入门简介 

# Small-x physics 101 (Brief introduction)

## ① Some terminologies:

In QCD, there are two complementary approaches:

DGLAP — 1. Collinear factorization ($\Rightarrow xq(x)$, $xg(x)$) neglected $k_\perp \ll Q$ | integrated $k_\perp$

both $Q^2, S \to \infty$, while $Q^2/S$ is fixed at large $x$.

BFKL — 2. TMD ($k_\perp$) factorization, $xq(x, k_\perp)$ $xg(x, k_\perp)$

BK, JIMWLK — $Q^2$ fixed, $S \to \infty$, $x = Q^2/S \to 0$ $k_\perp$ becomes important.

→ Non-linear, BFKL → linear. intrinsic $k_\perp$ not negligible

In Case 1, for observables not sensitive to $k_\perp$. If $k_\perp$ is measured, needs to improve CF by adding Collins-Soper-Sterman evolution. (TMD)

What is evolution? — Resummation of large logs.

What is resummation?

— Summation needs to be done once more.

What is the summation in the first place?

— In pQCD, expand $\sigma = \sigma_0 + \alpha_s \sigma_1 + \alpha_s^2 \sigma_2 + \cdots$

Sometimes, this is OK, Often, it is not sufficient since $\sigma_n$ can contain large logs

Lose predictive power!

Suppose $\alpha_s^n \sigma_n = C_n (\alpha_s L)^n + \cdot\, C(\alpha_s L)^{n-1} \alpha_s + \cdots$

LL NLL

Examples:

| | | |
|---|---|---|
| DGLAP | $(\alpha_s \ln \frac{Q^2}{Q_0^2})^n$ | physics comes from various diagrams |
| BFKL | $(\alpha_s \ln \frac{1}{x})^n$ | |
| CSS | $(\alpha_s \ln^2 \frac{Q}{k_\perp})^n +$ | |

Integrate over gluon phase-space ⇒ large logs

Gluon clouds are forced to be $\gamma^*$ produced!

purpose of resummation

obviously, can't do all order calculation in pQCD but with resummation, we can achieve two things.

① Restore predictive power.

② Describe physics and data better.

Example: dijet azimuthal angular correlation

☆ Why we go to coordinate space?

Joint resummation of CSS + small-x

Ref 1308.2993

| | | |
|---|---|---|
| (GeV $f_m$) | $(\frac{M}{p^+})L$, $(xp^+) \to \lambda = \frac{1}{xp^+}$ <br> $\lambda \gg \frac{m}{p^+}L \Rightarrow \frac{1}{xp^+} \gg \frac{mL}{p^+}$ $\quad x \leq \frac{1}{mL}$ | $= \frac{1}{5\times A^{1/3}}$ |

② Definitions of Gluon distributions (initial conditions)

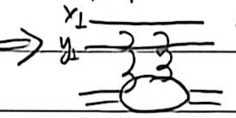
small-x UGDs ($k_\perp$) dependent

$\boxed{\text{gauge invariant}}$ Ref hep-ph/0307037 & 1101.0715

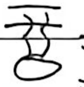
Dipole-gluon Weizsäcker-Williams ↗ Jackson QED Jackiw-Kabat-Ortiz 92

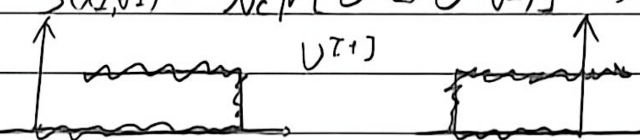
JP Ma ⓐ In small-x limit

TMD op. $xG_{DP} = 2\int \frac{d\xi^- d^2\xi_\perp}{(2\pi)^3 p^+} e^{-ik_\perp\cdot\xi_\perp - ixp^+\xi^-} \langle P| Tr[F^{i+}(\xi) U^{[-]\dagger} F^{i+}(0) U^{[+]}]|P\rangle$

use = to probe gluon $\Rightarrow$  $= \frac{k_\perp^2 N_c}{2\pi^2\alpha_s}\int d^2x_\perp \int d^2y_\perp e^{-ik_\perp(x_\perp - y_\perp)} S(x_\perp, y_\perp)$

$S(x_\perp, y_\perp) = \frac{1}{N_c} Tr[U(x_\perp) U^\dagger(y_\perp)] \Rightarrow e^{-\frac{1}{4}Q_s^2(x_\perp - y_\perp)^2}$

 $= 1 - S(x_\perp, y_\perp) = T(x_\perp - y_\perp)$



$U^{[+]}$

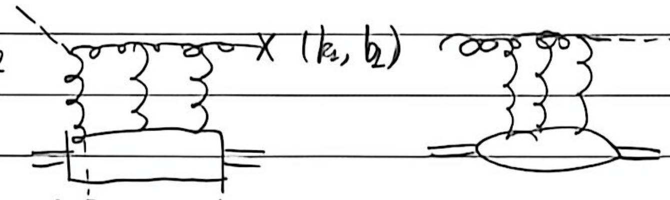
$\Rightarrow$ Wilson Loop $\Rightarrow$ $\partial_{x_\perp}\cdot\partial_{y_\perp}(S(x_\perp, y_\perp))$

$U(x_\perp) = P e^{-ig\int A^+(x_\perp, x^-)dx^-}$ ; $Q_c^2$ are fundamental

ⓑ $xG_{WW}$ defined with $U^{[+]}\cdot U^{[+]\dagger}$ or $U^{[-]}\, U^{[-]\dagger}$

only difference is in the gauge links

Only future



$x\ (k_\perp, b_\perp)$ Higgs only past

large nucleus

$0, z \qquad L$

1308.2993 $\quad \frac{dN_g}{dy\, d^2k_\perp} = xG_{WW}(x, k_\perp) = \int d^2b_\perp \int \frac{d^2x_\perp}{(2\pi)^2} e^{-ik_\perp\cdot x_\perp}\int_0^L dz\, \underbrace{\rho(xG_N(x))}_{\#\text{ density of gluons}}$

$e^{-\frac{1}{4}Q_s^2 x_\perp^2 \frac{(L-z)}{L}}$

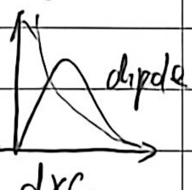
$\frac{4\pi^2\alpha_s N_c}{N_c^2-1}\rho\, xG(x) L$

keep the back-to-back configuration

$= \frac{(N_c^2-1)S_\perp}{\pi^2\alpha_s N_c}\int \frac{d^2x_\perp}{(2\pi)^2} e^{-ik_\perp\cdot x_\perp}\frac{1}{x_\perp^2}(1 - e^{-\frac{1}{4}Q_s^2 x_\perp^2})$

$= \int \frac{d^2x_\perp d^2x_\perp'}{(2\pi)^2} e^{-ik_\perp\cdot(x_\perp - x_\perp')} S_{WW}(x_\perp, x_\perp')$

$S_{WW} = \frac{N_c}{2\pi^2\alpha_s}\frac{1}{N_c}\langle Tr[i\partial_i U(x_\perp) U^\dagger(x_\perp') i\partial_i U(x_\perp') U^\dagger(x_\perp)]\rangle$



digamma function

| | | |
|---|---|---|
| | $\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$ | $\hbar c = 0.2$ GeV fm $-1$ |
| | $\gamma =$ | 1 GeV fm $= 5$ |

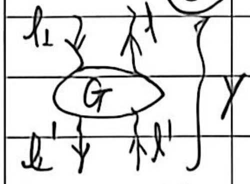


## 3 Small-x evolution equations

BFKL equation in the mom and coordinate spaces

(a) Original BFKL is regarding BFKL propagator

$G(l_\perp, l'_\perp, Y)$



$$\partial_Y G(l_\perp, l'_\perp, Y) = \frac{\bar\alpha_s}{\pi}\int \frac{d^2 q_\perp}{(q_\perp - l_\perp)^2}\left[G(q_\perp, l'_\perp; Y) - \frac{l^2}{2 q_\perp^2} G(l_\perp, l'_\perp, Y)\right]$$

$\bar\alpha_s = \frac{\alpha_s N_c}{\pi}$

$G_\gamma(Y) = G_\gamma(0)\, e^{\bar\alpha \chi(\gamma) Y}$ ; take $Y \to \infty$

saddle point approximation

key is to use dim-reg to handle singularities. 1607.04726 Appendix A

$$G(l_\perp, l'_\perp, Y) = \int \frac{d\gamma}{2\pi i}\left(\frac{l_\perp^2}{l'^2}\right)^{-\gamma} G_\gamma(Y)$$

$\Rightarrow \partial G_\gamma(Y) = \bar\alpha_s \chi(\gamma) G_\gamma(Y)$

$\Rightarrow G_\gamma(Y) = G_\gamma(0)\, e^{\bar\alpha_s \chi(\gamma) Y}$.

$Y \to \infty$. $\chi(\frac{1}{2}) = \boxed{4\ln 2}$ ? $G_\gamma(Y) \to \infty$ ?

Distribution? ☆ Comments LO BFKL too fast. too large

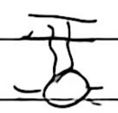
Not easy to see gluon! NLO BFKL turns negative $\boxed{\frac{\alpha_s N_c}{\pi} 4\ln 2}$

phenomenology Salam 2001. also need resummation 2.8

and running coupling.

(b) Formulate BFKL in coordinate space

Mueller's dipole model. define $T = 1 - S \ll 1$



$$\partial_Y T(x_{01}, Y) = \frac{\bar\alpha_s}{2\pi}\int d^2x_2 \frac{x_{10}^2}{x_{12}^2 x_{20}^2}\left[T(x_{12}, Y) + T(x_{20}) - T(x_{10})\right]$$

Define $T(x, Y) = \int_{c-i\infty}^{c+i\infty} \frac{d\gamma}{2\pi i}\left(\frac{x^2}{x_{10}^2}\right)^{\gamma} T_\gamma(Y)$

Note $\chi(\gamma) = \frac{1}{2\pi}\int dx_2 \frac{x_{10}^2}{x_{12}^2 x_{20}^2}\left[\left(\frac{x_{12}^2}{x_{10}}\right)^{\gamma} + \left(\frac{x_{20}^2}{x_{10}}\right)^{\gamma} - 1\right]$

$\partial T_\gamma = \bar\alpha_s \chi(\gamma) T_\gamma$

(1) equivalent to mom-space one

(2) related to gluons distribution more easily

(3) can be extended to include non-linear correction

$\int d^2 b_\perp \frac{\vec{b}_\perp}{b_\perp^2} e^{-ik_\perp \cdot b_\perp} = -2\pi i \frac{\vec{b}_\perp}{b_\perp^2}$ $\quad \int d^2 b_\perp e^{ik_\perp \cdot b_\perp} \int d^2 k_\perp \frac{\vec{k}_\perp}{k_\perp^2} e^{-ik_\perp \cdot b_\perp} = (2\pi)^2 \frac{\vec{k}_\perp}{k_\perp^2} = \int d^2 b_\perp e^{(-2\pi i)} \frac{\vec{b}_\perp}{b_\perp^2}$

$$\frac{\vec{k}_\perp}{k_\perp^2} = \frac{1}{2\pi i} \int \frac{d^2 b_\perp}{b_\perp^2} \vec{b}_\perp e^{+ik_\perp \cdot b_\perp} = \frac{1}{2\pi i} \int \frac{d^2 b_\perp}{b_\perp^2} \vec{b}_\perp e^{+ik_\perp \cdot b_\perp}$$

$$\int \frac{d^2 k_\perp k_\perp}{k_\perp^2 + \lambda^2} \quad 2\pi J_0(k_\perp b_\perp)$$



C. Non-linear BK evolution equation and saturation

when gluon density becomes high non-linear dynamics becomes important &

BK equation.

$$\frac{\partial}{\partial y} S^{(2)}(x_\perp, y_\perp) = -\frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 b_\perp (x_\perp - y_\perp)^2}{(x_\perp - b_\perp)^2 (y_\perp - b_\perp)^2} \left[ S^{(2)}(x_\perp - y_\perp) - S^{(4)}(x_\perp, b_\perp, y_\perp) \right]$$

① Approximately, $S^{(4)} \doteq S^{(2)}(x_\perp, b_\perp)\, S^{(2)}(b_\perp, y_\perp)$

mean field, large nucleus. $\frac{1}{N_c}$ suppressed

② closed equation, with stable fixed point at $S=0$, unstable fixed point at $S=1$

③ $$\frac{(x_\perp - y_\perp)^2}{(x_\perp - b_\perp)^2 (b_\perp - y_\perp)^2} = \underbrace{\frac{1}{(x - b_\perp)^2} + \frac{1}{(y_\perp - b_\perp)^2}}_{\text{virtual}} + \underbrace{\frac{2(x_\perp - b_\perp)(b_\perp - y_\perp)}{(x_\perp - b_\perp)^2 (b_\perp - y_\perp)^2}}_{\text{real}}$$

④ $S^{(2)}$ no non-linear scattering

$S^{(4)}$ non-lineal multiple scattering

⑤ Reduces to BFKL $S = 1 - T$ when $T$ is small

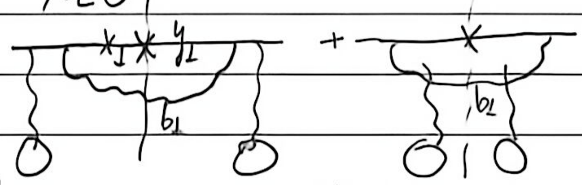
non-linear term decrease the exponential growth, make gluon distribution rise less rapidely. Leading to saturation & $T \to 1$, $S \to 0$

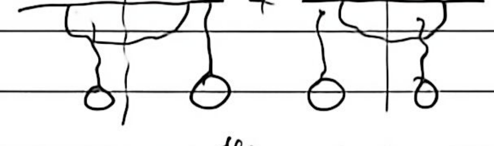
⑥ Deriving BK equation from a real process.

LO

$F_0(k_\perp)$

NLO,



$$\frac{2(x_\perp - b_\perp)\cdot(y_\perp - b_\perp)}{(x_\perp - b_\perp)^2 (y_\perp - b_\perp)^2} S^{(2)}(x_\perp - y_\perp)$$

$- S^{(4)}(x_\perp, b_\perp, y_\perp)$

$$F(k_\perp) = F_0(k_\perp) - \frac{\alpha_s N_c}{2\pi^2} \int \frac{d\xi}{1-\xi} \int \frac{d^2 x_\perp d^2 y_\perp d^2 b_\perp}{(2\pi)^2} K(x, y, b_\perp) e^{-ik\cdot(x_\perp - y_\perp)} (S^{(2)} - S^{(4)})$$

Rapidity divergence $\int_0^{1-x} \frac{d\xi}{1-\xi} = \ln \frac{1}{x}$