Constructing Massive Vector Amplitudes From Consistency Conditions

谌俊谋 (暨南大学)

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1. Introduction
Constructing Amplitudes and Theories

1. Introduction

Traditional View of QFT

- Lagrangian --> Quantization&LSZ -> S-matrix
- Feynman rules --> amplitudes

Problem:
1. The concept of field is often redundant, especially for gauge theory
2. S-matrix is more directly related to observables.

Modern View

- 1. Starting from S-matrix/Amplitudes Directly
- 2. Lorentz symmetry and unitarity
- 3. Fix amplitudes.
- 4. Extract the underlying theory

Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass*

STEVEN WEINBERG†
Physics Department, University of California, Berkeley, California
(Received 13 April 1964)

S. Weinberg, Phys.Rev. 135 (1964) B1049–B1056

We give a purely S-matrix-theoretic proof of the conservation of charge (defined by the strength of soft photon interactions) and the equality of gravitational and inertial mass. Our only assumptions are the Lorentz invariance and pole structure of the S matrix, and the zero mass and spin 1 and 2 of the photon and graviton. We also prove that Lorentz invariance alone requires the S matrix for emission of a massless particle of arbitrary integer spin to satisfy a "mass-shell gauge invariance" condition, and we explain why there are no macroscopic fields corresponding to particles of spin 3 or higher.

Modern View Beginning

- Lorentz symmetry (+ unitaity)--> Massless spin-1: $\epsilon^\mu_\pm \to \epsilon^\mu_\pm + \xi k^\mu$
- Leading to

1. On-shell gauge symmetry: $k^\mu \mathcal{M}_\mu = 0$
2. Charge conservation(equivalent principle for spin-2)
3. No non-trivial interactions for spin > 2

Modern View: Yang–Mills

- Lorentz sym –> little group scaling

$$A_3(1_a^{h_1}, 2_b^{h_2}, 3_c^{h_3}) = \begin{cases} g^+_{abc}[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2} & , \text{for} \sum_{i=1}^{3} h_i > 0\,, \\ g^-_{abc}\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1} & , \text{for} \sum_{i=1}^{3} h_i < 0\,, \end{cases}$$

- Unitarity&locality –> *Consistent factorization* for $A(1_{a_1}, 2_{a_2}, 3_{a_3}, 4_{a_4})$

$$f^{a_1 a_2 e} f^{e a_3 a_4} + f^{a_2 a_3 e} f^{e a_1 a_4} + f^{a_1 a_3 e} f^{e a_4 a_2} = 0$$

Jacob identity
Construct Yang–Mills theory

- **Key: Lorentz sym&Unitarity&locality encoded in amplitudes**

Modern View: Massive-1

J. M. Cornwall, D. N. Levin, and G. Tiktopoulos, Phys. Rev. Lett. 30, 1268 (1973)
J. M. Cornwall, D. N. Levin, and G. Tiktopoulos,Phys. Rev. D 10, 1145 (1974),
C.H. Llewellyn Smith, Phys.Lett.B 46 (1973) 233-236

- Earliest similar attempts for massive amplitudes started in 1970s
- Basic method: perturbative unitarity
  - write down all possible Lagrangian terms for interactions with massive vector bosons and scalars
  - Perturbative unitarity constrains the couplings
  - Result: gauge theory with spontaneous symmetry breaking.

Modern View: Massive-2

After spinor formalism

- Classify all 3-point amplitudes
  match with massless limits(UV/IR)

3-point
E. Conde and A. Marzolla, 1601.08113
S. Y. Choi and J. H. Jeong, 2111.08236
N. Arkani-Hamed, T.-C. Huang, and Y.-t. Huang, 1709.04891

4-point
B. Bachu and etc. 2305.02502&1912.04334
D. Liu and Z. Yin, 2204.13119
H.-Y. Lai, D. Liu, and J. Terning, 2312.11621

- Construct 4-point amplitudes
  apply perturbative unitarity
  obtain gauge theories with Higgs mechanism

- Generally corresponds to the method with Lagrangian

Motivation for Massive Case

- Status quo: the main method is still perturbative unitarity, the same as in 50 years ago
    - ➢ 3-point amplitudes are relative complete
    - ➢ 4-point requires computation off-shell(energy increasing)

- Problem: the core of Higgs mechanism not taken into account:

  Goldstone eaten by gauge boson(at least for 4-point)

    - ➢ Why additional degree of freedom from massless?
    - ➢ Precise cancellation indicates symmetry, what symmetry?

Goal of This Project/Talk

- Propose Consistency Conditions for Massive Amplitudes

    1. Originating from first principles
    2. Particles are elementary

- Construct amplitudes for 3-point and 4-point

- Extract underlying theories

2. Consistent Conditions for Massive Amplitudes

Consistent Conditions for Massive Amplitudes

- Condition-1: On-shell Gauge Symmetry(OGS)/Ward identity

$$k^\mu \mathcal{M}_\mu = \pm i m_V \mathcal{M}(\varphi)$$

Goldstone Boson

$\frac{E}{m_V} \to \infty$, reduce to Goldstone Equivalence Theorem

- Condition-2: Strong Massive-Massless Continuation

$$\lim_{m_i \to 0} \mathcal{M}(\{p_i, \epsilon_i, m_i\}) = \text{finite}$$

Selects Elementary particles

Physical origin of OGS: Constructing Vector Boson

- The vector representation of Poincare group: 4 d.o.f. including a time-like scalar, giving negative norm
- Massive spin-1: 3 d.o.f. only
- The standard construction

Deficiency: no smooth massless limit:

$$\epsilon_L \propto \frac{k^\mu}{m} \propto \infty$$

- Gauge Condition

$$k \cdot \epsilon = 0/\ \partial_\mu V^\mu = 0$$

- Eliminate time–like d.o.f.

$$k^\mu \frac{k_\mu}{m} = m \neq 0$$

- No negative norm

Physical Origin of OGS: Constructing Vector Boson

Our Construction: introduce an auxiliary scalar φ

- **Gauge Condition**

$$k_\mu \epsilon_s^\mu = i m_V \epsilon^\varphi / \ \partial^\mu V_\mu = m_V \varphi,$$

- Identify $|V_0\rangle + |\varphi\rangle$ as a state

$$(\langle V_0| + \langle\varphi|)(|V_0\rangle + |\varphi\rangle)$$

$$= (-1)\frac{k^\mu}{m_V} \cdot \frac{k_\mu}{m_V} + 1 = -1 + 1 = 0,$$

No negative norm

- **Condition invariant under**

$$\epsilon^\mu \to \epsilon^\mu + \xi k^\mu / m_V, \quad \epsilon^\varphi \to \epsilon^\varphi \pm \xi i.$$

gauge symmetry.
also gives another condition

- Final d.o.f.:

4+1−2=3

Smooth Massless limit of The New Construction

- Gauge Condition:

$$k_\mu \epsilon_s^\mu = i m_V \epsilon^\varphi$$

- Gauge symmetry

$$\epsilon^\mu \to \epsilon^\mu + \xi k^\mu / m_V, \quad \epsilon^\varphi \to \epsilon^\varphi \pm \xi i.$$

- Polarization vector(longitudinal)

$$\text{initial: } \epsilon_L^M = (\epsilon_n^\mu, -i)^T$$

$$\xrightarrow{m \to 0}$$

- Gauge Condition:

$$k_\mu \epsilon_s^\mu = 0 \quad (\varphi \text{ decouples})$$

- Gauge symmetry

$$\epsilon^\mu \to \epsilon^\mu + \lambda k^\mu$$

- Pol. vec.

$$\epsilon_L^M \to \epsilon^\varphi$$

Physical origin of OGS: Lagrangian

- Construct kinematic Lagrangian with both V^{μ} and φ have mass m_V and the condition $\partial^\mu V_\mu = m_V \varphi,$
- The resulting Lagrangian

$$\mathcal{L}_{V^2} = -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 + \frac{1}{2} m_V^2 V_\mu^2$$

$$\mathcal{L}_{V-\varphi} = m_V V_\mu \partial^\mu \varphi$$

$$\mathcal{L}_{\varphi^2} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi$$

- Exactly the Lagrangian of S.S.B
- φ has no mass term, i.e. Goldstone boson, only obtaining its mass through mixing with V.

Consistent Condition-2: Elementary Particle

- It's crucial to differentiate Higgs mechanism, which particles are elementary, from Composite Higgs scenario.

- Elementary particle:couplings remain constant at high energy(tree)
- Composite particle: couplings blow up at high energy(in EFT)

- **Strong massive-massless continuation: the amplitude remains finite as one of the particle goes to massless.**

$$\lim_{m_i \to 0} \mathcal{M}(\{p_i, \epsilon_i, m_i\}) = \text{finite}$$

This selects amplitudes of elementary particles

5-component formalism for amplitudes

- Combine gauge and Goldstone components, define 5-components field/polarization vector:

$$V^M = (V^\mu, \varphi)$$

$$\text{initial: } \epsilon_\pm^M = (\epsilon_\pm^\mu, 0)^T \quad \text{final: } \epsilon_\pm^{*M} = (\epsilon_\pm^{*\mu}, 0)^T$$

$$\text{initial: } \epsilon_L^M = (\epsilon_n^\mu, -i)^T \quad \text{final: } \epsilon_L^{*M} = (\epsilon_n^\mu, +i)^T$$

$$k^M: \; k^M = (k^\mu, \pm i m_V)$$

$$k^M k_M^* = 0$$

- Gauge conditions

on-shell condition $k^{*M}\epsilon_M^s(k) = 0$ gauge condition $r \cdot \epsilon(k, r) = 0 \quad r^\mu = (1, -\frac{\vec{k}}{|\vec{k}|})$

- Massive vector boson amplitudes written as:

$$\mathcal{M}(p_1, p_2, ...) = \epsilon_{s_1}^{M_1}(p_1)\epsilon_{s_2}^{M_2}(p_2)...\mathcal{M}_{M_1 M_2 ...}(p_1, p_2, ...)$$

- **On-shell gauge symmetry(OGS)**

$$k^M \mathcal{M}_M = 0$$

Methods of Constructing Amplitudes

- Construct massive amplitudes from massless limits

$$\mathcal{M}_s(V) = \epsilon^M_{\pm,L}\mathcal{M}_M(p,m_V) \xrightarrow{m_V\to 0} \begin{cases} \epsilon^\mu_s \mathcal{M}_\mu(p) & s=\pm \\ \mathcal{M}^\phi(p) & s=L \end{cases} \Rightarrow \mathcal{M}^V_s(p,m_V) = a\ \epsilon^\mu_s \mathcal{M}^V_\mu(p) + b\ \mathcal{M}^\phi(p)$$

- Fix parameters with on-shell gauge symmetry& strong massive-massless continuation

No complexification of momenta

Off-shell continuation

- 3-point amplitudes -> 4-point amplitudes

$$\mathcal{M}_4 \xrightarrow{k_a^2\to m_a^2} \sum_{s,a} \mathcal{M}^s_{3,a}\frac{i}{k_a^2-m_a^2}\mathcal{M}^{-s}_{3,a}$$

$$\sum_{s=\pm,0}\epsilon^M_s\epsilon^{*N}_s = -g^{MN} + \frac{k_i^M n^N + n^M k_i^{*N}}{n\cdot k_i} \xrightarrow{\text{O.G.S}} -g^{MN}$$

- Finally $\displaystyle\sum_{s,a} \mathcal{M}^s_{3,a}\frac{i}{k_a^2-m_a^2}\mathcal{M}^{-s}_{3,a} \xrightarrow{\text{off-shell}} \mathcal{M}_4 = \sum_a \mathcal{M}^M_{3,a}\frac{-ig_{MN}}{k_a^2-m_a^2}\mathcal{M}^N_{3,a}$

3. Constructing Massive Amplitudes

Construct 3-point: massless

- Masslss gauge symmetry $\mathcal{M}(1,2,...,n)=\epsilon_1^{\mu_1}\epsilon_2^{\mu_2}...\ \epsilon_n^{\mu_n}\mathcal{M}(1,2...,n)_{\mu_1\mu_2...\mu_n}$

$$k_1^{\mu_1}\epsilon_2^{\mu_2}...\epsilon_n^{\mu_n}\mathcal{M}(1,2...,n)_{\mu_1\mu_2...\mu_n}=0.$$

dim=1(tree level)

- Massless SSV

$$\mathcal{M}_0(S_1S_2V_3)=(a_1k_1+a_2k_2)\cdot\epsilon_3$$

OSG-> $\mathcal{M}_0(S_1S_2V_3)=(k_1-k_2)\cdot\epsilon_3$

Boson sym. $\mathcal{M}_0(S_1^aS_2^bV_3)=T_{ab}\mathcal{M}_0(S_1S_2V_3)\quad T_{ab}=-T_{ba}$

➢ Massless VVV

$$\mathcal{M}_0(V_1V_2V_3)=(a_{13}k_1+a_{23}k_2)\cdot\epsilon_3\ \epsilon_1\cdot\epsilon_2+\text{cyclic}$$

OSG&Boson sym:

$$\mathcal{M}_0(V_1^aV_2^bV_3^c)=f^{abc}\mathcal{M}_0(V_1V_2V_3)=f^{abc}\left[(\epsilon_1\cdot\epsilon_2)\ (k_1-k_2)\cdot\epsilon_3+\text{cyclic}\right]$$

f^{abc} is a totally anti-symmetric factor,

- "Massless" SVV

$$\mathcal{M}_0(S_1V_2V_3)=m\epsilon_2\cdot\epsilon_3\qquad \dim(m)=1$$

- Doesn't satisfy O.S.G

Should be considered as part of massive SVV

Construct Massive SSV

- Amplitude of SSV

$$\mathcal{M}_0(SSV) \qquad \mathcal{M}_0(SS\varphi)$$

$$\mathcal{M}(S_1S_2V_3) = (k_1 - k_2)\cdot\epsilon_3 - \lambda_{SS\phi}\epsilon_3^4$$

- O.S.G. $\mathcal{M}(S_1S_2V_3)|_{\epsilon_3^M \to k_3^M} = 0 \quad \rightarrow \quad \lambda_{S_1S_2\phi} = -i\frac{(m_1^2 - m_2^2)}{m_V}$

- Massive–massless continuation

$$\lim_{m_i \to 0} \lambda_{SS\phi}(m_1, m_2, m_3) \neq \infty$$

$$m_1^2 = c^{(0)} + c_1^{(1)} m_V + \mathcal{O}(m_V^2) \qquad m_2^2 = c^{(0)} + c_2^{(1)} m_V + \mathcal{O}(m_V^2) \qquad c^{(0)} = 0 \text{ if } \lambda_{SS\phi} \to 0, \text{ as } m_V \to 0$$

Particle masses have the same origin

- Special case: $m_1 = m_2 \rightarrow \lambda_{SS\varphi} = 0$ No dynamic Goldstone(Stueckelberg theory)

Construct Massive SVV

- General amplitude of SVV

$$\mathcal{M}(S_3V_1V_2) = m(\epsilon_1 \cdot \epsilon_2) - a_1(k_3 - k_1) \cdot \epsilon_2 \; \epsilon_1^4 - a_2(k_3 - k_2) \cdot \epsilon_1 \; \epsilon_2^4 + a_3\epsilon_1^4\epsilon_2^4$$

- O.S.G gives

$$a_1 = -\frac{im}{2m_1} \qquad a_2 = -\frac{im}{2m_2} \qquad a_3 = \frac{mm_3^2}{2m_1m_2}$$

- Massive–massless continuation

$$m = c_{1m}m_1 + \mathcal{O}(m_1^2) \qquad m_2 = c_{12}m_1 + \mathcal{O}(m_1^2) \qquad m_3 = c_3^{(13)}m_1 + \mathcal{O}(m_1^2)$$

Construct Massive VVV

- $$\mathcal{M}(V_1V_2V_3) = [(\epsilon_1 \cdot \epsilon_2 + a_{12}\epsilon_1^4\epsilon_2^4)\ (k_1 - k_2)\cdot\epsilon_3 + \text{cyclic}] + [b_{12}m_3\epsilon_1\cdot\epsilon_2\epsilon_3^4 + \text{cyclic}] + c_{123}\epsilon_1^4\epsilon_2^4\epsilon_3^4$$

$$\mathcal{M}(V_1V_2V_3) \longrightarrow \begin{cases} \mathcal{M}_0(V_1^TV_2^TV_3^T) \\ \mathcal{M}_0(V_1^0V_2^TV_3^T) + \text{cyclic} \\ \mathcal{M}_0(V_1^0V_2^0V_3^T) + \text{cyclic} \\ \mathcal{M}_0(V_1^0V_2^0V_3^0) \end{cases}$$

- O.S.G. gives

1. If $m_1 \neq 0, m_2 \neq 0, m_3 \neq 0$:

$$b_{12} = -i\frac{m_1^2 - m_2^2}{m_3^2} \qquad b_{23} = -i\frac{m_2^2 - m_3^2}{m_1^2} \qquad b_{31} = -i\frac{m_3^2 - m_1^2}{m_2^2}$$

$$a_{12} = -\frac{m_1^2 + m_2^2 - m_3^2}{2m_1m_2} \qquad a_{23} = -\frac{m_2^2 + m_3^2 - m_1^2}{2m_2m_3} \qquad a_{31} = -\frac{m_3^2 + m_1^2 - m_2^2}{2m_3m_1}$$

$$c_{123} = 0$$

2. If $m_1 = 0, m_2 \neq 0, m_3 \neq 0$:

$$b_{12} = i \qquad b_{23} = 0 \qquad b_{31} = -i$$

$$a_{12} = 0 \qquad a_{23} = -1 \qquad a_{31} = 0$$

$$c_{123} = 0$$

$$m_2 = m_3$$ New

3. If $m_1 = 0, m_2 = 0, m_3 \neq 0$, then there is no non-trivial solution.

Construct Massive VVV: Special cases

- Take two masses to be equal: $m_2 = m_3 \neq m_1,$

$m_1 \neq 0$

$$b_{12} = i\left(1 - \frac{m_1^2}{m_3^2}\right) \qquad b_{23} = 0 \qquad b_{31} = -b_{12}$$

$$a_{12} = -\frac{m_1}{2m_3} \qquad a_{23} = -\left(1 - \frac{m_1^2}{m_3^2}\right) \qquad a_{31} = -a_{12}$$

$m_1 = 0$

$$b_{12} = i \qquad b_{23} = 0 \qquad b_{31} = -i$$

$$a_{12} = 0 \qquad a_{23} = -1 \qquad a_{31} = 0$$

$$c_{123} = 0$$

- Parameterization: $m_1 > m_3,$ $\quad \cos\theta_W \equiv \frac{m_3}{m_1}$

$m_1 \neq 0$	WWZ	SM
$m_1 = 0$	WWA	

Construct 4-point Amplitude

- Classify amplitudes by number of scalars and vector bosons

4-point

- $n_V \leq 2$
  - $n_V = 1, n_S \geq 2$: SSV
  - $n_V \leq 2$, $n_S = 1$: VVS
  - $n_V \leq 2$, $n_S \geq 2$: VVS, SSV

VVVV, VVSS;

- $n_V \geq 3$
  - $n_V \geq 3, n_S = 0$: VVV
  - $n_V \geq 3, n_S = 1$: VVV, VVS
  - $n_V \geq 3, n_S \geq 2$: VVV, VVS and SSV

VVVV,VVSS;

$n_V \leq 2$: VVVV from VVS

- Set $\boldsymbol{n_V} = 1$, $\boldsymbol{n_S} = 1$

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_c$$

$$\mathcal{M}_s(V_1V_2V_3V_4) \xrightarrow{p_{12}^2\to m_S^2} \mathcal{M}(V_1V_2S_{12})\frac{1}{p_{12}^2-m_S^2}\mathcal{M}(-S_{12}V_3V_4)$$

$$\mathcal{M}_t(V_1V_2V_3V_4) \xrightarrow{p_{13}^2\to m_V^2} \sum_{s_{13}=\pm,0}\mathcal{M}(V_1V_3S_{13})\frac{1}{p_{13}^2-m_V^2}\mathcal{M}(-S_{13}V_2V_4)$$

$$\mathcal{M}_u(V_1V_2V_3V_4) \xrightarrow{p_{14}^2\to m_V^2} \sum_{s_{14}=\pm,0}\mathcal{M}(V_1V_3S_{14})\frac{1}{p_{14}^2-m_V^2}\mathcal{M}(-V_{14}V_2S_3)$$

$$\mathcal{M}_s(V_1V_2V_3V_4) = \mathcal{M}(V_1V_2S_{12})\frac{1}{p_{12}^2-m_S^2}\mathcal{M}(-S_{12}V_3V_4)$$

$$\mathcal{M}_t(V_1V_2V_3V_4) = M(V_1V_3S_{13})\frac{1}{p_{13}^2-m_V^2}\mathcal{M}(-S_{13}V_2V_4)$$

$$\mathcal{M}_u(V_1V_2S_3S_4) = \mathcal{M}(V_1V_3S_{14})\frac{1}{p_{14}^2-m_V^2}\mathcal{M}(-S_{14}V_2V_3)$$

- OSG

$$a_1 = a_2 = a_3 = 0$$

$$b_{11} = b_{12} = b_{21} = b_{22} = b_{31} = b_{31} = g_{VVS}^2$$

$$c \equiv \lambda_{\phi^4} = \frac{3}{4}g_{VVS}^2\frac{m_S^2}{m_V^2}$$

$$\begin{aligned}\mathcal{M}_c = {} & a_1\epsilon_1\cdot\epsilon_2\ \epsilon_3\cdot\epsilon_4 + a_2\epsilon_1\cdot\epsilon_3\ \epsilon_2\cdot\epsilon_4 + a_3\epsilon_1\cdot\epsilon_4\ \epsilon_3\cdot\epsilon_2 \\ & +b_{11}\epsilon_1\cdot\epsilon_2\ \epsilon_3^4\cdot\epsilon_4^4 + b_{12}\epsilon_1^4\cdot\epsilon_2^4\ \epsilon_3\cdot\epsilon_4 \\ & +b_{21}\epsilon_1\cdot\epsilon_3\ \epsilon_2^4\cdot\epsilon_4^4 + b_{22}\epsilon_1^4\cdot\epsilon_3^4\ \epsilon_2\cdot\epsilon_4 \\ & +b_{31}\epsilon_1\cdot\epsilon_4\ \epsilon_2^4\cdot\epsilon_3^4 + b_{32}\epsilon_1^4\cdot\epsilon_4^4\ \epsilon_2\cdot\epsilon_3 \\ & +c\ \epsilon_1^4\epsilon_2^4\epsilon_3^4\epsilon_4^4\end{aligned}$$

all couplings fixed by masses (up to an overall coupling)

$n_V \leq 2$: VVSS from VVS and SSS

- $$\mathcal{M}_{\text{tot}} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_c$$

$$\mathcal{M}_s(S_1^a S_2^a V_3 V_4) = \sum_{i=a,b} M(S_1^a S_2^a S_{12}^i) \frac{1}{p_{12}^2 - m_i^2} \mathcal{M}(S_{12}^i V_3 V_4)$$

$$\mathcal{M}_c(S_1^a S_2^a V_3 V_4) = g_{S^a S^a V V}\ \epsilon_3 \cdot \epsilon_4 + \lambda_{SS\varphi^a\varphi^a}\ \epsilon_3^4 \epsilon_4^4$$

$$\begin{aligned}\mathcal{M}_t(S_1^a S_2^a V_3 V_4) &= \mathcal{M}(S_1^a V_3 S_{13}^b) \frac{1}{p_{13}^2 - m_b^2} \mathcal{M}(S_{13}^b S_2^a V_4) \\ &+ \mathcal{M}(S_1^a V_3 V_{13})^{M_{13}} \frac{-g_{M_{13}N_{13}}}{p_{13}^2 - m_V^2} \mathcal{M}^{N_{13}}(V_{13} S_2^a V_4^{s_4}) \qquad)\end{aligned}$$

$$\begin{aligned}\mathcal{M}_u(S_1^a S_2^a V_3 V_4) &= \mathcal{M}(S_1^a V_4 S_{14}^i) \frac{1}{p_{14}^2 - m_i^2} \mathcal{M}(S_{14}^i S_2^a V_3) \\ &+ \mathcal{M}(S_1^a V_4 V_{14})^{M_{14}} \frac{-g_{M_{14}N_{14}}}{p^2 - m^2} \mathcal{M}^{N_{14}}(V_{14} S_2^a V_3^{s_3}) \qquad)\end{aligned}$$

$$\mathcal{M}_{\text{tot}}^{\epsilon_3^M \to p_3^M} = 0$$

$$g_{S^a S^a VV} = -2\left(g_{S^a S^b V}^2 - \frac{g_{VVS^a}^2}{4}\right)$$

$$\frac{1}{2}(\lambda_{S^a S^a S^a} g_{VVS^a} + \lambda_{S^a S^a S^b} g_{VVS^b}) - \lambda_{S^a S^a \varphi\varphi} m_V = g_{S^a S^b V}^2 \frac{m_a^2 - m_b^2}{m_V} - \frac{m_a^2}{2m_V} g_{VVS^a}^2$$

All couplings fixed if scalar self-couplings are input.

$n_V \leq 2$: VVSS from VVS、SSV&SSS

- Limit-1: $g_{S^a S^b V} = 0$ and $\lambda_{S^a S^a S^b} = 0.$

obtain

$$g_{SSVV} = \frac{1}{2} g_{VVS}^2$$

$$\lambda_{\varphi\varphi SS} = g_{VVS} \frac{\lambda_{SSS}}{m_V} + \frac{m_S^2}{2m_V^2} g_{VVS}^2$$

$$\lim_{m_V \to 0} \lambda_{SSS} = 0$$

Scalar QED

scalar self-couplings not fixed

- Limit-2: $g_{VVS} = 0$ $\quad m_a = m_b.$

obtain $\quad g_{S^a S^a VV} = -2g_{S^a S^b V}^2 \quad \lambda_{S^a S^a \varphi\varphi} = 0$

Stueckelberg theory

Combined with 3-point: $m_1 = m_2 \to \lambda_{SS\varphi} = 0$

- The general case: U(1) version of SHDM
needs more conditions(e.g. mixing) to fix the parameters

$n_V > 2$: VVVV

- VVSS is the same as $\boldsymbol{n_V \leq 2}$
- VVVV: choose all masses to be equal; construct from V only first

$$\mathcal{M}_{\rm tot}(V_1^a V_2^b V_3^c V_4^d) = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_c$$

- parameters: set g_{VVV}=1

$$\begin{aligned}\mathcal{M}_c = {} & a_{12,34}\ \epsilon_1 \cdot \epsilon_2\ \epsilon_3 \cdot \epsilon_4 + a_{13,24}\ \epsilon_1 \cdot \epsilon_3\ \epsilon_2 \cdot \epsilon_4 + a_{14,23}\ \epsilon_1 \cdot \epsilon_4\ \epsilon_2 \cdot \epsilon_3 \\ & + b_{12,34}\ \epsilon_1 \cdot \epsilon_2 \epsilon_3^4 \epsilon_4^4 + b_{13,24}\ \epsilon_1 \cdot \epsilon_3\ \epsilon_2^4 \epsilon_4^4 + b_{14,23}\ \epsilon_1 \cdot \epsilon_4\ \epsilon_2^4 \epsilon_3^4 \\ & + b_{34,12}\ \epsilon_3 \cdot \epsilon_4 \epsilon_1^4 \epsilon_2^4 + b_{24,13}\ \epsilon_2 \cdot \epsilon_4\ \epsilon_1^4 \epsilon_3^4 + b_{23,14}\ \epsilon_2 \cdot \epsilon_3\ \epsilon_1^4 \epsilon_4^4 \\ & + \lambda_{\phi^4} \epsilon_1^4 \epsilon_2^4 \epsilon_3^4 \epsilon_4^4\end{aligned}$$

- O.S.G. on helicities of 1TTT:

$$f^{abe} f^{cde} + f^{ace} f^{dbe} + f^{ade} f^{bce} = 0$$

Massless Yang–Mills theory

$n_V > 2$: VVVV

- 1LLT: no-solution for O.S.G (vector boson only)
- Solution: add a scalar (and VVS amplitudes)

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_c + \mathcal{M}_s^S + \mathcal{M}_t^S + \mathcal{M}_u^S$$

- Then applying O.S.G on 1LTT

$$b_{34,12} = \frac{1}{2} g_{V^a V^b S} g_{V^c V^d S}$$

$b_{ij,kl}$ by permutation

Solution:

- 1LLT:

$$f^{abe} f^{cde} = g_{V^a V^c S} g_{V^b V^d S} - g_{V^a V^d S} g_{V^b V^c S}$$

$$f^{ace} f^{dbe} = -g_{V^a V^b S} g_{V^c V^d S} + g_{V^a V^d S} g_{V^b V^c S}$$

In addition:

$$g_{V^b V^a S} = g_{V^a V^b S}.$$

for equal masses

- 1LLL:

$$\lambda_{\varphi^4} = \frac{1}{4} \frac{m_S^2}{m_V^2} (g_{V^a V^b S} g_{V^c V^d S} + g_{V^a V^c S} g_{V^d V^b S} + g_{V^a V^d S} g_{V^b V^c S})$$

$n_V > 2$: VVVV

- Adding a (mild) condition:

$$g \equiv g_{V^aV^bS} = g_{V^cV^dS} = g_{V^aV^cS} = g_{V^bV^dS} = g_{V^aV^dS} = g_{V^bV^cS}.$$

- The only nontrivial solution

$$g_{V^aV^bS} = 0 \quad \text{or} \quad g_{V^cV^dS} = 0 \qquad f^{abe}f^{cde} = 0 \qquad g^2_{VVS} = f^{ace}f^{dbe} = f^{ade}f^{bce}$$

$$g_{VVS} \equiv g_{V^aV^cS} = g_{V^bV^dS} = g_{V^aV^dS} = g_{V^bV^cS}$$

has only s and t channels (i.e. plannar diagrams only)

- Example: SU(2), agree in basis $W^{\pm}$

Conclusion

- We propose two consistent conditions to construct amplitudes with elementary massive vector bosons and scalars.
  - On-shell gauge symmetry(from Lorentz symmetry)
  - strong massive-massless continuation

- We construct possible 3-point&4-point amplitudes, including Goldstone-related.
- All couplings can be fixed by particle masses, except for scalar self-couplings
- Underlying theories:
  - Gauge theory with S.S.B. (e.g. Higgs mechanism)
  - Stuckelberg mechanism: no dynamic Goldstone