Hidden-heavy Baryonium States in QCD Sum Rules

Xuan-Heng Zhang (张轩珩)[1]

School of Physical Sciences, University of Chinese Academy of Sciences

Collaborator: Prof. Cong-Feng Qiao

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[1]zhangxuanheng22@mails.ucas.ac.cn

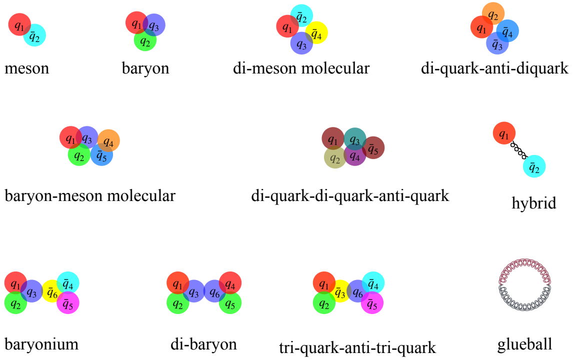
Outline

Section 1

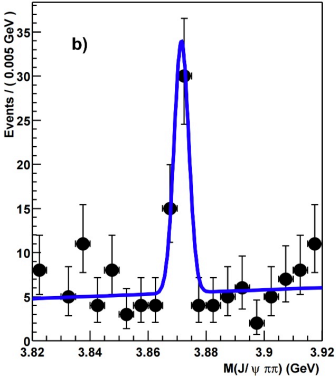
Review

Hadronic States

- Color confinement and Asymptotic freedom – Hadrons – Non-perturbative effects
- Traditional Hadrons in Quark Model:
  - Mesons $(q\bar{q})$;
  - Baryons (qqq).
- QCD allows for hadrons beyond Quark Model – Exotic states:



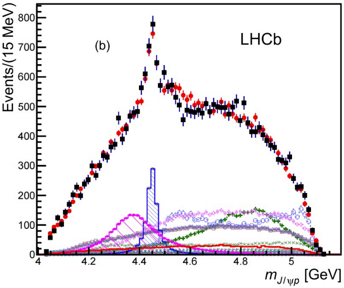
Observation of Exotic States



The Tetraquark states $X(3872)$

$M_{X(3872)} = (3871.69 \pm 0.17)\text{MeV}$

[Belle.2003]



The Pentaquark states

$P_c(4380)^+, \ P_c(4450)^+$

$M_{4380} = (4380 \pm 8 \pm 29)\text{MeV}$

$M_{4450} = (4449.8 \pm 1.7 \pm 2.5)\text{MeV}$

[LHCb.2015]

Theoretical Methods

- Various methods :
  - Quark Model
  - MIT bag model
  - Chiral effective theories
  - Lattice QCD
  - NRQCD
  - AdS/QCD
  - QCD sum rules (QCDSR)
  - Light-cone sum rules (LCSR)
  - Inverse Problems
  - ...

[SVZ, 1979]

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QCD AND RESONANCE PHYSICS. THEORETICAL FOUNDATIONS

M.A. SHIFMAN, A.I. VAINSHTEIN * and V.I. ZAKHAROV
Institute of Theoretical and Experimental Physics, Moscow, 117259, USSR

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QCD AND RESONANCE PHYSICS. APPLICATIONS

M.A. SHIFMAN, A.I. VAINSHTEIN * and V.I. ZAKHAROV
Institute of Theoretical and Experimental Physics, Moscow, 117259, USSR

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- Advantages of QCDSR:

1. Based on the first principle of QCD.
2. Gives the analytical results.
3. Sum rules of perturbative and non-perturbative effects.
4. An effective and proven method on researching the properties of hadrons. Including conventional hadrons, tetraquarks and pentaquarks.

Hexaquark States

- Experimental candidates: $X(1835)$, $X(1860)$, $X(1880)$, $X(2075)$, $X(2085)$, $Y(4260)$, $Y(4660)$, etc.
- Light hexaquarks:
  - Deuteron: $J^P = 1^+$, $E_B = 2.225\mathrm{MeV}$ pn dibaryon bound state.
  - Dihyperon states: [R.L.Jaffe. 1977] etc. ;
  - Diproton states: [P.J.Mulders, et.al. 1980];
  - Light Baryoniums: [B.S.Zou, H.C.Chiang. 2004], [S.L.Zhu, C.S.Gao. 2006], [Z.G.Wang, et.al. 2006],
    [**ZXH**, Zhang and Qiao. EPJC 85 (2025) 6, 693], [**ZXH**, Zhang and .Qiao. PRD 113, 034022 (2026)] etc.
- Heavy hexaquarks:
  - $\Lambda_c\Lambda_c$ and $\Sigma_c\Sigma_c$: [Z.G.Wang et.al. 2021], [Z.G.Wang et.al. 2022];
  - Hidden-charm or hidden bottom: [C.F.Qiao, 2005/2007], [Chen, et.al. 2006], [Z.G.Wang, et.al. 2021], [Z.G.Wang et.al. 2022], [B.D.Wan, L.Tang and C.F.Qiao. 2019];
  - Full-heavy hexaquark states: [Z.G.Wang. 2022], [Chen.et.al. 2026] , etc.

Hidden-heavy Baryoniums in QCDSR

[S.Q.Zhang and C.F.Qiao, 2025]

State	J^{PC}	QCDSR
Λ_c-$\bar{\Lambda}_c$	0^+	5.00 [560], 5.19±0.24 [561], $5.11^{+0.15}_{-0.12}$ [562]
	0^-	$4.66^{+0.10}_{-0.06}$ [562]
	1^+	4.89 [560], $4.99^{+0.10}_{-0.09}$ [562]
	1^-	4.78±0.23 [561], $4.68^{+0.08}_{-0.08}$ [562]
	2^+	5.15 [560]
	2^-	4.83 [560]
	3^+	5.68 [560]
	3^-	5.04 [560]
Σ_c-$\bar{\Sigma}_c$	0^+	$5.23^{+0.07}_{-0.07}$ [563]
	0^-	$4.88^{+0.08}_{-0.08}$ [563]
	1^+	$5.31^{+0.07}_{-0.07}$ [563]
	1^-	$4.88^{+0.09}_{-0.08}$ [563]
Λ_b-$\bar{\Lambda}_b$	0^+	11.84±0.22 [561]
	0^-	
	1^+	
	1^-	11.72±0.26 [561]

560 : [W.Chen, H.X.Chen, S.L.Zhu et.al, EPJC, 2016]

561 : [B.D.Wan, L.Tang and C.F.Qiao. EPJC, 2019]

562 : [X.W.Wang, Z.G.Wang and G.L.Yu. EPJA, 2021]

563 : [X.W.Wang, Z.G.Wang. AHEP, 2021]

- No obvious results of $\Lambda_Q\bar{\Sigma}_Q$ ($\mathscr{B}\bar{\mathscr{B}}'$) !
- $\Lambda_c\bar{\Sigma}_c$ masses may larger than 5 GeV.
  $\Lambda_b\bar{\Sigma}_b$ masses may larger than 12 GeV.

$\Lambda_c\bar{\Sigma}_c$ States

- The BESIII results: [BESIII, 2025]

We search for a possible $\Lambda_c\bar{\Sigma}_c$ bound state, denoted as $H_c^{\pm}$, via the $e^+e^- \to \pi^+\pi^-\Lambda_c^+\bar{\Lambda}_c^-$ process for the first time. This analysis utilizes 207.8 and 159.3 pb^{-1} of e^+e^- annihilation data at the center-of-mass energies of 4918.02 and 4950.93 MeV, respectively, collected with the BESIII detector at the BEPCII collider. No statistically significant signal is observed. The upper limits of the product of Born cross section and branching fraction $\sigma(e^+e^- \to \pi^+H_c^- + c.c.) \times \mathcal{B}(H_c^- \to \pi^-\Lambda_c^+\bar{\Lambda}_c^-)$ at a 90% confidence level are reported at each energy point and for various H_c mass hypotheses (4715, 4720, 4725, 4730, and 4735 MeV/c^2) and widths (5, 10, or 20 MeV), with the upper limits ranging from 1.1 pb to 6.4 pb.

- Near-threshold bound states or Resonances ?
- Different theoretical methods should be further examined.

Section 2

Theoretical Framework

Interpolating Currents

The chiral limit $m_u = m_d \to 0$ is taken to simplify the currents.

- Color singlet Baryonic currents

$$\eta_{1\mathscr{B}}(x) = \varepsilon_{abc}\left[q_a^{iT}(x)\mathcal{C}q_b^j(x)\right]\gamma_5 q_c^k(x)\ ,$$

$$\eta_{2\mathscr{B}}(x) = \varepsilon_{abc}\left[q_a^{iT}(x)\mathcal{C}\gamma_5 q_b^j(x)\right] q_c^k(x)\ ,$$

where $(i, j, k) = (u, d, Q)$ for Λ_Q, (u, Q, d) for Σ_Q, $Q = c, b$.

- Baryonium currents of the $\Lambda_Q\bar{\Sigma}_Q$ states with $J^P = 0^+, 0^-, 1^-, 1^+$

$$j^{0^-}(x) = \mathrm{i}\bar{\eta}_{\mathscr{B}'}(x)\gamma_5\eta_{\mathscr{B}}(x),$$
$$j^{0^+}(x) = \bar{\eta}_{\mathscr{B}'}(x)\eta_{\mathscr{B}}(x),$$
$$j^{1^-}_\mu(x) = \bar{\eta}_{\mathscr{B}'}(x)\gamma_\mu\eta_{\mathscr{B}}(x),$$
$$j^{1^+}_\mu(x) = \bar{\eta}_{\mathscr{B}'}(x)\gamma_\mu\gamma_5\eta_{\mathscr{B}}(x).$$

Quark Side

- 2-Point correlation functions

$$\text{For } J=0, \quad \Pi(q^2) = \mathrm{i}\int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}q\cdot x} \langle\Omega|\, \mathbb{T}\{j(x), j^\dagger(0)\}\, |\Omega\rangle\,,$$

$$\text{For } J=1, \quad \Pi(q^2) = -\frac{1}{3}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Pi_{\mu\nu}(q^2),$$

$$\Pi_{\mu\nu}(q^2) = \mathrm{i}\int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}q\cdot x} \langle\Omega|\, \mathbb{T}\{j_\mu(x), j_\nu^\dagger(0)\}\, |\Omega\rangle\,,$$

Dispersion Relation $\Downarrow$ $\rho(s) = \frac{1}{\pi}\,\mathrm{Im}\,\Pi(s)$

$$\Pi^{\text{QCD}}_{X,J^P}(q^2) = \int_{s_{\min}}^{\infty} \mathrm{d}s\, \frac{\rho^{\text{OPE}}_{X,J^P}(s)}{s-q^2}, \qquad s_{\min} = (2m_Q)^2$$

- Do contractions, using full propagators, we obtain the OPE of spectral density, which should be cut off at dimension 12

$$\rho^{\text{OPE}}_{X,J^P} = \rho^{\text{pert}} + \sum_{n=3}^{12} \rho^{\langle\mathcal{O}_n\rangle}.$$

Hadron Side

- Phenomenological framework

$$\rho^{\text{Phen}}_{X,J^P}(s) = \lambda^2_{X,J^P}\,\delta\left(s - m^2_{X,J^P}\right) + \theta(s - s_0)\,\rho_{X,J^P}(s),$$

$$\lambda_{X,J^P} \equiv \langle\Omega|\, j(x)\,|H_0\rangle\,,$$
$$\lambda_{X,J^P}\epsilon^\mu \equiv \langle\Omega|\, j^\mu(x)\,|H_0\rangle\,.$$

$$\Pi^{\text{Phen}}_{X,J^P}(q^2) = \frac{\lambda^2_{X,J^P}}{m^2_{X,J^P} - q^2} + \int_{s_0}^{\infty} \mathrm{d}s\, \frac{\rho_{X,J^P}(s)}{s - q^2},$$

pole of ground state

s_0: continuum threshold

Quark-Hadron duality ⇓ Borel transformation

$$\lambda^2_{X,J^P}\mathrm{e}^{-m^2_{X,J^P}/M_B^2} = \int_{s_{\min}}^{s_0} \mathrm{d}s\ \rho^{\text{OPE}}_{X,J^P}(s)\mathrm{e}^{-s/M_B^2},$$

which is the sum rule of the correlation function of the ground state hexaquarks.

- Extract the mass and decay constant

$$L_{X,J^P,0}(s_0, M_B^2) = \int_{s_{\min}}^{s_0} \mathrm{d}s\,\rho^{\text{OPE}}(s)\,\mathrm{e}^{-s/M_B^2},$$

$$L_{X,J^P,1}(s_0, M_B^2) = \frac{\partial}{\partial(M_B^{-2})} L_{X,J^P,0}(s_0, M_B^2).$$

$$m_{X,J^P}(s_0, M_B^2) = \sqrt{-\frac{L_{X,J^P,1}(s_0, M_B^2)}{L_{X,J^P,0}(s_0, M_B^2)}},$$

$$\lambda_{X,J^P}(s_0, M_B^2) = \sqrt{\mathrm{e}^{m^2_{X,J^P}(s_0,M_B^2)/M_B^2}\, L_{X,J^P,0}(s_0, M_B^2)}.$$

2 additional parameters s_0 and M_B are inserted!

Criteria of Choosing s_0 and M_B

- Pole Dominate:

$$R^{\text{PC}}_{X,J^P} = \frac{L_{X,J^P,0}(s_0, M_B^2)}{L_{X,J^P,0}(\infty, M_B^2)} \geqslant 50\%/40\%/30\%/20\%/15\%.$$

Large power of s suppress this value in hexaquarks. [Chen, Zhu, et.al, PRC, 2015]
$\Rightarrow$ Complex structure of the multiquark spectra.

- OPE convergence:

$$R^{\text{OPE}}_{X,J^P} = \left| \frac{L^{\langle O_{12}\rangle}_{X,J^P,0}(s_0, M_B^2)}{L_{X,J^P,0}(s_0, M_B^2)} \right| \lesssim 10\% \quad \text{in this work,}$$

where

$$L^{\langle \mathcal{O}_{12}\rangle}_{X,J^P,0}(s_0, M_B^2) = \int_{s_{\text{min}}}^{s_0} \mathrm{d}s \; \rho^{\langle \mathcal{O}_{12}\rangle}(s) \mathrm{e}^{-s/M_B^2}.$$

Also, the dimension-13 term should not spoil the hierarchical structure.

- M_B^2 stability: Observables shouldn't depend on the additional parameters.
$\Rightarrow$ Find a platform: Borel Window.

Section 3

Numerical Results

Numerical Setup

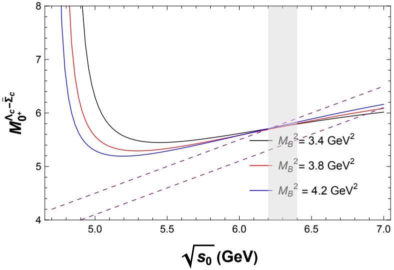
- Parameters form lattice and PDG :

$$\langle\bar{q}q\rangle = -(0.24\pm0.01)^3\ \mathrm{GeV}^3, \qquad \langle g_s^2G^2\rangle = (0.88\pm0.25)\ \mathrm{GeV}^4,$$
$$\langle g_s^3G^3\rangle = (0.045\pm0.013)\ \mathrm{GeV}^6, \quad \langle\bar{q}g_s\sigma\cdot Gq\rangle = m_0^2\langle\bar{q}q\rangle,$$
$$\bar{m}_c(\bar{m}_c) = 1.273\pm0.0028\ \mathrm{GeV}, \qquad \bar{m}_b(\bar{m}_b) = 4.183\pm0.004\ \mathrm{GeV},$$
$$m_0^2 = (0.8\pm0.1)\ \mathrm{GeV}^2.$$

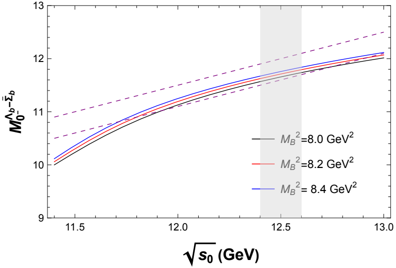
- Reliable region of s_0 should be $\sqrt{s_0}\sim m_X+\delta$, where $\delta\approx0.5$ GeV for conventional hadrons. In this work, the range $0.5\leqslant\delta\leqslant0.9$ GeV is chosen to accommodate these heavier states.
- We allow $\sqrt{s_0}$ to vary within a range of $\pm0.1\mathrm{GeV}$.
- A stable physical state: M_B^2 within the Borel window exceeds $0.5\,\mathrm{GeV}^2$, ensuring the stability of the results.

Scanning from Threshold

- Thresholds: $4.71\ \mathrm{GeV}$ for $\Lambda_c\bar{\Sigma}_c$ and $11.43\ \mathrm{GeV}$ for $\Lambda_b\bar{\Sigma}_b$.



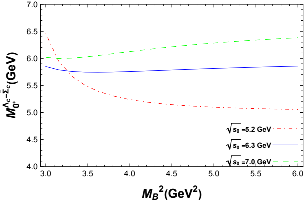
(a) Type-2 0^+ $\Lambda_c\bar{\Sigma}_c$



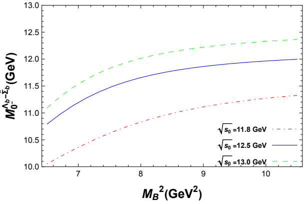
(b) Type-1 0^- $\Lambda_b\bar{\Sigma}_b$

- Unstable behaviors (a larger sensitivity $\partial M_X/\partial\sqrt{s_0}$, even vertical dropping) show the instability near threshold. Threshold effects may pollute the spectrum.

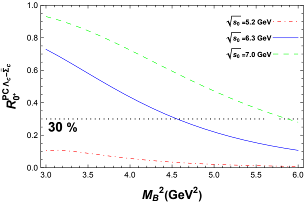
Behaviors at Different s_0



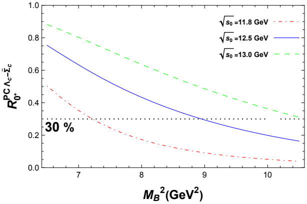
(c) m_X for Type-2 0^+ $\Lambda_c\bar{\Sigma}_c$



(d) m_X for Type-1 0^- $\Lambda_b\bar{\Sigma}_b$



(e) R^{PC} for Type-2 0^+ $\Lambda_c\bar{\Sigma}_c$



(f) R^{PC} for Type-1 0^- $\Lambda_b\bar{\Sigma}_b$

Results of $\Lambda_Q\bar{\Sigma}_Q$ States

J^P	State	$\sqrt{s_0}$(GeV)	M_B^2(GeV^2)	M_X(GeV)	λ_X(GeV^8)	$R^{\rm PC}$(%)	$\lvert R^{\langle\mathcal{O}_{12}\rangle}\rvert$(%)
0^-	$\Lambda_b\bar{\Sigma}_b$	12.5 ± 0.1	$7.8-8.5$	11.68 ± 0.18	$(5.8\pm1.7)\times10^{-3}$	$32-51$	$4.8-10.4$
1^-	$\Lambda_b\bar{\Sigma}_b$	12.5 ± 0.1	$7.9-8.6$	11.70 ± 0.18	$(5.8\pm1.7)\times10^{-3}$	$31-50$	$4.5-10.0$

TABLE I: The related numerical results of the Type-I currents

J^P	State	$\sqrt{s_0}$(GeV)	M_B^2(GeV^2)	M_X(GeV)	λ_X(GeV^8)	$R^{\rm PC}$(%)	$\lvert R^{\langle\mathcal{O}_{12}\rangle}\rvert$(%)
0^-	$\Lambda_c\bar{\Sigma}_c$	6.3 ± 0.1	$4.1-4.7$	5.72 ± 0.08	$(1.7\pm0.3)\times10^{-3}$	$31-54$	$0.5-1.2$
	$\Lambda_b\bar{\Sigma}_b$	12.6 ± 0.1	$9.1-10.5$	11.86 ± 0.10	$(1.4\pm0.2)\times10^{-2}$	$30-54$	$0.4-1.0$
0^+	$\Lambda_c\bar{\Sigma}_c$	6.3 ± 0.1	$3.3-4.3$	5.77 ± 0.06	$(1.4\pm0.1)\times10^{-3}$	$32-69$	$1.5-8.3$
	$\Lambda_b\bar{\Sigma}_b$	12.5 ± 0.1	$7.5-9.3$	11.89 ± 0.07	$(9.2\pm1.3)\times10^{-3}$	$30-63$	$1.7-9.5$
1^-	$\Lambda_c\bar{\Sigma}_c$	6.4 ± 0.1	$4.2-4.9$	5.79 ± 0.09	$(2.0\pm0.3)\times10^{-3}$	$31-56$	$0.4-1.0$
	$\Lambda_b\bar{\Sigma}_b$	12.7 ± 0.1	$9.1-10.6$	11.92 ± 0.11	$(1.5\pm0.2)\times10^{-2}$	$33-57$	$0.3-1.0$
1^+	$\Lambda_c\bar{\Sigma}_c$	6.4 ± 0.1	$3.3-4.5$	5.82 ± 0.09	$(1.4\pm0.3)\times10^{-3}$	$31-73$	$1.2-8.6$
	$\Lambda_b\bar{\Sigma}_b$	12.6 ± 0.1	$7.6-9.5$	11.95 ± 0.07	$(1.0\pm0.4)\times10^{-2}$	$31-65$	$1.5-8.5$

TABLE II: The related numerical results of the Type-II currents

Consistent with the BESIII results!

Section 4

Conclusion

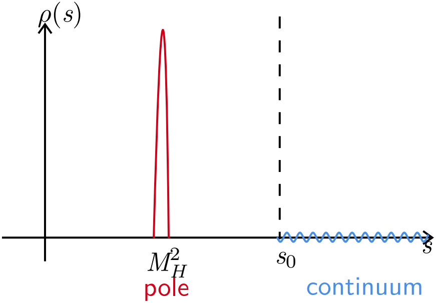
Conclusion

- The masses of the $\Lambda_Q\bar{\Sigma}_Q$ ground states $J^P = 0^-, 0^+, 1^-, 1^+$ are evaluated by two independent currents, all extracted masses lie above the corresponding baryon–antibaryon thresholds. Some of the $0^+, 1^+$ hidden-charm baryonium masses predicted in Chen.et.al lie in the $5.5 - 6.0\ \mathrm{GeV}$ region, which are consistent with our results.
- Our results do not contradict with the null result of BESIII in $4715 - 4735\ \mathrm{GeV}$. We suggest the experiments could increase the center-of-mass energy and explore the region $5.5 - 6.0\ \mathrm{GeV}$ to search for the $\Lambda_c\bar{\Sigma}_c$ states.
- The states $\Lambda_b\bar{\Sigma}_b$ we have calculated would be detected in experiments such as STCF, LHCb, ATLAS, BelleII, and others. They can serve as candidates for hidden-bottom resonances in the $12\ \mathrm{GeV}$ region.
- NLO corrections: the one-gluon exchange provides the color Coulomb interaction between Q and $\bar{Q}$, yields an important short-range attractive force. [Wu, Wang, Meng, Ma and Chao, JHEP, 2023]. A fully reliable insight into the nonperturbative hadronization process, requires further study.

Reliability of QCDSR for Multiquark States

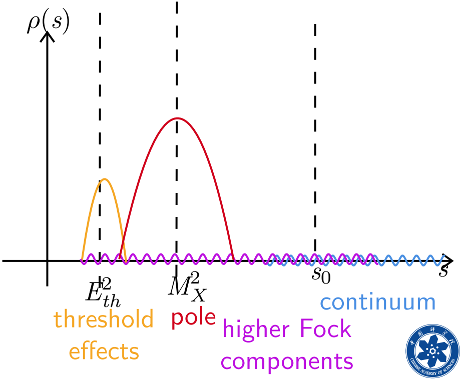
Conventional Hadrons

- Ground state usually well separated from excited states.
- Spectral function often dominated by a clear lowest pole.



Multiquark States

- Ground state and continuum are less clearly separated.
- Pole dominance is usually weaker.



Reliability of QCDSR for Multiquark States

Conventional Hadrons

- Compact quark configurations.
- OPE typically converges rapidly.
- Effective threshold $\sqrt{s_0}$ is usually close to the first excited state.
- QCDSR predictions are often quantitatively successful.

Multiquark States

- More spatially extended configurations.
- Higher-dimensional condensates may be more important.
- Choice of $\sqrt{s_0}$ is more subtle due to limited knowledge of excited spectra.
- Larger systematic uncertainties are expected.

Reliability of QCDSR for Multiquark States

- Existing QCDSR studies of multiquark systems show no obvious contradiction with current experimental observations.
- Typical uncertainties are of order $\Lambda_{\rm QCD}$, but still sufficient to distinguish near-threshold states from states well above threshold.
- If QCDSR obtains a near-threshold result, it should be carefully considered. It may be a molecular state, threshold effects, or 2-hadron scattering states. In the $\Lambda_c\bar{\Sigma}_c$, the near threshold result may also contribute to the $c\bar{c}q\bar{q}+\pi$ structure whose masses may lie in 4–5 GeV. [W.Chen, H.X.Chen, S.L.Zhu et.al, EPJC, 2016]
- Therefore, QCDSR remains a semi-quantitative but useful framework for exploring multiquark spectroscopy. The applicability remains an open question deserving further investigation, especially clarifying the interplay between the OPE and the complex spectral structure of such systems.

Thanks!

The Full Propagator of Quarks

- The full propagator of light quarks

$$
\begin{aligned}
\mathrm{i}\mathcal{S}_q^{jk}(x) =& \mathrm{i}\delta^{jk}\frac{\not{x}}{2\pi^2x^4} - \delta^{jk}m_q\frac{1}{4\pi^2x^2} - \mathrm{i}t_a^{jk}\frac{G^a_{\alpha\beta}}{32\pi^2x^2}\left(\sigma^{\alpha\beta}\not{x}+\not{x}\sigma^{\alpha\beta}\right) - \delta^{jk}\frac{\langle\bar{q}q\rangle}{12} \\
&+ \mathrm{i}\delta^{jk}\frac{\not{x}}{48}m_q\langle\bar{q}q\rangle - \delta^{jk}\frac{x^2}{192}\langle g_s\bar{q}\sigma\cdot Gq\rangle + \mathrm{i}\delta^{jk}\frac{x^2\not{x}}{1152}m_q\langle g_s\bar{q}\sigma\cdot Gq\rangle \\
&- t_a^{jk}\frac{\sigma_{\alpha\beta}}{192}\langle g_s\bar{q}\sigma\cdot Gq\rangle - \mathrm{i}t_a^{jk}\frac{1}{768}\left(\sigma_{\alpha\beta}\not{x}+\not{x}\sigma_{\alpha\beta}\right)m_q\langle g_s\bar{q}\sigma\cdot Gq\rangle .
\end{aligned}
$$

- The full propagator of heavy quarks

$$
\begin{aligned}
\mathcal{S}_Q^{jk}(p) =& \frac{\mathrm{i}\delta^{jk}\left(\not{p}+m_Q\right)}{p^2-m_Q^2} - \frac{\mathrm{i}}{4}\frac{t_a^{jk}G^a_{\alpha\beta}}{\left(p^2-m_Q^2\right)^2}\left[\sigma^{\alpha\beta}\left(\not{p}+m_Q\right)+\left(\not{p}+m_Q\right)\sigma^{\alpha\beta}\right] \\
&+ \frac{\mathrm{i}\delta^{jk}m_Q\left\langle g_s^2G^2\right\rangle}{12\left(p^2-m_Q^2\right)^3}\left[1+\frac{m_Q\left(\not{p}+m_Q\right)}{p^2-m_Q^2}\right] \\
&+ \frac{\mathrm{i}\delta^{jk}}{48}\left\{\frac{\left(\not{p}+m_Q\right)\left[\not{p}\left(p^2-3m_Q^2\right)+2m_Q\left(2p^2-m_Q^2\right)\right]\left(\not{p}+m_Q\right)}{\left(p^2-m_Q^2\right)^6}\right\}\left\langle g_s^3G^3\right\rangle
\end{aligned}
\tag{6.1}
$$

The Spectral Density

- Using the Wick's theorem,

$$\Pi_{(\mu\nu)}(q^2) = -\mathrm{i}\varepsilon_{abc}\varepsilon_{a_1b_1c_1}\varepsilon_{def}\varepsilon_{d_1e_1f_1}\int_X\int_P \mathrm{Tr}\left[\mathcal{S}_d^{aa_1}(-x)\Gamma_1\Gamma_{(\mu)}\Gamma_1\mathcal{S}_Q^{f_1f}(p_1)\Gamma_1\Gamma_{(\nu)}\Gamma_1\right] \times \mathrm{Tr}\left[\mathcal{C}\mathcal{S}_u^{Tcc_1}(-x)\mathcal{C}\Gamma_2\mathcal{S}_Q^{bb_1}(-p_2)\Gamma_2\right] \times \mathrm{Tr}\left[\mathcal{C}\mathcal{S}_u^{Td_1d}(x)\mathcal{C}\Gamma_2\mathcal{S}_d^{e_1e}(x)\Gamma_2\right].$$

$$\int_X\int_P = \int \mathrm{d}^4x\int\frac{\mathrm{d}^4p_1}{(2\pi)^4}\int\frac{\mathrm{d}^4p_2}{(2\pi)^4}$$

- The spectral density can be expressed with the Källén-Lehmann spectral representation $\rho(s) = \dfrac{1}{\pi}\,\mathrm{Im}\,\Pi(s)$,

$$\rho^{\mathrm{OPE}}(s) = \rho^{\mathrm{pert}}(s) + \rho^{\langle\bar{q}q\rangle}(s) + \rho^{\langle G^2\rangle}(s) + \rho^{\langle\bar{q}Gq\rangle}(s) + \rho^{\langle\bar{q}q\rangle^2}(s) + \rho^{\langle G^3\rangle}(s) + \rho^{\langle\bar{q}q\rangle\langle G^2\rangle} + \rho^{\langle\bar{q}q\rangle\langle\bar{q}Gq\rangle}(s) + \rho^{\langle\bar{q}q\rangle^3}(s) + \rho^{\langle\bar{q}Gq\rangle\langle G^2\rangle}(s) + \rho^{\langle\bar{q}q\rangle^2\langle G^2\rangle}(s) + \rho^{\langle\bar{q}Gq\rangle^2}(s) + \rho^{\langle\bar{q}q\rangle^2\langle\bar{q}Gq\rangle}(s) + \rho^{\langle\bar{q}q\rangle^4}(s). \tag{6.2}$$

- The loop integrals of the relevant Feynman diagrams can be evaluated by the Schwinger parametrization method, and the ultraviolet divergences arising from these loop integrals are removed through renormalization in the $\overline{\mathrm{MS}}$ scheme.

Borel Transformation

- Definition

$$B\left[\Pi\left(q^2\right)\right] = \lim_{\substack{q^2, n\to\infty \\ -q^2/n = M_B^2}} \frac{\left(-q^2\right)^{n+1}}{n!}\left(\frac{\partial}{\partial q^2}\right)^n \Pi\left(q^2\right).$$

- Useful Relations

$$\mathcal{B}\left[\left(q^2\right)^k\right] = 0,$$

$$\mathcal{B}\left[\frac{1}{\left(s-q^2\right)^k}\right] = \frac{1}{(k-1)!}\left(\frac{1}{M_B^2}\right)^{k-1} \mathrm{e}^{-s/M_B^2}.$$

- Borel transformation is used for suppressing the contribution of higher excited states and the continuum spectra.