PRECISION FRONTIER OF PQCD WITH NNLOJET

第八届全国重味物理与量子色动力学研讨会

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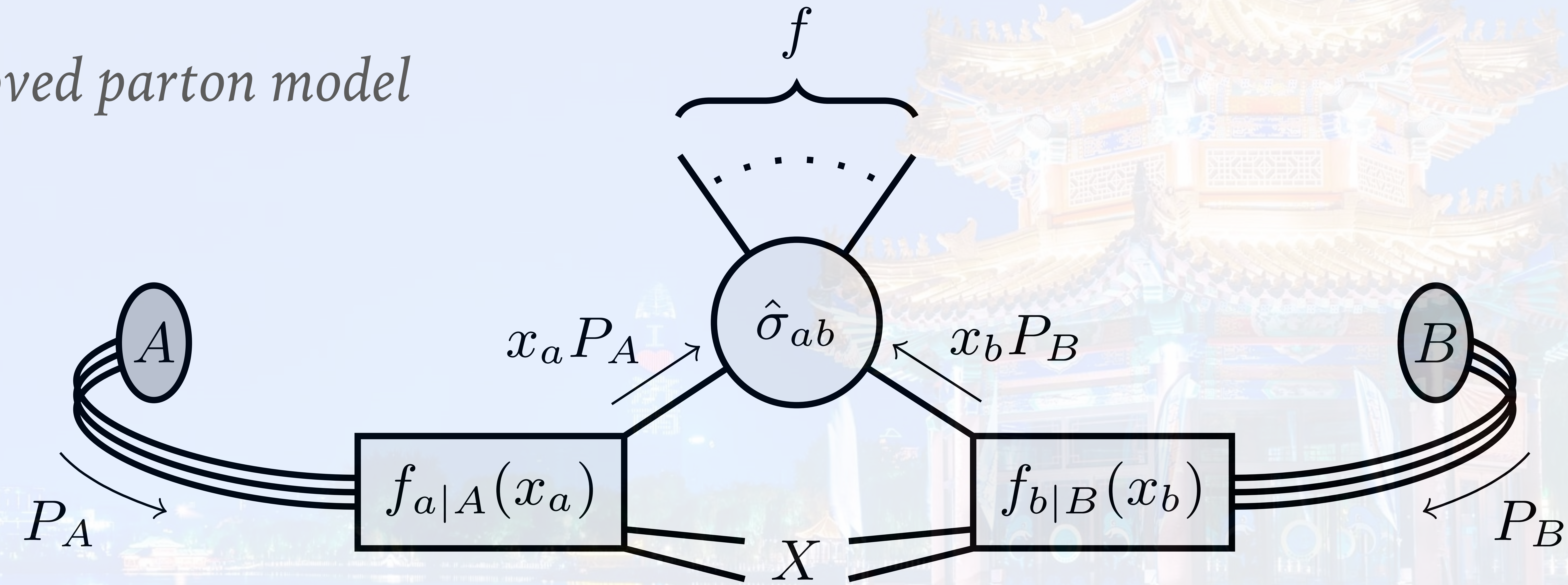
Shandong University

Chongqing, 27 April, 2026

Precision Predictions at Hadron Collider

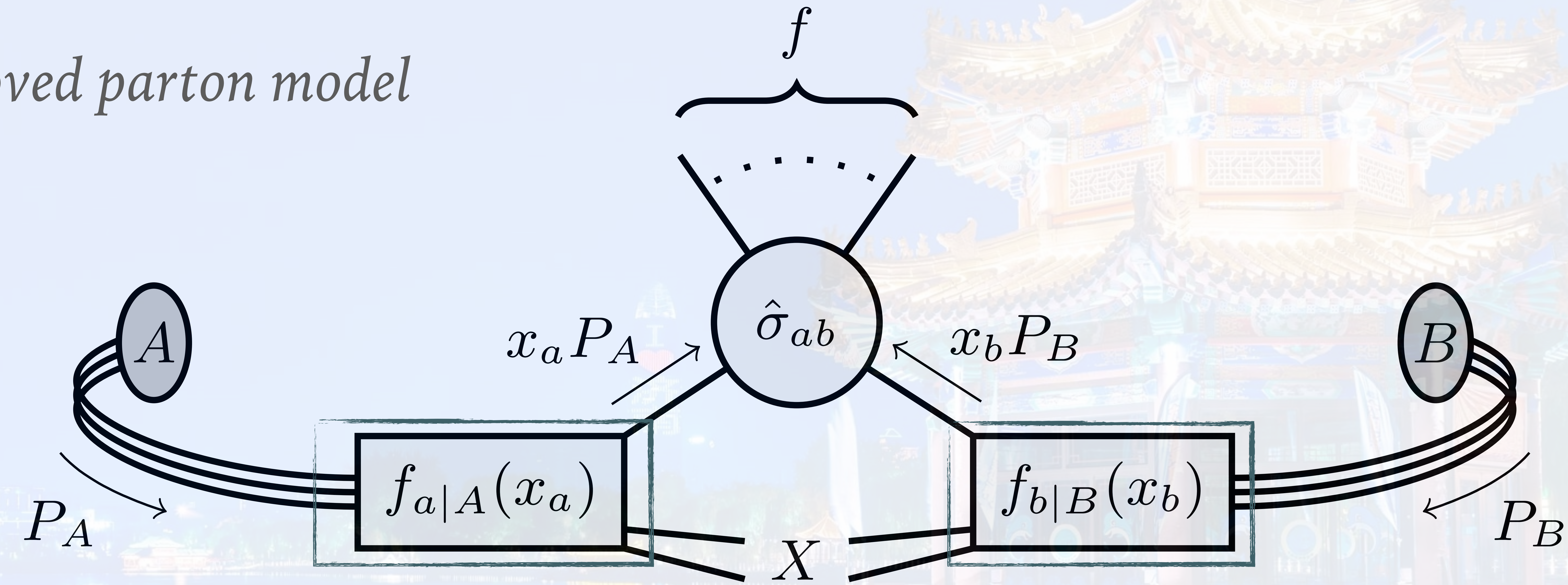
QCD improved parton model



$$\sigma_{AB} = \sum_{ab} \int_0^1 \mathrm{d}x_a \int_0^1 \mathrm{d}x_b \; f_{a|A}(x_a) \; f_{b|B}(x_b) \; \hat{\sigma}_{ab}(x_a, x_b) \; \big(1 + \mathcal{O}(\Lambda_{\mathrm{QCD}}/Q)\big)$$

Precision Predictions at Hadron Collider

QCD improved parton model



$$\sigma_{AB} = \sum_{ab} \int_0^1 \mathrm{d}x_a \int_0^1 \mathrm{d}x_b \; f_{a|A}(x_a)\; f_{b|B}(x_b)\; \hat{\sigma}_{ab}(x_a, x_b)\; \left(1+\mathcal{O}(\Lambda_{\mathrm{QCD}}/Q)\right)$$

Parton distribution functions
(Energy evolution from all exp.)
± 3-5 % at LHC energy

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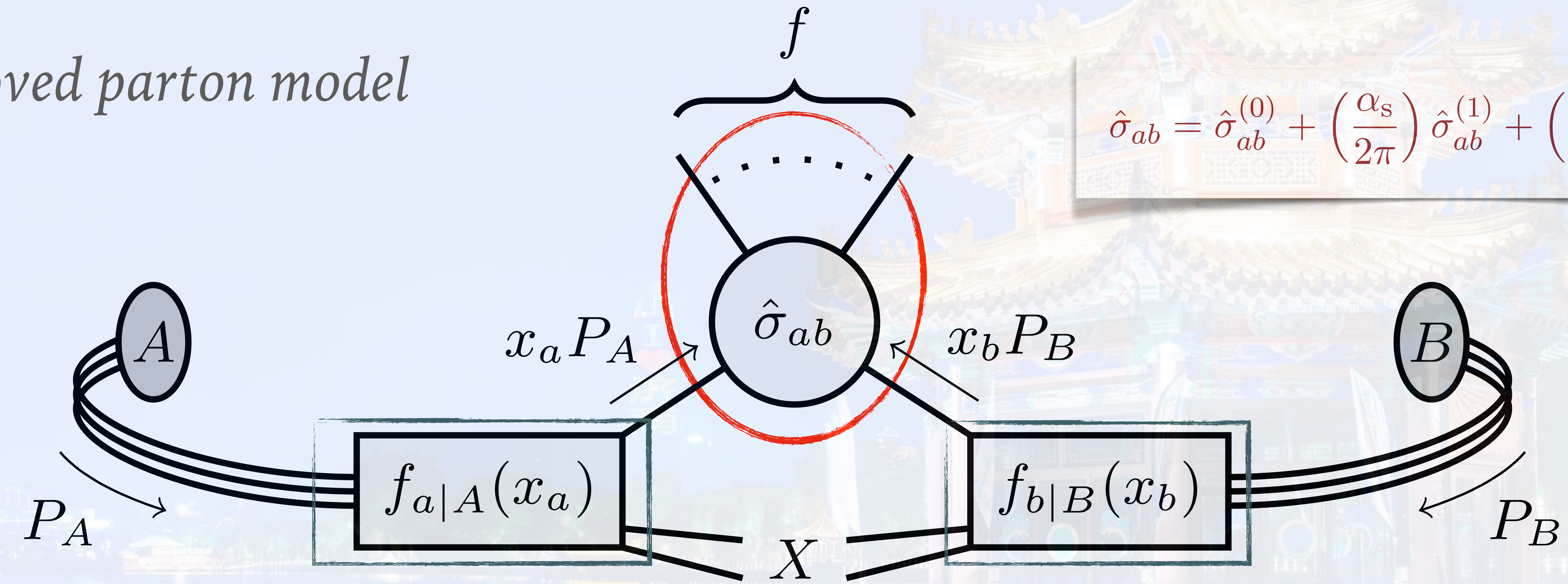


$$\hat{\sigma}_{ab} = \hat{\sigma}_{ab}^{(0)} + \left(\frac{\alpha_s}{2\pi}\right)\hat{\sigma}_{ab}^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 \hat{\sigma}_{ab}^{(2)} + \dots$$

$$\sigma_{AB} = \sum_{ab} \int_0^1 \mathrm{d}x_a \int_0^1 \mathrm{d}x_b \, f_{a|A}(x_a)\, f_{b|B}(x_b)\, \hat{\sigma}_{ab}(x_a, x_b)\, \left(1 + \mathcal{O}(\Lambda_{\mathrm{QCD}}/Q)\right)$$

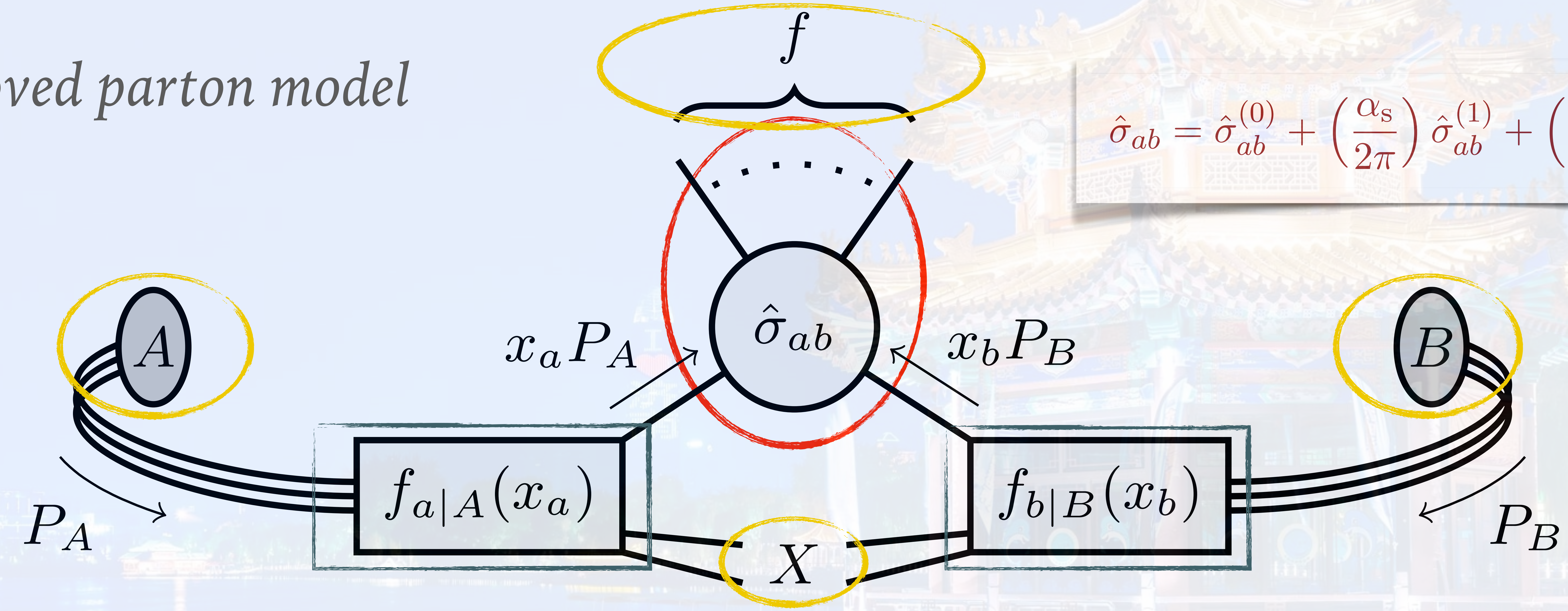
Parton distribution functions
(Energy evolution from all exp.)
± 3-5 % at LHC energy

Hard scattering
(Perturbative quantum field theory)
± 1~3 % level!

Precision Predictions at Hadron Collider

QCD improved parton model

$$\hat{\sigma}_{ab} = \hat{\sigma}^{(0)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right)\hat{\sigma}^{(1)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right)^2\hat{\sigma}^{(2)}_{ab} + \ldots$$



$$\sigma_{AB} = \sum_{ab}\int_0^1 \mathrm{d}x_a \int_0^1 \mathrm{d}x_b \, f_{a|A}(x_a)\, f_{b|B}(x_b)\, \hat{\sigma}_{ab}(x_a, x_b)\, \left(1+\mathcal{O}(\Lambda_{\mathrm{QCD}}/Q)\right)$$

Parton distribution functions
(Energy evolution from all exp.)
± 3-5 % at LHC energy

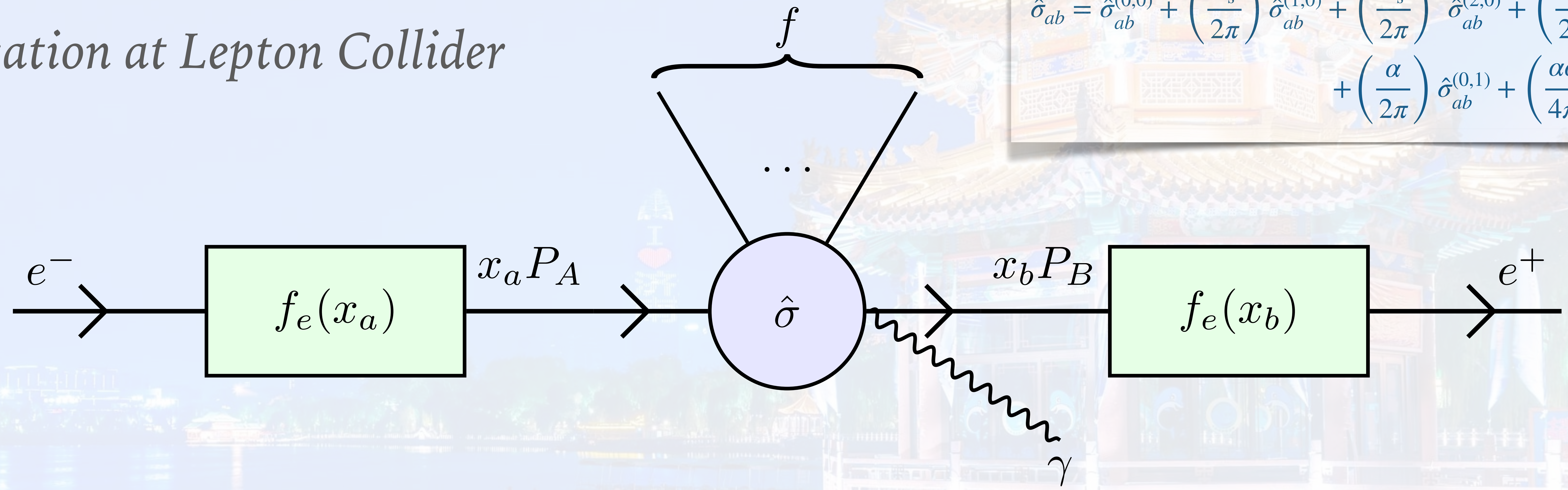
Hard scattering
(Perturbative quantum field theory)
± 1~3 % level!

non-perturbative effects
(Fragmentation, hadronisation)
± Λ QCD/ $\sqrt{\hat{s}}$

PRECISION PREDICTIONS AT LEPTON COLLIDER

Factorisation at Lepton Collider

$$\hat{\sigma}_{ab} = \hat{\sigma}^{(0,0)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right)\hat{\sigma}^{(1,0)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right)^2\hat{\sigma}^{(2,0)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right)^3\hat{\sigma}^{(3,0)}_{ab} + \cdots$$
$$+ \left(\frac{\alpha}{2\pi}\right)\hat{\sigma}^{(0,1)}_{ab} + \left(\frac{\alpha\alpha_s}{4\pi^2}\right)\hat{\sigma}^{(1,1)}_{ab} + \cdots$$



$$\sigma_{AB} = \sum_{ab}\int_0^1 \mathrm{d}x_a \int_0^1 \mathrm{d}x_b\, f_{a|A}(x_a)\, f_{b|B}(x_b)\, \hat{\sigma}_{ab}(x_a, x_b)\,(1+\mathcal{O}(\Lambda_{\mathrm{QCD}}/Q))$$

Electron PDF at NLO+NLL
(Determined by first principle)
$\pm$ 0.1~1 % at $\mathcal{O}(10^2)$ GeV

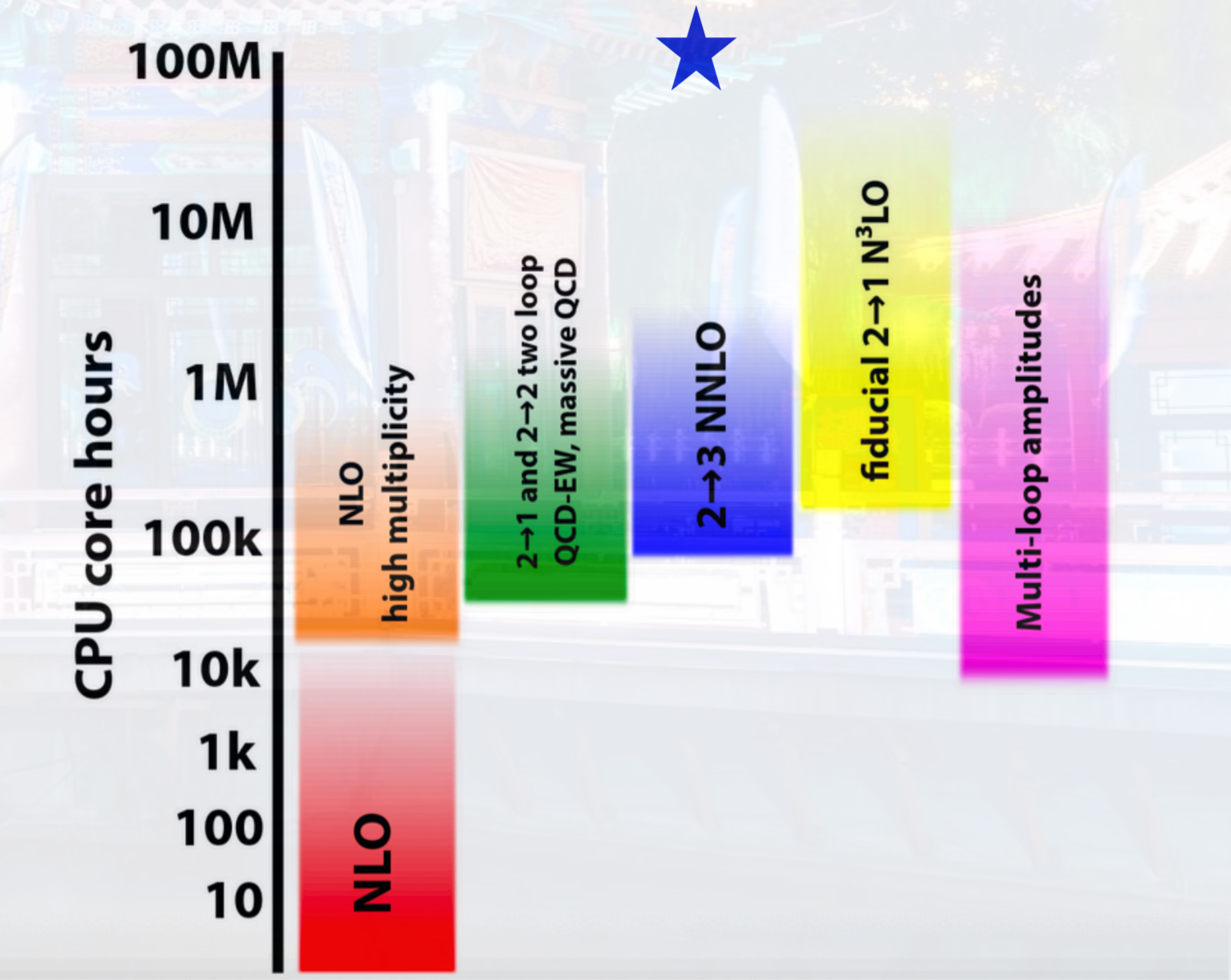
Hard scattering *(Perturbative QFT)*
$\pm$ 5~10 % for jets
$\pm$ 0.01~2 % for leptons

non-perturbative effects
(Fragmentation, hadronisation)
$\pm\, \Lambda/\sqrt{\hat{s}},\ \delta(\mathcal{O} - c_{\mathcal{O}}/\sqrt{\hat{s}})$

State-of-the-Art QCD Calculations @ NNLO

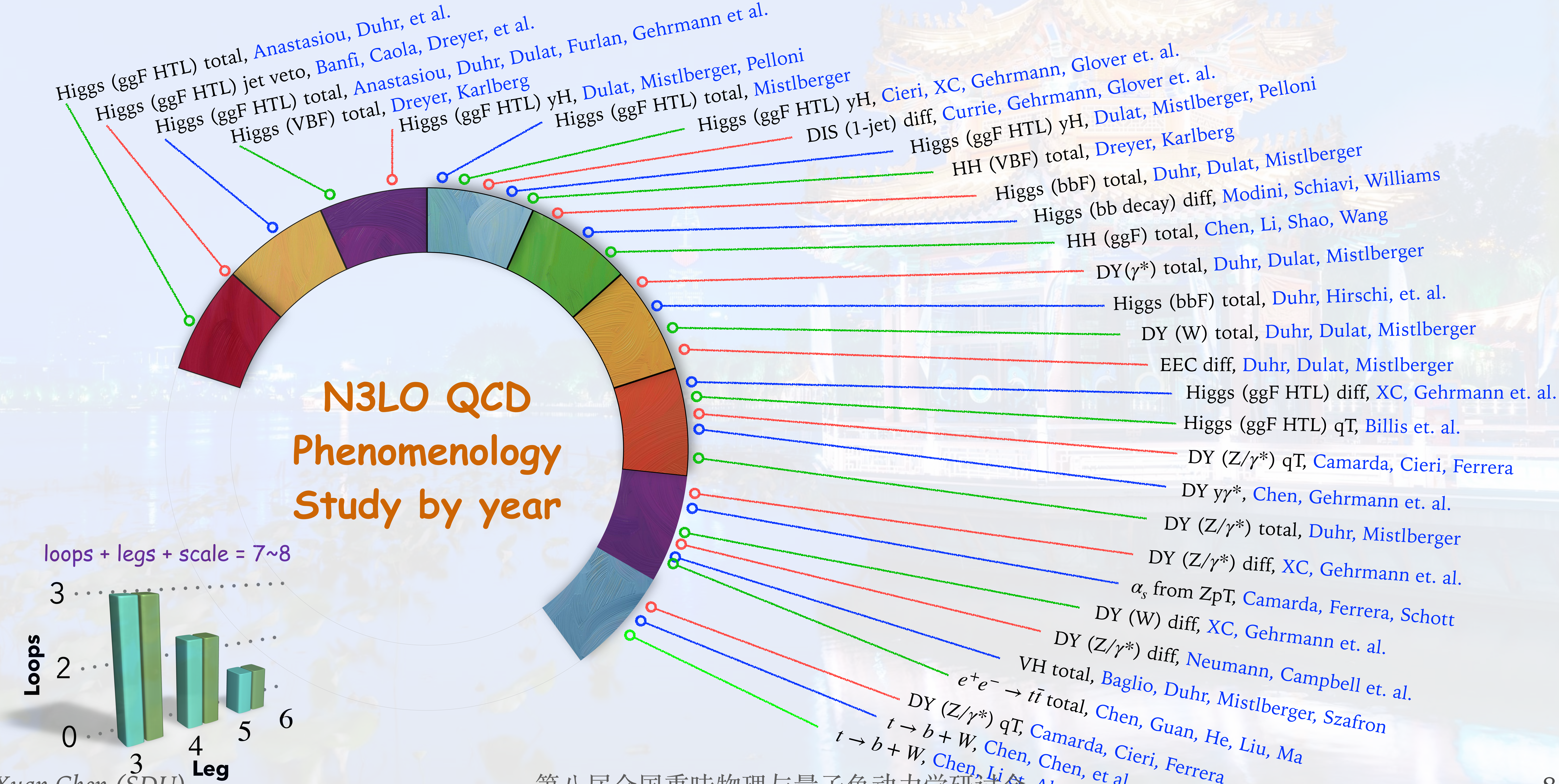
- NNLO QCD predictions for 2 → 2 processes (NNLO revolution since 2015)
  - Accomplished during past 10 years on case-by-case basis
  - As parton-level event generators (fully differential final state information)
  - Current frontier at NNLO 2 → 3
- Typical size of corrections and uncertainty
  - NLO corrections: 10~100%, uncertainty: 10~30%
  - NNLO corrections: 2~15%, uncertainty: 3~8%
  - expect N3LO to yield uncertainty at level of 1%
- So, is NNLO **solved**?
  - In principle yes: STRIPPER, given the relevant amplitudes and enough computational resources, the NNLO calculation is streamlined.
  - But:
    - Prohibitive computational cost (loop AMP, IR subtraction)
    - Missing cross-validation (many years between 1st and 2nd)
    - Still a long way to automated NNLO event generation

pp → jjj event shapes with STRIPPER

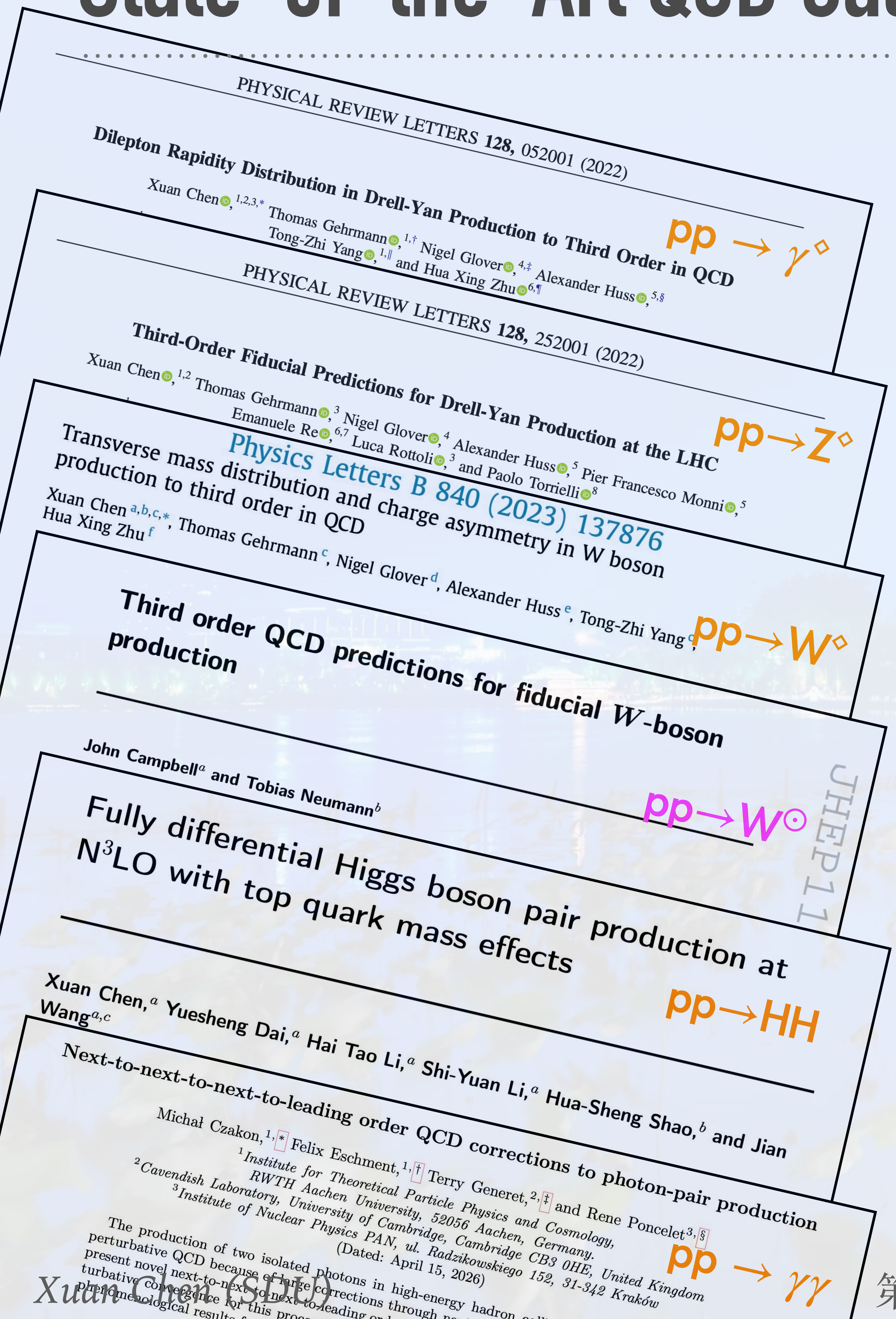


Snowmass White Paper, Comput. Softw. Big Sci. 6 (2022)

Perturbative QCD @ N3LO



State-of-the-Art QCD Calculations @ N3LO



➤ Several phenomenologically relevant results despite the extreme complexity.

➤ Available techniques are applicable to limited cases with high quality EXP data.

➤ New approaches must be developed for more complicated scattering.

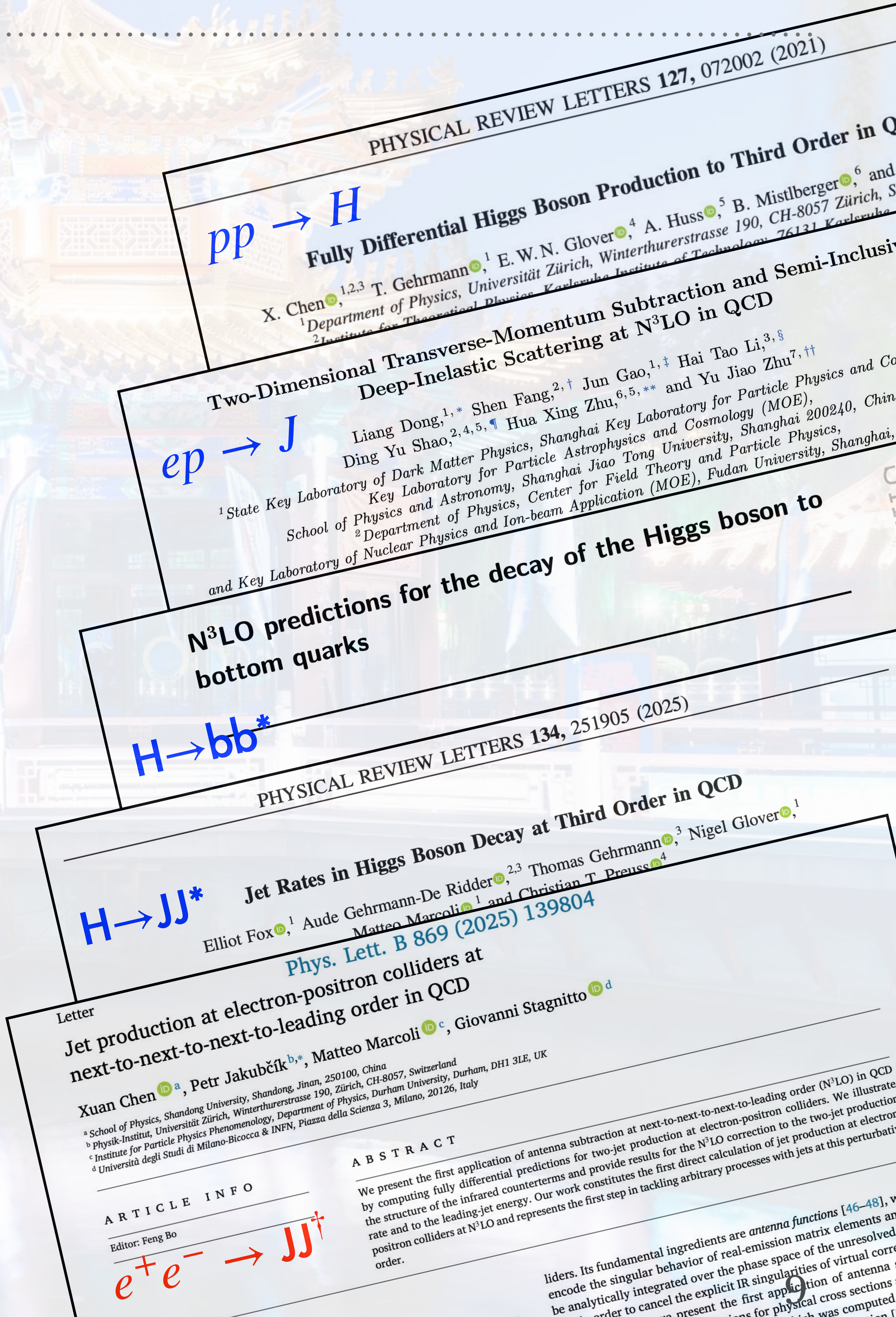
(10 M → X00 k CPU hours)

◇ qT slicing

⊙ τ slicing

* Projection-2-Born

† Antenna subtraction



NNLOJET: Parton Level Event Generator

A parton-level event generator for jet cross sections at NNLO QCD accuracy

About NNLOJET is a parton-level event generator for jet cross sections using the antenna subtraction method. It can be used to compute a large number of jet cross sections and related observables in e^+e^-, ep and pp collisions at next-to-next-to-leading order in QCD. NNLOJET contains routines for Monte Carlo phase-space integration, event handling and analysis.

Citation If you are using NNLOJET for a scientific paper, please cite:

A. Huss et al. (NNLOJET Collaboration)
NNLOJET: a parton-level event generator for jet cross sections at NNLO QCD accuracy
arXiv:2503.22804 [INSPIRE]

Please also cite the revelant references for each processes (as included in the .bib file which is automatically written when running NNLOJET through the automatic workflow)

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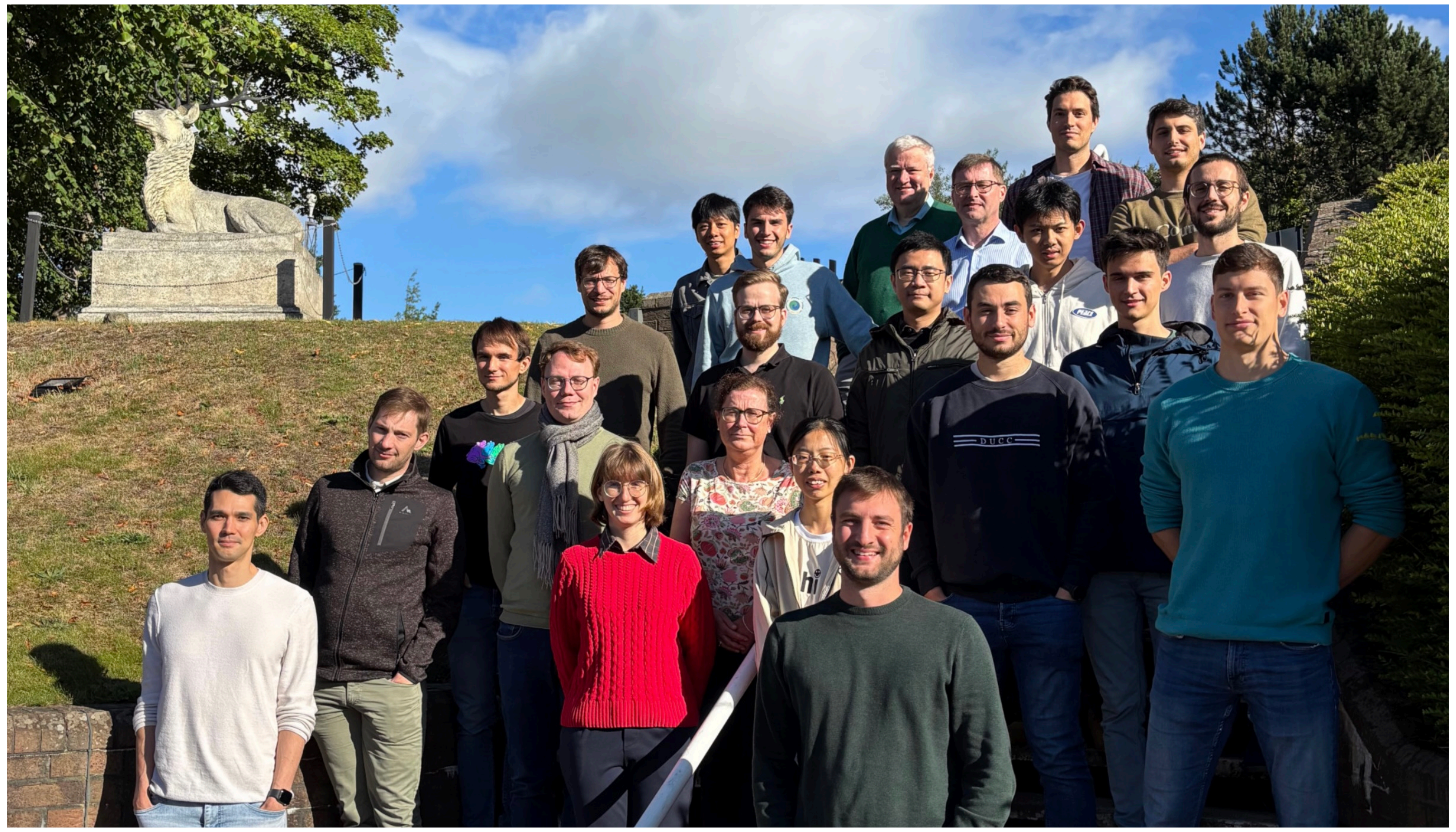
Contact Please send comments, questions and suggestions to nnlojet-support@cern.ch

https://nnlojet.hepforge.org/index.html



A.Huss, L.Bonino, O.Braun-White, S.Caletti, XC, J.Cruz-Martinez, J.Currie, Y,S, Dai, W.Feng, G.Fontana, E.Fox, R.Gauld, A.Gehrmann-De Ridder, T. Gehrmann, E.W.N.Glover, M.Höfer, P.Jakubcik, M.Jaquier, M.Löchner, F.Lorkowski, I.Majer, M.Marcoli, F. Merlotti, P.Meinzinger, J.Mo, T. Morgan, J.Niehues, J.Pires, C.Preuss, A.Rodriguez Gracia, K.Schönwald, R.Schürmann, V.Sotnikov, G.Stagnitto, H.S. Sun, D.Walker, J.Whitehead, T.Z.Yang, H.Zhang,

- NNLO parton level event generator
  - Based on antenna subtraction
- Provides infrastructure
  - Process management
  - Phase space, histogram routines
  - Validation and testing
  - Parallel computing (MPI) support
  - Typical runtimes: 60 k ~ 250 k core-hours

NNLOJET: Parton Level Event Generator

➤ Processes implemented:

e^+e^- scattering Jet production

- $e^+e^- \to$ 2jets
- $e^+e^- \to$ 3jets

***ep* scattering** Jet production

- ep $\to$ lepton + 1jet
- ep $\to$ lepton + 2jets

***pp* scattering** Jet production

- pp $\to$ 1jet + X
- pp $\to$ 2jets

Vector boson (+ jet) production

- pp $\to$ $(\gamma^* Z)$ + 0jet
- pp $\to$ $(\gamma^* Z)$ + 1jet
- pp $\to$ $W^\pm$ + 0jet
- pp $\to$ $W^\pm$ + 1jet

Photon (+ jet) production

- pp $\to \gamma$ + X
- pp $\to \gamma$ + 1jet
- pp $\to \gamma\gamma$

Higgs (+ jet) production

- pp $\to$ H + X
- pp $\to$ H + 1jet

➤ Open-source code release: NNLOJET v1.0.2
  - ➤ Analytic matrix elements and subtraction
  - ➤ Download from nnlojet.hepforge.org

➤ Runcard options:
  - ➤ Process/sub-process selection
  - ➤ Generic histogramming
  - ➤ Multi-run feature: e.g. jet radius
  - ➤ Example runcards for published studies

➤ Cluster workflow management: Dokan
  - ➤ Automated resource allocation
  - ➤ Works with slurm and htcondor (lxplus)
  - ➤ Combination of results, quality control

https://github.com/aykhuss/dokan

dokan (土管)

pypi v1.0.0 | python missing

A pipeline for automating the NNLOJET workflow

#	LO	R	V	RR	RV	VV
1	PRD A[0] D[1]	PRD A[0] D[1]	PRD A[0] D[1]	WRM A[0] D[4]	PRD A[1] D[0]	PRD A[0] D[1]
2	PRD A[0] D[1]	PRD A[1] D[0]	PRD A[0] D[1]	PRD A[0] D[1]	WRM A[1] D[3]	PRD A[0] D[1]
3	-	PRD A[0] D[1]	PRD A[0] D[1]	PRD A[0] D[0]	WRM A[1] D[3]	PRD A[0] D[1]
4	-	PRD A[0] D[1]	PRD A[0] D[1]	PRD A[0] D[1]	PRD A[0] D[1]	PRD A[0] D[1]
5	-	PRD A[0] D[1]	PRD A[0] D[1]	PRD A[0] D[0]	PRD A[0] D[1]	PRD A[0] D[1]
6	-	PRD A[0] D[0]	PRD A[0] D[1]	PRD A[0] D[1]	PRD A[0] D[1]	PRD A[0] D[1]
7	-	-	-	PRD A[0] D[0]	PRD A[0] D[1]	PRD A[0] D[1]
8	-	-	-	PRD A[1] D[0]	PRD A[0] D[1]	PRD A[0] D[1]
9	-	-	-	PRD A[0] D[0]	PRD A[1] D[0]	PRD A[0] D[1]
10	-	-	-	PRD A[1] D[0]	PRD A[0] D[1]	PRD A[0] D[1]
11	-	-	-	PRD A[0] D[0]	PRD A[0] D[1]	PRD A[0] D[1]
12	-	-	-	PRD A[0] D[1]	-	-
13	-	-	-	PRD A[0] D[0]	PRD A[0] D[1]	PRD A[0] D[1]
14	-	-	-	PRD A[0] D[0]	-	-
15	-	-	-	PRD A[0] D[1]	WRM A[1] D[3]	PRD A[0] D[1]
16	-	-	-	PRD A[1] D[0]	WRM A[1] D[3]	PRD A[0] D[1]
17	-	-	-	PRD A[0] D[0]	WRM A[1] D[3]	PRD A[0] D[1]
18	-	-	-	PRD A[1] D[0]	-	-
19	-	-	-	PRD A[0] D[1]	-	-
20	-	-	-	PRD A[0] D[1]	-	-
21	-	-	-	WRM A[1] D[3]	-	PRD A[0] D[1]
22	-	-	-	PRD A[0] D[1]	-	PRD A[0] D[1]
23	-	-	-	PRD A[1] D[0]	-	PRD A[0] D[1]
24	-	-	-	WRM A[1] D[3]	-	PRD A[0] D[1]
25	-	-	-	WRM A[1] D[3]	-	-
26	-	-	-	PRD A[1] D[0]	-	-
27	-	-	-	PRD A[0] D[1]	-	-

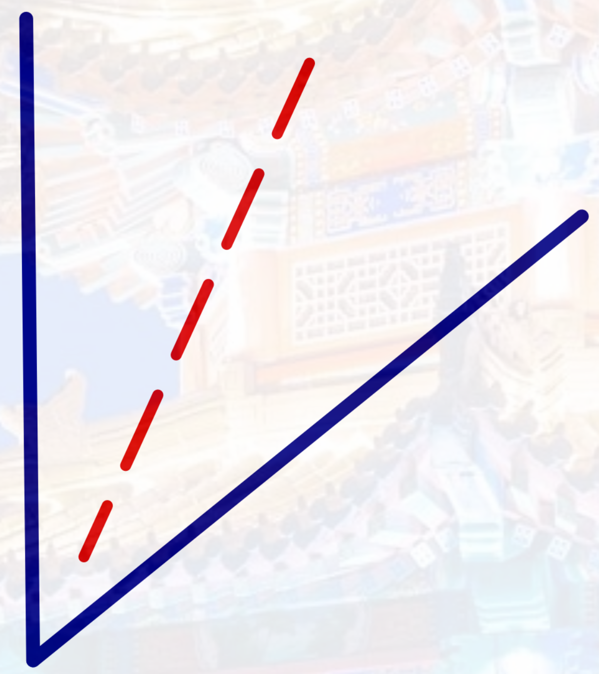
Development of Antenna Subtraction Method

Based on Marcoli's slide @ Loop Summit 2

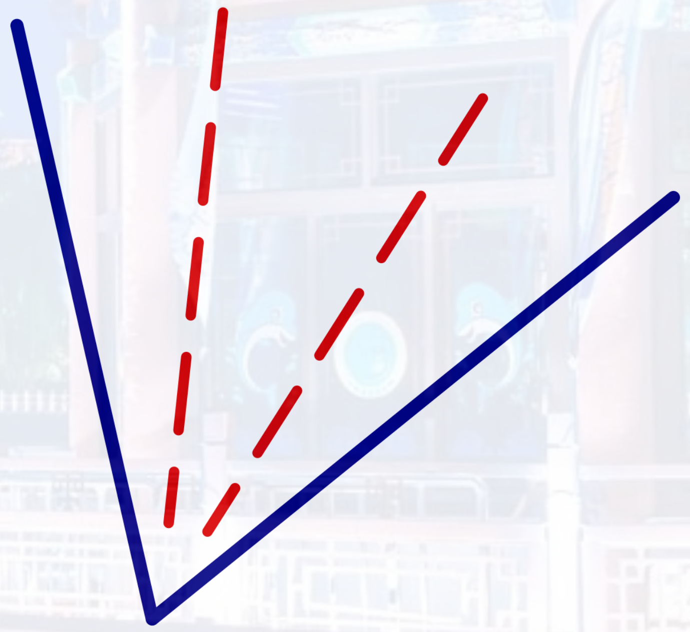
Successfully applied at NNLO to a variety of processes within the ***NNLOJET*** Monte Carlo framework

Antenna Subtraction @ N3LO

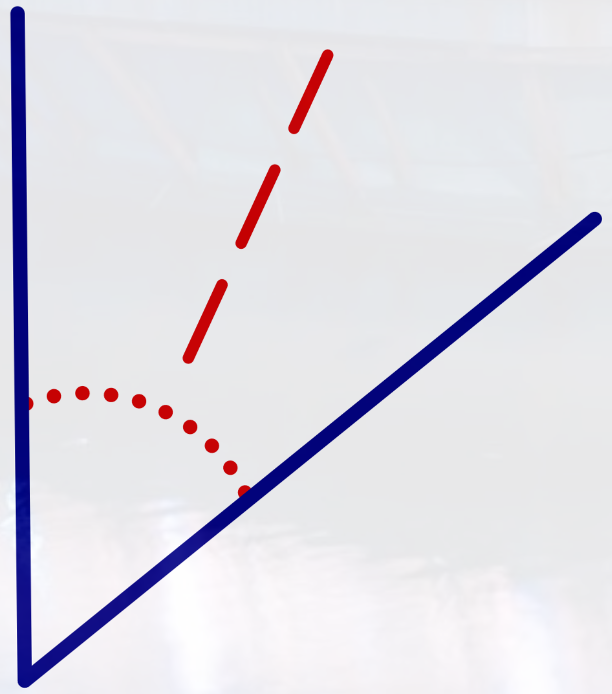
	NLO	NNLO
RRR	$X_3^0 \mathcal{M}_{h+2}^0$	$(X_4^0 - X_3^0 X_3^0)\mathcal{M}_{h+1}^0$
RRV	$\mathcal{X}_3^0 \mathcal{M}_{h+2}^0$	$(X_3^1 - \mathcal{X}_3^0 X_3^0)\mathcal{M}_{h+1}^0$ $+ X_3^0 \mathcal{M}_{h+1}^1$
RVV		$(\mathcal{X}_4^0 + \mathcal{X}_3^1 - \mathcal{X}_3^0 \mathcal{X}_3^0)\mathcal{M}_{h+1}^0$ $+ \mathcal{X}_3^0 \mathcal{M}_{h+1}^1$
VVV	• Reduced Matrix Elements $\mathcal{M}_{hard\ partons}^{loops}$ • Unintegrated Antenna $X_{emissions\ +2}^{loops}$ • Integrated Antenna $\mathcal{X}_{emissions\ +2}^{loops}$	



$$X_3^0 = \frac{|\mathcal{M}_3^0(i,j,k)|^2}{|\mathcal{M}_2^0(\tilde{I},\tilde{K})|^2}$$



$$X_4^0 = \frac{|\mathcal{M}_4^0(i,j,k,l)|^2}{|\mathcal{M}_2^0(\widetilde{ijk},\widetilde{jkl})|^2}$$



$$X_3^1 = \frac{|\mathcal{M}_3^1(i,j,k)|^2}{|\mathcal{M}_2^0(\tilde{I},\tilde{K})|^2} - X_3^0 \frac{|\mathcal{M}_2^1(\tilde{I},\tilde{K})|^2}{|\mathcal{M}_2^0(\tilde{I},\tilde{K})|^2}$$

Antenna Subtraction @ N3LO

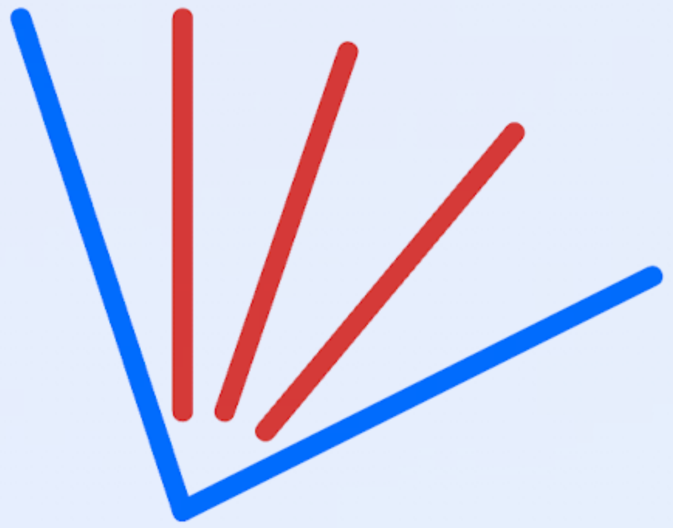
	NLO	NNLO	N3LO
RRR	$X_3^0\mathcal{M}^0_{h+2}$	$(X_4^0 - X_3^0X_3^0)\mathcal{M}^0_{h+1}$	$(X_5^0 - X_4^0X_3^0 - X_3^0X_4^0 + X_3^0X_3^0X_3^0)\mathcal{M}^0_h$
RRV	$\mathcal{X}_3^0\mathcal{M}^0_{h+2}$	$(X_3^1 - \mathcal{X}_3^0X_3^0)\mathcal{M}^0_{h+1}$ $+ X_3^0\mathcal{M}^1_{h+1}$	$(X_4^1 - X_3^1X_3^0 + \mathcal{X}_3^0X_3^0X_3^0 + \mathcal{X}_3^0X_4^0)\mathcal{M}^0_h$ $+ (X_4^0 + X_3^0X_3^0)\mathcal{M}^1_{h+1}$
RVV		$(\mathcal{X}_4^0 + \mathcal{X}_3^1 - \mathcal{X}_3^0\mathcal{X}_3^0)\mathcal{M}^0_{h+1}$ $+ \mathcal{X}_3^0\mathcal{M}^1_{h+1}$	$(X_3^2 + \mathcal{X}_3^1X_3^0 + \mathcal{X}_3^0X_3^1 + \mathcal{X}_4^0X_3^0 + \mathcal{X}_3^0X_3^0X_3^0)\mathcal{M}^0_h$ $+(-X_3^1 + \mathcal{X}_3^0X_3^0)\mathcal{M}^1_h$ $+ X_3^0\mathcal{M}^2_h$
VVV	• Reduced Matrix Elements $\mathcal{M}^{loops}_{hard\ partons}$ • Unintegrated Antenna $X^{loops}_{emissions\ +2}$ • Integrated Antenna $\mathcal{X}^{loops}_{emissions\ +2}$		$(\mathcal{X}_5^0 + \mathcal{X}_4^1 + \mathcal{X}_3^2 + \mathcal{X}_4^0\mathcal{X}_3^0 + \mathcal{X}_3^1\mathcal{X}_3^0 + \mathcal{X}_3^0\mathcal{X}_3^0\mathcal{X}_3^0)\mathcal{M}^0_h$ $(+\mathcal{X}_4^0 + \mathcal{X}_3^1 + \mathcal{X}_3^0\mathcal{X}_3^0)\mathcal{M}^1_h$ $+ \mathcal{X}_3^0\mathcal{M}^2_h$

Antenna Subtraction @ N3LO

Topology of X_5^0, X_4^1, X_3^2 antenna functions:

- Build from scattering matrix elements
- Capture genuine N3LO QCD divergences
- Retain colour connection
- Multiple unresolved limits in one antenna

X_5^0 :



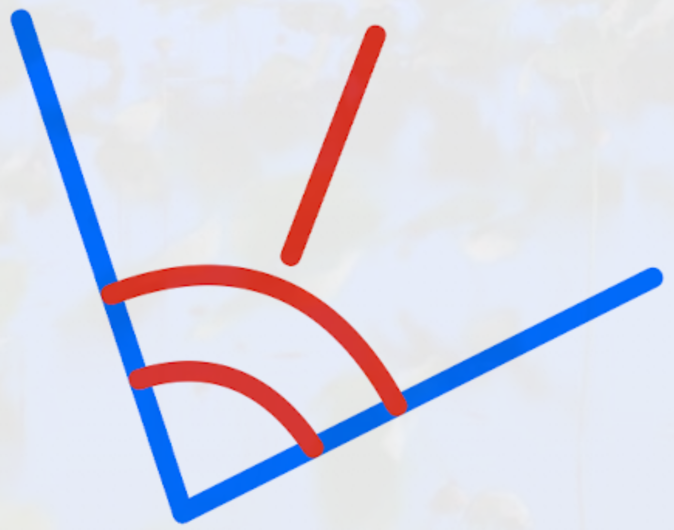
$= \mathcal{M}_5^0 / \mathcal{M}_2^0$

X_4^1 :

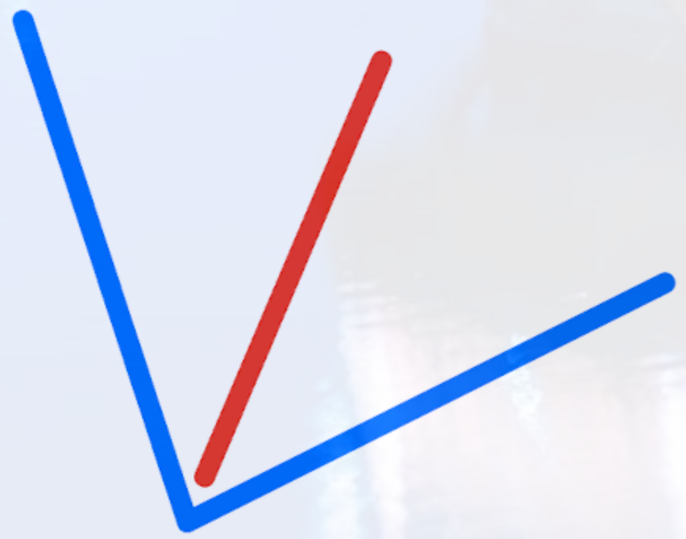
$= \Big(\mathcal{M}_4^1 -$

$\mathcal{M}_2^1 \Big) / \mathcal{M}_2^0$

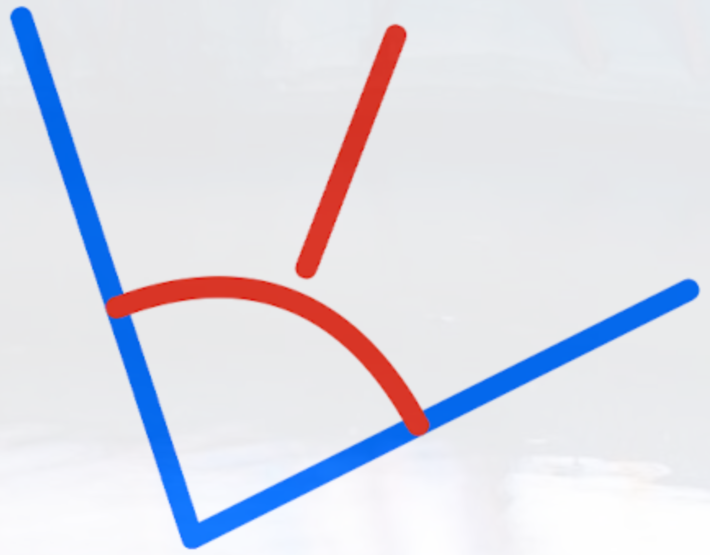
X_3^2 :



$= \Big(\mathcal{M}_3^2 -$



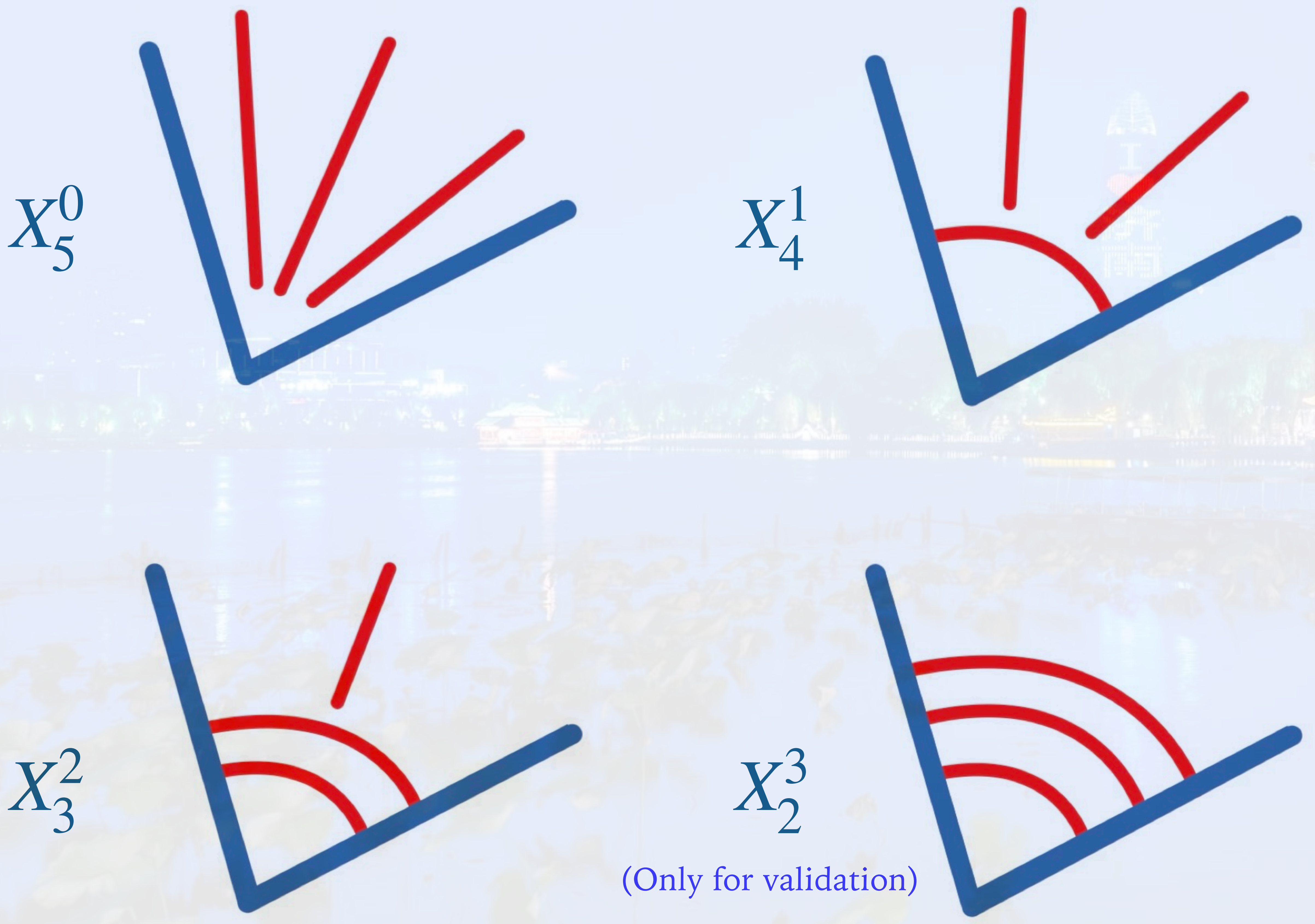
$\mathcal{M}_2^2 -$



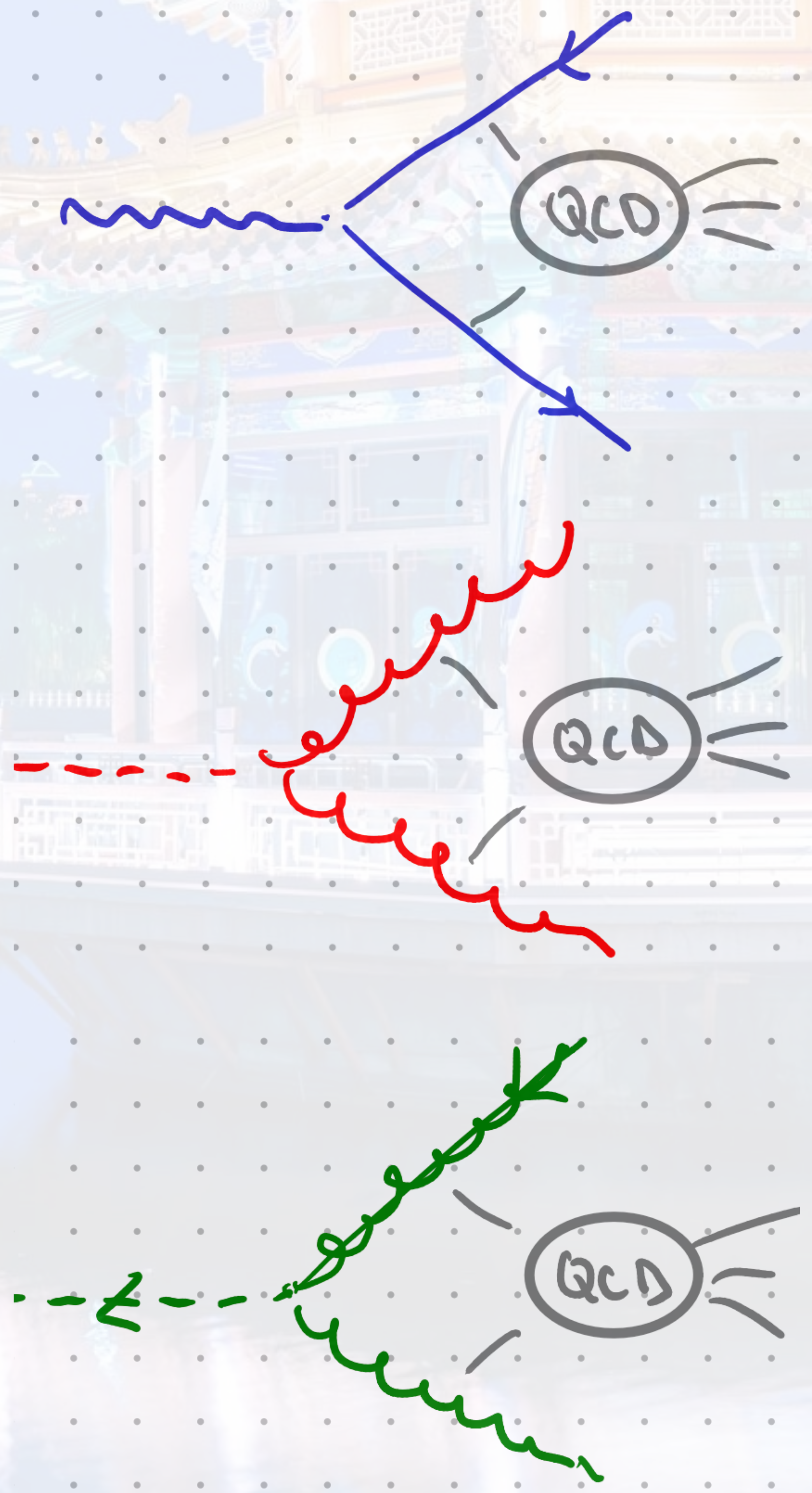
$\mathcal{M}_2^1 \Big) / \mathcal{M}_2^0$

Antenna Subtraction @ N3LO

Topology of X_5^0, X_4^1, X_3^2 antenna functions:



Integration of X_5^0, X_4^1, X_3^2 finished for all final states:



$\gamma^* \to q\bar{q}$

Jakubcik, Marcoli, Stagnitto

JHEP 01 (2023) 168

$H \to gg$

XC, Jakubcik, Marcoli, Stagnitto

JHEP 06 (2023) 192

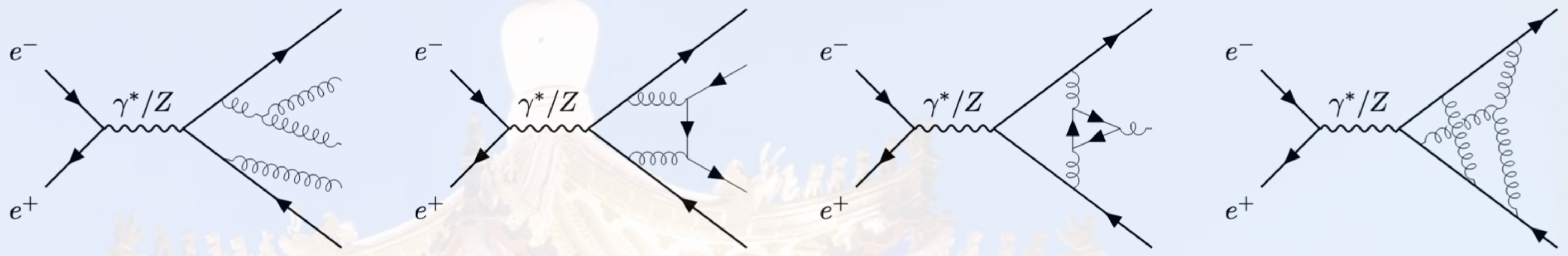
$\chi \to \tilde{g}g$

XC, Jakubcik, Marcoli, Stagnitto

JHEP 12 (2023) 198

$e^+e^- \rightarrow$ JJ @ N3LO

- Simple process:
  - Only $\gamma^* \rightarrow q\bar{q}$ N3LO antenna functions
  - Only dipole-like correlations at N3LO
  - Recycle ingredients from $e^+e^- \rightarrow JJJ$ @ NNLO
- Goals:
  - Establish N3LO antenna subtraction framework
    - Extension of NNLO framework
    - Introduce sector antenna mapping to remove the requirement of sub-antenna functions
  - Exploration of numerical challenges:
    - One-loop double-unresolved regions
    - Two-loop single-unresolved regions
    - Rescue-system to trigger:
      1, Quadruple precision
      2, Taylor expansion of special functions
  - Preparation of computational framework:
    - Spike tests of multiple unresolved IR limits
    - Phase space generators
    - Code generation for N3LO MC.



(a) triple-real (b) double-real-virtual (c) real-double-virtual (d) triple-virtual

$$\mathrm{d}\sigma_{N^3LO} = \int_n [\mathrm{d}\sigma^{VVV} - \mathrm{d}\sigma^{W}] + \int_{n+1} [\mathrm{d}\sigma^{RVV} - \mathrm{d}\sigma^{U}]$$

triple-virtual subtraction term

double-virtual real subtraction term

$$+ \int_{n+2} [\mathrm{d}\sigma^{RRV} - \mathrm{d}\sigma^{T}] + \int_{n+3} [\mathrm{d}\sigma^{RRR} - \mathrm{d}\sigma^{S}]$$

double-real-virtual subtraction term

triple-real subtraction term

$$\mathrm{d}\sigma^{S} = \mathrm{d}\sigma^{S_1} + \mathrm{d}\sigma^{S_2} + \mathrm{d}\sigma^{S_3}$$

$$\mathrm{d}\sigma^{T} = \mathrm{d}\sigma^{V_1S_1} + \mathrm{d}\sigma^{V_1S_2} - \int_1 \mathrm{d}\sigma^{S_1}$$

$$\mathrm{d}\sigma^{U} = \mathrm{d}\sigma^{V_2S_1} - \int_1 \mathrm{d}\sigma^{V_1S_1} - \int_2 \mathrm{d}\sigma^{S_2}$$

$$\mathrm{d}\sigma^{W} = -\int_1 \mathrm{d}\sigma^{V_2S_1} - \int_2 \mathrm{d}\sigma^{V_1S_2} - \int_3 \mathrm{d}\sigma^{S_3}$$

XC, Jakubcik, Marcoli, Stagnitto, Phys. Lett. B. *869 (2025) 139804*

XC, Marcoli, EPJC. 86 (2026) 136

$e^+e^- \rightarrow$ JJ @ N3LO

➤ Basic checks: inclusive cross section

N3LO coefficient:

$$\sigma^{(3)} = \sigma^{(0)}\left(\frac{\alpha_s}{2\pi}\right)^3(-105 \pm 11)$$

Monte Carlo error:

Not so small for inclusive quantities due to large cancellations.

Not the most clever way to compute inclusive cross sections.

XC, Jakubcik, Marcoli, Stagnitto, Phys. Lett. B. 869 (2025) 139804

$$\sigma^{(3)} = \sigma^{(0)}\left(\frac{\alpha_s}{2\pi}\right)^3(-102.14\cdots)$$

Chetyrkin, Künn, Kwiatkowski, Phys. Rept. 277 (1996) 189

N3LO 2-jet rate:

Exclusive n-jet rate @ N3LO: $R_n^{(3)}(y_{cut}) = \dfrac{\Gamma^{(3)}_{\gamma^*\to n\ jets}(y_{cut})}{\Gamma^{(3)}_{\gamma^*\to hadrons}}$

For back-to-back QCD emissions, we have at least two jets $\rightarrow$ n $\geq$ 2

$$R_2^{(3)}(y_{cut})\Gamma^{(3)}_{\gamma^*\to hadrons} = \int_0^{y_{cut}} \frac{d\sigma}{dy_{23}}dy_{23} \qquad \text{with} \quad y_{ij} = \frac{2\min(E_i^2, E_j^2)}{Q^2}(1-\cos\theta_{ij})$$

$$R_3^{(3)}(y_{cut})\Gamma^{(3)}_{\gamma^*\to hadrons} = \int_{y_{cut}}^{1} \frac{d\sigma}{dy_{23}}dy_{23} - \int_{y_{cut}}^{1} \frac{d\sigma}{dy_{34}}dy_{34}$$

$$R_4^{(3)}(y_{cut})\Gamma^{(3)}_{\gamma^*\to hadrons} = \int_{y_{cut}}^{1} \frac{d\sigma}{dy_{34}}dy_{34} - \int_{y_{cut}}^{1} \frac{d\sigma}{dy_{45}}dy_{45}$$

$$R_5^{(3)}(y_{cut})\Gamma^{(3)}_{\gamma^*\to hadrons} = \int_{y_{cut}}^{1} \frac{d\sigma}{dy_{45}}dy_{45}$$

$\Rightarrow \displaystyle\sum_{n=2}^{m+2} R_n^{(m)}(y_{cut}) = 1$

$R_n^{(\alpha_s)}$	α_s^0	α_s^1	α_s^2	α_s^3	α_s^4
n=2		1977	2004, 2004, 2006	2008, 2025 **This Work**	
n=3		1976	1980, 1981, 1981, 1982	2008, 2008, 2014, 2025	
n=4			1979, 1980	1996, 1997, 1998, 1999	2026
n=5				1989, 1989, 1989	2010, 2011
n=6					1999

FO: N4LO, N3LO, NNLO, NLO, LO

$e^+e^- \to$ JJ @ N3LO

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XC, Jakubcik, Marcoli, Stagnitto, Phys. Lett. B. 869 (2025) 139804

$$\sigma^{(3)} = \sigma^{(0)}\left(\frac{\alpha_s}{2\pi}\right)^3(-102.14\cdots)$$

Chetyrkin, Künn, Kwiatkowski, Phys. Rept. 277 (1996) 189

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For back-to-back QCD emissions, we have at least two jets $\to n \geq 2$

$$R_2^{(3)}(y_{cut})\Gamma^{(3)}_{\gamma^*\to hadrons} = \int_0^{y_{cut}} \frac{d\sigma}{dy_{23}} dy_{23} \quad \text{with} \quad y_{ij} = \frac{2\min(E_i^2, E_j^2)}{Q^2}(1-\cos\theta_{ij})$$

$$R_3^{(3)}(y_{cut})\Gamma^{(3)}_{\gamma^*\to hadrons} = \int_{y_{cut}}^1 \frac{d\sigma}{dy_{23}} dy_{23} - \int_{y_{cut}}^1 \frac{d\sigma}{dy_{34}} dy_{34}$$

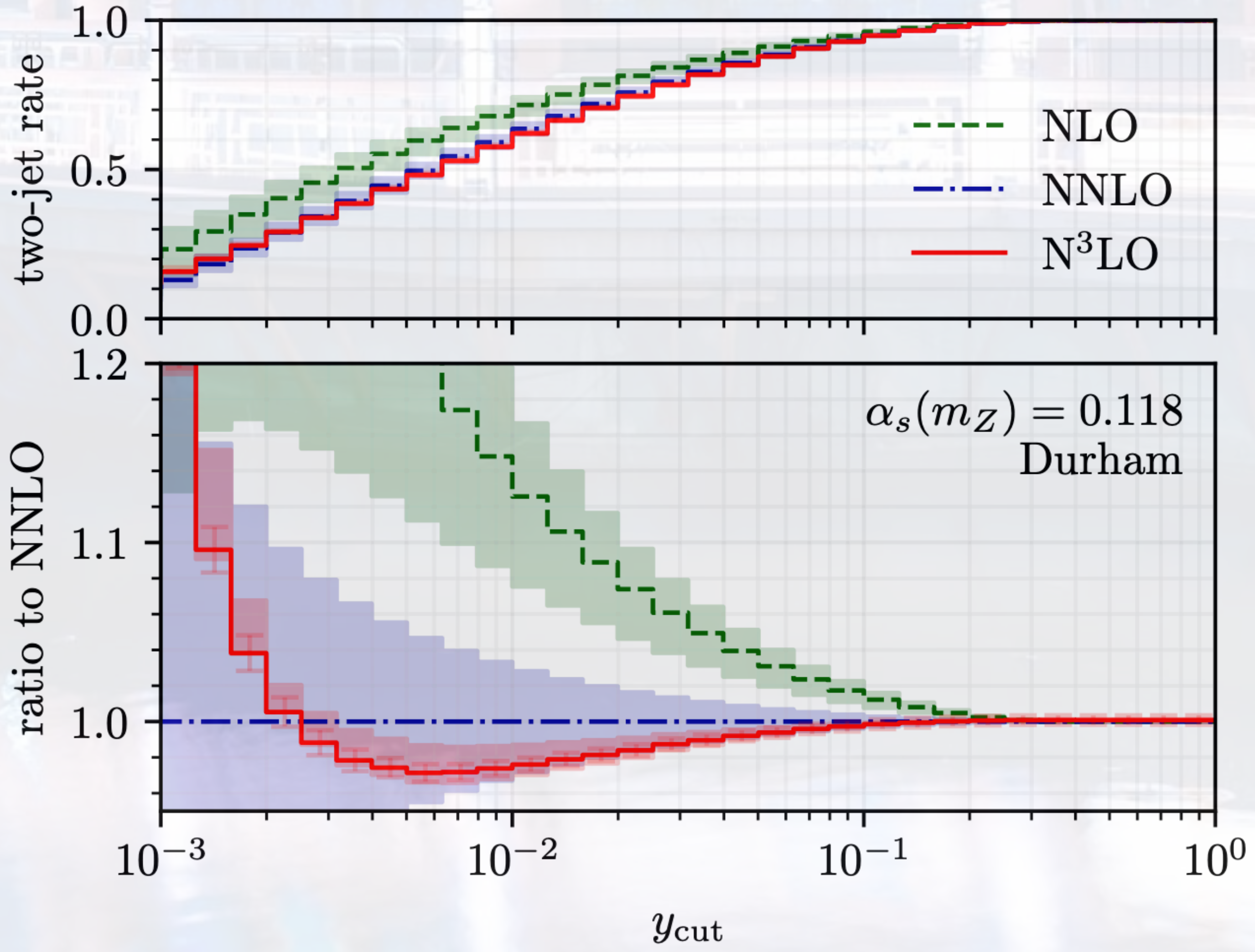
$$R_4^{(3)}(y_{cut})\Gamma^{(3)}_{\gamma^*\to hadrons} = \int_{y_{cut}}^1 \frac{d\sigma}{dy_{34}} dy_{34} - \int_{y_{cut}}^1 \frac{d\sigma}{dy_{45}} dy_{45}$$

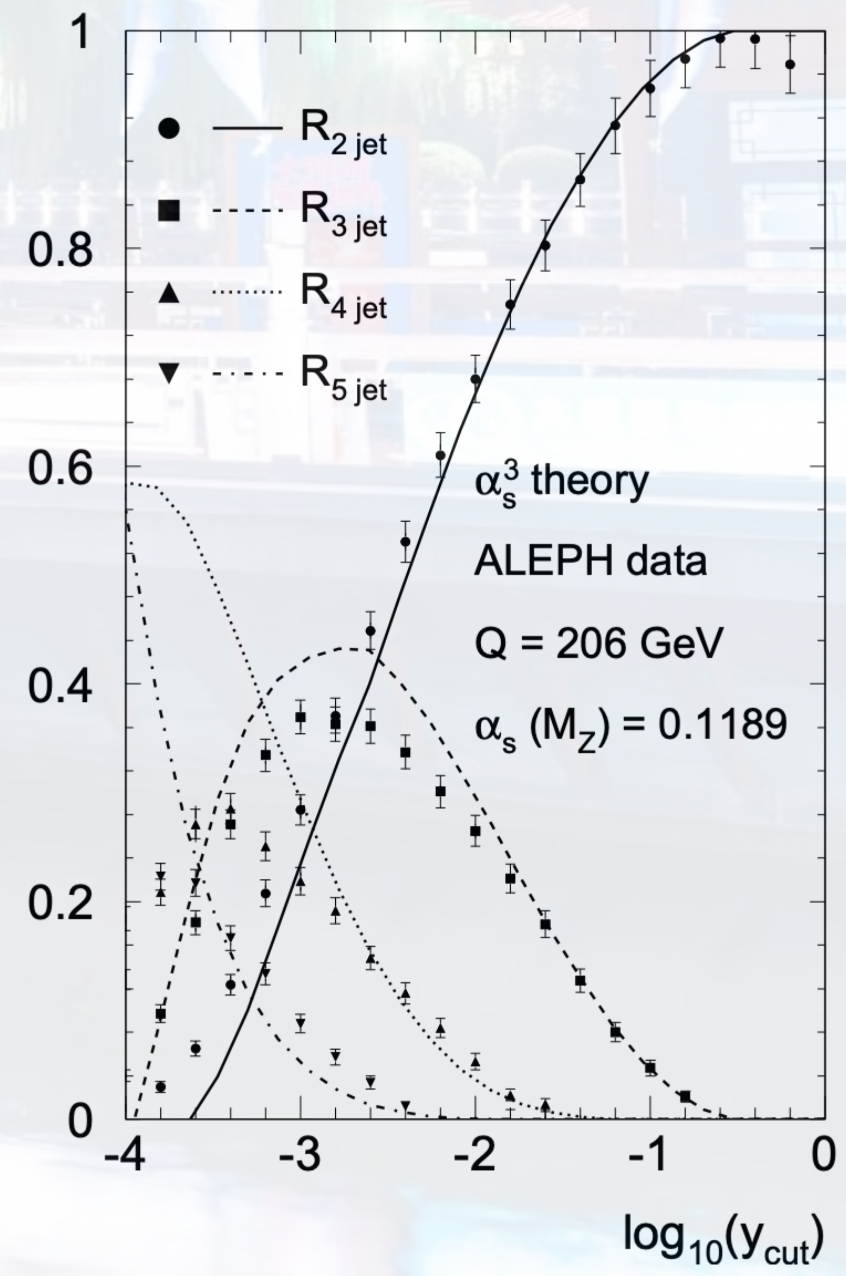
$$R_5^{(3)}(y_{cut})\Gamma^{(3)}_{\gamma^*\to hadrons} = \int_{y_{cut}}^1 \frac{d\sigma}{dy_{45}} dy_{45}$$

$$\Rightarrow \sum_{n=2}^{m+2} R_n^{(m)}(y_{cut}) = 1$$

Gehrmann-De Ridder, Gehrmann, Glover, Heinrich, Phys. Rev. Lett. 100 (2008) 172001

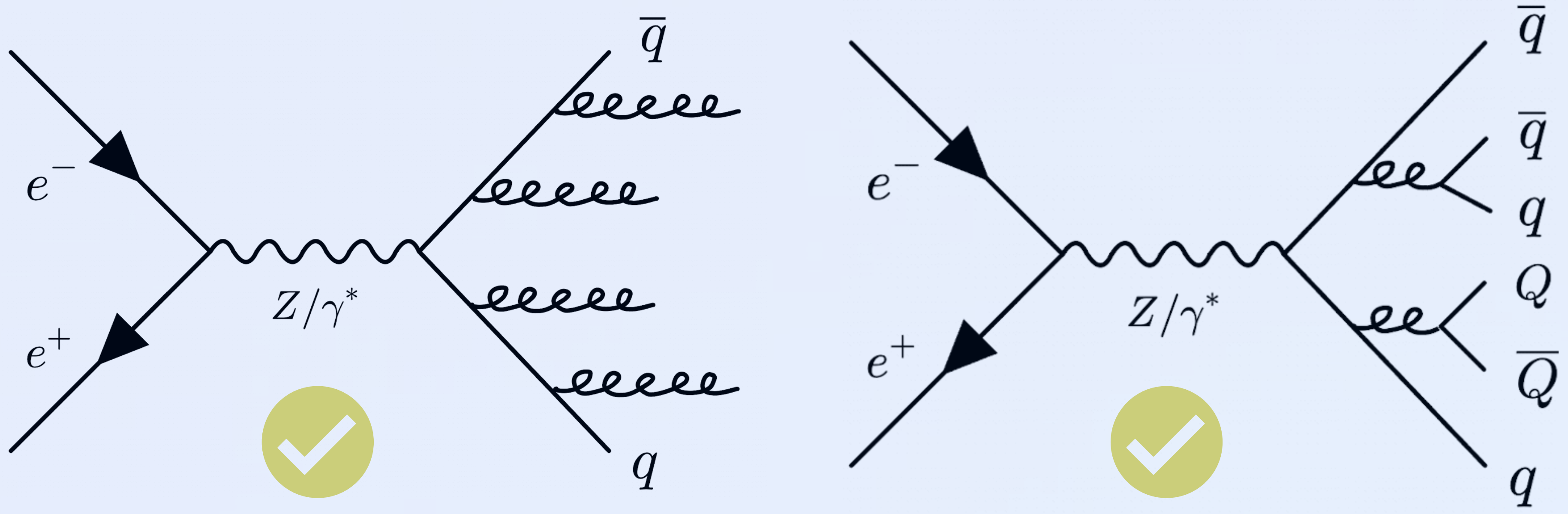
Full agreement between direct and indirect calculation of $R_2^{(3)}(y_{cut})$



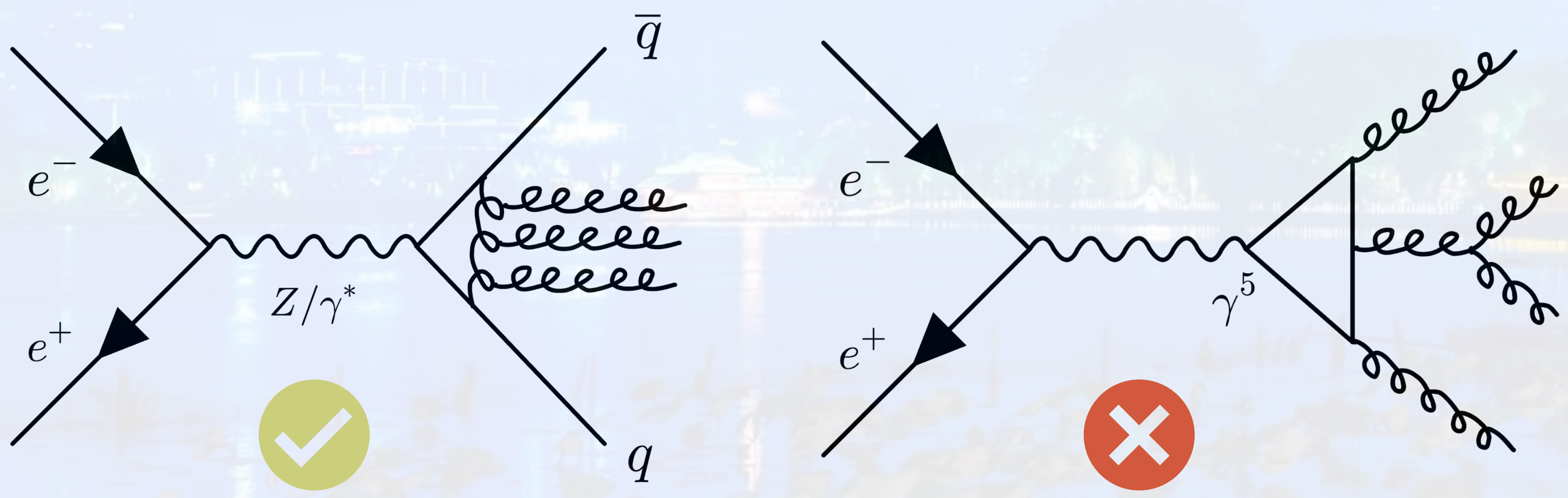


$e^+e^- \to$ JJJJ @ NNLO

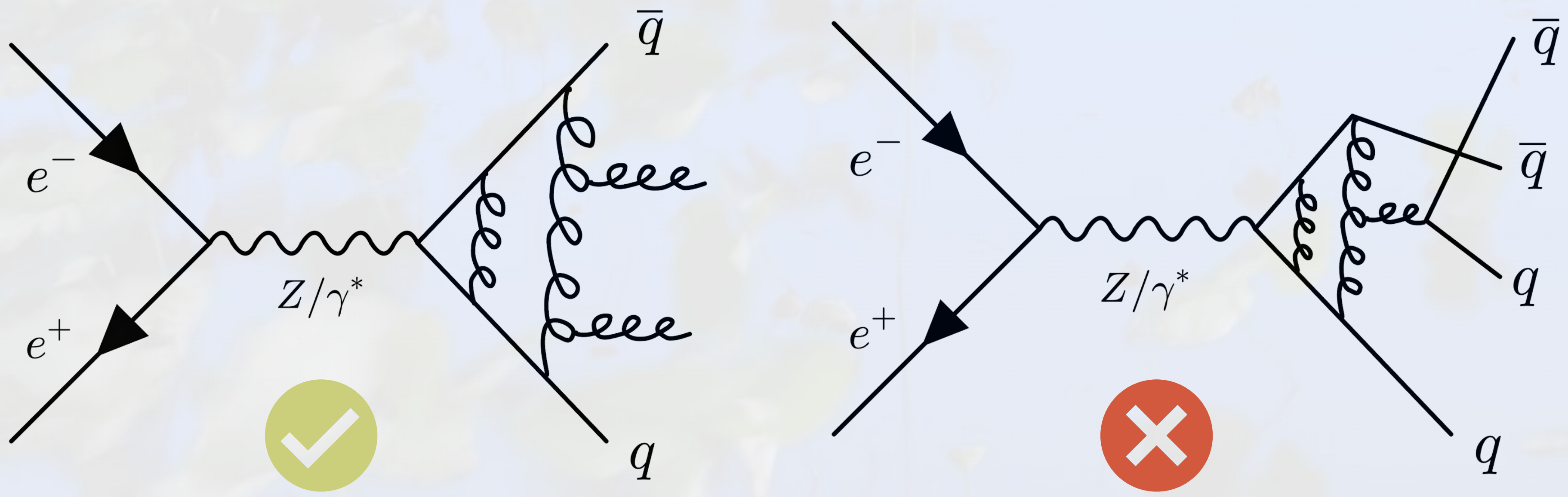
- Double-real matrix element examples:



- Real-virtual matrix element examples:



- Double-virtual matrix element examples



- Massless quark with $n_f = 5$ flavour scheme
- Tree and one-loop matrix elements are obtained from **OpenLoops2**: *Buccioni, Lang, Lindert, Maierhoefer et. al. , EPJC. 79 (2019) 10, 866*
  - Efficient and stable numerical evaluations in IR regions
  - Axial-vector coupling of the Z to closed quark loops ignored
- Two-loop matrix elements finite remainders obtained from **PentagonFunctions++** and **FivePointAmplitudes++**:

De Laurentis, Ita, Page, Sotnikov, JHEP. 06 (2025) 093 *Abreu, et. al., Phys. Rev. Lett. 132 (2024) 141601*

  - Solve the canonical differential equations with one-mass pentagon functions and cross kinematics to decay channel
  - Use leading colour approximation in each partonic channel
  - Vector coupling with singlet contribution (d_{abc}) ignored
  - Finite remainders defined in Catani's scheme
- $e^+e^- \to 5$ Jets @ NLO validation with SHERPA:

$y_{\rm cut}$	NNLOJET	Sherpa	Deviation
$10^{-2.5}$	$-1.13(2)\cdot 10^2$(RS) $6.96(1)\cdot 10^2$(VI) $5.84(2)\cdot 10^2$	$1.843(8)\cdot 10^3$(RS) $-1.262(3)\cdot 10^3$(VI) $5.84(4)\cdot 10^2$	0.11σ
$10^{-3.5}$	$-5.87(6)\cdot 10^3$(RS) $1.419(3)\cdot 10^4$(VI) $8.32(7)\cdot 10^3$	$3.42(3)\cdot 10^4$(RS) $-2.60(1)\cdot 10^4$(VI) $8.35(9)\cdot 10^3$	0.26σ

$e^+e^- \to$ JJJJ @ NNLO

➤ Application to jet rates @ α_s^4:

$$R_n^{(4)}(y_{cut}) = \frac{\Gamma^{(4)}_{Z/\gamma^*\to n\ jets}(y_{cut})}{\Gamma^{(4)}_{Z/\gamma^*\to hadrons}}$$

$$R_4^{(4)}(y_{cut})\Gamma^{(4)}_{Z/\gamma^*\to hadrons} = \int_{y_{cut}}^{1}\frac{d\sigma}{dy_{34}}dy_{34} - \int_{y_{cut}}^{1}\frac{d\sigma}{dy_{45}}dy_{45}$$

$R_n^{(\alpha_s)}$	α_s^0	α_s^1	α_s^2	α_s^3	α_s^4	
n=2		1977	2004, 2004, 2006	2008, 2025		**FO**
n=3		1976	1980, 1981, 1981, 1982	2008, 2008, 2014, 2025		N4LO
n=4			1979, 1980	1996, 1997, 1998, 1999	**2026 This work**	N3LO
n=5				1989, 1989, 1989	2010, 2011	NNLO
n=6					1999	NLO
						LO

Table 4.2: Wish-list for calculations of missing higher-order perturbative QCD $\mathcal{O}(\alpha_s^n)$ and/or EW $\mathcal{O}(\alpha^n)$ to match the expected experimental uncertainty at future e^+e^- and $e\,p$ colliders.

Observable	Missing higher-order & power-suppressed corrections
Hadronic Z width	$\mathcal{O}(\alpha_s^5)$, $\mathcal{O}(\alpha_s^6)$, $\mathcal{O}(\alpha^3)$, $\mathcal{O}(\alpha_s\alpha^3)$, $\mathcal{O}(\alpha_s^2\alpha^2)$
Hadronic W width	$\mathcal{O}(\alpha_s^5)$, $\mathcal{O}(\alpha^2)$, $\mathcal{O}(\alpha_s^2\alpha)$
Hadronic τ width	$\mathcal{O}(\alpha_s^5)$
Hadronic event shapes (Z, W, H decays)	N^3LO differential, $N^{3,4}LL$ resummation, power corrections
Inclusive jet rates	3-jet cross-sections at N^3LO, 4-jets at N^2LO, 5-jets at NLO
Lattice QCD results	$\mathcal{O}(\alpha_s^6)$ β-function; $\mathcal{O}(\alpha_s^5)$ heavy quark decoupling; $\mathcal{O}(\alpha_s^4)$ static potential
(α_s extr.; quark masses m_c, m_b)	$\mathcal{O}(\alpha_s^3)$ lattice perturbation theory matching (lattice coupling to $\alpha_s^{\overline{MS}}$ etc.)
$\sigma(e^+e^-\to W^+W^-)$ vs. $\sqrt{s}$	EW N^2LO: $\mathcal{O}(\alpha^2)$, Mixed EW-QCD: $\mathcal{O}(\alpha_s\alpha^2)$, $\mathcal{O}(\alpha_s^2\alpha)$
$\sigma(e^+e^-\to t\bar{t})$ vs. $\sqrt{s}$	NRQCD: $\mathcal{O}(\alpha_s^5)$, Non-resonant: $\mathcal{O}(\alpha_s^5)$, $\mathcal{O}(\alpha_s^3)$ differential; QED: $\mathcal{O}(\alpha^3)$ at NNLL
$H\to b\bar{b}$ width	N^4LO ($m_b \neq 0$); N^4LO differential ($m_b = 0$)
$H\to gg$ width	N^5LO (heavy-top limit), N^4LO ($m_t \neq 0$); N^4LO differential, N^3LO differential ($m_t \neq 0$)
MC simulations for $e^+e^-\to X$ processes	$N^{2,3}LO$ matched to $N^{2,3}LL$ PS. Permille control of non-pQCD effects (hadronization, CR, ...)
$ep\to$ hadrons (PDF and α_s determ.)	$N^{3,4}LO$ evolution equations and inclusive cross-sections
$ep\to$ jets (α_s determ.)	N^3LO cross-sections

Physics Briefing Book: Input for the 2026 update of the European Strategy for Particle Physics

➤ JJJJ@NNLO is also genuine NNLO contribution to event shapes:

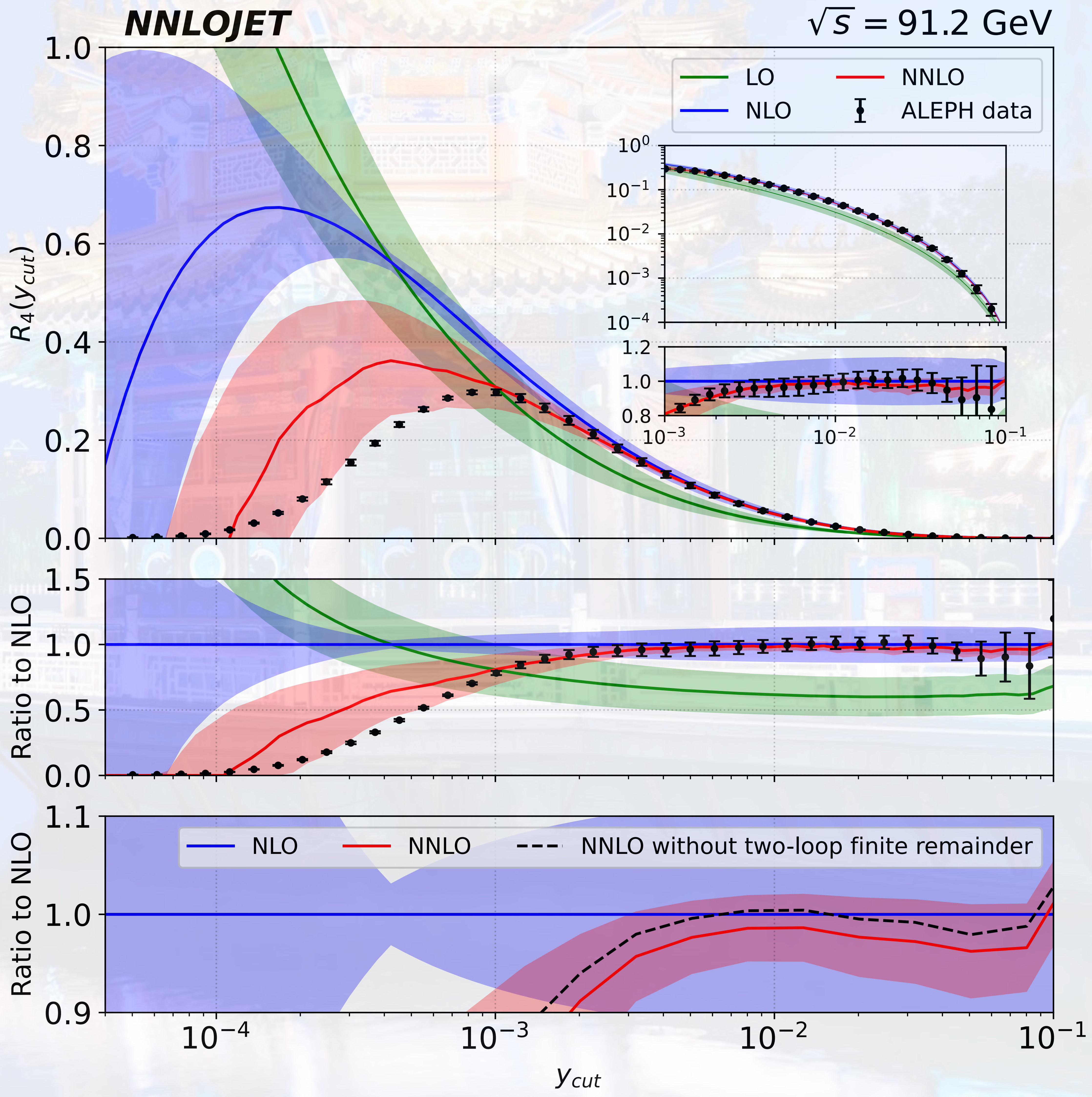
$$D = 27\lambda_1\lambda_2\lambda_3,\quad C = 3\lambda_1\lambda_2\lambda_3,\quad \text{from}\quad \Theta^{ij} = \frac{\sum_k \frac{p_k^i p_k^j}{|\vec{p}_k|}}{\sum_k |\vec{p}_k|}$$

$$\frac{M_L^2}{s} = \frac{1}{s}\min\left\{\Big(\sum_{\vec{k}_k\in H_1} p_k\Big)^2, \Big(\sum_{\vec{k}_k\in H_2} p_k\Big)^2\right\}$$

$$A = \frac{3}{2}\lambda_3,\quad \text{from}\quad \Phi^{ij} = \frac{\sum_k p_k^i p_k^j}{\sum_k |\vec{p}_k|^2}$$

$e^+e^- \rightarrow$ JJJJ @ NNLO

- Application to 4 jet rate up to NNLO:
  - Computational setup:
    - $m_Z = 91.2$ GeV, $\alpha_s(m_Z) = 0.118$
    - $\mu_R^c = \sqrt{s}$ with $\mu_R \in [\sqrt{s}/2, 2\sqrt{s}]$
    - $\Gamma^{\alpha_s}_{Z/\gamma^* \to hadrons}$ with matching α_s order.
  - Perturbative region $10^{-3} \leq y_{\mathrm{cut}} \leq 10^{-1}$ for $\sqrt{s} = m_Z$:
    - Remarkable convergence with NLO.
    - Excellent agreement with ALEPH data.
    - Scale uncertainty reduced to $\pm(3\% - 5\%)$ smaller than LEP systematic errors.
    - Symmetrization of scale uncertainty.
    - Two-loop finite reminder has size above -2% ~ -5%
    - LCA is expected to have ~ 0.5% modification
  - Resummation and none-perturbative region:
    - NNLO peak moves to larger y_{cut} value than NLO
    - Jet rate turns negative around $y_{\mathrm{cut}} = 10^{-4}$



XC, Chicherin, Fox, Glover, Marcoli, Sotnikov, Sun, Zhang, Zoia, 2602.18185 accepted by PRL

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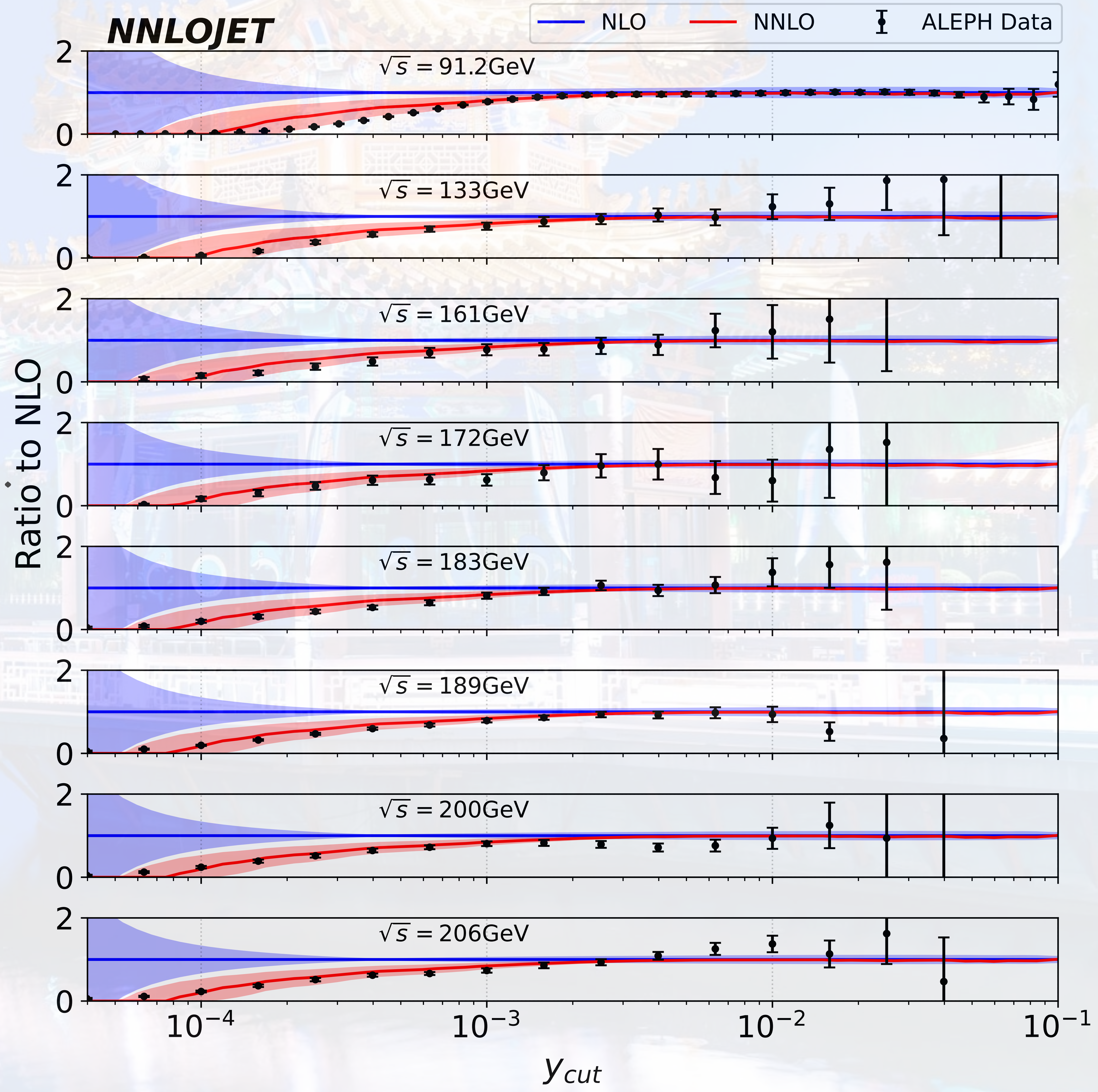
    - $\Gamma^{\alpha_s}_{Z/\gamma^* \rightarrow hadrons}$ with matching α_s order.
  - Perturbative region:
    - $R_4(y_{\mathrm{cut}})$ dependence of $\sqrt{s}$ is only via running of $\alpha_s(\mu_R)$.
    - FO prediction extends to smaller y_{cut} as $\sqrt{s}$ increase.
    - Excellent agreement with ALEPH data for $y_{\mathrm{cut}} \sim 10^{-4}$
    - Two-loop finite remainder is largely insensitive to $\sqrt{s}$.
    - Further validation with data:
      - Reanalyzing existing measurements
      - Future electron-position colliders
  - Resummation and none-perturbative region:
    - NLO and NNLO results deviate for $y_{\mathrm{cut}} \leq 10^{-3}$
    - Need resummation to reduce dominant scale uncertainty



XC, Chicherin, Fox, Glover, Marcoli, Sotnikov, Sun, Zhang, Zoia, 2602.18185 accepted by PRL

HH Production at N3LO QCD

➤ **Feynman diagrams in the SM:**

- Earlier Studies: **SM LO (1988)** ⇒ · · · ⇒ **NLO (2016)** ⇒
  Nucl. Phys. B 309 (1988) 282-294
  JHEP 10 (2016) 107
  PRL 117 (2016) 1, 012001

$\sqrt{s}$ (TeV)	NLO_{m_t}(fb)	NLO_{∞}(fb)
13	$27.56^{+14\%}_{-13\%}$	$25.81^{+18\%}_{-15\%}$
14	$32.64^{+14\%}_{-12\%}$	$31.89^{+18\%}_{-15\%}$
27	$126.2^{+12\%}_{-10\%}$	$183.0^{+16\%}_{-14\%}$
100	$1119^{+13\%}_{-13\%}$	$3724^{+13\%}_{-11\%}$

➤ **Heavy Top Limit (HTL):** Integrate out top quarks ($m_t \to \infty$)

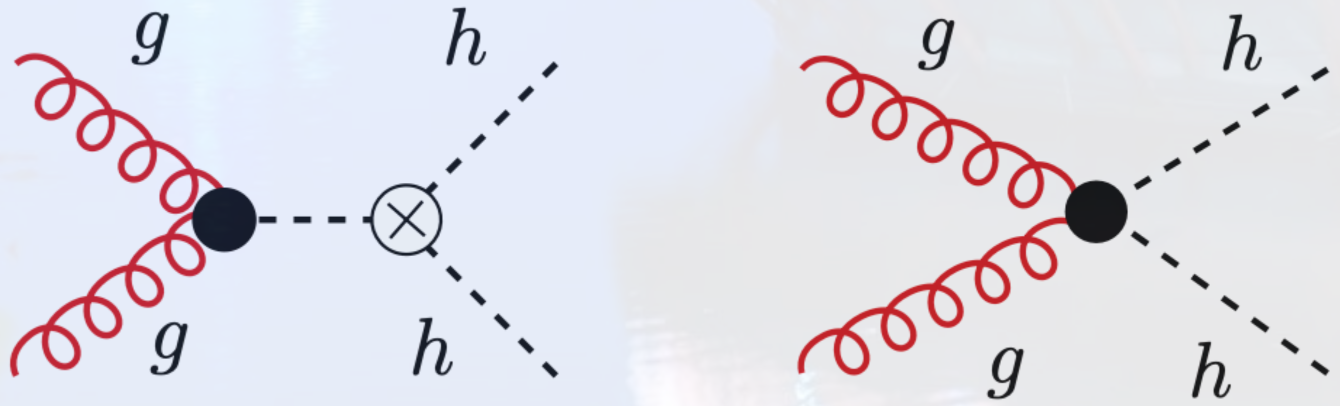
➤ **Effective Lagrangian:** $\mathscr{L}_{\mathrm{eff}} = -\frac{1}{4}\left(C_h \frac{h}{v} - C_{hh}\frac{h^2}{2v^2}\right) G^a_{\mu\nu} G^{a\mu\nu} + \mathcal{O}\left(h^k, k \geq 3\right)$

- Couplings: $C_h = -\frac{\alpha_s}{3\pi}(1+\delta), \quad C_{hh} = -\frac{\alpha_s}{3\pi}(1+\delta+\eta)$
- Wilson Coefficients: $\delta = \sum_{i=0}\left(\frac{\alpha_s}{4\pi}\right)^i \delta^{(i)}, \quad \eta = \sum_{i=0}\left(\frac{\alpha_s}{4\pi}\right)^i \eta^{(i)}$

HTL NLO (1998) ⇒ **NNLO (2013)**
Phys. Rev. D 58 (1998) 115012
Phys. Rev. Lett. 111 (2013) 201801
Nucl. Phys. B 888 (2014) 17-29

⇒ **NNLO FTa (2018)** ⇒ **N3LO (2020)**
JHEP 05 (2018) 059
Phys. Lett. B 803 (2020) 135292
JHEP 03 (2020) 072

➤ **Feynman diagrams in the HTL:**



- HH production: $2m_H < \sqrt{\hat{s}}$
- HTL Valid condition: $\sqrt{\hat{s}} \ll 2m_T$

→ **Top-quark mass effects**

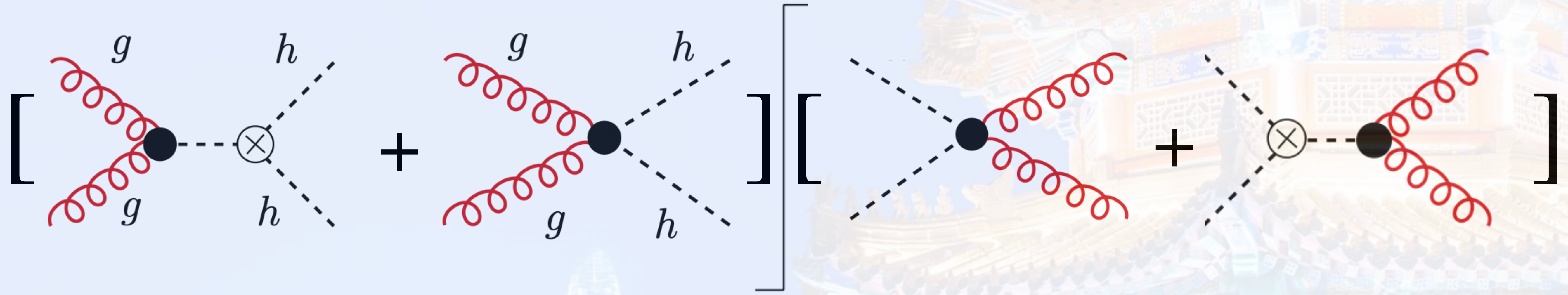
$\sqrt{s}$	13 TeV	14 TeV	27 TeV	100 TeV
LO	$13.80^{+31\%}_{-22\%}$	$17.06^{+31\%}_{-22\%}$	$98.22^{+26\%}_{-19\%}$	$2015^{+19\%}_{-15\%}$
NLO	$25.81^{+18\%}_{-15\%}$	$31.89^{+18\%}_{-15\%}$	$183.0^{+16\%}_{-14\%}$	$3724^{+13\%}_{-11\%}$
NNLO	$30.41^{+5.3\%}_{-7.8\%}$	$37.55^{+5.2\%}_{-7.6\%}$	$214.2^{+4.8\%}_{-6.7\%}$	$4322^{+4.2\%}_{-5.3\%}$
N³LO	$31.31^{+0.66\%}_{-2.8\%}$	$38.65^{+0.65\%}_{-2.7\%}$	$220.2^{+0.53\%}_{-2.4\%}$	$4439^{+0.51\%}_{-1.8\%}$

Total cross section + m_{hh} distribution

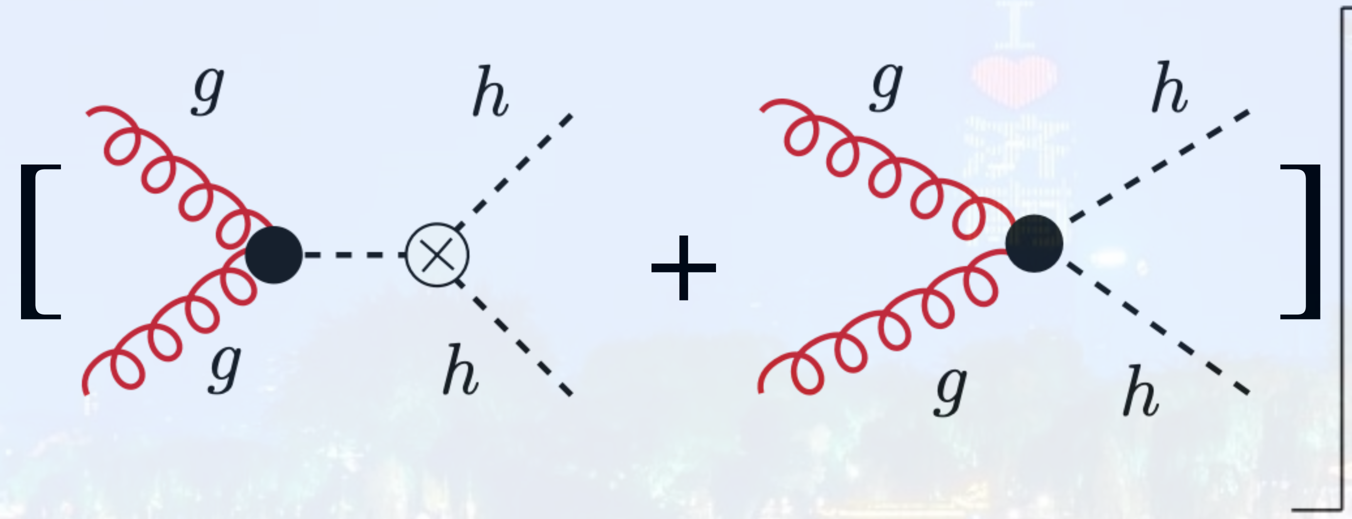
HH Production at N3LO QCD

- **According to the number of effective couplings:** $d\sigma_{hh} = d\sigma^{a}_{hh} + d\sigma^{b}_{hh} + d\sigma^{c}_{hh}$

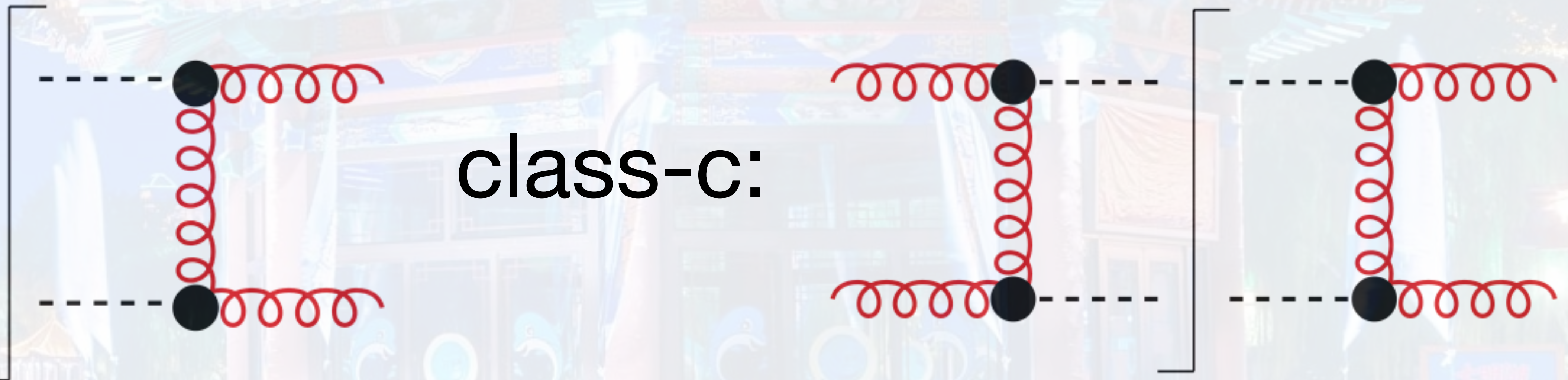
class-a:



class-b:



class-c:



- **Scheme of each class (fully differential):**

XC, Y.H. Dai, H.T. Li, S.Y. Li, H.S. Shao, J. Wang in 2601.19990 accepted by JHEP

	LO	NLO	NNLO	N^{3}LO
total	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^3)$	$\mathcal{O}(\alpha_s^4)$	$\mathcal{O}(\alpha_s^5)$
class-a	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^3)$	$\mathcal{O}(\alpha_s^4)$	$\mathcal{O}(\alpha_s^5)$
class-b	0	$\mathcal{O}(\alpha_s^3)$	$\mathcal{O}(\alpha_s^4)$	$\mathcal{O}(\alpha_s^5)$
class-c	0	0	$\mathcal{O}(\alpha_s^4)$	$\mathcal{O}(\alpha_s^5)$

- **Usage of packages:**

→ SCET+NNLOJET

→ SCET+MadGraph5

→ MadGraph5

NNLOJET (derived from single H) **qt-slicing:** $d\sigma_{hh} = d\sigma_{hh}\Big|_{p_T^{hh} < p_T^{veto}} + d\sigma_{hh}\Big|_{p_T^{hh} > p_T^{veto}}$

HH Production at N3LO QCD

➤ Differential XS calculation setup:

- Fiducial cuts:

$$p_T^{h\text{1st}} > 30\ \text{GeV} \quad p_T^{h\text{2nd}} > 20\ \text{GeV} \quad |y_h| < 2.4$$

- Centre of mass energy: $\sqrt{s} = 14\ \text{TeV}$
- PDF: NNPDF40_an3lo_as_01180
- SM Parameters:

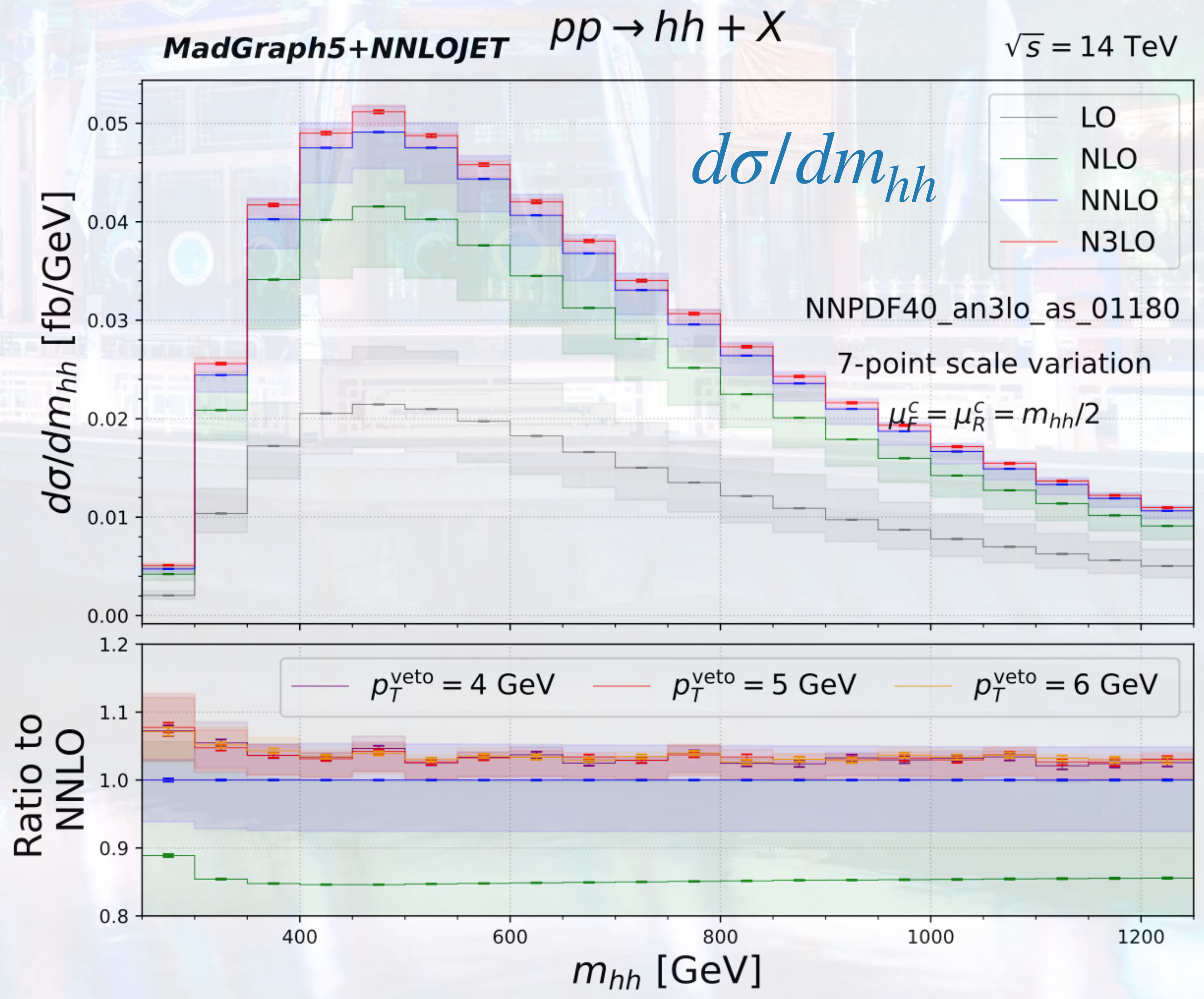
$$v = 246.2\ \text{GeV} \quad m_h = 125\ \text{GeV} \quad m_t = 173\ \text{GeV}$$

- Central scale: $\mu_F = \mu_R = m_{hh}/2$
- Scale variation: $\mu_R \to k_{\mu_R}\mu_R, \ \ \mu_F \to k_{\mu_F}\mu_F$

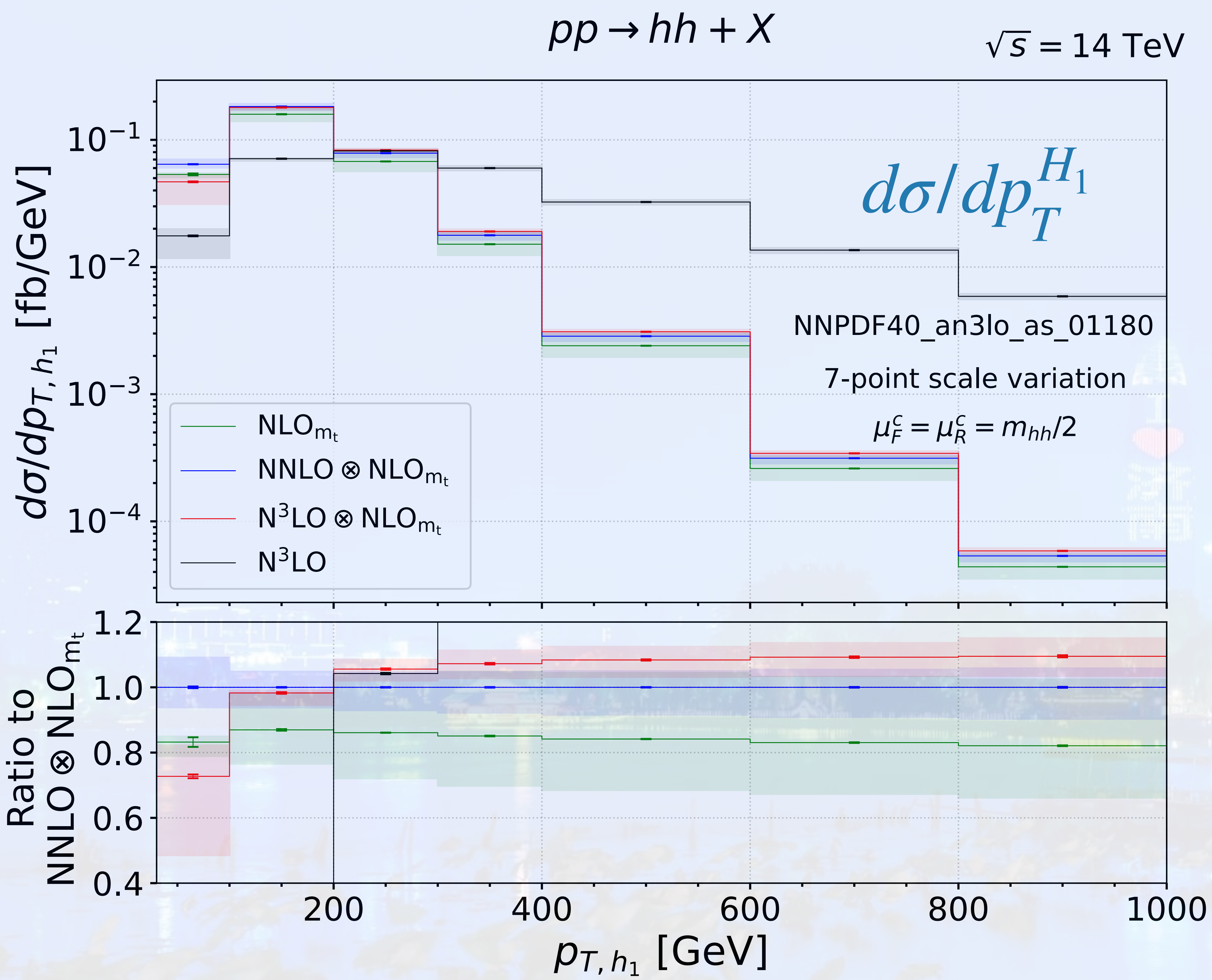
$$(k_{\mu_R}, k_{\mu_F}) \in \{(1,1),(1,1/2),(1,2),(1/2,1),(2,1),(1/2,1/2),(2,2)\}$$

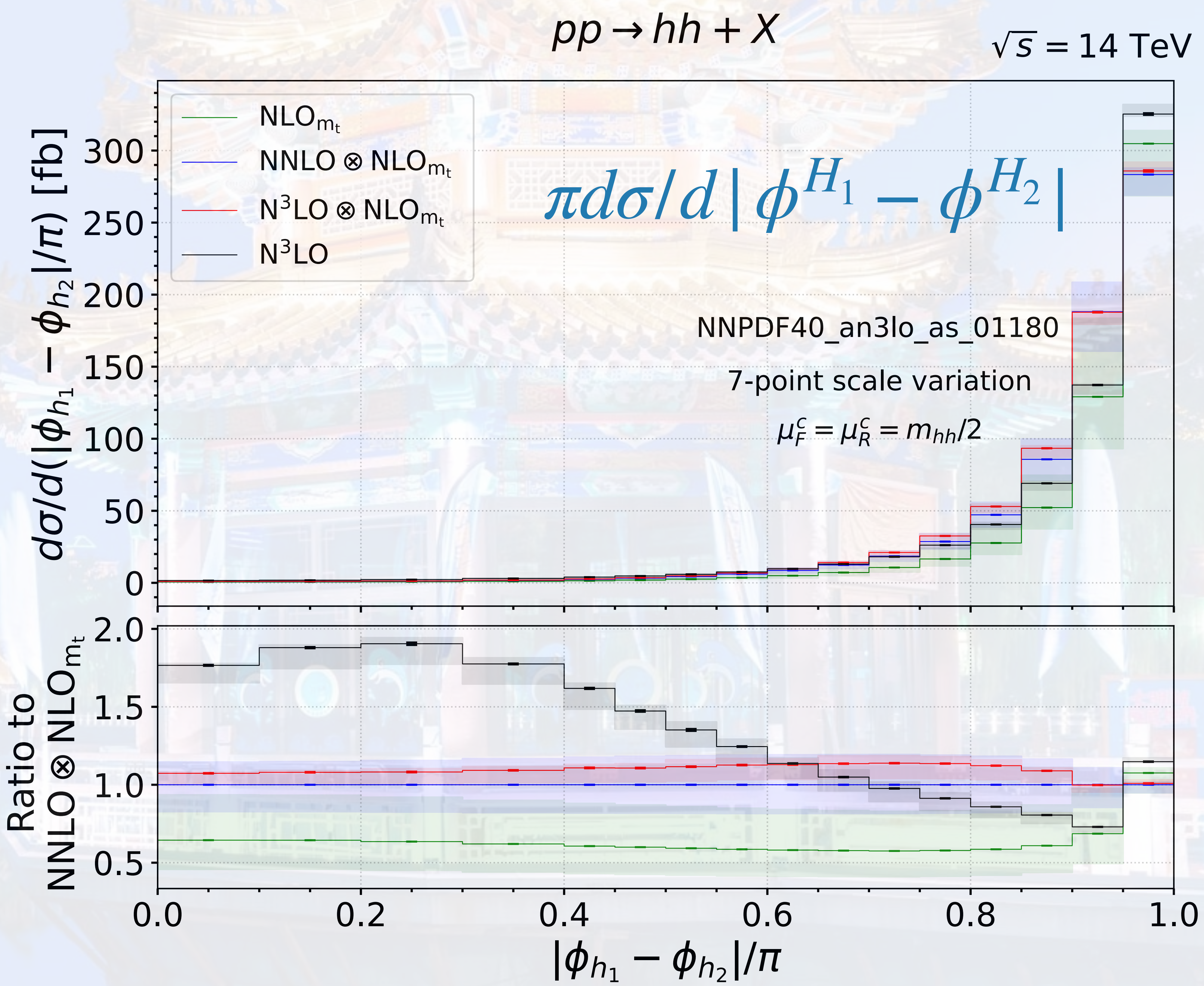
XC, Y.H. Dai, H.T. Li, S.Y. Li, H.S. Shao, J. Wang 2601.19990 accepted by JHEP

	σ_{hh}^{LO} [fb]	σ_{hh}^{NLO} [fb]	$\sigma_{hh}^{\text{NNLO}}$ [fb]	$\sigma_{hh}^{\text{N}^3\text{LO}}$ [fb]
Sum	$14.85^{+30\%}_{-22\%}$	$27.87^{+18\%}_{-15\%}$	$32.74^{+5.2\%}_{-7.3\%}$	$34.00(4)^{+1.4\%}_{-2.9\%} \pm 0.05\%$
Class-a	$14.85^{+30\%}_{-22\%}$	$28.25^{+18\%}_{-15\%}$	$33.62^{+5.7\%}_{-7.7\%}$	$35.20(4)^{+1.7\%}_{-3.4\%} \pm 0.02\%$
Class-b	0	$-0.3758(2)^{+42\%}_{-28\%}$	$-0.8854(2)^{+29\%}_{-12\%}$	$-1.216(3)^{+13\%}_{-14\%} \pm 1\%$
Class-c	0	0	$0.003065(3)^{+55\%}_{-34\%}$	$0.008865(3)^{+41\%}_{-28\%}$



HH Production at N3LO QCD





- Fully differential predictions from event generators with nontrivial N3LO corrections to the shape of distributions.
- Excellent fixed order convergence with scale uncertainties reduced by more than 60%.
- Reference p_T^{veto}= 5 GeV with stable predictions by varying p_T^{veto} at 4 and 6 GeV (negligible p_T^{veto} dependence)
- Top mass effects at NLO included by reweighting (available from ggxy package) *Davies et. al. Comput.Phys.Commun. 320 (2026) 109933*
- Included in LHCHWG YR5 report as inclusive and differential benchmarks

CONCLUSIONS AND FUTURE PROSPECTS

- Precision phenomenology requires improvements in multiple frontier (PDFs, α_s, quark mass etc.)
- **Event generators** serve as Swiss knives to push the prediction power to new level of accuracy.
- First few applications of **fully-differential N3LO QCD calculation** are available with nontrivial corrections.
- NNLO QCD predictions maybe solved, but still not easily accessible. The **public release of NNLOJET** aims to improve the accessibility.

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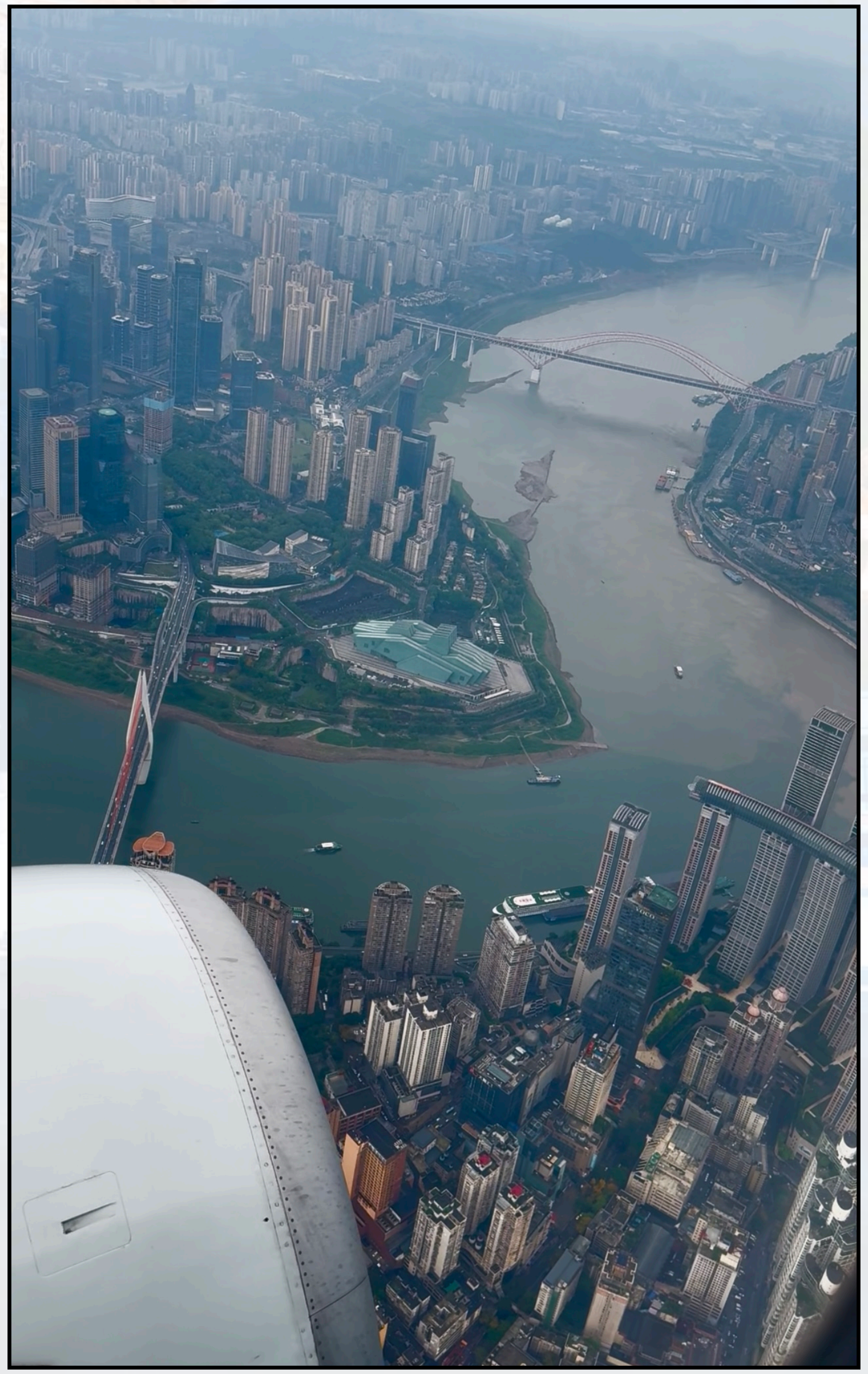
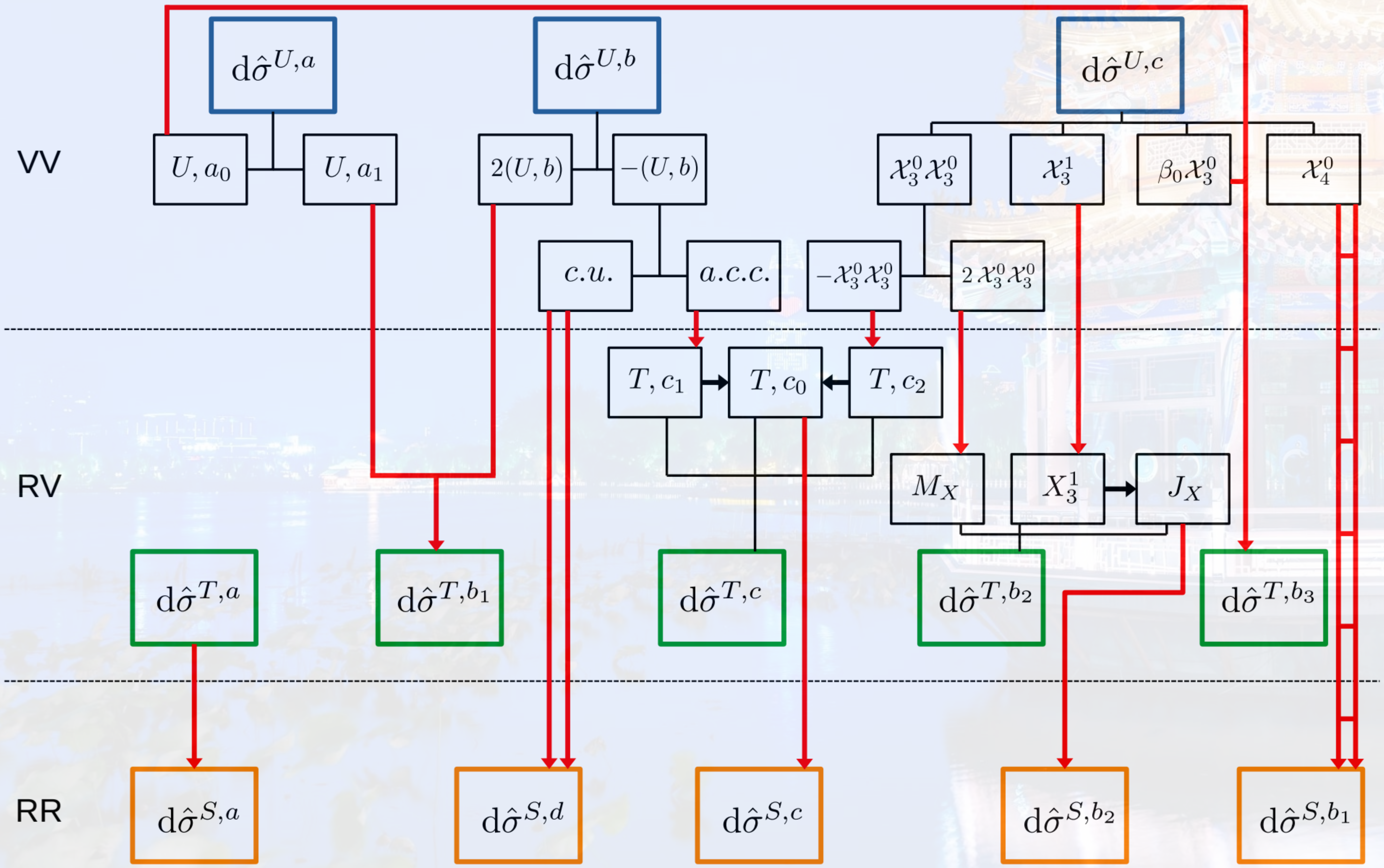
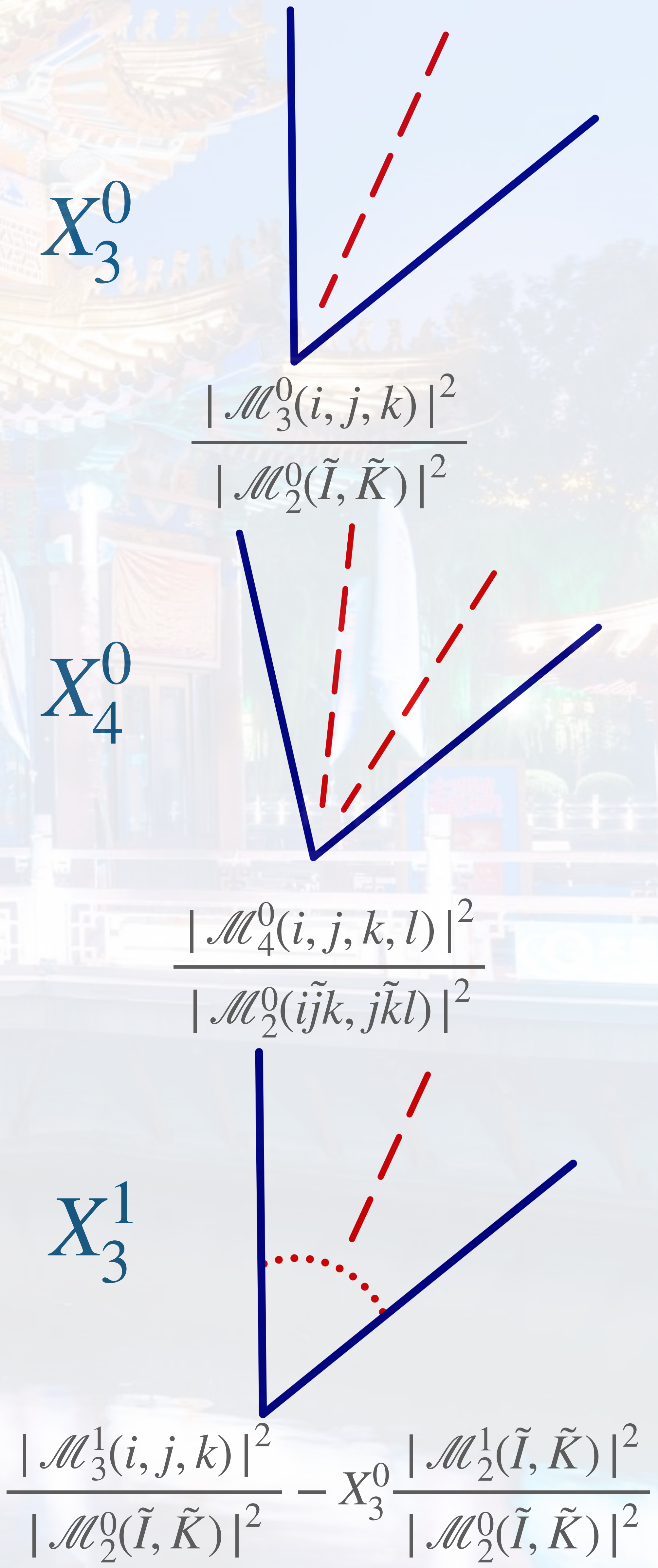


Thank You for Your Attention

BACK UP SLIDES

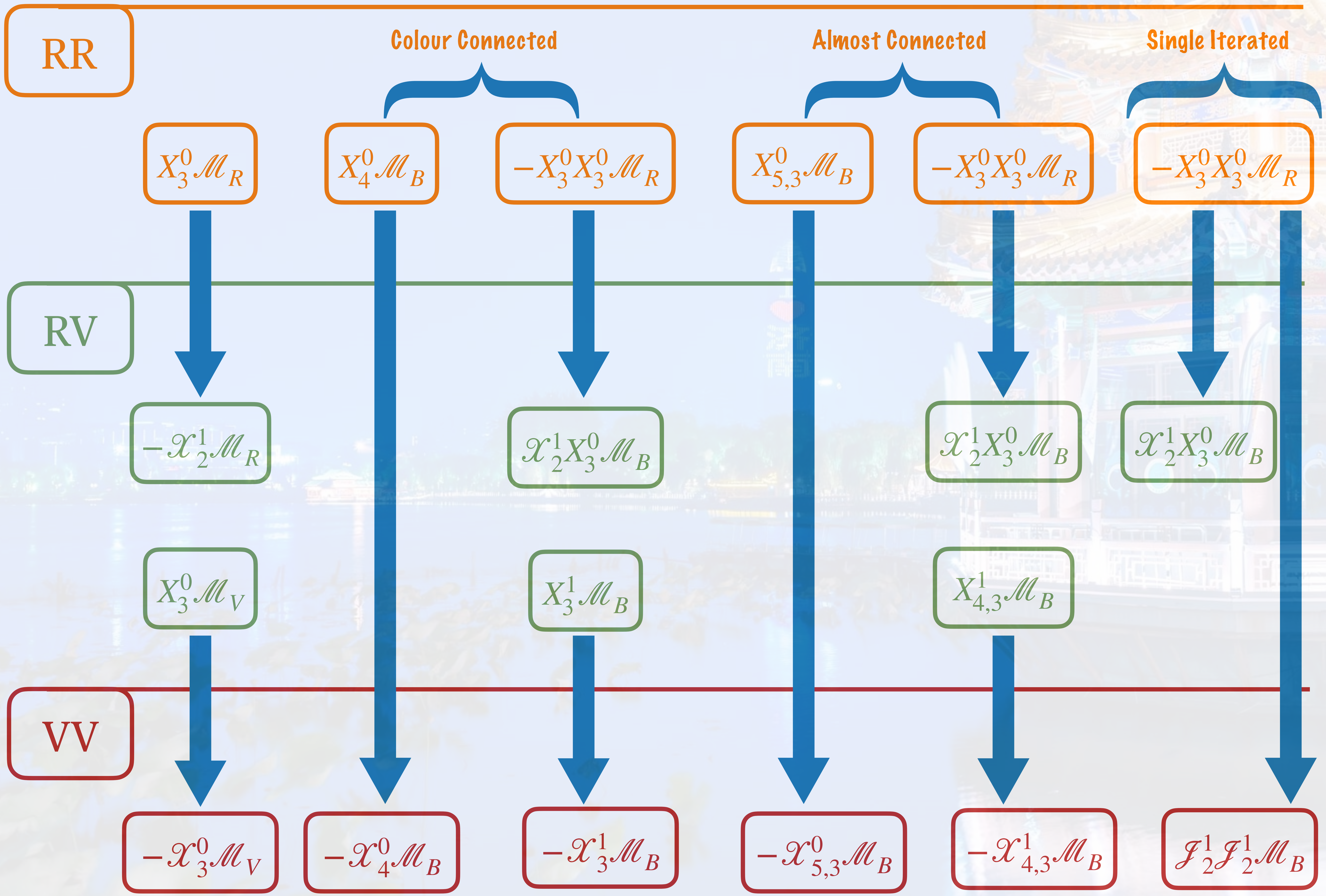
Traditional Antenna Subtraction @ NNLO

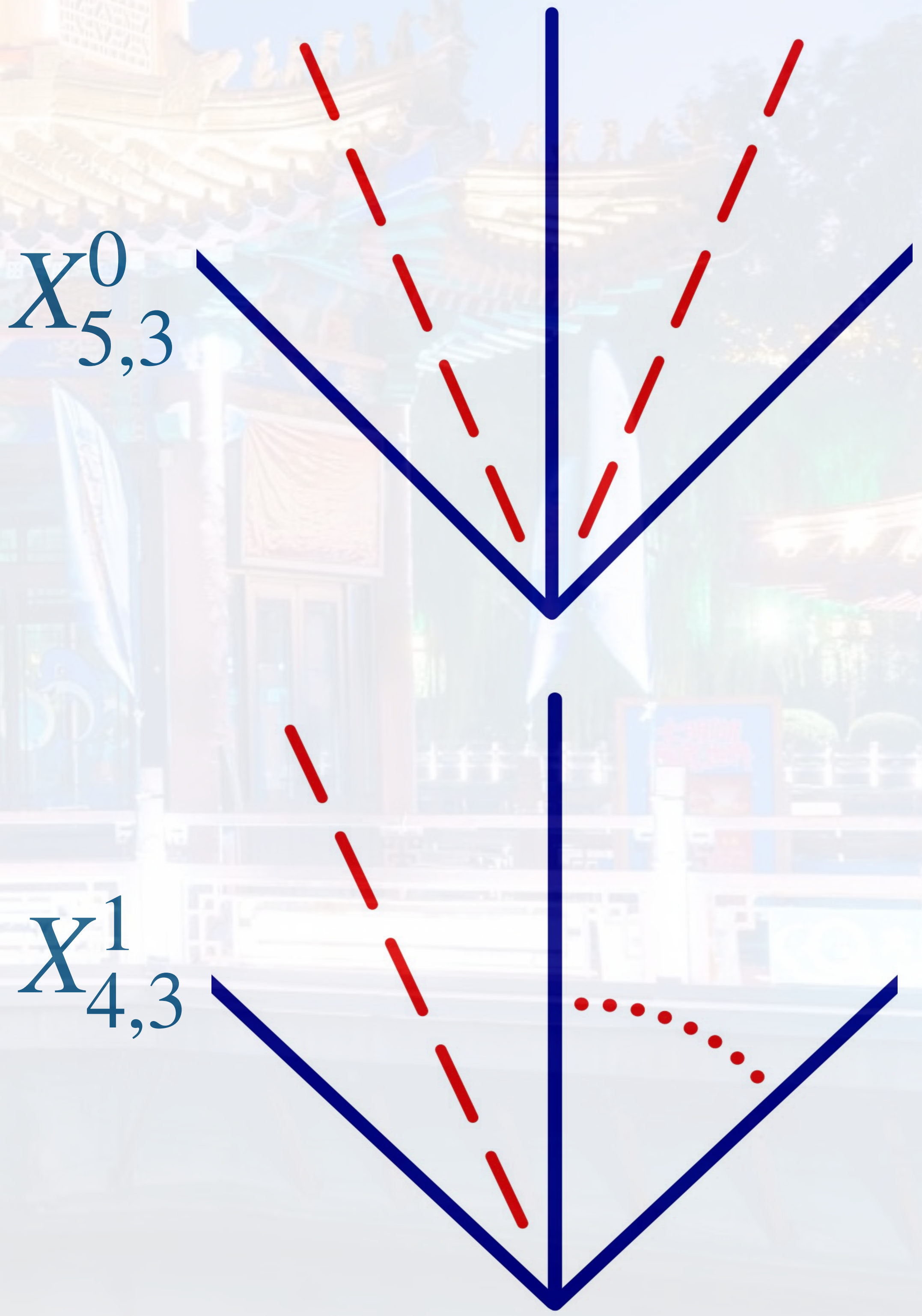




XC, Gehrmann, Glover, Huss, Marcoli JHEP 10 (2022) 099

Generalized Antenna Subtraction @ NNLO





$e^+e^- \to 3$ jets @ NNLO

Efficiency boost by a factor of 10

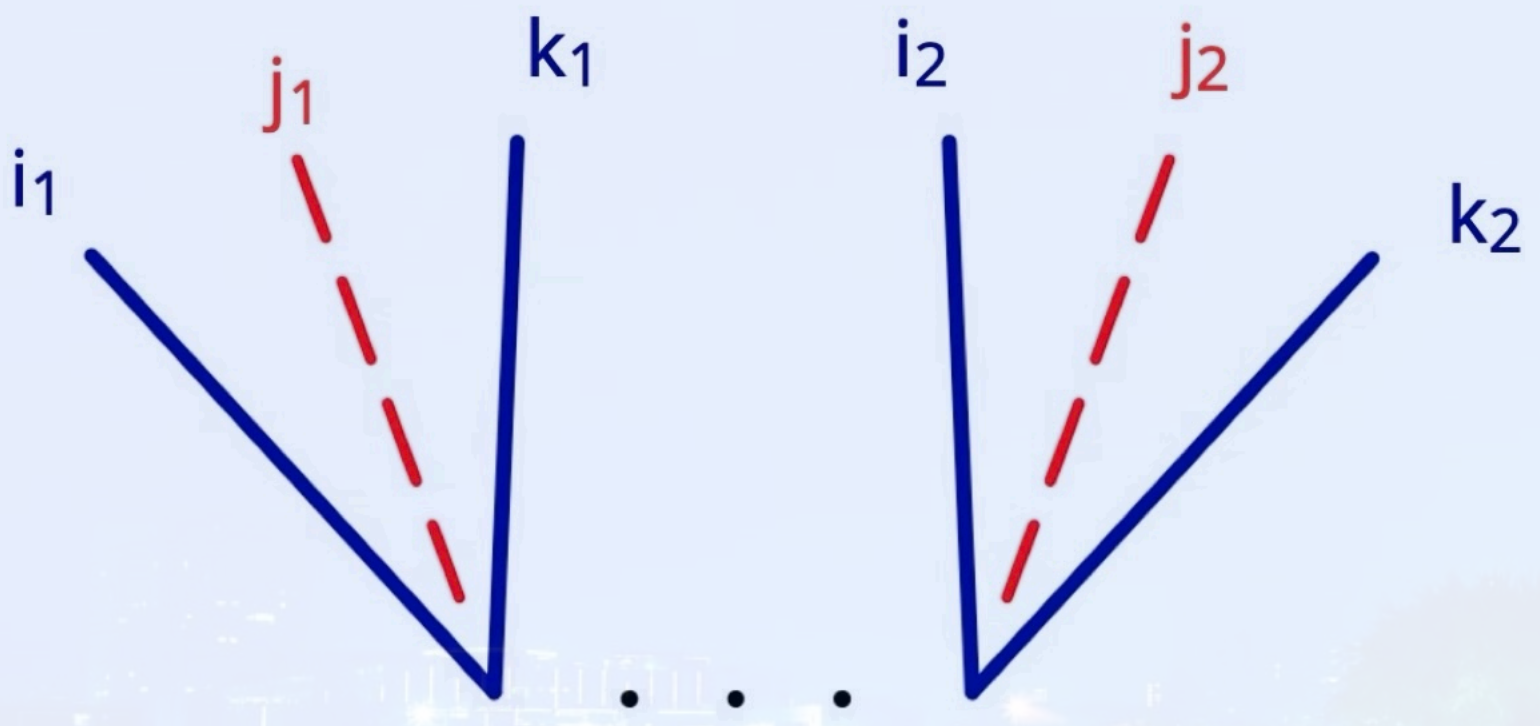
Fox, Glover, Marcoli, JHEP 12 (2024) 225

Antenna Subtraction @ NNLO

Based on Marcoli's slide @ Loop Summit 2

NNLO: two unresolved emissions → multiple topologies

colour-unconnected emissions: no shared hard radiatior



$$M^0_{n+2}(\ldots, i_1, j_1, k_1, \ldots, i_2, j_2, k_2, \ldots)$$

$$X^0_3(i^h_1, j_1, k^h_1)X^0_3(i^h_2, j_2, k^h_2)M^0_n(\ldots, \widetilde{(i_1 j_1)}, \widetilde{(j_1 k_1)}, \ldots, \widetilde{(i_2 j_2)}, \widetilde{(j_2 k_2)}, \ldots)$$

fully **iterated** structure

3→2 momentum mapping

colour-connected emissions: both hard radiators shared

$$M^0_{n+2}(\ldots, i, j, k, l, \ldots)$$

4→2 momentum mapping

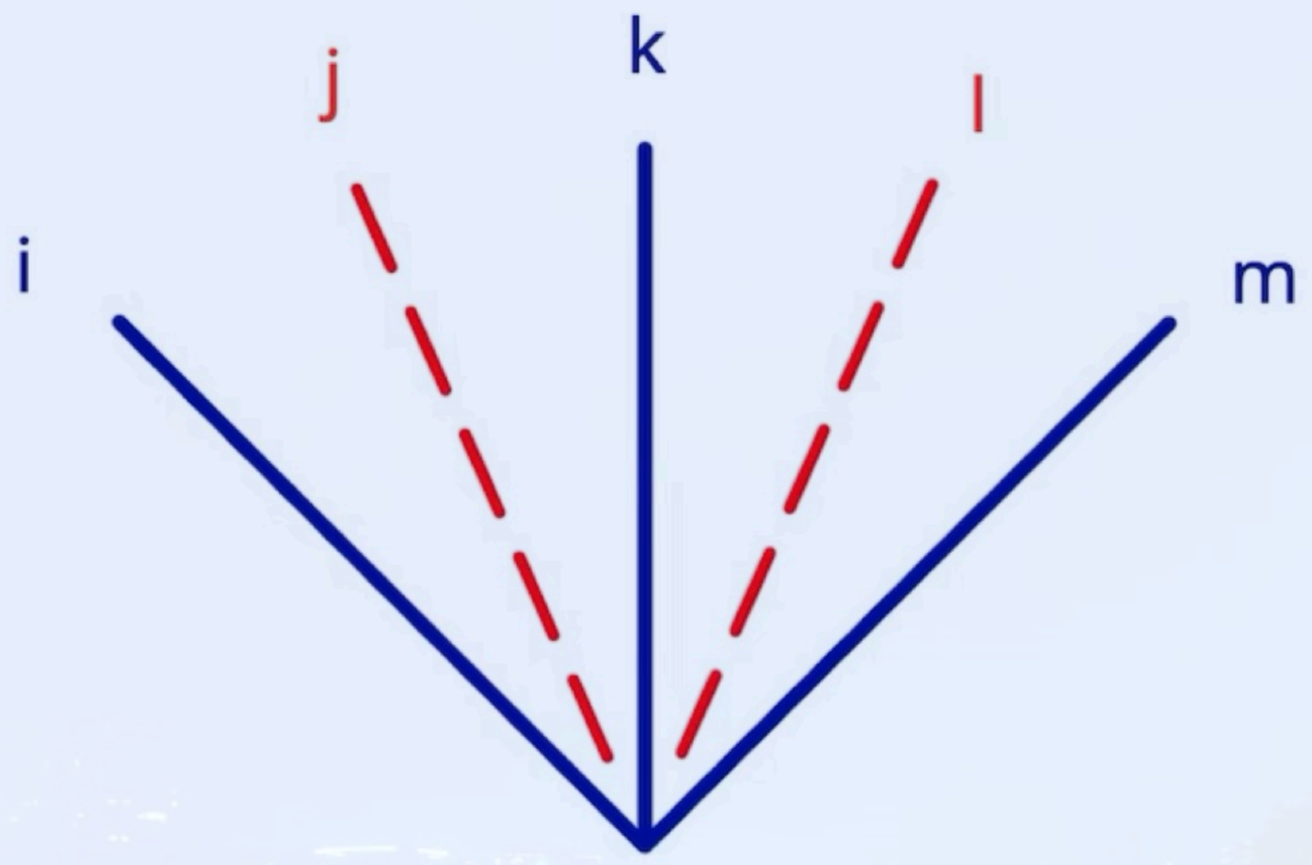
$$X^0_4(i^h, j, k, l^h)M^0_n(\ldots, \widetilde{(ijk)}, \widetilde{(jkl)}, \ldots)$$

$$X^0_3(i,j,k) = \frac{|\mathcal{M}^0_3(i,j,k)|^2}{|\mathcal{M}^0_2(\tilde{I},\tilde{K})|^2} \quad X^0_4(i,j,k,l) = \frac{|\mathcal{M}^0_4(i,j,k,l)|^2}{|\mathcal{M}^0_2(\widetilde{ijk},\widetilde{jkl})|^2}$$

Generalized Antenna @ NNLO

Based on Marcoli's slide @ Loop Summit 2

There is more ... **almost colour-connected** emissions: only one shared hard radiator



NOT fully iterated: the two emissions "feel" each other through the recoil on the shared radiator

traditional antenna functions can be used, but a **very complicated** sequence of iterated structures is needed, plus **Large-Angle-Soft-Terms**

[Gehrmann-De Ridder, Gehrmann, Glover, Heinrich '07]

[Weinzierl '08] [Currie, Glover, Wells '13]

most complicated and inefficient sector of antenna subtraction

```
-1/2*d30FF(l,i,j)*d30FF(m,k,[j,i])*B1g0Z([l,i],[[j,i],k], [m,k],1,2)*JET33([l,i],[[j,i],k], [m,k])*a31
+1/2*d30FF(m,i,j)*d30FF([m,i],k,[j,i])*B1g0Z(l,[[j,i],k],[[m,i],k],1,2)*JET33(
l,[[j,i],k],[[m,i],k])*a32
-1/2*A30FF(l,i,m)*d30FF([m,i],k,j)*B1g0Z([l,i],[j,k],[[m,i],k],1,2)*JET33([l,i],[j,k],[[m,i],k])*a33
-1/2*(SFF([i,j],i,[m,i])-SFF([k,[i,j]],i,[k,[m,i]])-SFF(l,i,[i,j])+SFF(l,i,[k,[i,j]])-SFF(
l,i,[m,i])+SFF(l,i,[k,[m,i]]))*d30FF([m,i],k,[i,j])*B1g0Z(l,[k,[i,j]],[k,[m,i]],1,2)*JET33(
l,[k,[i,j]],[k,[m,i]])*a34

-1/2*d30FF(m,k,j)*d30FF(l,i,[j,k])*B1g0Z([l,i],[[j,k],i], [m,k],1,2)*JET33([l,i],[[j,k],i], [m,k])*a35
+1/2*d30FF(
l,k,j)*d30FF([l,k],i,[j,k])*B1g0Z([[l,k],i],[[j,k],i],m,1,2)*JET33([[l,k],i],[[j,k],i],m)*a36
-1/2*A30FF(l,k,m)*d30FF([l,k],i,j)*B1g0Z([[l,k],i],[j,i], [m,k],1,2)*JET33([[l,k],i],[j,i], [m,k])*a37
-1/2*(SFF([k,j],k,[l,k])-SFF([i,[k,j]],k,[i,[l,k]])-SFF(m,k,[k,j])+SFF(m,k,[i,[k,j]])-SFF(
m,k,[l,k])+SFF(m,k,[i,[l,k]]))*d30FF([l,k],i,[k,j])*B1g0Z([i,[l,k]],[i,[k,j]],m,1,2)*JET33(
m,[i,[k,j]],[i,[l,k]])*a38

-1/2*d30FF(m,i,j)*d30FF(l,k,[j,i])*B1g0Z([l,k],[[j,i],k], [m,i],1,2)*JET33([l,k],[[j,i],k], [m,i])*a39
+1/2*d30FF(
l,i,j)*d30FF([l,i],k,[j,i])*B1g0Z([[l,i],k],[[j,i],k],m,1,2)*JET33([[l,i],k],[[j,i],k],m)*a40
-1/2*A30FF(l,i,m)*d30FF([l,i],k,j)*B1g0Z([[l,i],k],[j,k], [m,i],1,2)*JET33([[l,i],k],[j,k], [m,i])*a41
-1/2*(SFF([i,j],i,[l,i])-SFF([k,[i,j]],i,[k,[l,i]])-SFF(m,i,[i,j])+SFF(m,i,[k,[i,j]])-SFF(
m,i,[l,i])+SFF(m,i,[k,[l,i]]))*d30FF([l,i],k,[i,j])*B1g0Z([k,[l,i]],[k,[i,j]],m,1,2)*JET33(
m,[k,[i,j]],[k,[l,i]])*a42

-1/2*d30FF(l,k,j)*d30FF(m,i,[j,k])*B1g0Z([l,k],[[j,k],i], [m,i],1,2)*JET33([l,k],[[j,k],i], [m,i])*a43
+1/2*d30FF(m,k,j)*d30FF([m,k],i,[j,k])*B1g0Z(l,[[j,k],i],[[m,k],i],1,2)*JET33(
l,[[j,k],i],[[m,k],i])*a44
-1/2*A30FF(l,k,m)*d30FF([m,k],i,j)*B1g0Z([l,k],[j,i],[[m,k],i],1,2)*JET33([l,k],[j,i],[[m,k],i])*a45
-1/2*(SFF([k,j],k,[m,k])-SFF([i,[k,j]],k,[i,[m,k]])-SFF(l,k,[k,j])+SFF(l,k,[i,[k,j]])-SFF(
l,k,[m,k])+SFF(l,k,[i,[m,k]]))*d30FF([m,k],i,[k,j])*B1g0Z(l,[i,[k,j]],[i,[m,k]],1,2)*JET33(
l,[i,[k,j]],[i,[m,k]])*a46

+1/2*d30FF(l,k,j)*A30FF([l,k],i,m)*B1g0Z([[l,k],i],[j,k], [m,i],1,2)*JET33([[l,k],i],[j,k], [m,i])*a47
+1/2*d30FF(m,k,j)*A30FF(l,i,[m,k])*B1g0Z([l,i],[j,k],[[m,k],i],1,2)*JET33([l,i],[j,k],[[m,k],i])*a48
-1/2*A30FF(
l,k,m)*A30FF([l,k],i,[m,k])*B1g0Z([[l,k],i],j,[[m,k],i],1,2)*JET33([[l,k],i],j,[[m,k],i])*a49
+1/2*(-SFF([l,k],k,j)-SFF([m,k],k,j)+SFF([l,k],k,[m,k])+SFF([[l,k],i],k,j)+SFF([[m,k],i],k,j)-SFF([[l,
k],i],k,[[m,k],i]))*A30FF([l,k],i,[m,k])*B1g0Z([[l,k],i],j,[i,[m,k]],1,2)*JET33([[l,k],i],j,[i,[m,k]])
*a50

+1/2*d30FF(l,i,j)*A30FF([l,i],k,m)*B1g0Z([[l,i],k],[j,i], [m,k],1,2)*JET33([[l,i],k],[j,i], [m,k])*a51
+1/2*d30FF(m,i,j)*A30FF(l,k,[m,i])*B1g0Z([l,k],[j,i],[[m,i],k],1,2)*JET33([l,k],[j,i],[[m,i],k])*a52
-1/2*A30FF(
l,i,m)*A30FF([l,i],k,[m,i])*B1g0Z([[l,i],k],j,[[m,i],k],1,2)*JET33([[l,i],k],j,[[m,i],k])*a53
+1/2*(-SFF([l,i],i,j)-SFF([m,i],i,j)+SFF([l,i],i,[m,i])+SFF([[l,i],k],i,j)+SFF([[m,i],k],i,j)-SFF([[l,
i],k],i,[[m,i],k]))*A30FF([l,i],k,[m,i])*B1g0Z([[l,i],k],j,[k,[m,i]],1,2)*JET33([[l,i],k],j,[k,[m,i]])
*a54
```

23 additional subtraction terms for leading colour $\mathscr{M}^0_{RR}(q,g,g,g,\bar{q})$

Ideally we want:

$$M^0_{n+2}(\ldots,i,j,k,l,m\ldots)$$

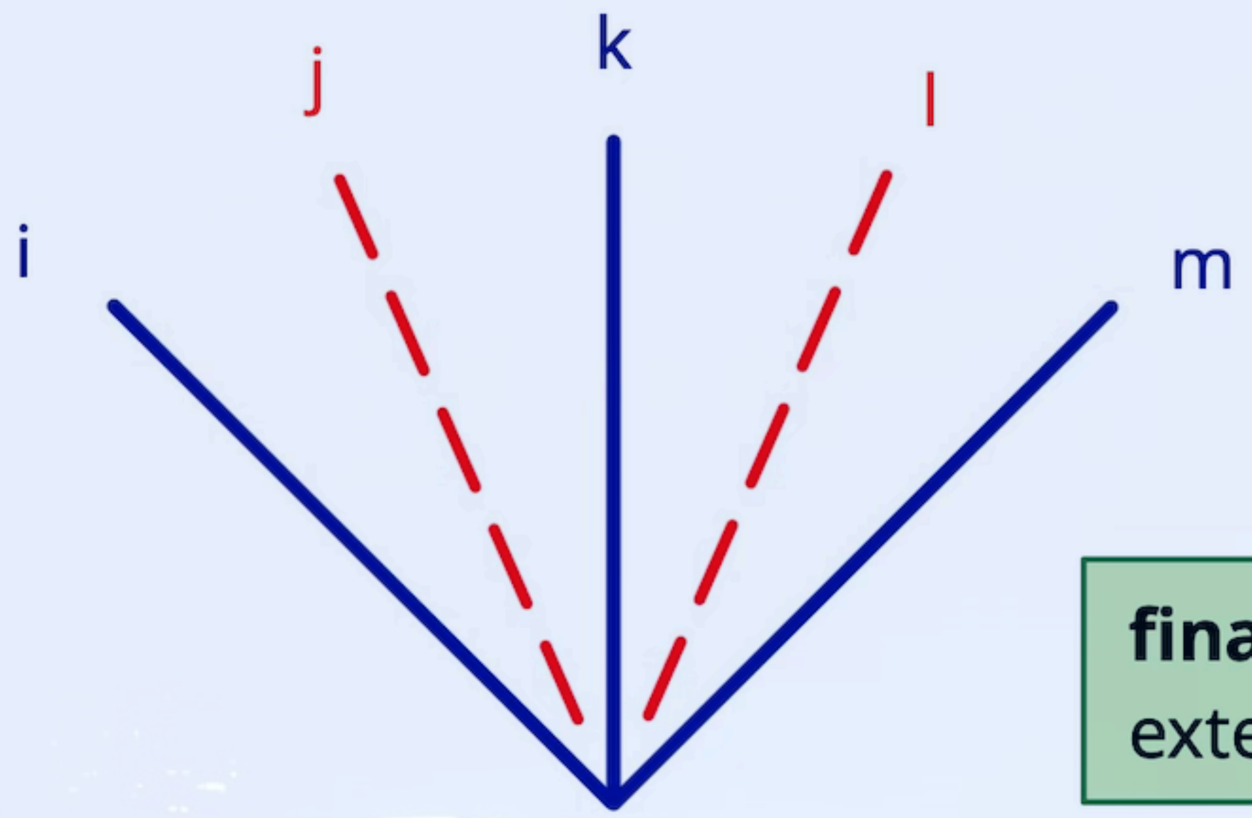
5→3 momentum mapping

$$X^0_{5,3}(i^h,j,k^h,l,m^h)M^0_n(\ldots,\widetilde{(ijk)},\widetilde{(ijklm)},\widetilde{(klm)},\ldots)$$

generalized three-hard-radiator antenna function

Generalized Antenna @ NNLO

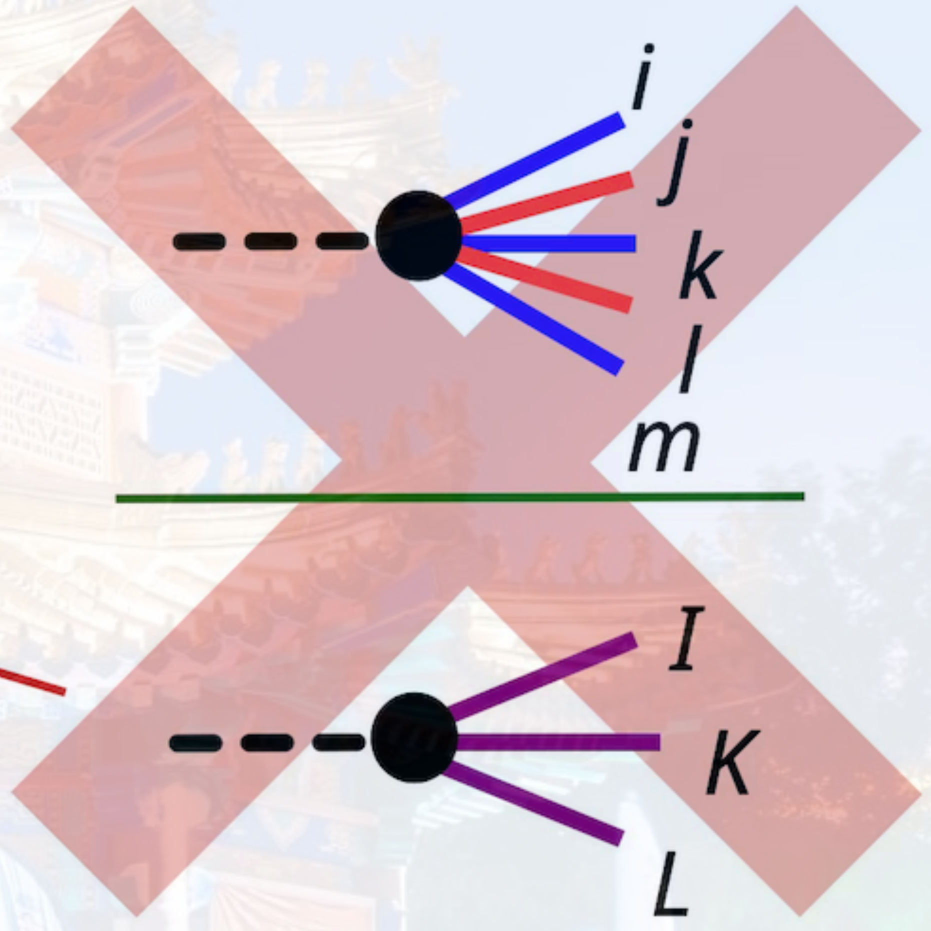
Based on Marcoli's slide @ Loop Summit 2



final-state radiation only, extension to ISR in progress

Not possible with matrix element-based antenna functions

non-trivial function of the three-particle phase space



With the **designer antenna algorithm**, it is possible to construct antenna functions with **more than two hard radiators: generalized antenna functions.**

[Fox, Glover, Marcoli '24]

Analytical integration made particularly simple thanks to a convenient choice of **5→3 momentum mapping**.

$$\text{map}_{5\to3}: \quad \begin{aligned} p_I &= p_i + p_j - \frac{s_{ij}}{s_{ik}+s_{jk}} p_k \\ p_K &= \left(1 + \frac{s_{ij}}{s_{ik}+s_{jk}} + \frac{s_{lm}}{s_{lk}+s_{mk}}\right) p_k \\ p_M &= p_l + p_m - \frac{s_{lm}}{s_{lk}+s_{mk}} p_k \end{aligned}$$

iterated dipole mapping

- Constructed with an iterative algorithm
  - From the desired IR limits (not from physical matrix-elements)
  - Using projector (up-down) to connect full phase space (antennae) and subspace (IR limits)
  - Can be integrated analytically (as in conventional method)

[Braun-White, Glover, Preuss '23]

[Braun-White, Glover, Preuss '23]

[Fox, Glover '23]

STATE-OF-THE-ART PREDICTIONS FOR $d\sigma_{N^3LO+N^{3(4)}LL}$

FO	α_s^n	$H(m_V,\mu)$	$I^{(n)}_{i/j}(x,b)$	$\ln W(x_a, x_b, m_V, \vec{b}, \mu = b_0/b) \sim \int_{\mu_h}^{\mu} d\bar{\mu}/\bar{\mu}\left(A(\alpha_s(\bar{\mu}))\ln\frac{m_V^2}{\bar{\mu}^2} + B(\alpha_s(\bar{\mu}))\right)$						
$\frac{d\hat{\sigma}^V_{NLO}}{dq_T}$	NLO	✓	✓	$\ln^2(b^2m_V^2)$	$\ln(b^2m_V^2)$	1				
$\frac{d\hat{\sigma}^V_{NNLO}}{dq_T}$	N2LO	✓	✓	$\ln^3(b^2m_V^2)$	$\ln^2(b^2m_V^2)$	$\ln(b^2m_V^2)$	1			
$\frac{d\hat{\sigma}^V_{N^3LO}}{dq_T}$	N3LO	✓	✓	$\ln^4(b^2m_V^2)$	$\ln^3(b^2m_V^2)$	$\ln^2(b^2m_V^2)$	$\ln(b^2m_V^2)$	1		
$\frac{d\hat{\sigma}^V_{N^4LO}}{dq_T}$	N4LO	✓	✗	$\ln^5(b^2m_V^2)$	$\ln^4(b^2m_V^2)$	$\ln^3(b^2m_V^2)$	$\ln^2(b^2m_V^2)$	$\ln(b^2m_V^2)$	1	
…	…			…	…	…	…		…	…
$\frac{d\hat{\sigma}^V_{N^kLO}}{dq_T}$	NKLO			$\ln^{k+1}(b^2m_V^2)$	$\ln^k(b^2m_V^2)$	$\ln^{k-1}(b^2m_V^2)$	$\ln^{k-2}(b^2m_V^2)$	$\ln^{k-3}(b^2m_V^2)$	…	…
…	…			…	…	…	…		…	…
Resum				LL	NLL	NNLL	N3LL	N4LL	…	$N^{k+1}LL$
A				A1 ✓	A2 ✓	A3 ✓	A4 ✓	A5 ✗	…	A_{k+2}
B					B1 ✓	B2 ✓	B3 ✓	B4 ✓	…	B_{k+1}

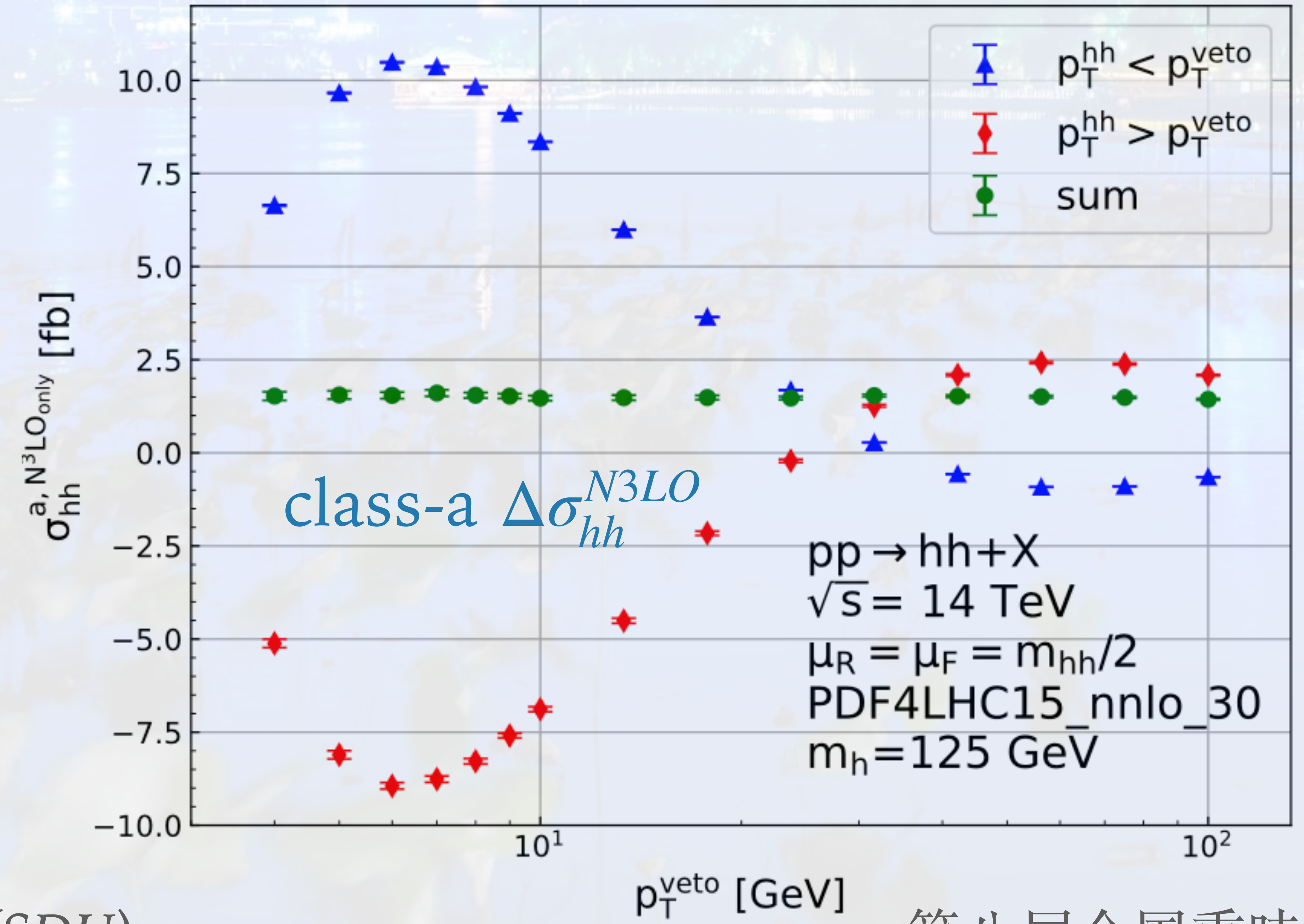
HH Production at N3LO QCD

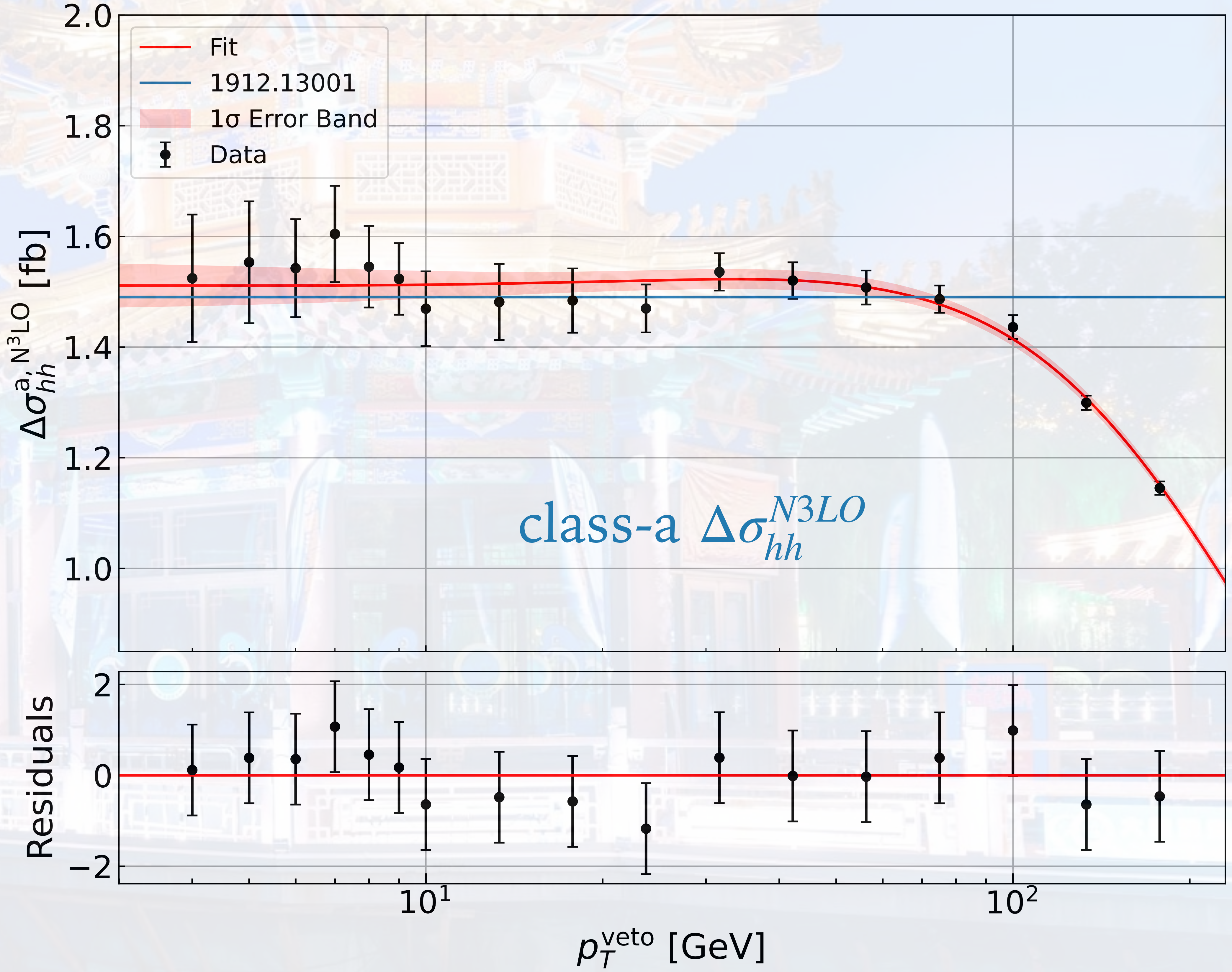
- Class-a contribution achieve N3LO accuracy via q_T-slicing:

$$d\sigma_{hh} = d\sigma_{hh}\Big|_{p_T^{hh}<p_T^{veto}} + d\sigma_{hh}\Big|_{p_T^{hh}>p_T^{veto}}$$

- Large cancellation between above and below p_T^{veto} cut.
- Fit the p_T^{veto} power correction with χ^2 and functional form:

$$\Delta\sigma^{\mathrm{N^3LO}}\left(p_T^{\mathrm{veto}}\right) \sim \Delta\sigma_0^{\mathrm{N^3LO}} + \frac{\alpha_s^3}{(2\pi)^3}\frac{p_T^{\mathrm{veto}2}}{m_{hh}^2}\sum_{n=0}^{5} a_n^{(3)} \ln^n \frac{m_{hh}}{p_T^{\mathrm{veto}}}$$





XC, Y.H. Dai, H.T. Li, S.Y. Li, H.S. Shao, J. Wang 2601.19990 accepted by JHEP

- p_T^{veto} plateau observed < 10 GeV.
- Excellent agreement with inclusive total XS:

  N3LO only ($p_T^{veto} \to 0$): 1.513(47) fb

  N3LO only reference: 1.4904(1) fb (based on iHixs 2)

F. Dulat, A. Lazopolulos, B. Mistlberger, Comput.Phys.Commun 233 (2018)

STATE-OF-THE-ART PREDICTIONS: $d\sigma_{N^3LO}$

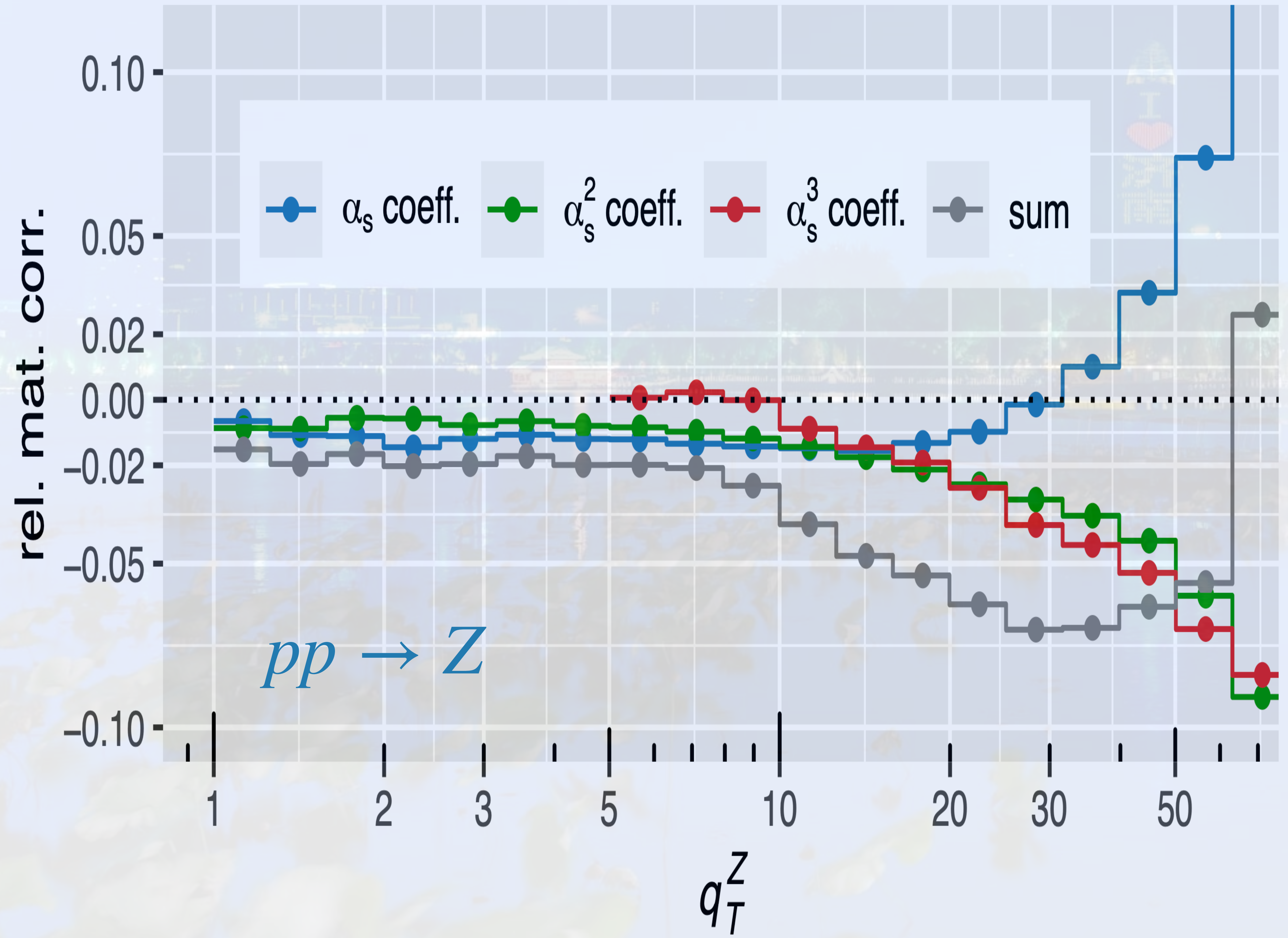
➤qT slicing at N3LO for neutral and charged current production (MCFM)

$$\sum d\sigma^{V}_{N3LO} \equiv \sum_{dp_{T,V}} d\sigma^{V+jet}_{NNLO}/dp_{T,V}|_{p_{T,V}>q_T^{cut}} + \sum_{dp_{T,V}} d\sigma^{V\ SCET}_{N^3LO}/dp_{T,V}|_{p_{T,V}\in[0,q_T^{cut}]}$$

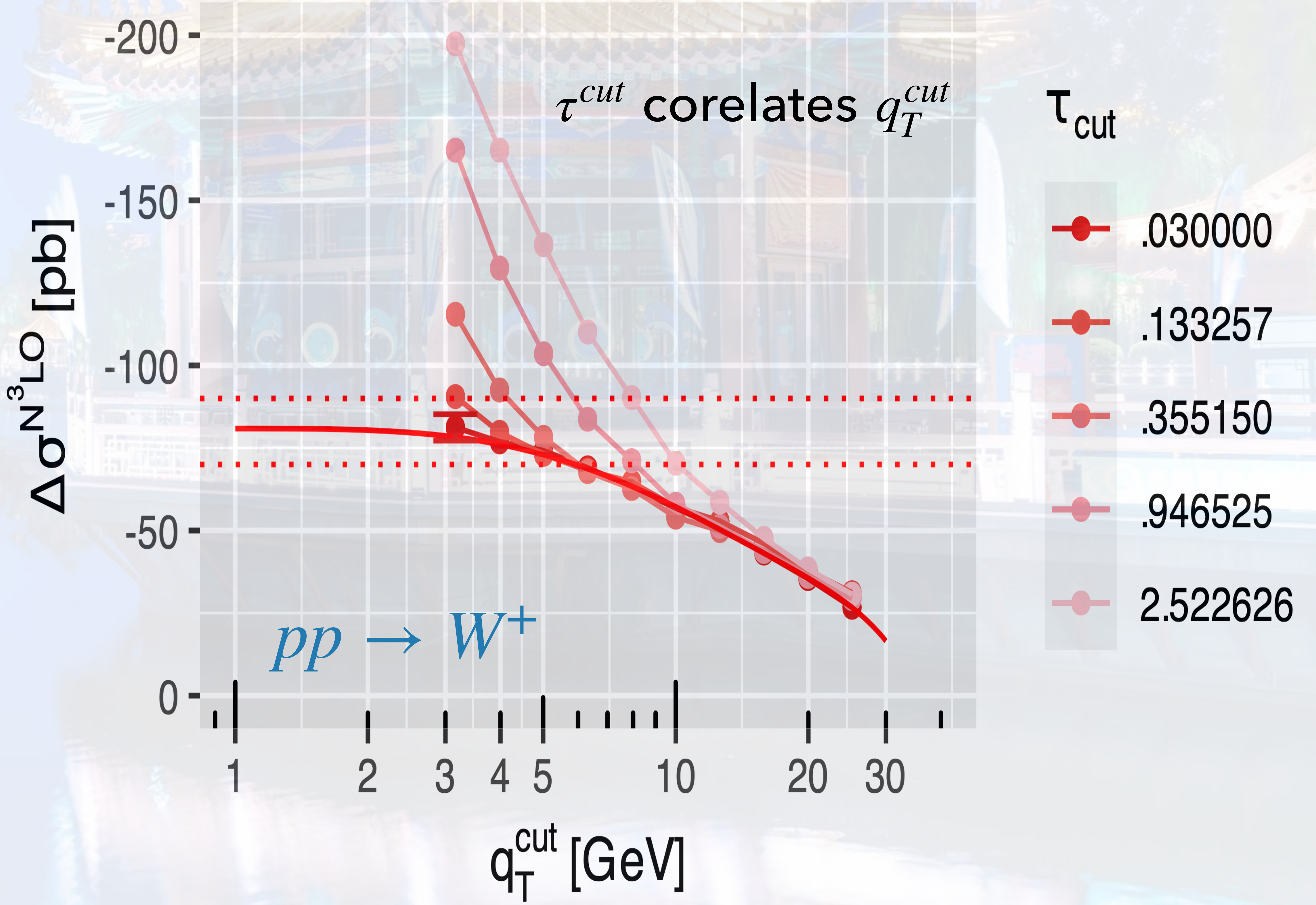
NC MCFM: $-22.6\,\text{pb} \pm 1.4\,\text{pb}\,(\text{num.}) \pm 1\,\text{pb}\,(\text{slicing})$

NC NNLOJET: $-18.7\,\text{pb} \pm 1.1\,\text{pb}\,(\text{num.}) \pm 0.9\,\text{pb}\,(\text{slicing})$

CC agree to inclusive XS within ± 60% uncertainty of $\Delta(\alpha_s^3)$



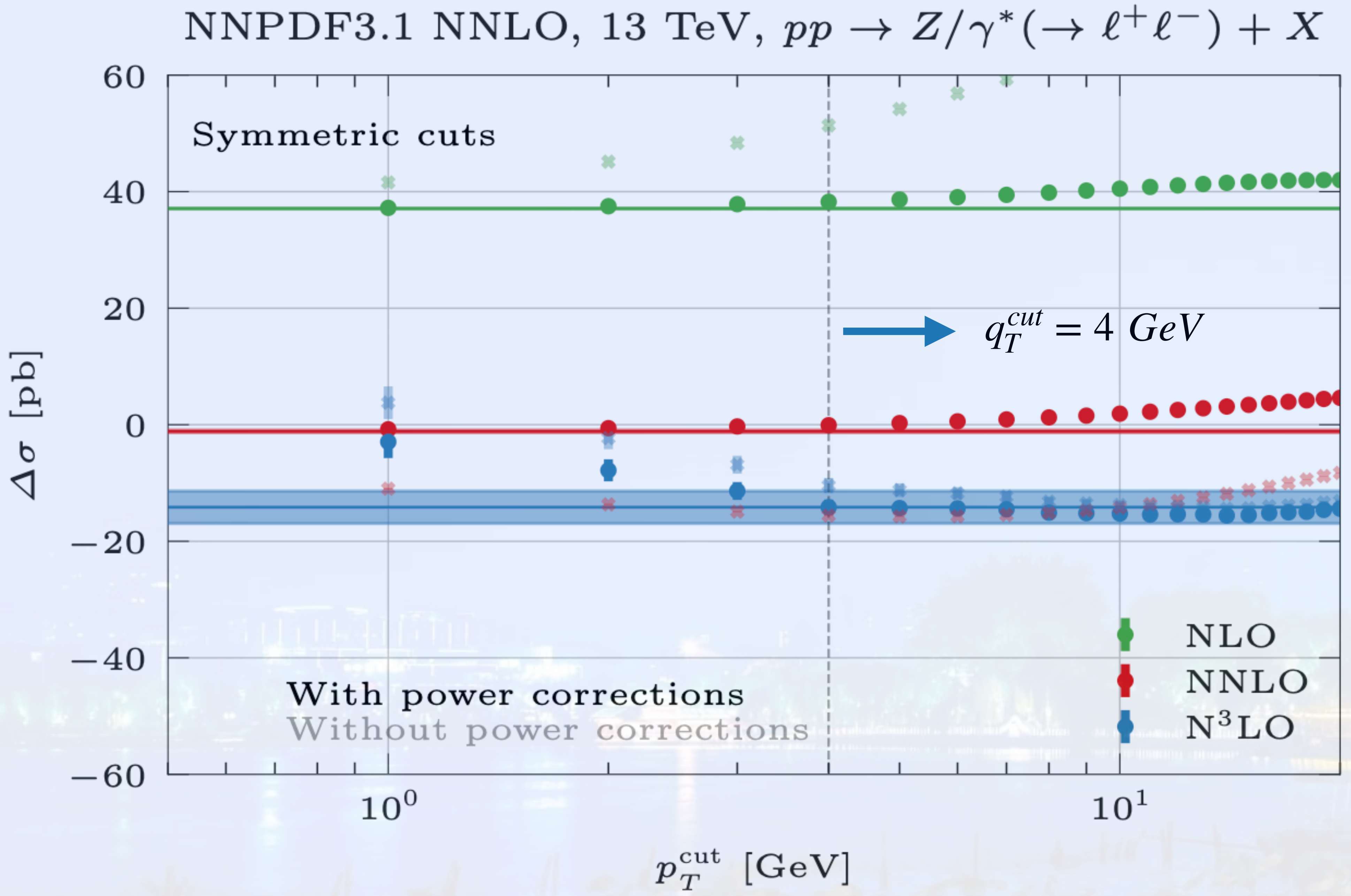
Neumann and Campbell *Phys.Rev.D* 107 (2023) 1



Neumann and Campbell *JHEP 11 (2023) 127*

Precision Predictions at Hadron Collider

$2 \to 1$ @ N3LO (+ N3LL) QCD



XC, T. Gehrmann, N. Glover, et. al. PRL 128, 252001 (2022)

- Solid horizontal lines: NLO, NNLO at 1 GeV, N3LO at 4 GeV with MC error.
  - N3LO shows no plateau in 1905.05171
- Pale dots are values used by DYTurbo in 2103.04974 and 2303.12781 (taken from 1905.05171).
  - Fiducial power corrections are not included.
  - Leads to 30% difference of N3LO coefficients at $q_T^{cut} = 4\ GeV$.
- Solid dots are corrected values with fiducial power correction.
  - Central value shifts 2 pb starting from NLO (the dominant error).
  - ± 2.1 pb uncertainty from MC and q_T^{cut} (estimated from [3,5] GeV region).
  - Not consistent with DYTurbo update result of ± 0.7 pb uncertainty.

DYTurbo result with fiducial power correction

Order	N^{3}LO
q_T subtr. ($q_T^{\rm cut} = 4\,{\rm GeV}$)	$747.1 \pm 0.7\,{\rm pb}$
recoil q_T subtr.	$745.7 \pm 0.7\,{\rm pb}$

S. Camarda, L. Cieri, G. Ferrera Eur.Phys.J.C 82 (2022) 6

DYTurbo result without fiducial power correction cited in ATLAS α_s fitting

Order	NLO	NNLO	N^{3}LO
$\sigma(pp \to Z/\gamma^* \to l^+l^-)$ [pb]	766.3 ± 1	757.4 ± 2	746.1 ± 2.5
Order	NLL+NLO	NNLL+NNLO	N^{3}LL+N^{3}LO
$\sigma(pp \to Z/\gamma^* \to l^+l^-)$ [pb]	773.7 ± 1	759.8 ± 2	749.6 ± 2.5

*S. Camarda, L. Cieri, G. Ferrera Phys. Rev. D **104**, L111503 (2021)*