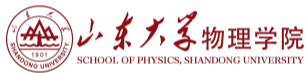


Triple Differential Semileptonic Heavy-to-light Decays @ $\mathcal{O}(\alpha_s^3)$

陈龙 (Long Chen)



第八届全国重味物理与量子色动力学研讨会

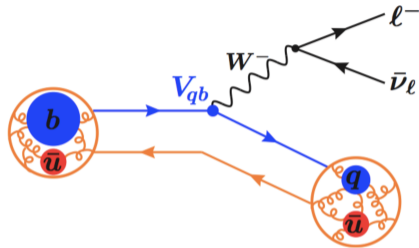
2026.04.24–28, 重庆

Based on 2602.11537, 2602.11879, 2509.03580

Why Precision Study of Semileptonic B-meson Decay

- Measurement of $|V_{ub}|$ (and $|V_{cb}|$) at the unprecedented $\mathcal{O}(0.01)$ level @ Belle II
- Clarifying the long-standing discrepancy between the *inclusive* and *exclusive* measurements of $|V_{ub}|$
- Precision determination of the (global) non-perturbative parameters and *shape* functions for B-mesons
-

[→See talk by Q.D.Zhou]



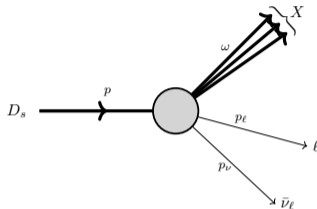
	Statistical	Systematic (reducible, irreducible)	Total expt.	Theory	Total
1 ab^{-1}	2.5	(2.9, 1.6)	4.1	2.5 – 4.5	4.8 – 6.1
5 ab^{-1}	1.1	(1.3, 1.6)	2.3	2.5 – 4.5	3.4 – 5.1
50 ab^{-1}	0.4	(0.4, 1.6)	1.7	2.5 – 4.5	3.0 – 4.8

Precise measurements of both **inclusive** and **differential** observables of $B \rightarrow X_u \ell \bar{\nu}_\ell$ are crucial for these goals.

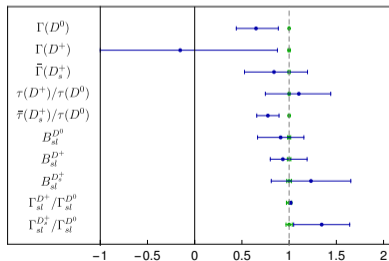
Why Precision Study of Semileptonic D-meson Decay

- Test of CKM mechanism, and check our understanding of hadron dynamics
(e.g. heavy-quark flavor-spin symmetry)
- Interesting interplay with B-meson physics
(e.g. shed light to the tension between inc. and excl. values of V_{cb}, V_{ub})
- Unique opportunity and stress-test for both pert. and non-pert. QCD methods @ $m_c \sim 1 \sim 2 \text{ GeV}$
(e.g. Heavy-Quark Expansion, Lattice-QCD ...)
- Considerable truncation errors and imprecise theory parameters in decays of D^0, D^+, D_s^+ to be improved

[→See talks by Li, Zhang and Shao]



[De Santis et al. 25]



[King et al. 21]

Precision measurement of Semileptonic Decays of $D^{0/+}$

Experimental precision of the semileptonic Γ_D and electron-energy moments for D_s^+ now reaches the level of a few (2 ~ 3) %

- From CLEO: [Asner et al. 10]

$$\mathcal{B}(D_s^+ \rightarrow X e^+ \nu_e) = (6.52 \pm 0.39 \pm 0.15)\%$$

- From BES-III: [Ablikim et al. 21]

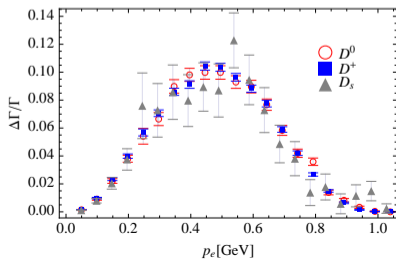
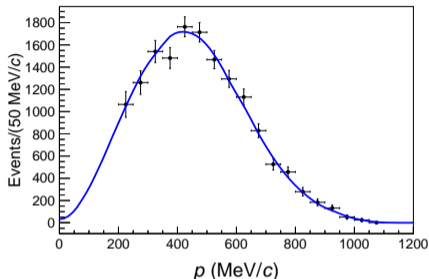
$$\mathcal{B}(D_s^+ \rightarrow X e^+ \nu_e) = (6.30 \pm 0.13 \text{ (stat.)} \pm 0.10 \text{ (syst.)})\%$$

- From Lattice-QCD: [De Santis et al. 25]

$$\Gamma = 8.72(56) \times 10^{-14} \text{ GeV} ,$$

$$M_1 = 0.456(22) \text{ GeV} ,$$

$$M_2 = 0.227(10) \text{ GeV}^2 .$$

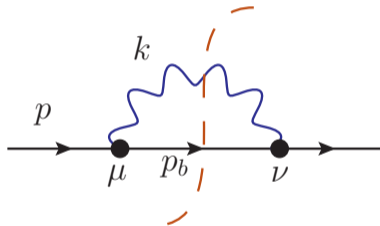


Cut Diagrams for Heavy-to-light Decay

Γ_{Q2q} in terms of the **semi-inclusive** $\mathcal{W}_{Q2q}^{\mu\nu}$

$$\Gamma_{Q2q} = \frac{1}{2m_Q} \int \frac{d^{d-1}k}{(2\pi)^{d-1}2E} \mathcal{W}_{Q2q}^{\mu\nu} \sum_{\lambda}^{L,R,0} \varepsilon_{\mu}^*(k, \lambda) \varepsilon_{\nu}(k, \lambda),$$

$$\left| \mathcal{M}_{Q \rightarrow q+W} \right|^2 \Rightarrow$$

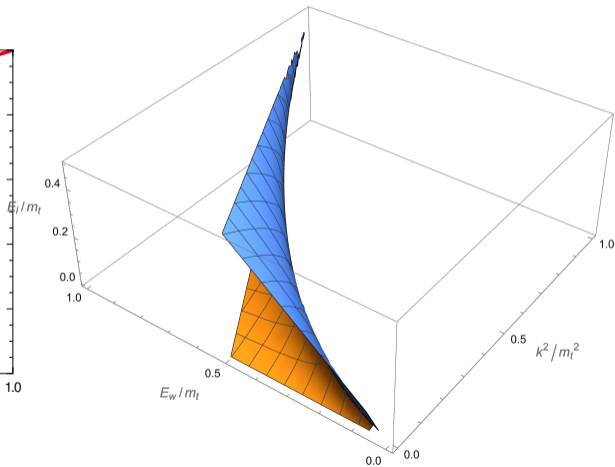
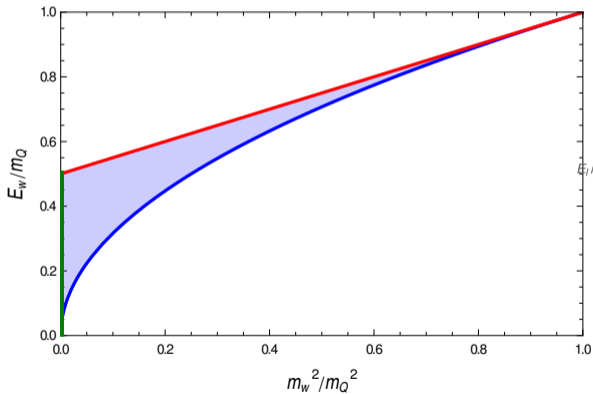


$$\begin{aligned} \mathcal{W}_{Q2q}^{\mu\nu}(p, k) = & W_1 g^{\mu\nu} + W_2 p^{\mu} p^{\nu} + W_3 k^{\mu} k^{\nu} \\ & + W_4 (p^{\mu} k^{\nu} + k^{\mu} p^{\nu}) + W_5 i \epsilon^{\mu\nu\rho\sigma} p_{\rho} k_{\sigma}, \end{aligned}$$

Selection Criteria: the **cut diagrams** of t -quark self-energy function with exactly one (**cut**) W propagator interacting with the *external* t -quark plus (up to 3) QCD loops

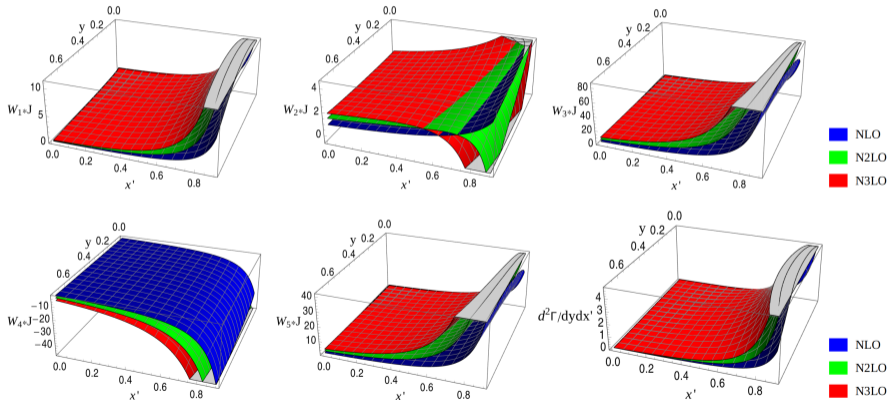
Phase-space for Structure functions $W_i(k^2, E_w)$

Triple-differential dist.: $\left. \frac{d^3\Gamma_{sl}}{dE_l dE_w dk^2} \right|_{m_W \gg m_q} \propto -k^2 W_1/m_q^2 - 1/2(4E_l^2 - 4E_l E_w + k^2) W_3 + (2E_l - E_w) k^2 W_5/m_q$



Heavy-to-light Structure Functions @ N3LO

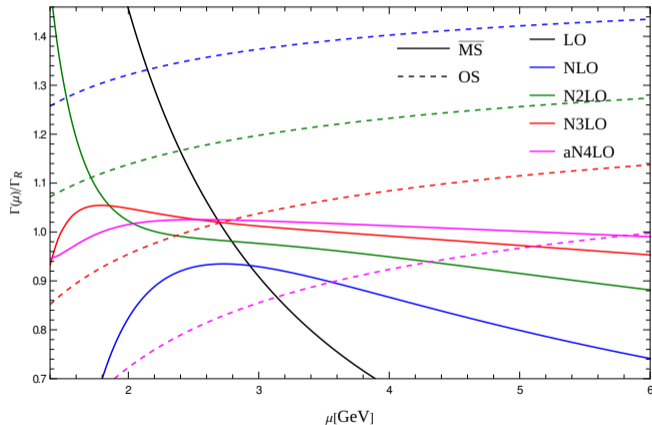
- First complete $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$ corrections to all H2L-structure functions underlying the *triple-differential distribution* $\frac{d^3\Gamma_{sl}}{dE_l dE_w dk^2}$, hence all precision observables for (inclusive) D and (charmless) B meson decays [LC, X.Chen, X.Guan, Y.Q. Ma] [2602.11537; 2602.11879]



- Our results for W_1, W_3 and W_5 at $\mathcal{O}(\alpha_s^2)$ agree with the recent fit [Broggio, Capdevila, Ferroglia, Gambino, 26]

Results for Free b -quark Decay $\Gamma_{\text{sl}}^{b \rightarrow u}$ in $\overline{\text{MS}}$ scheme

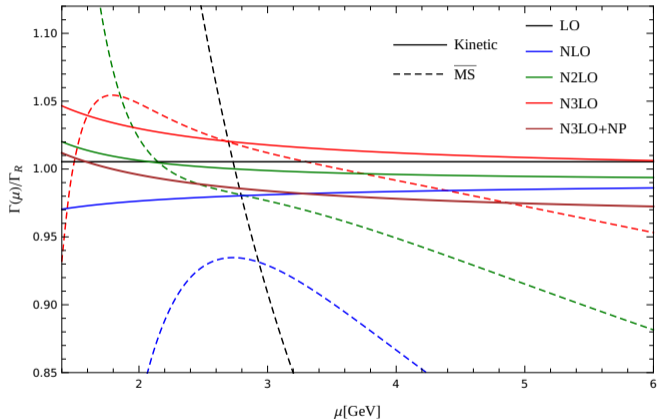
$$\Gamma_{\text{sl}}^{b \rightarrow u} \equiv \Gamma_0(m_b) \mathbf{C}_p = \bar{\Gamma}_0 \left(1 + 0.3036075 + 0.1365820 + 0.06841766 + 0.034184 \right)$$



- Every one more perturbative order higher in the $\overline{\text{MS}}$ result, the term is reduced roughly by $1/2$
- However, the scale uncertainty of $\Gamma_{\text{sl}}^{b \rightarrow u}$ at N3LO is still $[+6\%, -8\%]$ (!) for $\mu/\bar{m}_b \in [1/2, 2]$

Result for $\Gamma(B \rightarrow X_u \ell \bar{\nu}_\ell)$ in Kinetic-mass Scheme

In Heavy Quark Expansion theory $\Gamma(B \rightarrow X_u \ell \bar{\nu}_\ell) = \Gamma_0(m_b) \left[C_P \left(1 - \frac{\mu_\pi^2 - \mu_G^2}{2m_b^2} \right) - 2 \frac{\mu_G^2}{m_b^2} + \mathcal{O}(\Lambda_{\text{QCD}}^3/m_b^3) \right]$.

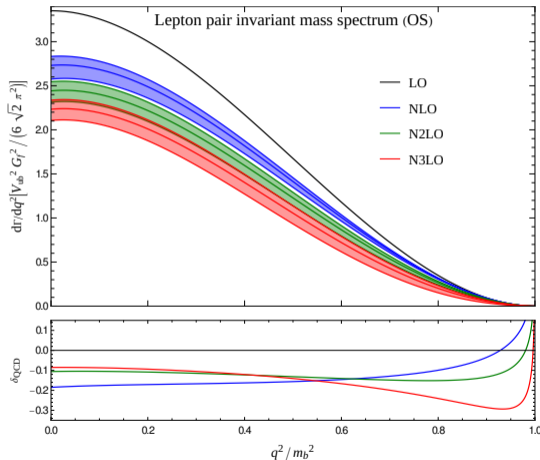


We obtain the **most accurate theoretical prediction for the inclusive $B \rightarrow X_u \ell \bar{\nu}_\ell$** (kine-scheme):

$$\Gamma(B \rightarrow X_u \ell \bar{\nu}_\ell) = \frac{|V_{ub}|^2}{|3.82 \times 10^{-3}|^2} (6.53 \pm 0.12 \pm 0.13 \pm 0.03) \times 10^{-16} \text{ GeV}.$$

Lepton-pair Invariant-mass Spectrum $d\Gamma_{sl}^{b \rightarrow u} / dq^2$

In the OS scheme, δ_{QCD} is large and does not converge quickly.



- $\mathcal{O}(\alpha_s^3)$ **Leading-color integrated** form agrees with [Chen, Li, Li, Wang, Wang, Wu 23] .
- $\mathcal{O}(\alpha_s^3)$ **integrated** fermionic part agrees with a recent evaluation [Fael, Usovitsch 23] .

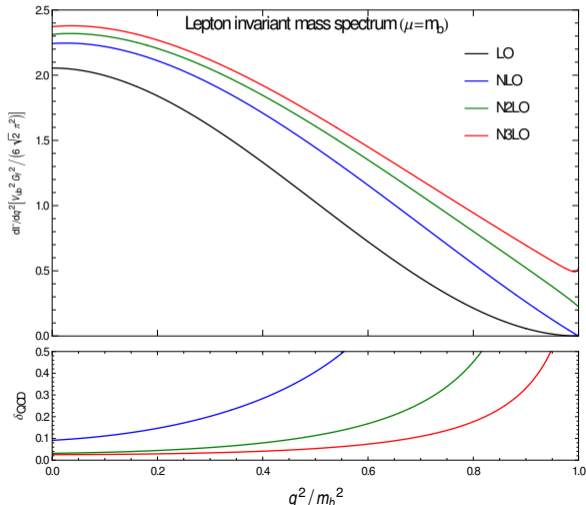
Subtlety with Re-expanding non-inclusive Observables

Rewrite $\Gamma_{sl}^{b \rightarrow u} = \int_0^{m^2} f(q^2, m^2) dq^2$ using an alternative \bar{m} is much more involved than changing mass-renormalization schemes for internal particles

- Consistency for the integral:

$$\int_0^{m^2} f(q^2, m^2) dq^2 = \int_0^{\bar{m}^2 + dm} f(q^2, \bar{m}^2 + dm) dq^2$$

- q^2 -range bounded by $\bar{m}(\mu)$ (phase-space expanded too)
- Not (necess.) 0 at **new** ends (non-0 high-order derivatives)
- Dirac- δ type **boundary**(!) term (histogramming in bins)



A scheme/scale-invariant trace-anomaly-subtracted σ -mass

Take-home messages: [LC, Li, Niggetiedt 25; LC, Zhao 25]

- A novel **diagrammatic proof of the identity**

$$\langle \vec{p} | -\frac{\epsilon}{2} [F_{\rho\sigma}^a F^{a\rho\sigma}]_B + \sum_f [m_f \bar{\psi}_f \psi_f]_B | \vec{p} \rangle = \bar{u}(\vec{p}) m_{\text{os}} u(\vec{p})$$

to **any loops** in perturbative gauge theories (and all orders in ϵ) **without** any ref. to pre-laid *operator* renormalization conditions.

- A **discovery**: **Leading IR-renormalon** div. in m_{os} of massive quarks resides entirely in the **EMT-trace anomaly** contribution!
- A trace-anomaly-subtracted **σ -mass definition** for heavy quarks, scheme/scale-invariant and free from the leading IR-renorm. ambiguity

- Portions Z_{TA} of the trace-anomaly contribution to m_{os} and m_σ of elementary fermions: [LC, Li, Niggetiedt 25]

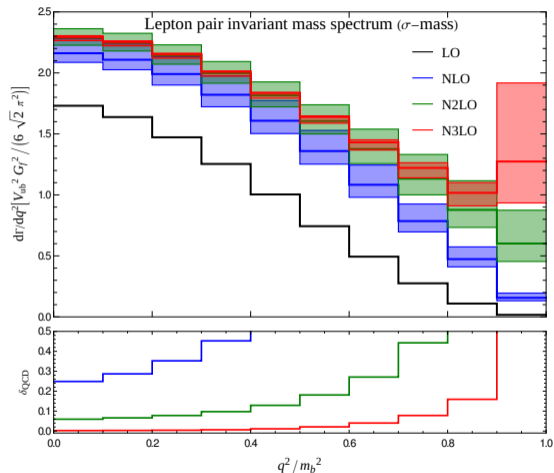
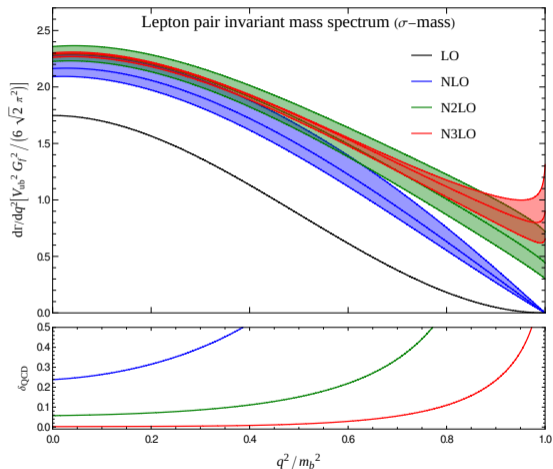
	electron	t -quark	b -quark	c -quark
Z_{TA}	0.347%	7.9%	20.4%	34.3%
m_σ	0.509 MeV	159.0 GeV	3.96 GeV	1.17 GeV

- ▶ Z_{TA} increase for lighter heavy quarks
- A **formula**: [LC, Zhao, arXiv:2509.10786]

$$Z_\sigma = \frac{m_\sigma}{m_{\text{os}}} = \frac{1 + 2\beta \frac{\partial \ln(C_{\overline{m}}(\alpha_s, \mu/\overline{m}))}{\partial \ln(\alpha_s)}}{1 - 2\gamma_m},$$

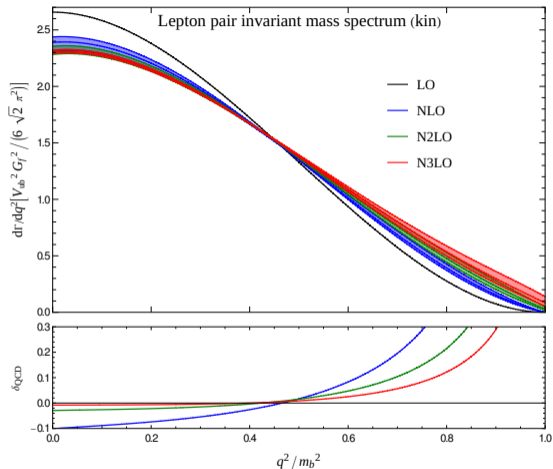
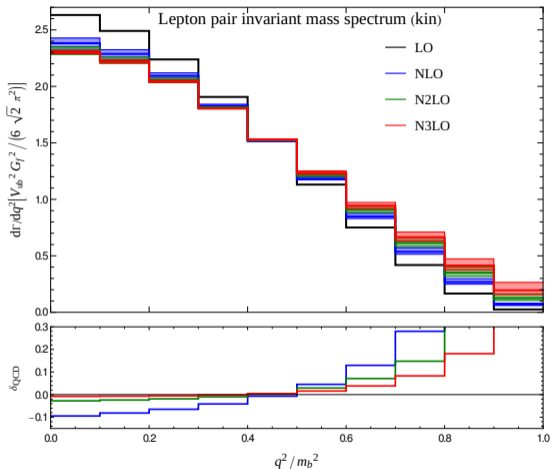
i.t.o. AMDs of α_s and \overline{m} and the pole-to- $\overline{\text{MS}}$ ratio $C_{\overline{m}}$, **useful for deriving 5-loop Z_σ with 4-loop $C_{\overline{m}}$.**

$d\Gamma_{sl}^{b \rightarrow u} / dq^2$ in σ -mass Scheme



- The curve at N3LO does **not** vanish at $q^2 \rightarrow (m_b^\sigma)^2$, even **without** the boundary-term included in the continuous plot.

$d\Gamma_{sl}^{b \rightarrow u} / dq^2$ in Kinetic-mass Scheme



- δ_{QCD} exhibits the usual expected (regular) behavior
- The particular (**crossing**) **pattern** helps to “understand” the puzzle (scale-uncertainty of the total inclusive result not reduced from N2LO to N3LO)
- The boundary-term is not included in the continuous plot (changing the last of 10-bin histograms by $3 \sim 5\%$)

A Tentative Application to $\Gamma_{\text{sl}}^{c \rightarrow q}$ (3+1)

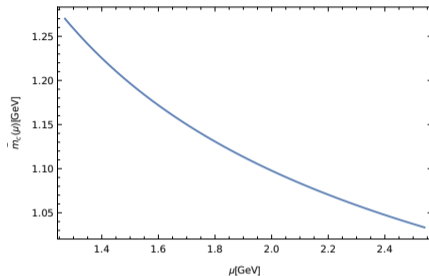
- Large QCD coupling at \bar{m}_c :

$$\alpha_s^{[4]}(\bar{m}_c) \approx 0.38 \approx 2 \alpha_s^{[5]}(\bar{m}_b) \approx 4 \alpha_s^{[6]}(m_t)$$

- Significant running-mass effect:

$$\bar{m}_c(\bar{m}_c) = 1.27 \text{ GeV}; \quad \bar{m}_c(\mu = 2 \text{ GeV}) = 1.10 \text{ GeV}$$

$$(\bar{m}_c + \Delta m)^5 \approx \bar{m}_c^5 \left(1 + 5 \frac{\Delta m}{\bar{m}_c} + \dots \right)$$



To get an idea of the sizes of the **perturbative QCD correction factors**:

$$\tilde{\Gamma}_0 \bar{m}_c^5(2 \text{ GeV}) \left(1 + 0.832 + 0.710 + 0.606 \right) = 5.01 \tilde{\Gamma}_0$$

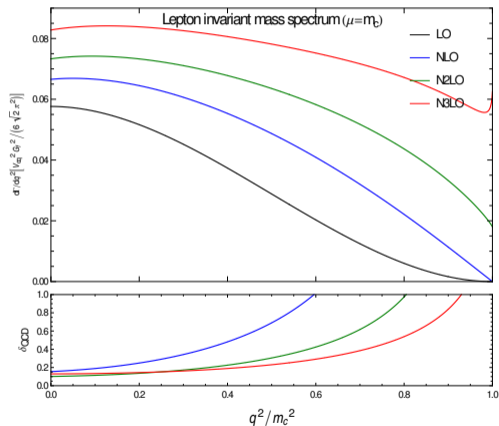
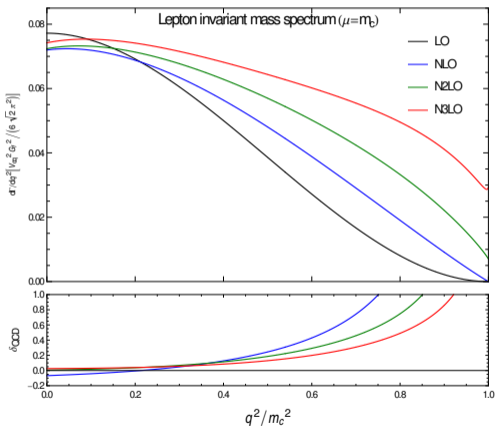
$$\tilde{\Gamma}_0 \bar{m}_c^5(2 \times 1.27 \text{ GeV}) \left(1 + 0.953 + 0.850 + 0.725 \right) = 4.16 \tilde{\Gamma}_0$$

At the ref. scale $\mu = 2 \text{ GeV}$ (typically chosen for \bar{m}_c), the $\mathcal{O}(\alpha_s^3)$ correction could increase the perturbative NNLO correction by $\sim 24\%$.

The **non-perturbative** corrections (to higher twists) are important

$d\Gamma_{sl}^{c \rightarrow q} / dq^2$ in $\overline{\text{MS}}$ and Kinetic Scheme

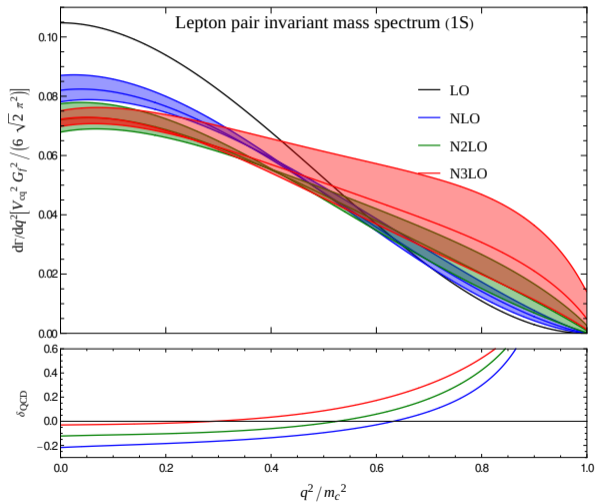
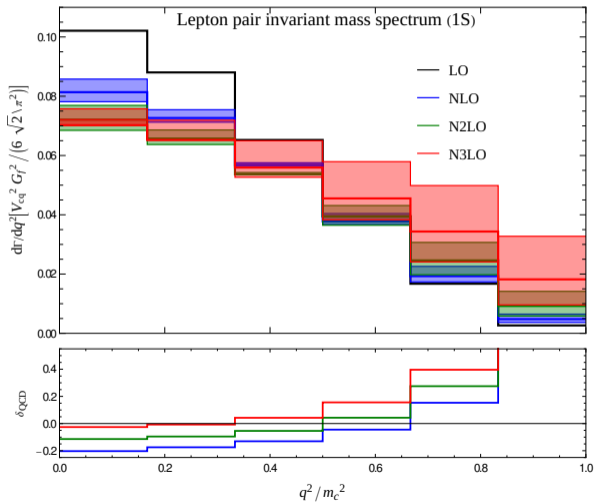
Neither $\overline{\text{MS}}$ nor kinetic mass, **unfortunately**, works well for $d\Gamma_{sl}^{c \rightarrow q} / dq^2$



The left is for the result with kinetic-mass, slightly better than the right with $\overline{\text{MS}}$ -mass, obtained using respectively standard parameter-settings employed in literature.

$d\Gamma_{sl}^{c \rightarrow q} / dq^2$ in 1S Scheme

But, there is still **hope!**



(To estimate the uncertainty, we used $[1 \text{ GeV}, 2 m_{1S}]$ as the usual choice $\mu = m_{1S}/2$ is too low)

$\Gamma_{\text{sl}}^{D_i}$ and e -energy Moments in 1S Scheme

[Shao, Huang, Qin 25 ; K. K. Shao, H. L. Feng, X. Y. Liu, Q. Qin, L. Sun, F. S. Yu 25]

- $\Gamma_{\text{sl}}^{c \rightarrow q}$ and e -energy moments for a *free* c -quark decay in **MS** scheme:

$$\Gamma_{D_i} = \hat{\Gamma}_0 \sum_{q=d,s} |V_{cq}|^2 \bar{m}_c^5(\mu) \left[1 + \frac{\alpha_s}{\pi} (5\bar{\ell} + 4.2536) + \frac{\alpha_s^2}{\pi^2} \left(\frac{425}{24} \bar{\ell}^2 + 38.3935\bar{\ell} + 29.8447 \right) \right],$$

$$\langle E_e \rangle_{D_i} = \frac{\hat{\Gamma}_0}{\Gamma_{D_i}} \sum_{q=d,s} |V_{cq}|^2 \bar{m}_c^6(\mu) \left[\frac{3}{10} + \frac{\alpha_s}{\pi} \left(\frac{9}{5} \bar{\ell} + 1.64052 \right) + \frac{\alpha_s^2}{\pi^2} \left(\frac{291}{40} \bar{\ell}^2 + 16.2359\bar{\ell} + 12.7981 \right) \right]$$

- $\Gamma_{\text{sl}}^{c \rightarrow q}$ and e -energy moments for a *free* c -quark decay in **1S** scheme (much improved convergence):

$$\Gamma_{D_i} = \hat{\Gamma}_0 \sum_{q=d,s} |V_{cq}|^2 m_{c,1S}^5 \left[1 + \epsilon \left(\frac{10\alpha_s^2}{9} - \frac{2\pi\alpha_s}{3} + \frac{25\alpha_s}{6\pi} \right) \right. \\ \left. + \epsilon^2 (2.9473\ell_\alpha \alpha_s^3 - 0.50936\ell_{1s} \alpha_s^2 + 0.74074\alpha_s^4 + 3.1352\alpha_s^3 - 2.3752\alpha_s^2) \right],$$

$$\langle E_e \rangle_{D_i} = \frac{\hat{\Gamma}_0}{\Gamma_{D_i}} \sum_{q=d,s} |V_{cq}|^2 m_{c,1S}^6 \left[\frac{3}{10} + \epsilon \left(\frac{2\alpha_s^2}{5} - \frac{\pi\alpha_s}{5} + \frac{1093\alpha_s}{900\pi} \right) \right. \\ \left. + \epsilon^2 (-0.16031\ell_{1s} \alpha_s^2 + 1.061\ell_\alpha \alpha_s^3 + 0.31111\alpha_s^4 + 1.1136\alpha_s^3 - 0.78025\alpha_s^2) \right]$$

- **First Inclusive** $\Gamma_{\text{sl}}^{D_i}$ and e -energy moments in **1S** scheme @ **N3LO** in ϵ : [LC, Chen, Guan, Ma, 26]

$$\langle E_e^0 \rangle|_{\text{unn.}} = 8.947 - 1.187 \epsilon - 0.447 \epsilon^2 - 0.051 \epsilon^3 + \mathcal{O}(\epsilon^4)$$

$$\langle E_e^1 \rangle|_{\text{unn.}} = 4.160 - 0.499 \epsilon - 0.131 \epsilon^2 + 0.086 \epsilon^3 + \mathcal{O}(\epsilon^4)$$

$$\langle E_e^2 \rangle|_{\text{unn.}} = 2.149 - 0.231 \epsilon - 0.0250 \epsilon^2 + 0.111 \epsilon^3 + \mathcal{O}(\epsilon^4),$$

Summary and Outlook

✓ First complete $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$ QCD corrections to all Heavy-to-light structure functions underlying the *triple-differential distribution* $\frac{d^3\Gamma_{sl}}{dE_l dE_w dk^2}$

✓ Combining our N³LO perturbative and non-perturbative information (relevant at %-level), an accurate theoretical prediction for the inclusive $B \rightarrow X_u \ell \bar{\nu}_\ell$ (using kinetic-mass scheme):

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✓ Our N³LO predictions for $d\Gamma_{sl}^{c \rightarrow q} / dq^2$ and inclusive moments of lepton-energy spectrum (using 1S-mass scheme) could be of great help for **extracting (non-perturbative) HQE-parameters in Charm sector**.

✓ The mixed QCD-EW corrections are needed for reliable sub-% accuracy [→See talks by Han and Li]

✓ Not only **cutting-edge precision results** but also **theoretically novel observations and fresh insights** are made regarding heavy-quark masses and decays!

Thank you for listening!

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