,q}^{\mathrm{GV},\,(l)}$$

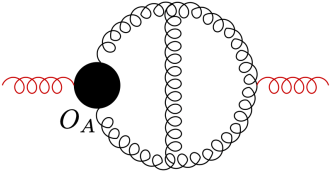
- $O_{ABC} = O_A + O_B + O_C$, O_A(gluon fields only), O_B(quark+gluon fields), O_C (ghost + gluon fields)
- $[ZO]_{g,q}^{\mathrm{GV},\,(l)}$: collection of higher-order counterterms
- Derive counterterm Feynman rules for O_{ABC} and $[ZO]_{g,q}^{\mathrm{GV},\,(l)}$ from off-shell OMEs

Working out non-physical contribution to determine γ_{singlet}

- Determine Feynman rules for non-physical operator from multi-loop multi-leg OMEs



- Insert non-physical Feynman rules into two-point diagrams



Summary and Outlook

- Timeline of splitting functions in QCD

1974	1980	2004	2026	???
1-loop	2-loop	3-loop	4-loop NS	4-loop singlet

- Four-loop non-singlet $\gamma_{\text{ns}}^{(3)\pm,V}$ Done
  - Direct multiplicative renormalization
  - This talk: the first complete, analytic results *New*
- Four-loop singlet $\gamma_{\text{singlet}}^{(3)}$ In progress
  - Renormalization: mix with new, unknown non-physical operators
  - This talk: systematic way to find non-phy. counterterm Feynman rules
  - First three-loop singlet splitting functions by off-shell method
  - $\gamma_{qq}^{(3)} = \gamma_{\text{ns}}^{(3)+} + \gamma_{\text{ps}}^{(3)}$, $\gamma_{\text{ps}}^{(3)}$ to appear
  - Others in order: $\gamma_{qg}^{(3)}, \gamma_{gq}^{(3)}, \gamma_{gg}^{(3)}$

$$\gamma_{\text{singlet}}^{(3)} = \begin{pmatrix} \gamma_{qq}^{(3)} & \gamma_{qg}^{(3)} \\ \gamma_{gq}^{(3)} & \gamma_{gg}^{(3)} \end{pmatrix}$$

OMEs required to derive four-loop splitting functions

Legs Loops	2	3	4	5	6
0	A.D.	$[ZO]_g^{\mathrm{GV},\,(3)}$	$[ZO]_g^{\mathrm{GV},\,(2)}$	O_{ABC}	$O_q,\, O_g$
1	$[ZO]_g^{\mathrm{GV},\,(3)}$	$[ZO]_g^{\mathrm{GV},\,(2)}$	O_{ABC}	O_g	
2	$[ZO]_g^{\mathrm{GV},\,(2)}$	O_{ABC}	O_g		
3	O_{ABC}	O_g			
4	$O_q,\, O_g$				

- 3-loop splitting functions (done)
- 1-loop five-point OMEs to extract Feynman rules of O_{ABC} (done)
- 2-loop 4-point OMEs to extract two-loop C.F.R (in progress)
- 4-loop two-point OMEs (in progress)

Complexities from multi-loop multi-leg OMEs

- In addition to the parameter x, there are many scales
  - 9 for four-point OMEs

$$p_1^2,\ p_2^2,\ p_3^2,\ p_1\cdot p_2,\ p_1\cdot p_3,\ p_2\cdot p_3,\ \Delta\cdot p_1,\ \Delta\cdot p_2,\ \Delta\cdot p_3$$

  - 14 for five-point OMEs
    $p_1^2, p_2^2, p_3^2, p_4^2, p_1\cdot p_2, p_1\cdot p_3, p_1\cdot p_4, p_2\cdot p_3, p_2\cdot p_4, p_3\cdot p_4, \Delta\cdot p_1, \Delta\cdot p_2, \Delta\cdot p_3, \Delta\cdot p_4$
- Many Lorentz structures for pure gluon OMEs
  - 5 for two-gluon OMEs: $\Delta^\mu\Delta^\nu,\ \Delta^\mu p^\nu,\ p^\mu\Delta^\nu,\ g^{\mu\nu},\ p^\mu p^\nu$.
  - 36 for three-gluon OMEs
  - 353 for four-gluon OMEs
  - 4400 for five-gluon OMEs

Derive Feynman rules from off-shell OMEs

- Key idea: derive Feynman rules instead of GV operators themselves
- Consider all-off-shell OMEs with $2j + m$-gluon external states

$$\langle j|O_g^{\mathsf{R}}|j + m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m} = \langle j|(Z_{gq}O_q^{\mathsf{B}} + Z_{gg}O_g^{\mathsf{B}})|j + m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m}$$
$$+ \langle j|Z_{gA}\,O_{ABC}^{\mathsf{B}}|j + m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m} + \langle j|\,[ZO]_g^{\mathrm{GV}}\,|j + m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m}\,,\ j = q, g \text{ or c}$$

- Expand OMEs order by order in loops and legs

$$\langle j|O|j + m\,g\rangle^{\mu_1\cdots\mu_m} = \sum_{l=1}^{\infty}\left[\langle j|O|j + m\,g\rangle^{\mu_1\cdots\mu_m,\,(l),\,(m)}\right]\left(\frac{\alpha_s}{4\pi}\right)^l g_s^m$$

- Left: UV renormalized and IR finite → no poles in ϵ
- Right: Each term is UV divergent, but the sum should be finite
- Finiteness $\longrightarrow$ counterterm Feynman rules order by order in α_s

Evaluations of multi-loop multi-point OMEs

- Constrain Lorentz structures of F.R. based on dimensional analysis
  - Feynman rules involving quarks or ghosts has one structure only

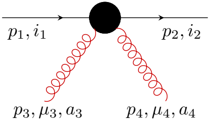
$$\langle j|O|j+m\,g\rangle_{\text{1PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m)} = c_m(\Delta\cdot p_i)\,\Delta^{\mu_1}\cdots\Delta^{\mu_m}\ \text{with}\ j=q,c$$

  - 31 Lorentz structures for 5-gluon F.R, instead of naively expected 4400
  - $a_m\Delta^{\mu_1}\cdots \to a_m$ is linear in or has no dependence on $p_i^2, p_i\cdot p_j$
- One-loop: only two types of integrals are needed, others are finite

$$I_1=\int\frac{d^dl}{i\pi^{d/2}}\frac{1}{(l-q_1)^2l^2},\quad I_2=\int\frac{d^dl}{i\pi^{d/2}}\frac{1}{(l-q_1)^2l^2\big(1-t\Delta\cdot(l+q_2)\big)}$$

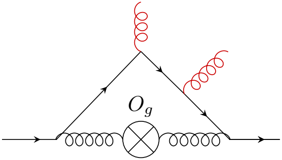
- Two-loop: difficult to evaluate the master integrals analytically
  - Derive differential equation (DE)[Gehrmann,Remiddi,1999] in t
  - Solve DE in $t\to 0$ limit $\longrightarrow$ fixed moments for the OMEs
  - Reconstruct the all-n counterterm Feynman rules

Sample results: Feynman rules for O_B



$$\rightarrow \frac{1+(-1)^n}{-8} g_s^2 \Delta^{\mu_3} \Delta^{\mu_4} \left(T^{a_3} T^{a_4} - T^{a_4} T^{a_3}\right)_{i_2 i_1} \not{\Delta} \sum_{j_1=0}^{n-3} \Bigg(3\left(\Delta \cdot (p_1+p_2)\right)^{-j_1+n-3} \left[(-\Delta \cdot p_3)^{j_1} - (-\Delta \cdot p_4)^{j_1} \right] - (-\Delta \cdot p_4)^{j_1} (\Delta \cdot p_3)^{-j_1+n-3} \Bigg)$$

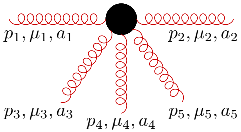
extracted from



$$\sum_n^{\infty} t^n (\text{all-}n \text{ Feynman rules}) \rightarrow \text{linear propagator in } t\text{-space}$$

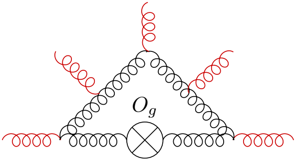
Feynman rules for O_{ABC} with five legs

[Falcioni, Herzog, Moch, Thurenhout, 24'; Gehrmann, Manteuffel, TZY, 24']



$$\to \frac{1+(-1)^n}{2}\, i\, g_s^3 \Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4} p_1^{\mu_5} \Bigg[\frac{1}{C_A} f^{aa_1a_2} d_4^{aa_3a_4a_5} \Bigg\{ \frac{3}{32} \sum_{j_1=0}^{n-4} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} (-\Delta\cdot p_1)^{n-4-j_1} (-\Delta\cdot(p_1+p_2))^{j_1-j_2} \times (\Delta\cdot(p_4+p_5))^{j_2-j_3} (\Delta\cdot p_5)^{j_3} + \cdots \Bigg\} + 11 \text{ color structures} \Bigg] + 30 \text{ Lorentz Structures}$$

extracted from

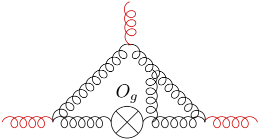


$$\sum_n^{\infty} t^n (\text{all-}n \text{ Feynman rules}) \to \text{linear propagator in } t\text{-space}$$

Two-loop C.F.R from two-loop three-leg OMEs

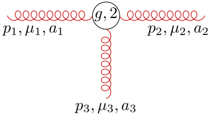
- Set all Mandelstam variables $p_1^2,\ p_2^2\ \cdots$ to numerical numbers and

$$\Delta \cdot p_1 = 1,\ \Delta \cdot p_2 = z_1$$



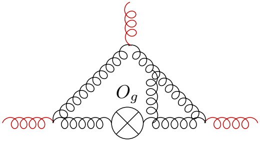
- Derive DE with respect to t
- Difficult to solve DE in terms of special functions
- Solve DE in $t \to 0$ limit with boundary conditions[Birthwright,Glover, Marquard, 2004]
- Determine two-loop counterterm Feynman rules to $n = 100$
- For a fixed n, the result is a polynomial in z_1
- Reconstruct the all-n two-loop C.F.R

Complexities from two-loop counterterm Feynman rules



$$\to 2ig_s C_A^2 f^{a_1 a_2 a_3} \frac{1+(-1)^n}{256n(n-1)} \frac{(\Delta \cdot p_1)^{n-2}}{\Delta \cdot p_2} \Big(\Delta^{\mu_2} \Delta^{\mu_3} p_1^{\mu_1} \Delta \cdot p_1 + \cdots \Big) \left\{ \frac{F_{-2,0}(\xi, z_1, n)}{\epsilon^2} + \frac{F_{-1,0}(\xi, z_1, n)}{\epsilon} \right\}, \; z_1 = \frac{\Delta \cdot p_2}{\Delta \cdot p_1}$$

extracted from



- $F_{-1,0}$ contains generalized harmonic sums[Moch,Uwer,Weinzierl,2002]

$$S_1\,(z_1+1; n) = \sum_{x_1=1}^{n} \frac{(1+z_1)^{x_1}}{x_1}, \; S_1(z_1+1; 2) = \frac{z_1^2}{2} + 2z_1 + \frac{3}{2}$$

$\sum_n^\infty t^n$(all-n Feynman rules) → polylogarithms in t-space

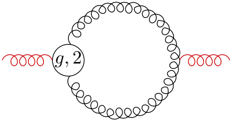
- IBP reductions with polylogarithms? (not feasible) *Need new idea*

Two-point OMEs with two-loop counterterm insertions

- Consider insertions of two-loop counterterms with 3-gluon vertex

$$ig_s f^{a_1 a_2 a_3} C_A^2 \Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3} p_1^2$$
$$\times \sum_{m=0}^{n-3} a_{mn}(\Delta\cdot p_1)^m(\Delta\cdot p_2)^{n-3-m} + \cdots$$

a_{mn} is rational number for any fixed m, n



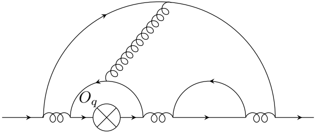
- *New idea*: replace a_{mn} by another tracing parameter t_1

$$h(t,t_1) = \sum_{n=3}^{\infty} t^n \sum_{m=0}^{n-3} t_1^m (\Delta\cdot p_1)^m(\Delta\cdot p_2)^{n-3-m} = \frac{t^3}{(1-t\,t_1\Delta\cdot p_1)(1-t\Delta\cdot p_2)}$$

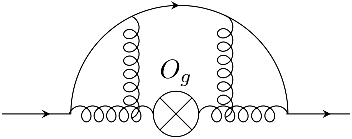
- Insert h into two-point diagrams: $\langle g|h(x,t)|g\rangle = \sum_{n=3}^{\infty} t^n \sum_{m=0}^{n-3} t_1^m c_{mn}$
- $\langle g|\sum_{m=0}^{n-3} a_{mn}(\Delta\cdot p_1)^m(\Delta\cdot p_2)^{n-3-m}|g\rangle = \sum_{m=0}^{n-3} a_{mn}\, c_{mn}$
- Evaluate OMEs to any fixed n efficiently
- Compute OMEs to $n=500$ and reconstruct the all-n result to ϵ^0

Sample Feynman diagrams for n_f^2 term of $\gamma_{\mathsf{ps}}^{(3)}$

- OMEs with physical operator insertions

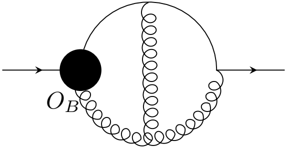


$\langle q | O_q | q \rangle^{(4)}$

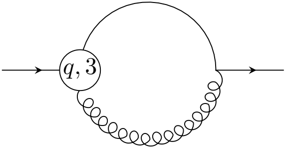


$\langle q | O_g | q \rangle^{(3)}$

- OMEs with GV operator or counterterm insertion



$\langle q | O_B | q \rangle^{(2)}$



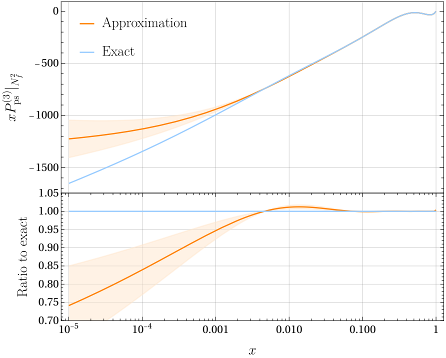
$\langle q | \left[ZO \right]_q^{\mathrm{GV},\,(3)} | q \rangle^{(1)} = 0$

New results for n_f^2 pure-singlet contributions

- New exact result, agree with the $n \leq 20$ results[G. Falcioni et al. 2023]
- Extract small-x_B result

$$P^{(3)}_{\text{ps}}(x_B)\big|_{N_f^2} = \frac{\log(x_B)^2}{x_B} \times 0 + \frac{\log(x_B)}{x_B}(\text{New results}) + \cdots$$

- Approximation: fitted from $n \leq 20$ results and previously known limits
- Large x-region: agree well with the exact result
- $x \sim 10^{-4}$, deviation ∼15%



Statistic data

- 549 top-topologies with 13 denominators + 5 ISPs
- Evolution of the amplitude

Amp.	Raw	Symmetry	IBP	Symmetry	Substi. master
Size	$\mathcal{O}(3GB)$	$\mathcal{O}(200MB)$	-	$\mathcal{O}(1GB)$	$\mathcal{O}(3MB)$
No. of int.	$\mathcal{O}(3M)$	$\mathcal{O}(220k)$	$\mathcal{O}(120k)$	$\mathcal{O}(6k)$	-

- $\mathcal{O}(40GB)$ for all $\mathcal{O}(300)$ DEs
- Series solutions for masters to $\mathcal{O}(4000)$ moments: $\mathcal{O}(100GB)$
- Final results: reconstruct $\gamma_{\text{ns}}^{(3)}$: $\mathcal{O}(200KB)$

Why 4-loop SFs for the evolutions of N3LO PDFs?

- Expand PDFs and SFs with $a_s = \alpha_s/(4\pi)$

$$f_{i|N} = f_{i|N}^{(0)} + f_{i|N}^{(1)} a_s + \cdots + f_{i|N}^{(3)} a_s^3 + \cdots$$

$$P_{ij} = P_{ij}^{(0)} a_s + \cdots + P_{ij}^{(3)} a_s^4 + \cdots$$

- Evolution of a_s

$$\frac{da_s}{d\ln\mu} = -2(a_s^2\beta_0 + a_s^3\beta_1 + \cdots)$$

- A consistent evolution of N3LO PDFs requires 4-loop SFs

$$f_{i|N}^{(3)} \frac{da_s^3}{d\ln\mu} = f_{i|N}^{(3)}(-6a_s^4\beta_0 + \cdots) = \sum_k P_{ik}^{(3)} a_s^4 \otimes f_{k|N}^{(0)} + \cdots$$

Decomposition of splitting functions

- The general structure of quark splitting functions,

$$P_{q_i q_k} = \delta_{ik} P^V_{qq} + P^S_{qq}, \quad P_{q_i \bar{q}_k} = \delta_{ik} P^V_{q\bar{q}} + P^S_{q\bar{q}}$$

- Non-singlet and singlet splitting functions

Non-singlet: $P^{\pm}_{\text{ns}} = P^V_{qq} \pm P^V_{q\bar{q}}, \quad P^V_{\text{ns}} = P^- + \overbrace{n_f(P^S_{qq} - P^S_{q\bar{q}})}^{P^s_{\text{ns}}}$

Singlet: $P_{qq} = P^+ + \underbrace{n_f(P^S_{qq} + P^S_{q\bar{q}})}_{P_{\text{ps}}}, P_{qg}, P_{gq}, P_{gg}$

- Evolution of PDFs

$$\frac{dT_i^{\pm}}{d\ln\mu} = 2P^{\pm}_{\text{ns}} \otimes T_i^{\pm}, \; \frac{d\sum_{k=1}^{n_f} q_k^-}{d\ln\mu} = 2P^V_{\text{ns}} \otimes \sum_{k=1}^{n_f} q_k^-\,, \; i = 3, 8, \cdots n_f^2 - 1$$

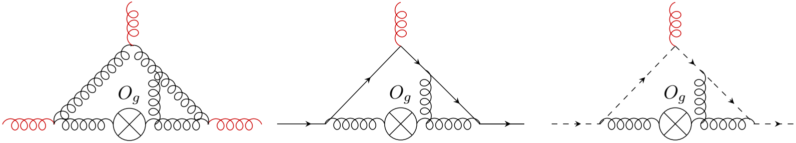
$$T_3^{\pm} = u^{\pm} - d^{\pm}, T_8^{\pm} = u^{\pm} + d^{\pm} - 2s^{\pm}, \cdots, q_k^{\pm} = q_k \pm \bar{q}_k\,,$$

$$\frac{d}{d\ln\mu}\begin{pmatrix} \Sigma \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix}, \quad \Sigma = \sum_{k=1}^{n_f} q_k^+$$

Determine Feynman rules for $[ZO]_g^{\mathrm{GV},\,(2)}$

- As an example, consider two ghosts + m-gluon external states and expand to two-loop order

$$
\begin{aligned}
\langle c|\,[ZO]_g^{\mathrm{GV},\,(2)}\,|c+m\,g\rangle_{\text{1PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m)} = -\Bigg\{\Bigg[& \langle c|O_g|c+m\,g\rangle_{\text{1PI}}^{\mu_1\cdots\mu_m,\,(2),\,(m),\,\text{B}} \\
& + \left(Z_c^{(1)} + \frac{mZ_g^{(1)}}{2} + Z_{gg}^{(1)} - \frac{\beta_0(m+2)}{2\epsilon}\right)\langle c|O_g|c+m\,g\rangle_{\text{1PI}}^{\mu_1\cdots\mu_m,\,(1),\,(m),\,\text{B}} \\
& + \left(Z_c^{(1)}Z_{gA}^{(1)} + \frac{1}{2}mZ_g^{(1)}Z_{gA}^{(1)} - \frac{\beta_0 mZ_{gA}^{(1)}}{2\epsilon} + Z_{gA}^{(2)}\right)\langle c|O_C|c+m\,g\rangle_{\text{1PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m),\,\text{B}} \\
& + Z_{gA}^{(1)}\,\langle c|O_{AC}|c+m\,g\rangle_{\text{1PI}}^{\mu_1\cdots\mu_m,\,(1),\,(m),\,\text{B}} + \cdots\Bigg]_{\text{div}}\Bigg\}
\end{aligned}
$$



Sample diagrams to extract Feynman rules for $[ZO]_g^{\mathrm{GV},\,(2)}$ with $m=1$

Renormalization of O_q to four loops in $q \to q$ channel

- Renormalization of two-point OMEs

$$\begin{aligned} \langle q| O_q^{\mathsf{R}} |q\rangle = {} & Z_{qq} \langle q| O_q^{\mathsf{B}} |q\rangle + Z_{qg} \langle q| O_g^{\mathsf{B}} |q\rangle \\ & + Z_{qA} \langle q| O_{ABC}^{\mathsf{B}} |q\rangle + \langle q| [ZO]_q^{\mathrm{GV}} |q\rangle, \\ Z_{qg} = {} & \mathcal{O}(a_s), Z_{qA} = \mathcal{O}(a_s^2), [ZO]_q^{\mathrm{GV}} = \mathcal{O}(a_s^3) \end{aligned}$$

- $\langle q| [ZO]_q^{\mathrm{GV}} |q\rangle = \mathcal{O}(a_s^5)$
  - Only $\langle q| [ZO]_q^{\mathrm{GV},\,(4)} |q\rangle^{(0)}$ and $\langle q| [ZO]_q^{\mathrm{GV},\,(3)} |q\rangle^{(1)}$ are relevant
  - Other operators (O_q, O_g, O_A, O_B) give all possible Lorentz structures of $q\bar{q}$, gg, $q\bar{q}g$ vertex Feynman rules
  - $\longrightarrow \langle q| [ZO]_q^{\mathrm{GV}} |q\rangle^{(0)} = 0$, $\langle g| [ZO]_q^{\mathrm{GV}} |g\rangle^{(0)} = 0$,

$$\langle q| [ZO]_q^{\mathrm{GV}} |qg\rangle^{(0)} = 0$$

Lorentz structures of a twist-two operator

- Based on the following two properties
  - A twist-two operator has spin-n and mass dimension $n+2$
  - Propagator-type Feynman rules like $1/p^2$ can not appear in a vertex
- A twist-2 operator involving quarks or ghosts has one Lorentz structure only

$$\langle q|O|q+m\,g\rangle^{\mu_1\cdots\mu_m,\,(0),\,(m)}_{\text{1PI}} = c_m\Delta^{\mu_1}\cdots\Delta^{\mu_m}$$

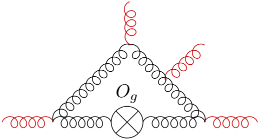
- A twist-two operator involving only gluons
  - Only $1+3/2m(m-1)$ Lorentz structures for m-gluon Feynman rules
  - $m=3$: $a_1\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3} + a_2\Delta^{\mu_1}\Delta^{\mu_2}p_1^{\mu_3} + \cdots + a_{10}\Delta^{\mu_3}g^{\mu_1\mu_2}$
  - 19 for $m=4$ and 31 for $m=5$
- Count the mass dimension of a_i: $[a_i] = x_i[\Delta\cdot p_j] + y_i[p_j\cdot p_k](y_i\geq 0)$
  $[a_1] = n-3+y_1[p_j\cdot p_k] = n+2-3 \to y_1 = 1($Linear in $p_1^2,\ p_1\cdot p_2\cdots)$
  $[a_2]+[p_1^{\mu_3}] = n-2+y_2[p_j\cdot p_k]+1 = n+2-3 \to y_2 = 0$
- Why not $a_{11}\Delta^{\mu_1}p_1^{\mu_2}p_2^{\mu_3}$

$$[a_{11}]+2+3 \geq n-1+y_{11}[p_j\cdot p_k]+2+3 = n+4(\text{if } y_{11}=0)$$

where 3 is mass dimension of the external 3 gluons. Twist-4 operators

Computations of single pole for one-loop multi-leg OMEs

- Set all Mandelstam variables $p_1^2,\ p_2^2\ \cdots$ to numerical numbers and reconstruct their linear dependence



- Only two types of integrals are needed, other integrals are finite

$$I_1 = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l-q_1)^2 l^2}, \quad I_2 = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l-q_1)^2 l^2 \big(1 - x\Delta\cdot(l+q_2)\big)}$$

- At most x-dependent logarithms appear in the single pole

$$I_2 = \frac{1}{\epsilon}\left[\frac{\ln(1 - x\Delta\cdot q_1 - x\Delta\cdot q_2) - \ln(1 - x\Delta\cdot q_2)}{-x\Delta\cdot q_1}\right] + \mathcal{O}(\epsilon^0)$$

- Logarithms in x-space → n-space

$$\ln(1 - x\Delta\cdot p_1 - x\Delta\cdot p_2) = \sum_{n=1}^{\infty} x^n \left[\frac{-1}{n}(\Delta\cdot p_1 + \Delta\cdot p_2)^n\right]$$

- Factoring out the overall factor $Z_{gA}^{(1)} = -\frac{C_A}{\epsilon}\frac{1}{n(n-1)}$

Reconstruct two-loop counterterm Feynman rules

- Obtain two-loop three-leg OMEs to x^{96} or $n = 96$
- For a fixed n, the result is a polynomial in z_1
- Construct full-x or full-n results from data to $n = 76$ based on ansatz
- Polylogarithms to weight-3, generalized Harmonic sums to weight-2

$$G(1,1,1/(1+z_1);x) = \sum_{n=1}^{\infty} x^n \left[\frac{S_1\left(z_1+1;n\right)}{n^2} + \frac{S_2\left(z_1+1;n\right)}{n} - \frac{S_{1,1}\left(1,z_1+1;n\right)}{n} - \frac{\left(z_1+1\right)^n}{n^3} \right]$$

where $S_{1,1}\left(1,z_1+1;n\right) = \sum_{t_1=1}^{n} \frac{1}{t_1} \sum_{t_2=1}^{t_1} \frac{(1+z_1)^{t_2}}{t_2}$

- Due to the generalized Harmonic sums, impossible to disentangle
  - renormalization constants (no z_1 dependence)
  - operator Feynman rules (no high-weight (≥ 1) functions)

A counterterm Feynman rule & infinite operator Feynman rules ($N_2 = \infty$)