

格点模拟对称性破缺过程及相关唯象学

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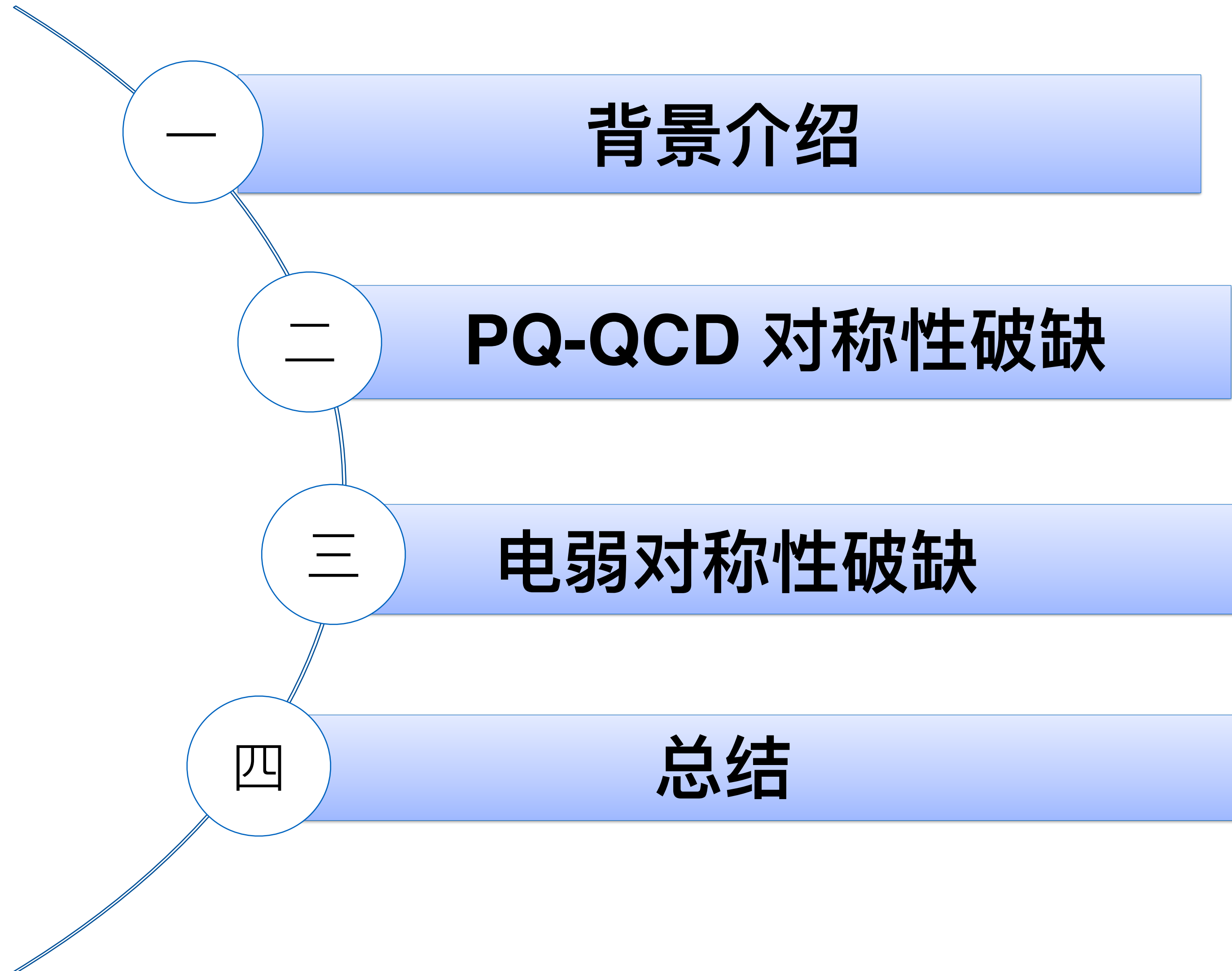
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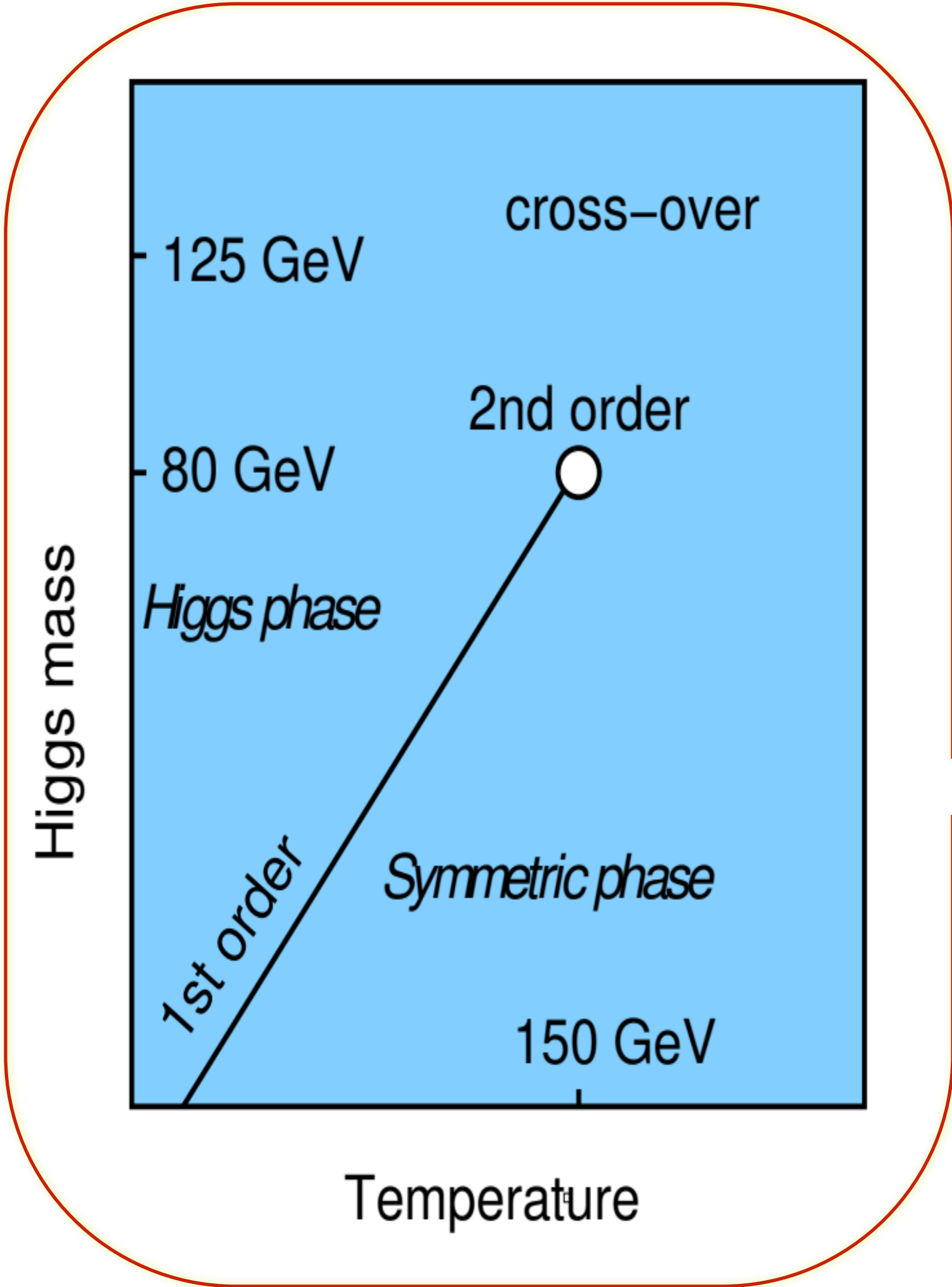
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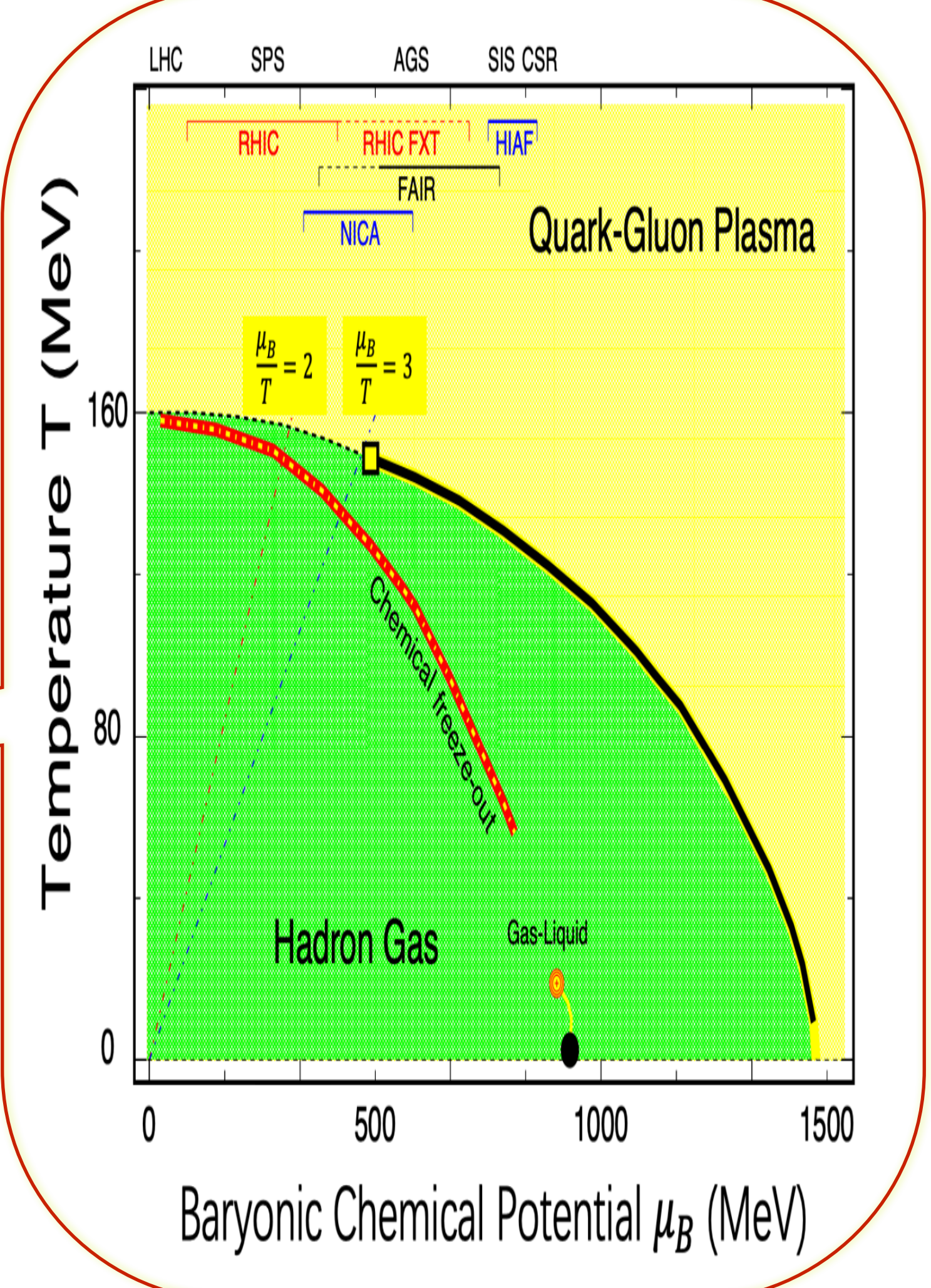
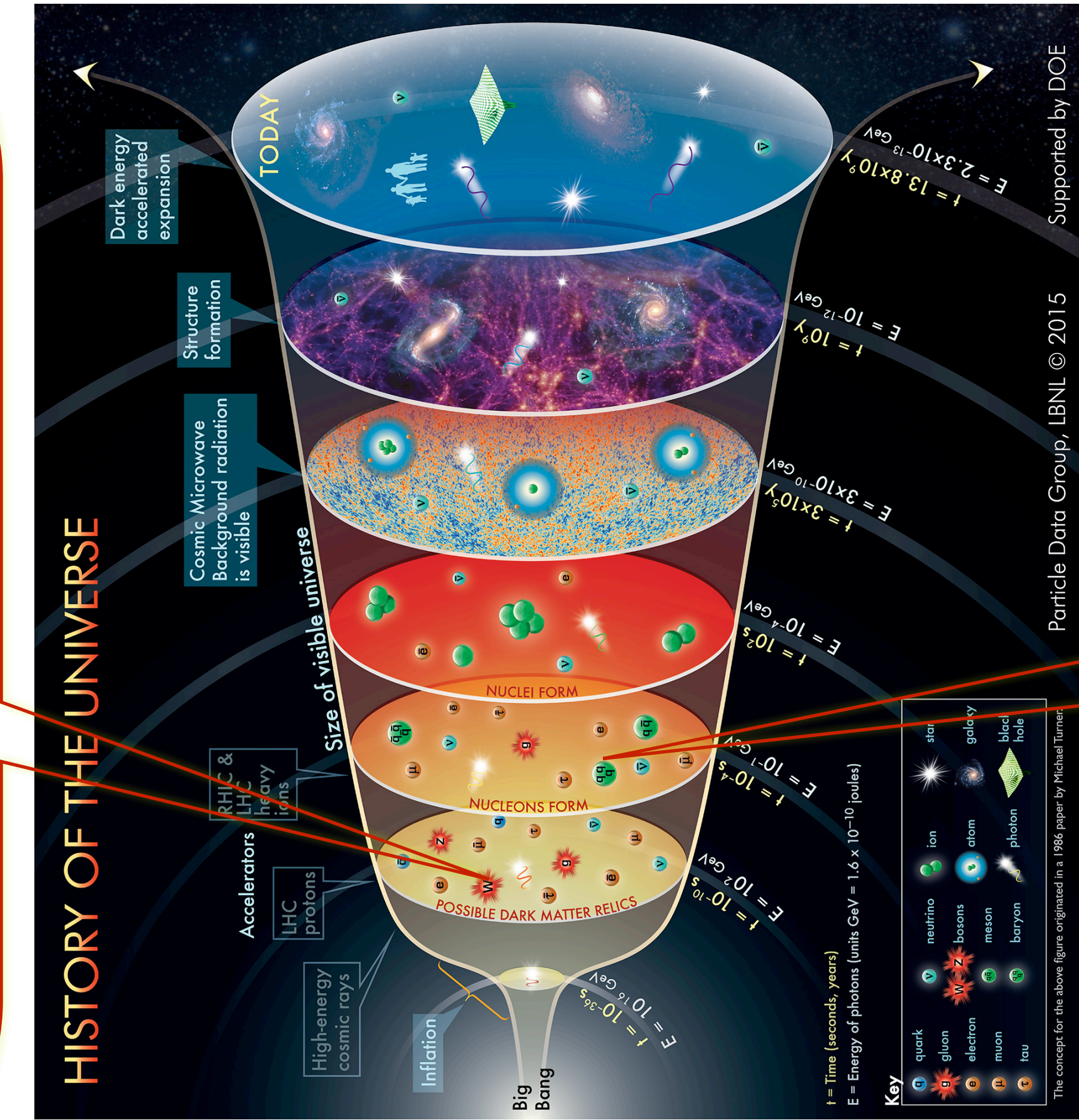
报告提纲



Symmetry Breaking in the early Universe

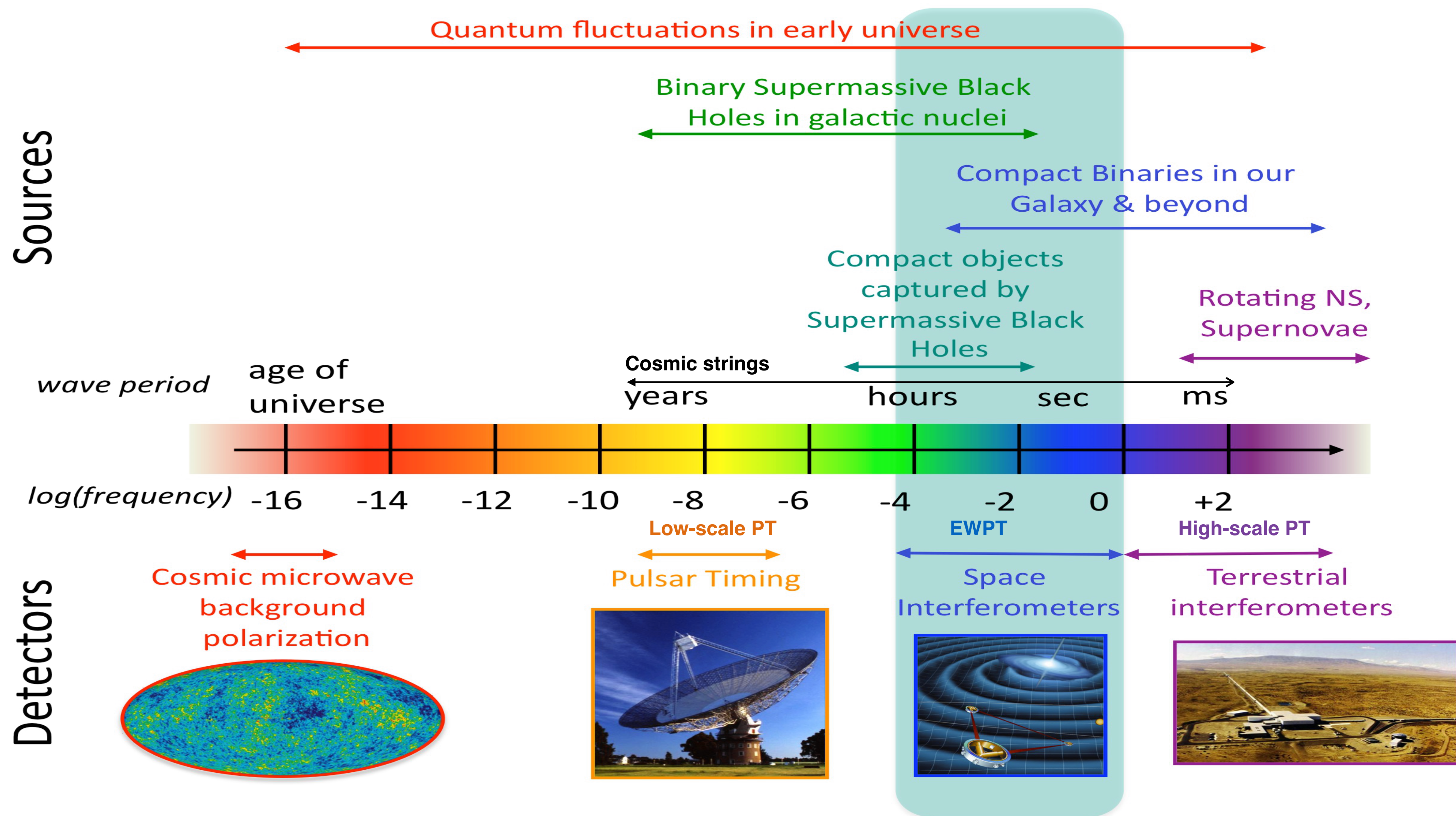


Electroweak Phase diagram



QCD Phase diagram

The Gravitational Wave Spectrum



► Cosmic string simulation: the DFSZ axion model

The action in expanding universe with spatially flat FLRW metric:

$$S = -\int d^4x \sqrt{-g} \left(g^{\mu\nu} \frac{1}{2} \partial_\mu \varphi^* \partial_\nu \varphi + V(\varphi) \right)$$

PQ complex scalar: $\varphi = \phi_1 + i\phi_2$

$$V(\varphi) = \underbrace{\frac{1}{4} \lambda (|\varphi|^2 - v^2)^2 + \frac{\lambda}{6} T^2 |\varphi|^2}_{\text{PQ era}} + \underbrace{\frac{m^2(T)v^2}{N_{DW}^2} (1 - \cos(N_{DW}\theta)) - \Xi v^3 (\varphi e^{-i\delta} + \text{h.c.})}_{\text{QCD era}}$$

min $\left[\frac{\alpha_a \Lambda^4}{f_a^2 (T/\Lambda)^{6.68}}, m_a^2 \right]$ axion field bias term

PQ era, PQ symmetry broken, second order phase transition, $T_c \sim 10^9\text{-}10^{11}\text{GeV}$

Axion(global) strings form and enters the scaling regime

QCD era, axion acquires a non-zero mass due to QCD non-perturbative effect, $T \sim 100\text{MeV}$

String-domain wall hybrid networks form and eventually decay

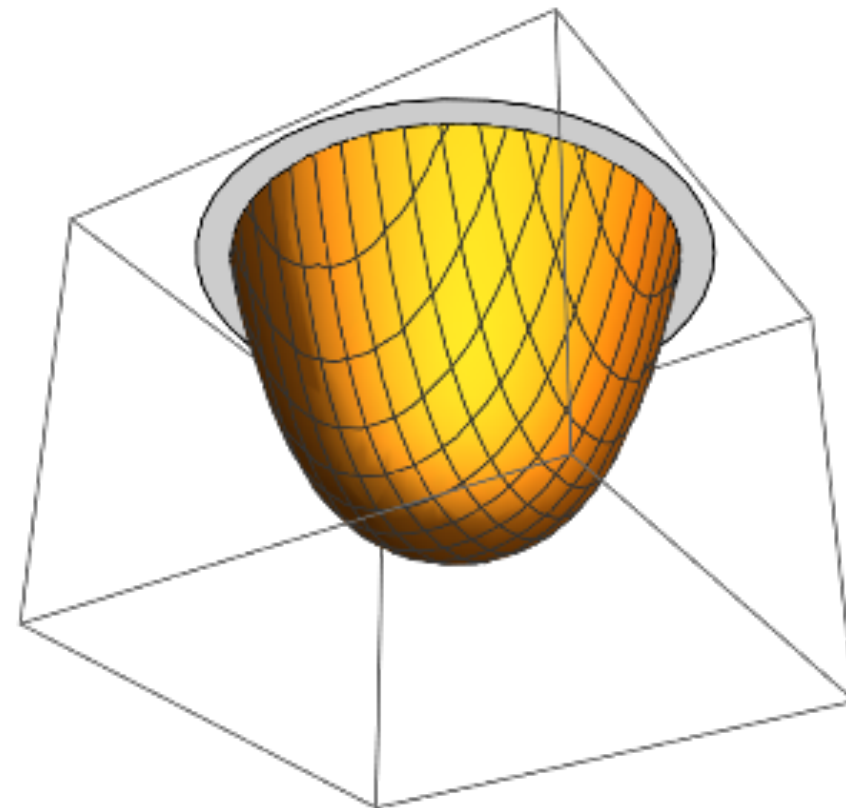
Gravitational waves and axion radiated by topological defects of two eras

→ Detection of axion dark matter

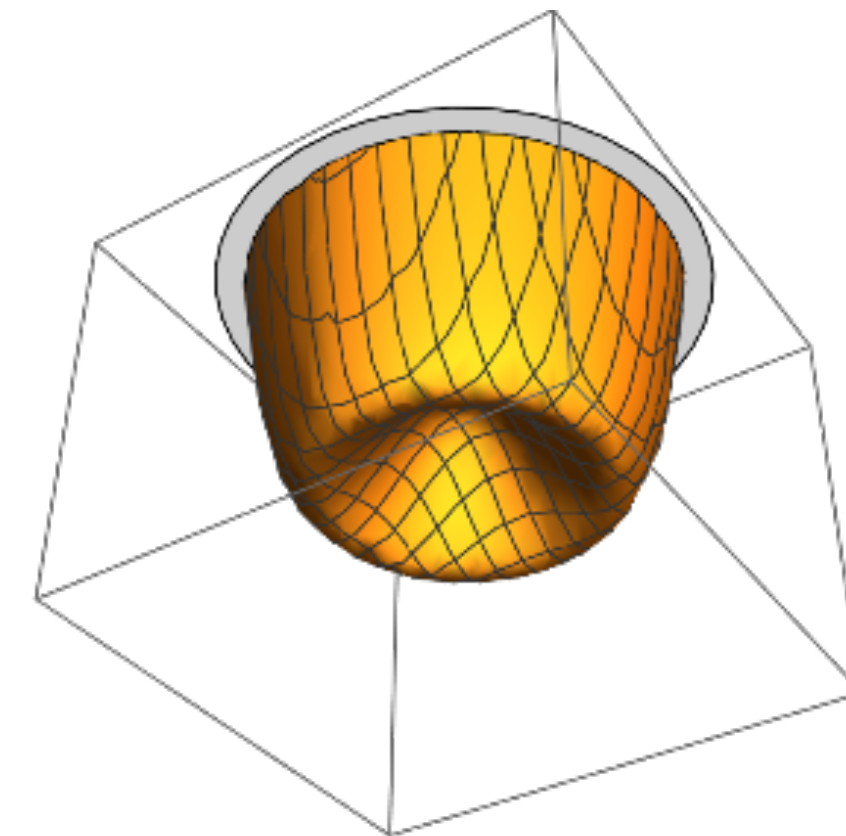
► Shape of the potential

PQ era:

before PQ transition



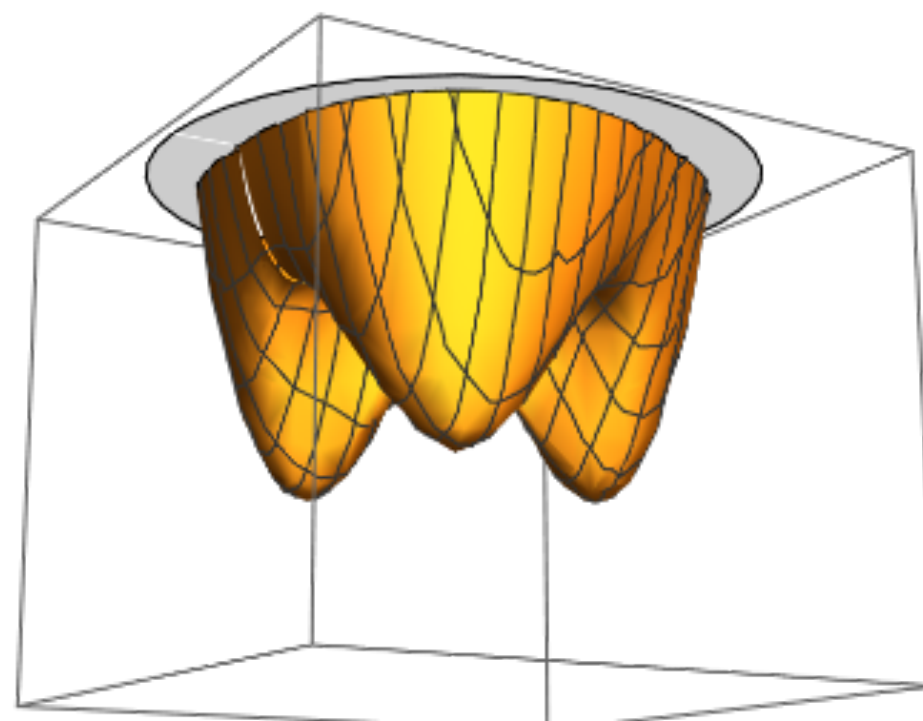
after PQ transition



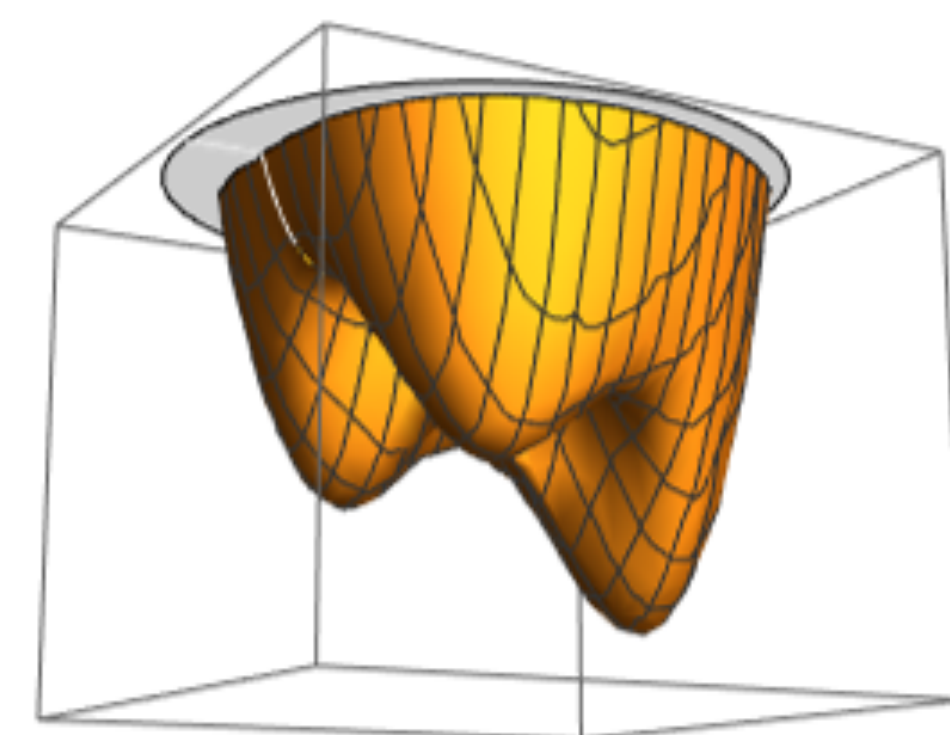
QCD era:

with nonzero axion mass

without bias term



with bias term



$N_{DW}=3$

► Simulation setups

Equations of motion

PQ era-the first stage

$$\begin{cases} \phi_1'' + 2\frac{a'}{a}\phi_1' - \nabla^2\phi_1 = -a^2[\lambda\phi_1(\phi_1^2 + \phi_2^2 - v^2 + \frac{1}{3}T^2) - \frac{m^2(T)v^2}{N_{\text{DW}}^2}(\cos\theta \cos N_{\text{DW}}\theta + N_{\text{DW}} \sin\theta \sin N_{\text{DW}}\theta) - 2\Xi v^3 \cos\delta] \\ \phi_2'' + 2\frac{a'}{a}\phi_2' - \nabla^2\phi_2 = -a^2[\lambda\phi_2(\phi_1^2 + \phi_2^2 - v^2 + \frac{1}{3}T^2) - \frac{m^2(T)v^2}{N_{\text{DW}}^2}(\sin\theta \cos N_{\text{DW}}\theta - N_{\text{DW}} \cos\theta \sin N_{\text{DW}}\theta) - 2\Xi v^3 \sin\delta] \end{cases}$$

Initial condition

thermal spectrum

$$\mathcal{P}_{\phi_1}(k) = \mathcal{P}_{\phi_2}(k) = \frac{n_k}{w_k} = \frac{1}{w_k} \frac{1}{e^{w_k/T} - 1}, \quad \mathcal{P}_{\dot{\phi}_1}(k) = \mathcal{P}_{\dot{\phi}_2}(k) = n_k w_k = \frac{w_k}{e^{w_k/T} - 1}$$

$$w_k = \sqrt{k^2/R^2 + m_{\text{eff}}^2} \quad m_{\text{eff}}^2 = \lambda(T^2/3 - v^2)$$

two-point correlation functions

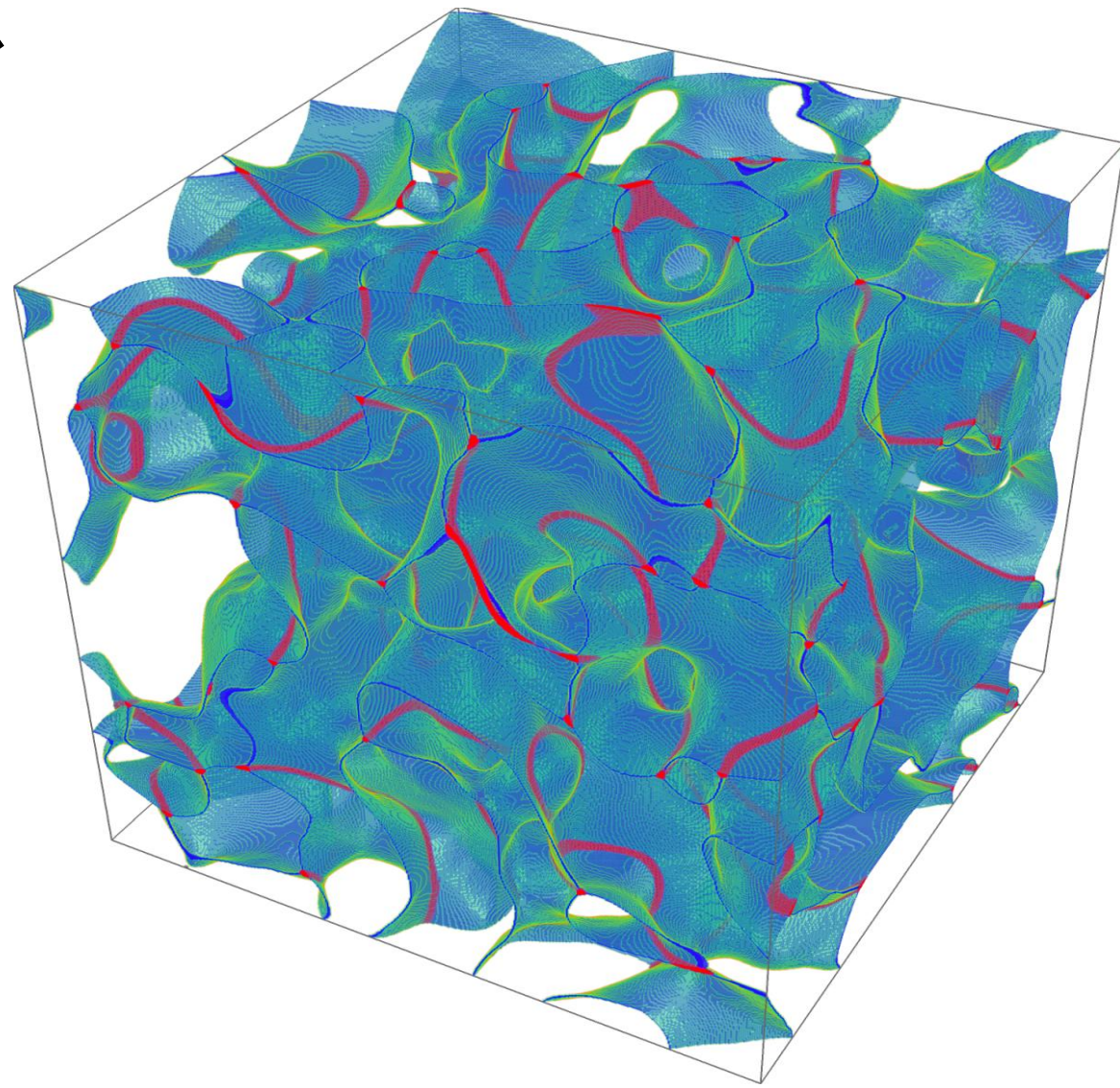
$$\begin{aligned} \langle \phi_i(\mathbf{k}) \phi_j(\mathbf{k}') \rangle &= (2\pi)^3 \mathcal{P}_\phi(k) \delta(\mathbf{k} - \mathbf{k}') \delta_{ij}, \\ \langle \dot{\phi}_i(\mathbf{k}) \dot{\phi}_j(\mathbf{k}') \rangle &= (2\pi)^3 \mathcal{P}_{\dot{\phi}}(k) \delta(\mathbf{k} - \mathbf{k}') \delta_{ij}, \\ \langle \phi_i(\mathbf{k}) \dot{\phi}_j(\mathbf{k}') \rangle &= 0. \end{aligned}$$

$$\begin{aligned} \langle |\phi_i(\mathbf{k})|^2 \rangle &= \left(\frac{N}{\delta x_{\text{phy}}} \right)^3 \mathcal{P}_{\phi_i}(k), & \langle \phi_i(\mathbf{k}) \rangle &= 0, \\ \langle |\dot{\phi}_i(\mathbf{k})|^2 \rangle &= \left(\frac{N}{\delta x_{\text{phy}}} \right)^3 \mathcal{P}_{\dot{\phi}_i}(k), & \langle \dot{\phi}_i(\mathbf{k}) \rangle &= 0, \end{aligned}$$

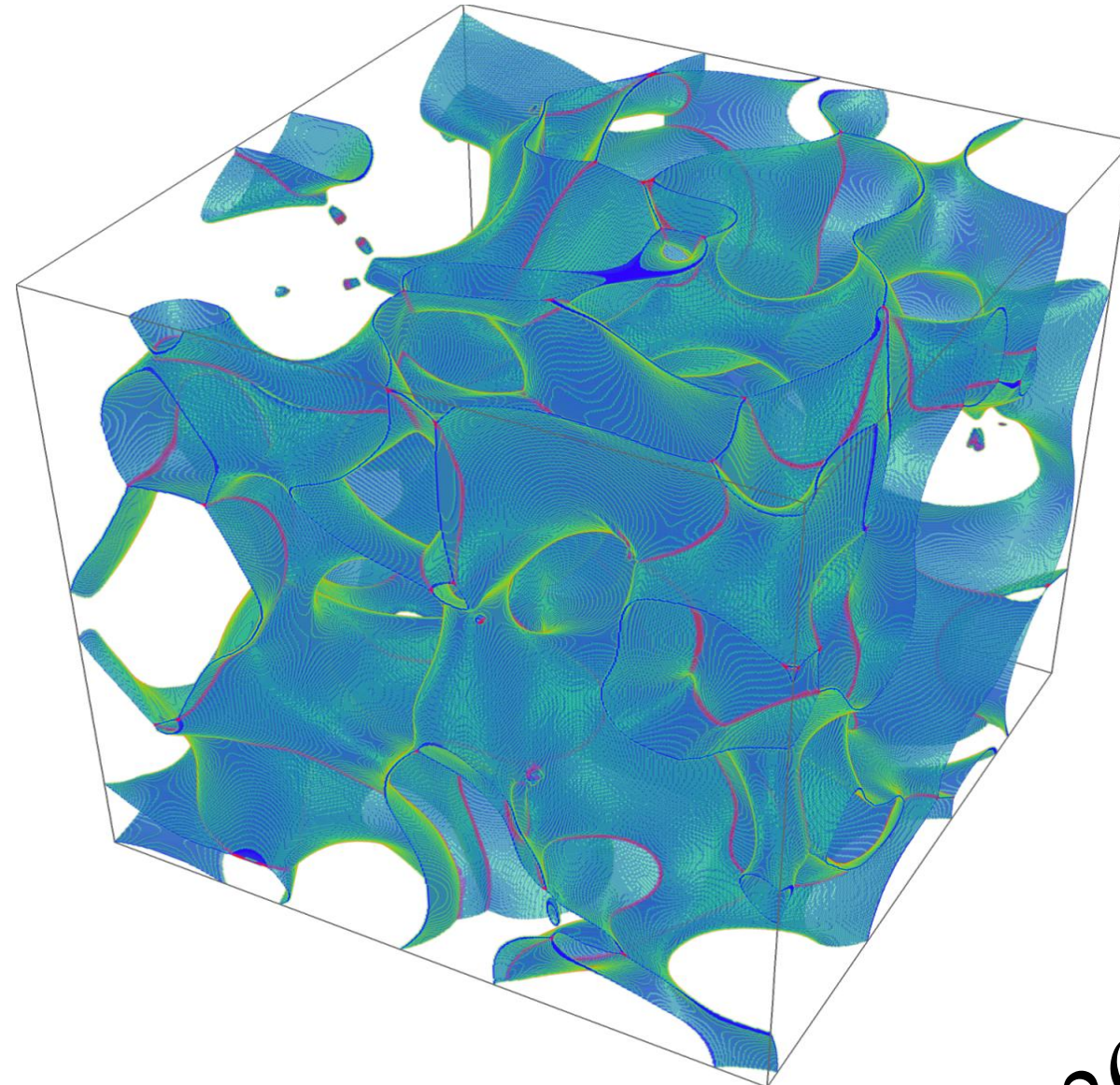
► String-wall evolution

Axion string-domain wall hybrid network destruction with gravitational waves and axions emission

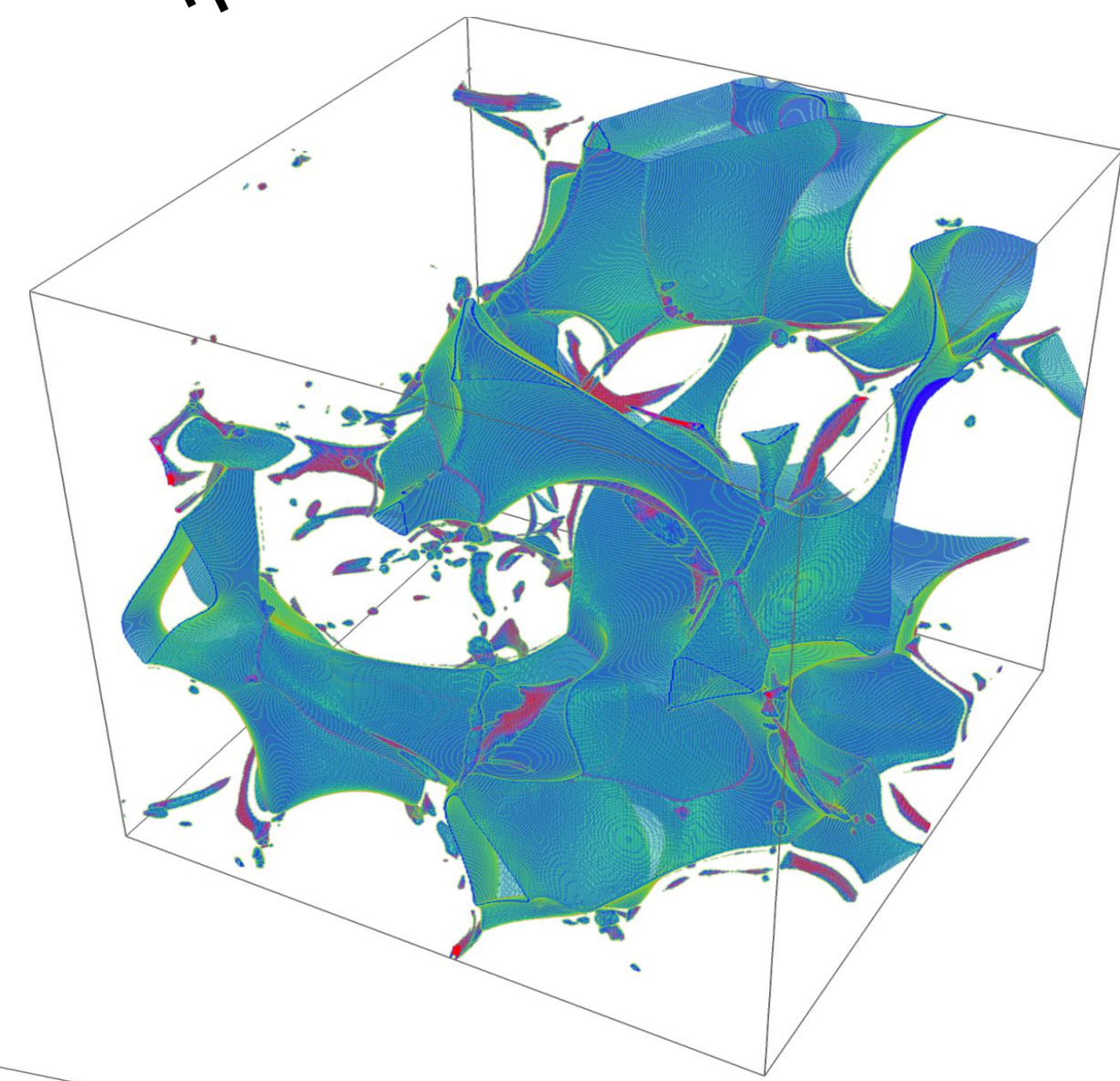
$\eta=4.6$



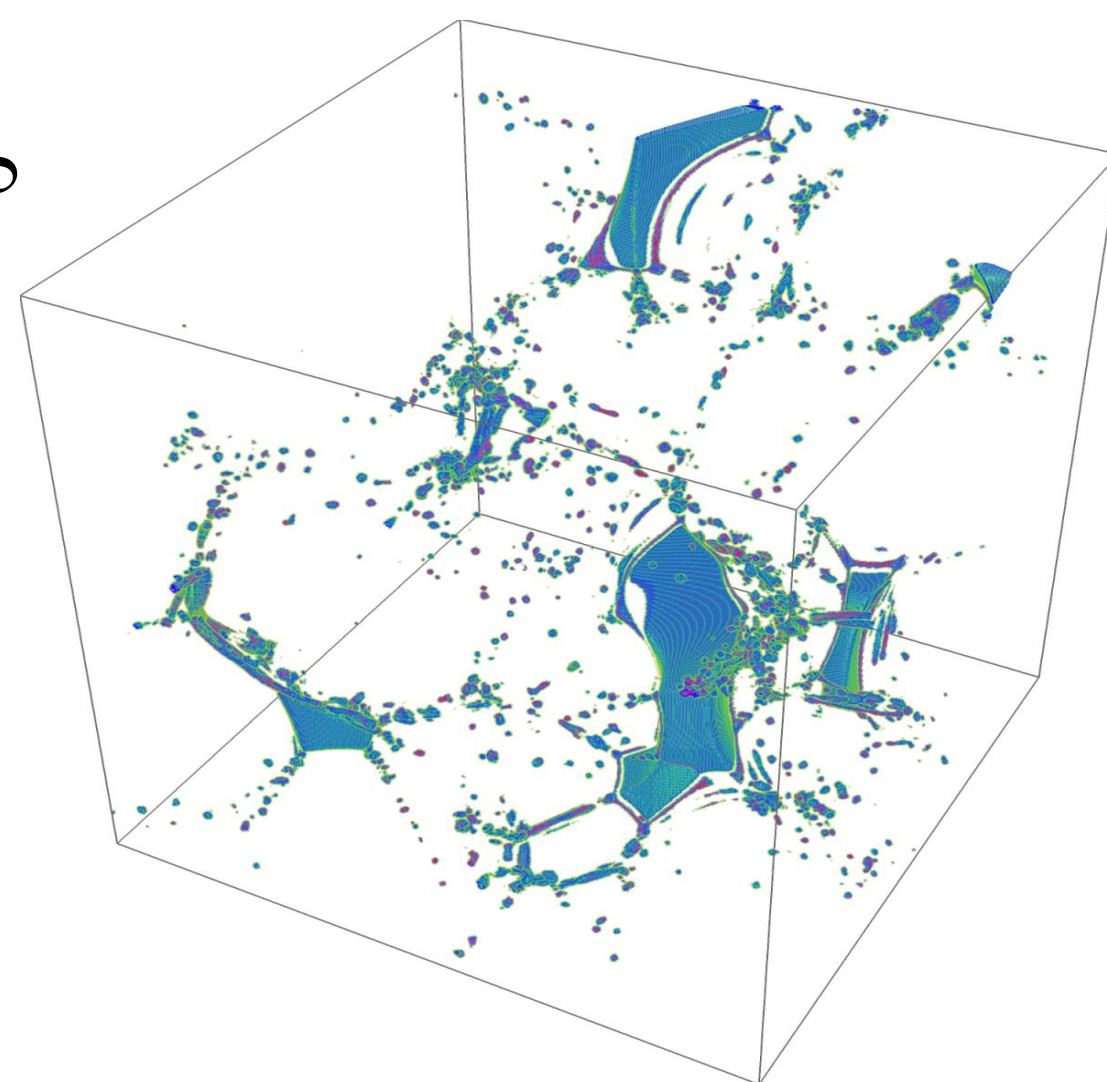
$\eta=10$



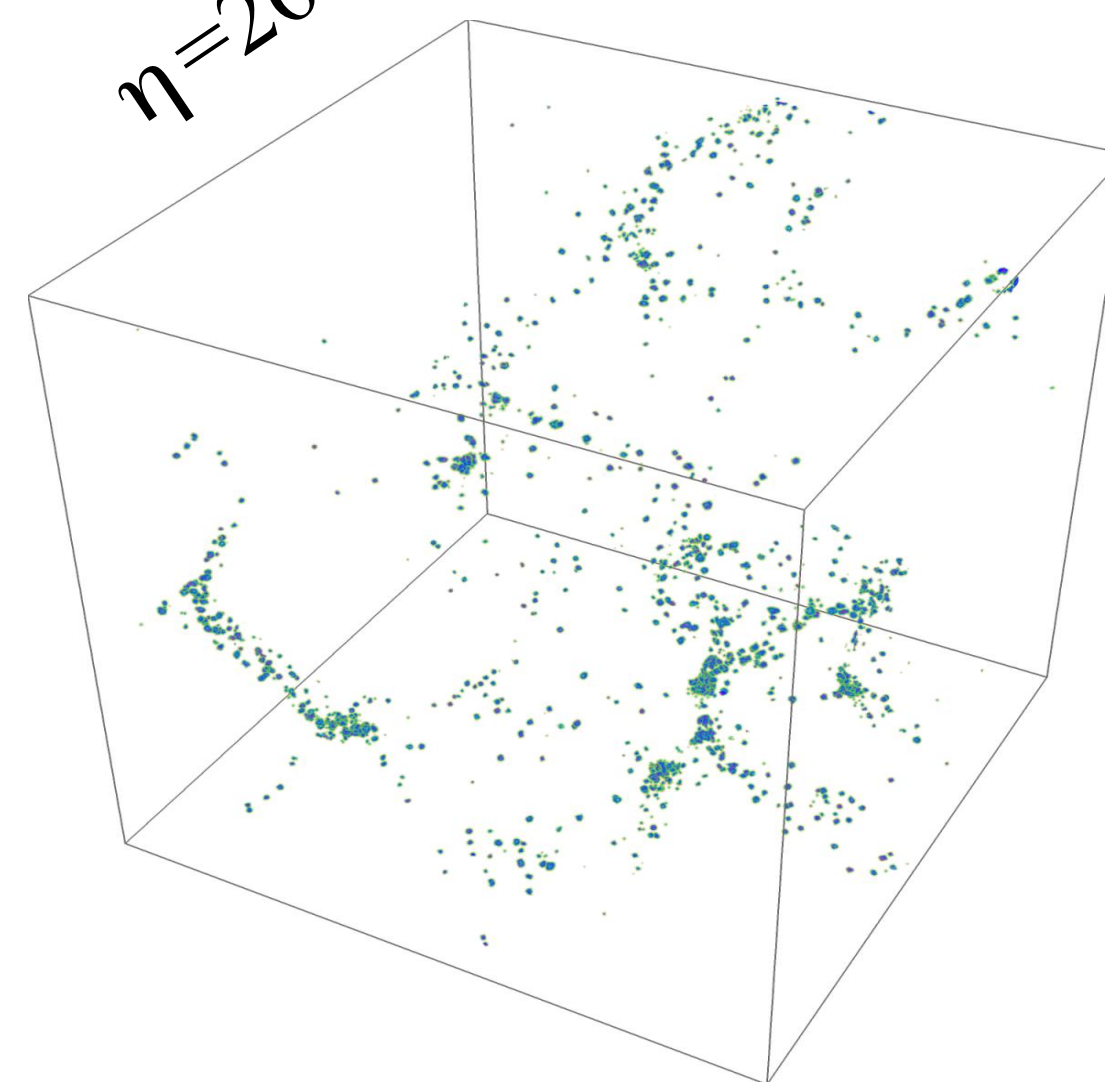
$\eta=15.4$



$\eta=20.8$

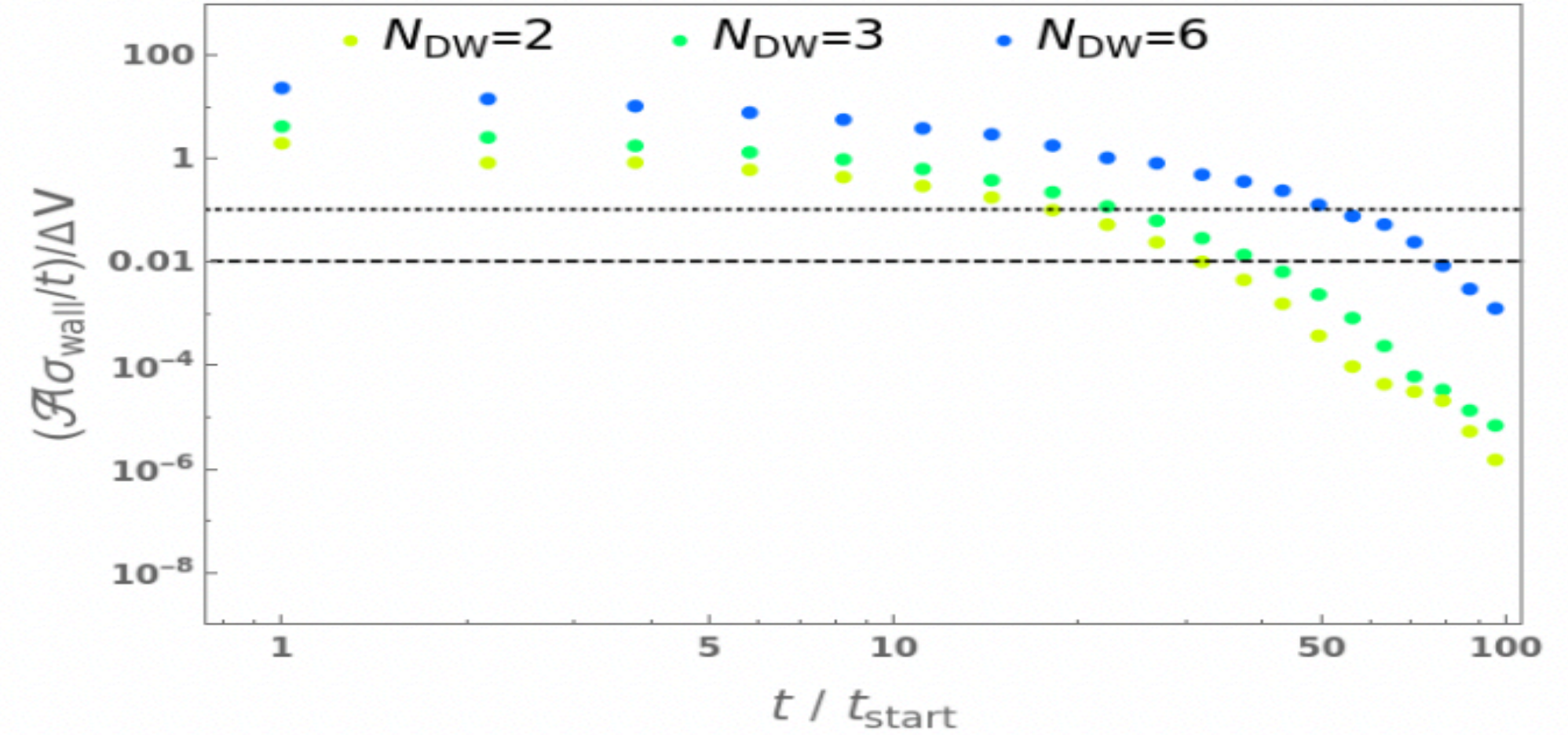
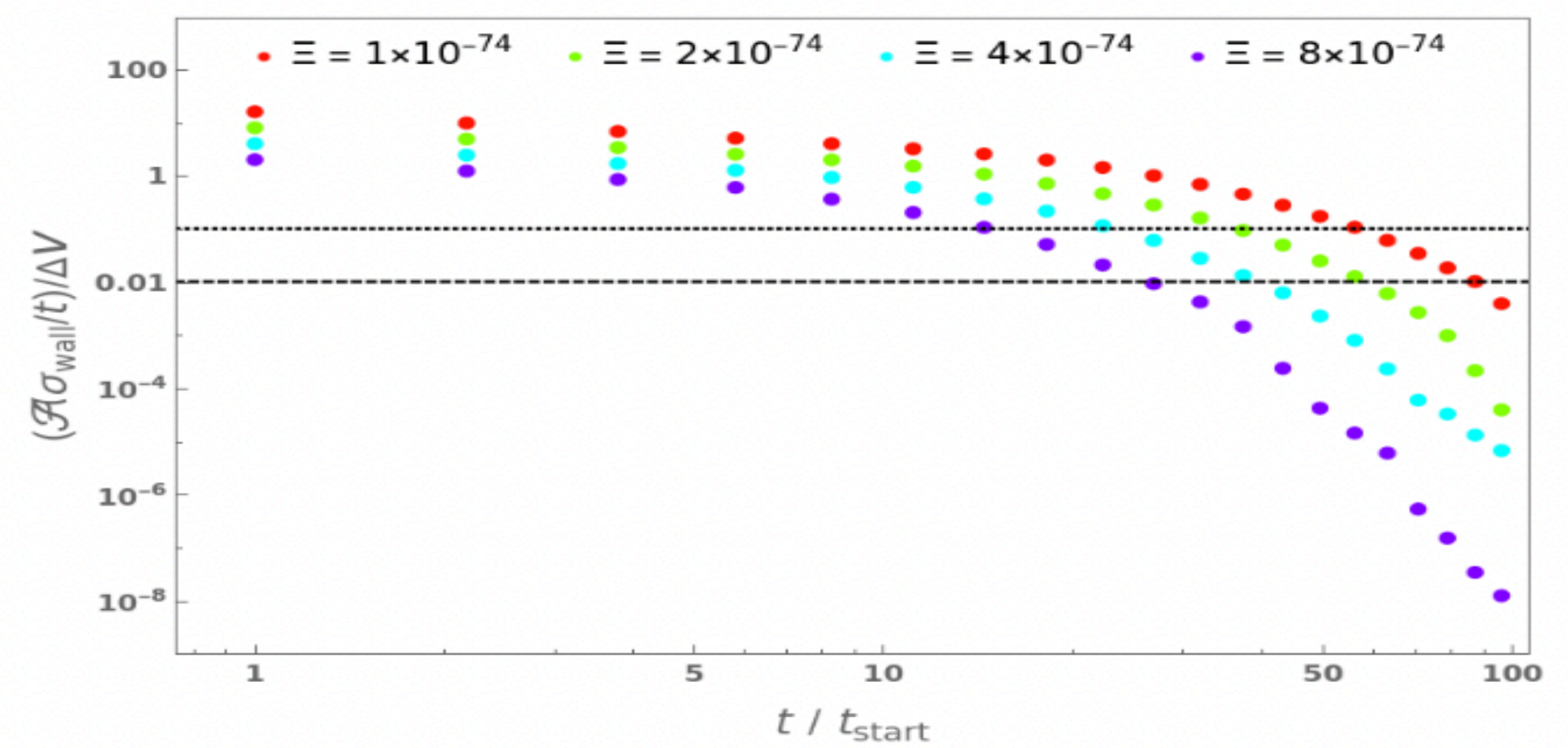
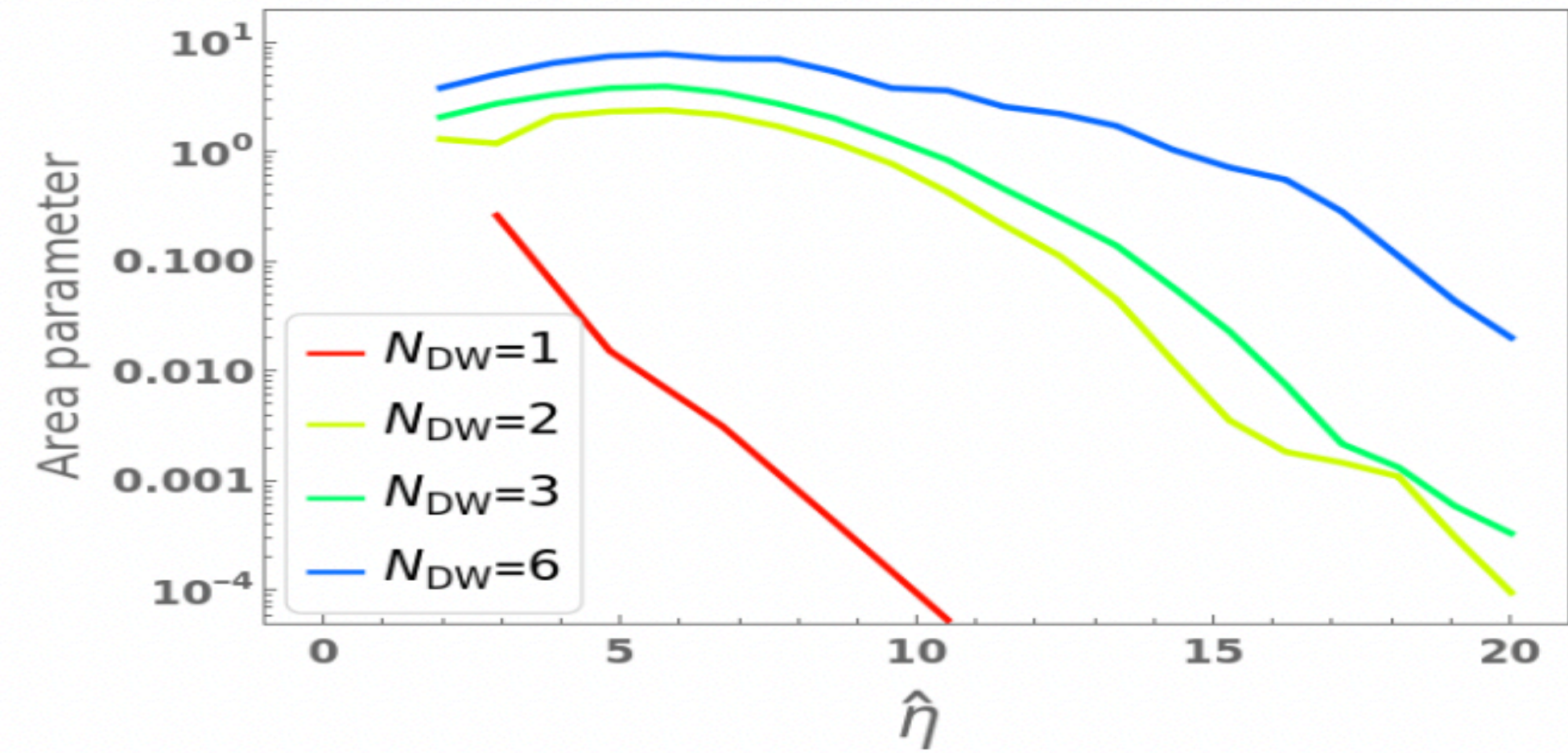
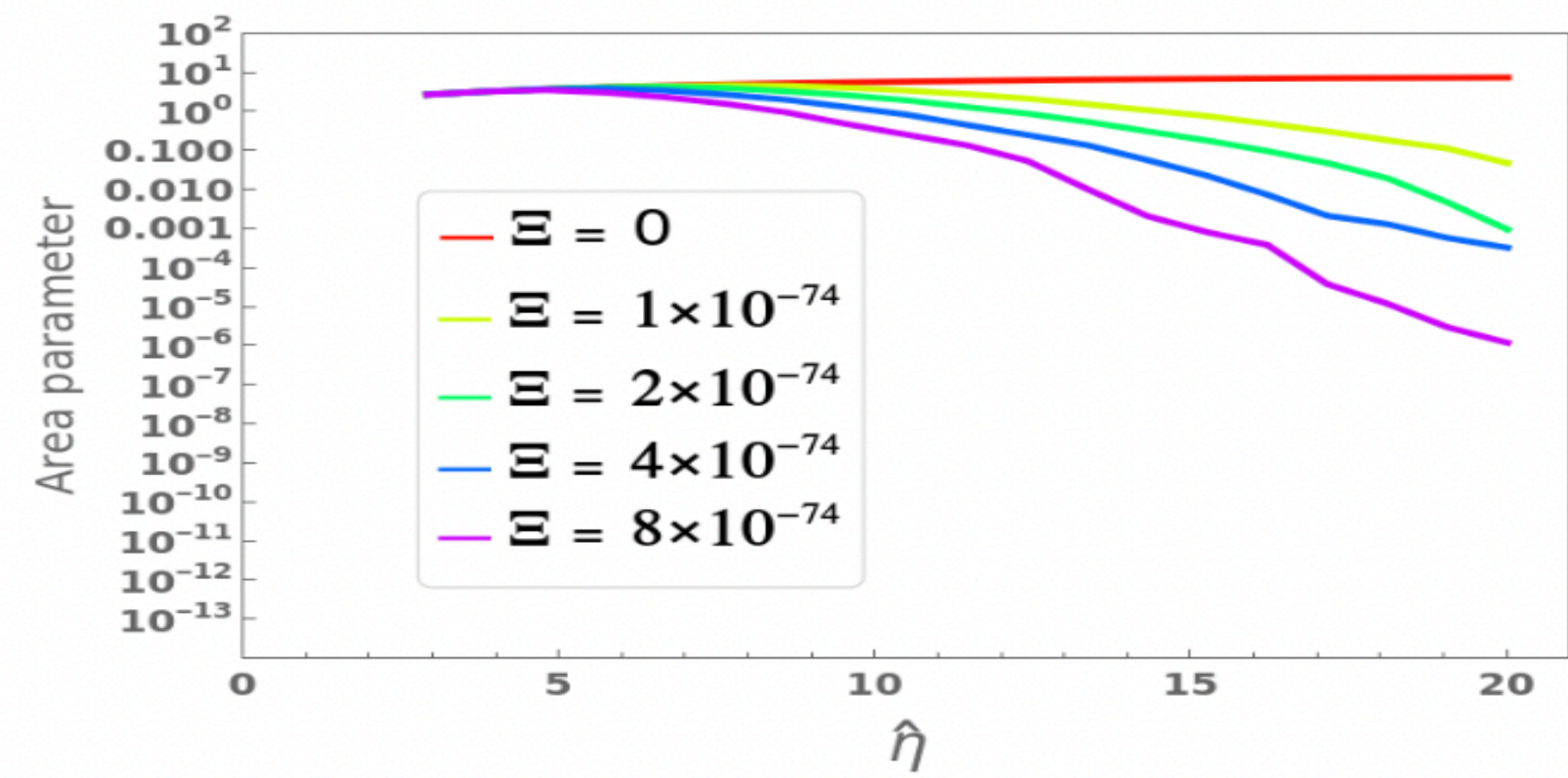


$\eta=26.2$



$$N_{\text{DW}} = 3$$

► Domain wall Area parameter



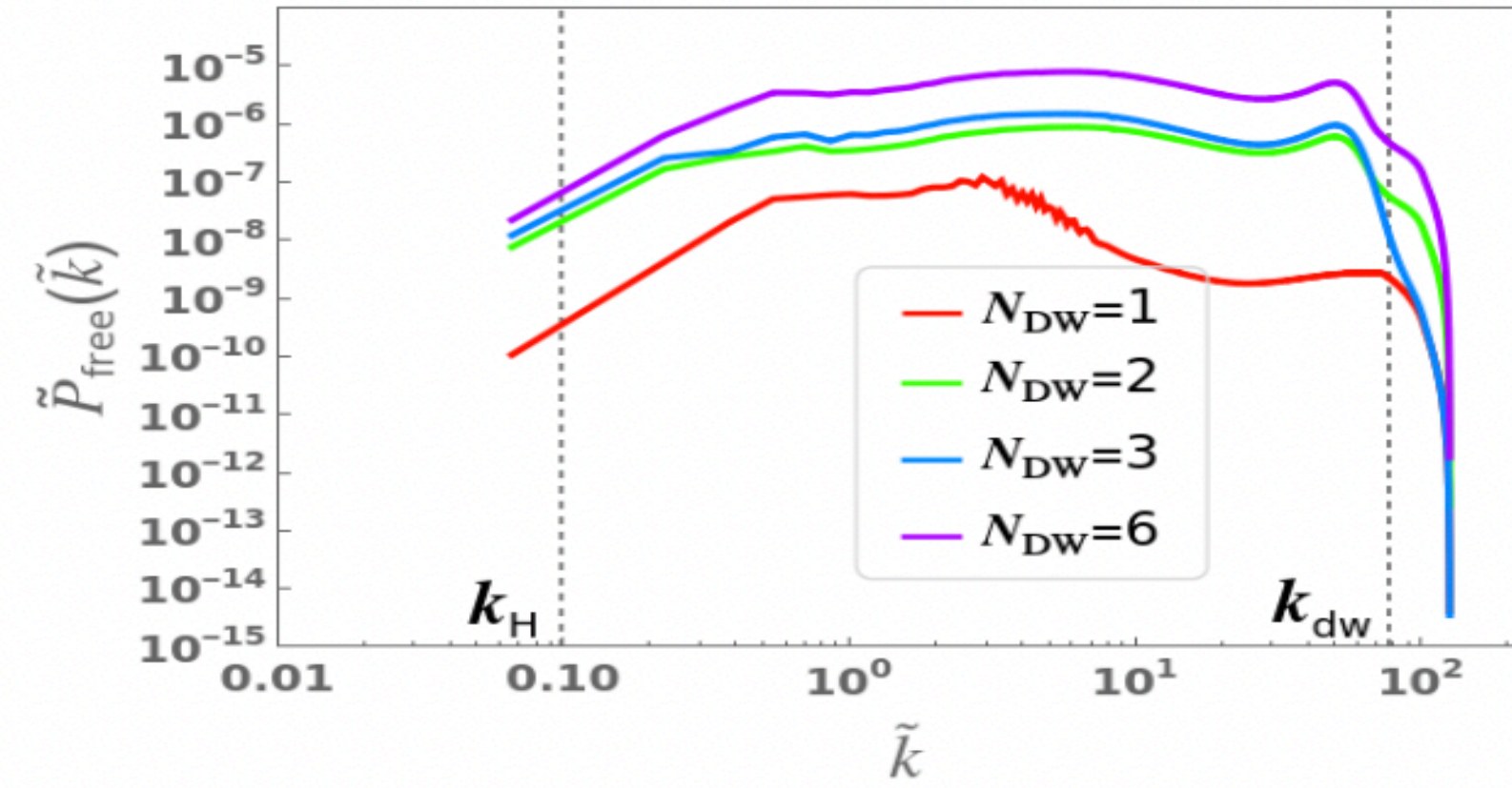
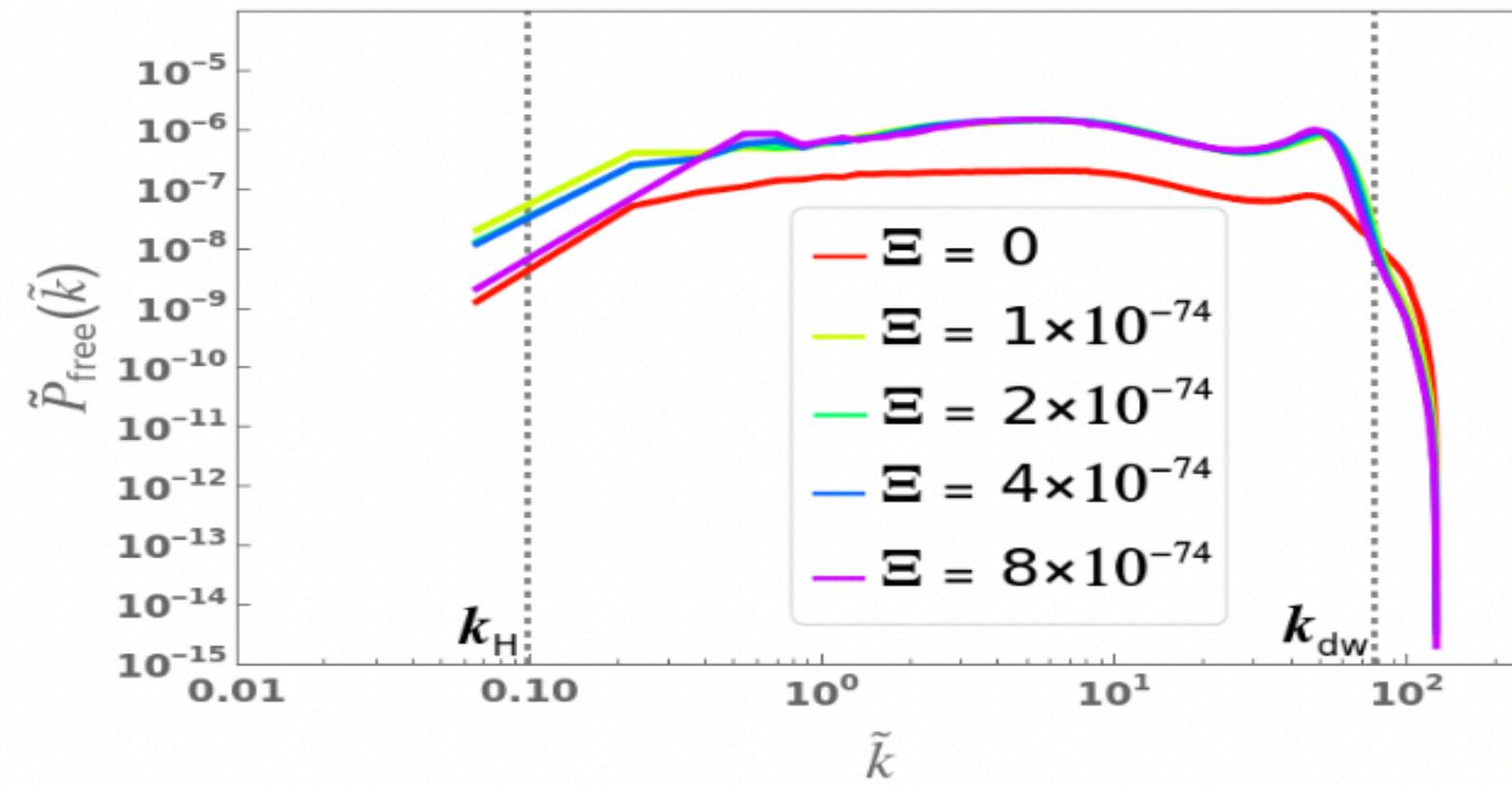
DW decay

$$\Delta V = 2\Xi v^4(1 - \cos(2\pi/N_{\text{DW}})) \sim \rho_{\text{wall}} = \mathcal{A}\sigma_{\text{wall}}/t$$

$$\Rightarrow t_{\text{dec}} = t_{\text{form}} \left(\frac{C_d \mathcal{A}_{\text{form}} \sigma_{\text{wall}}}{t_{\text{form}} \Xi N_{\text{DW}}^4 f_a^4 (1 - \cos(2\pi/N_{\text{DW}}))} \right)^{1/p}$$

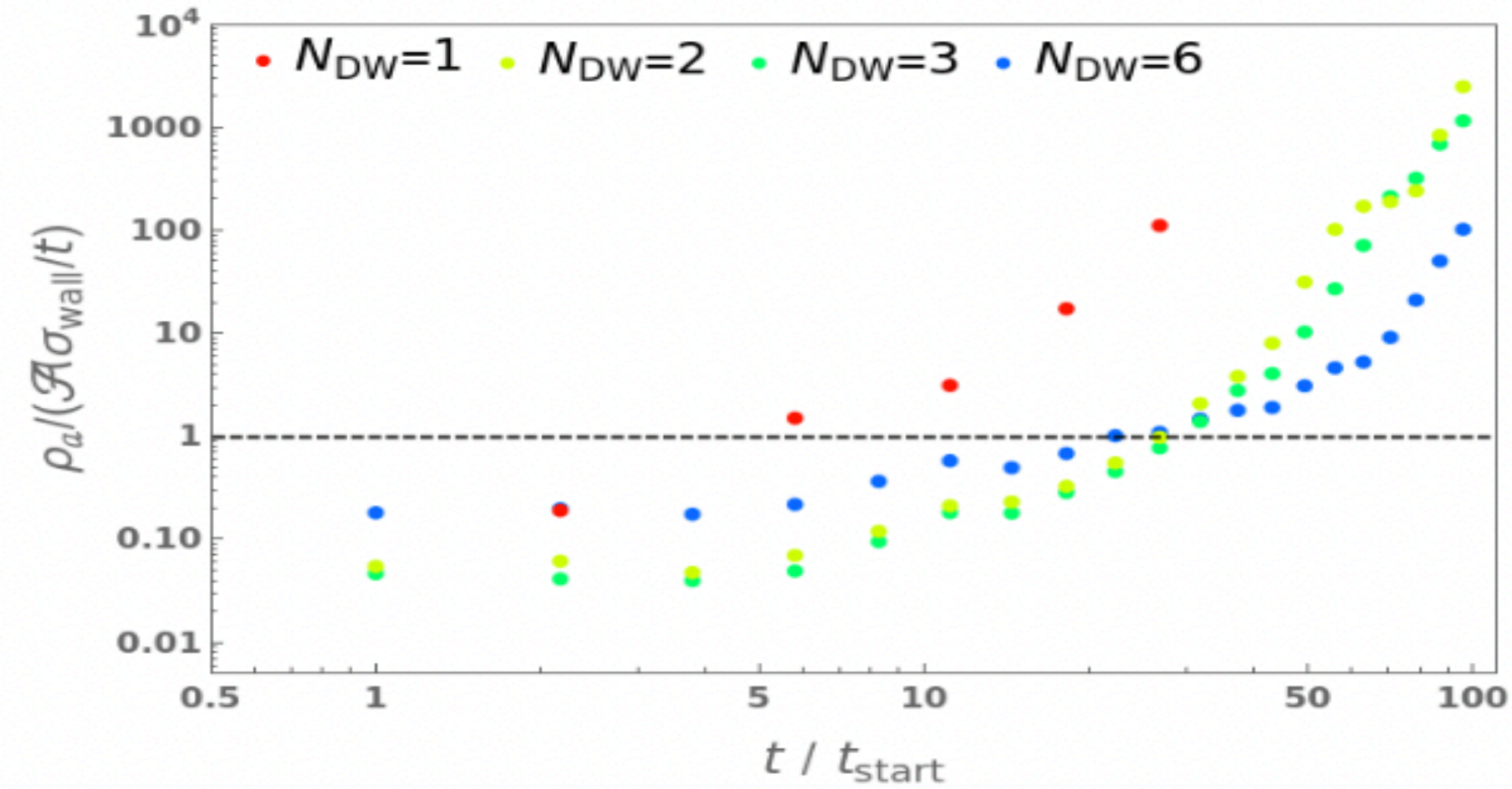
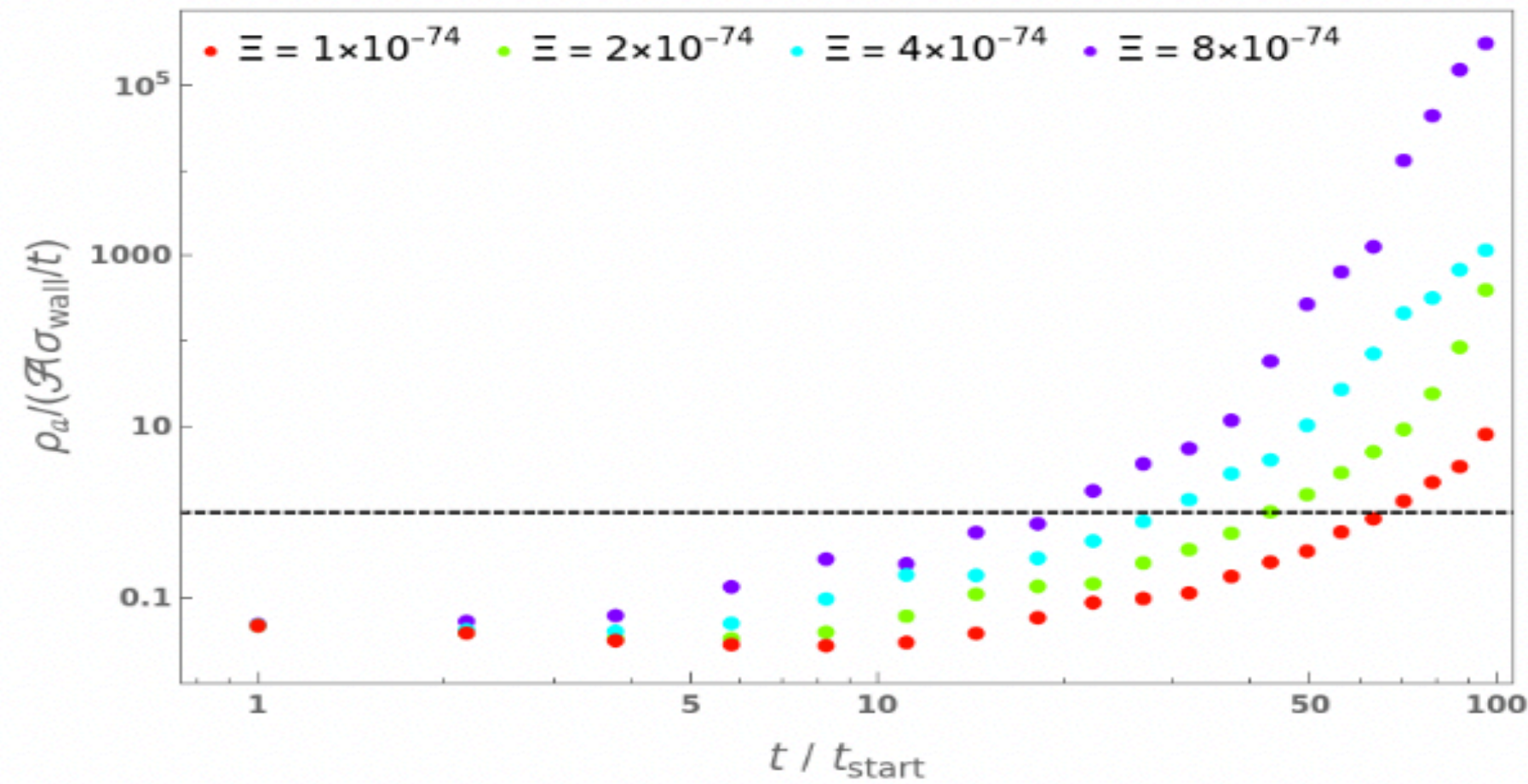
with $p = 1 + \ln C_d / \ln(t_{\text{dec}} / t_{\text{form}})$
 $C_d = \Delta V / (\mathcal{A}\sigma_{\text{wall}}/t) \sim \mathcal{O}(10^2)$

► Domain wall decay to free axions



$$\frac{1}{2} \langle \dot{a}_{\text{free}}(\mathbf{k})^* \dot{a}_{\text{free}}(\mathbf{k}') \rangle = (2\pi)^3 \frac{2\pi^2}{k^3} P_{\text{free}}(k) \delta(\mathbf{k} - \mathbf{k}') \quad W(\mathbf{x}) = W_{\text{st}}(\mathbf{x}) \times W_{\text{dw}}(\mathbf{x}), \quad \dot{a}_{\text{free}}(\mathbf{x}) = W(\mathbf{x}) \dot{a}(\mathbf{x}), \quad \dot{a}(\mathbf{x}) = f_a \frac{\phi_1 \dot{\phi}_2 - \dot{\phi}_1 \phi_2}{\phi_1^2 + \phi_2^2}$$

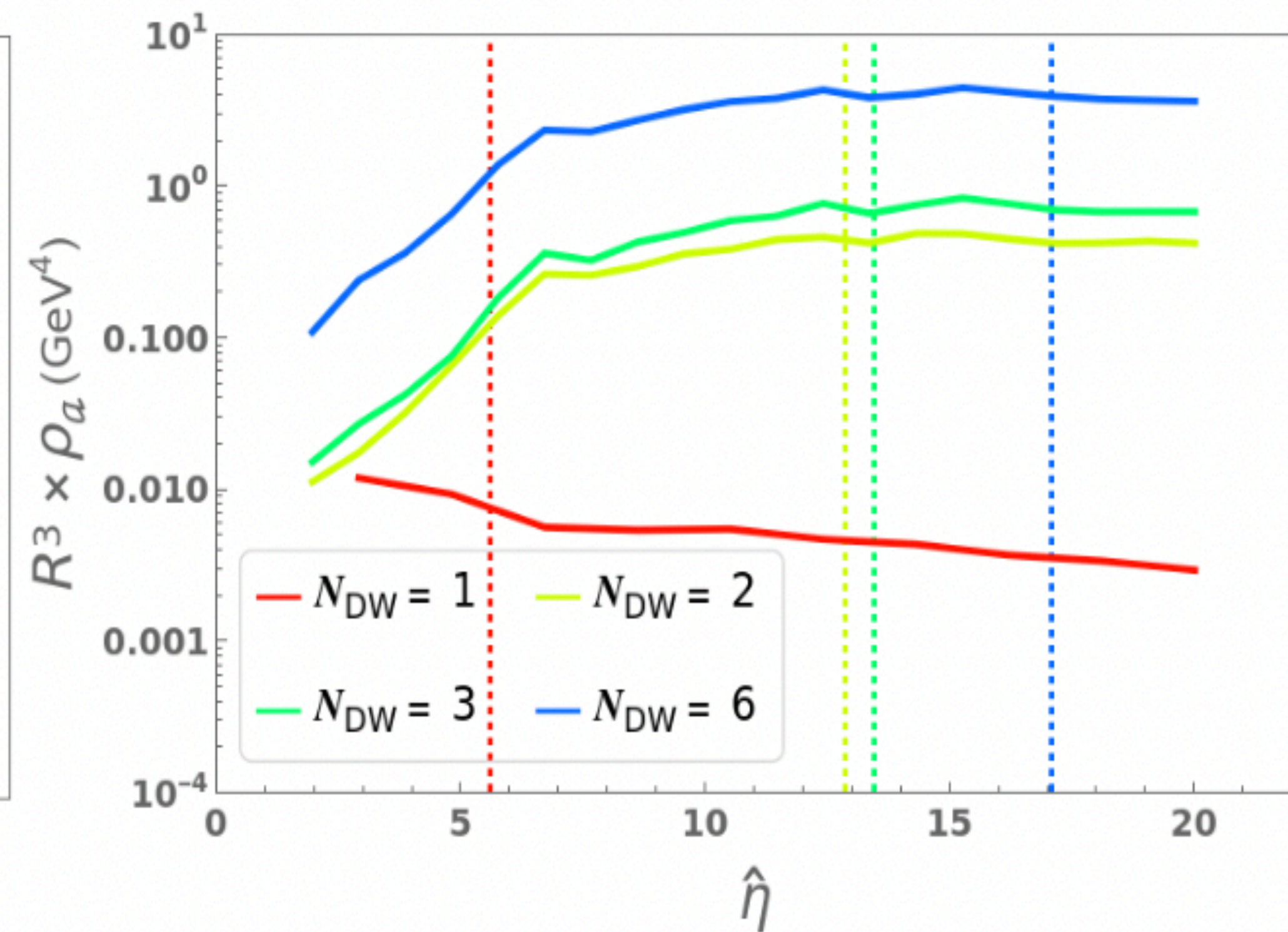
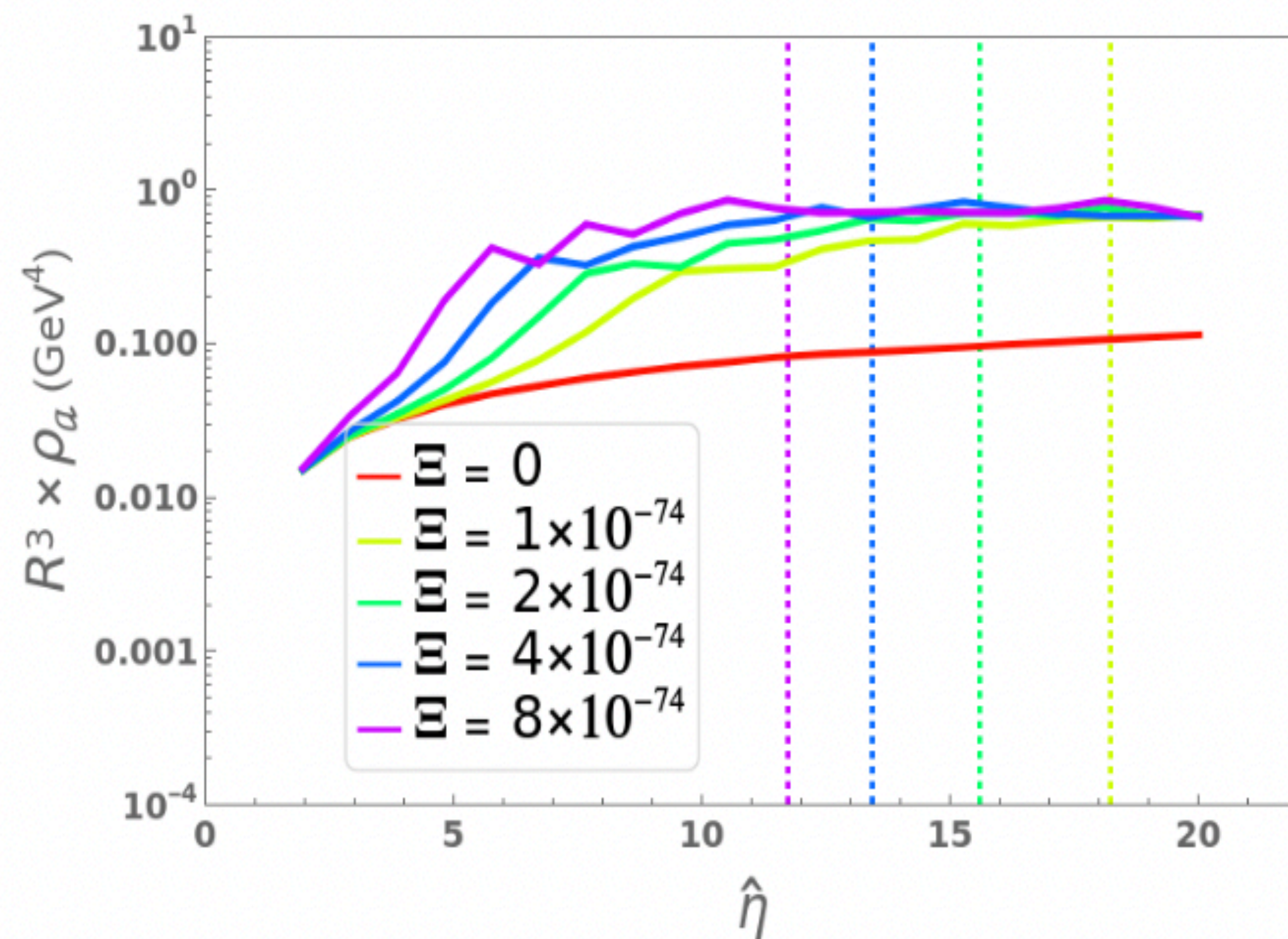
Window function $W_{\text{dw}}(\mathbf{x})$ will take 0 near the DW core and 1 near the true vacuum



$$\rho_a \simeq 2 \times \frac{1}{2} \langle \dot{a}_{\text{free}}^2(\mathbf{x}) \rangle$$

$$\rho_a \sim \mathcal{A} \sigma_{\text{wall}} / t \sim 8 \mathcal{A} m f_a^2 / t_{\text{dec}}$$

► Domain wall decay to free axions



$$\rho_a(t_{\text{dec}}) \simeq \frac{2(2p-1)f_a^4 N_{\text{DW}}^4}{(3-2p)C_d} \Xi \sin^2(\pi/N_{\text{DW}})$$

Axion energy density tends to be a constant at the final moment, and the energy density of the radiated free axion is (almost) proportional to the bias term and N_{DW}

► Domain wall decay to axion Dark matter

$$\Omega_{a,\text{new}}(t_0)h^2 = \Omega_a^{\text{DW}}h^2 + \Omega_{a,0}(t_0)h^2 + \Omega_{a,\text{st}}(t_0)h^2$$

Axion DM from Wall decay:

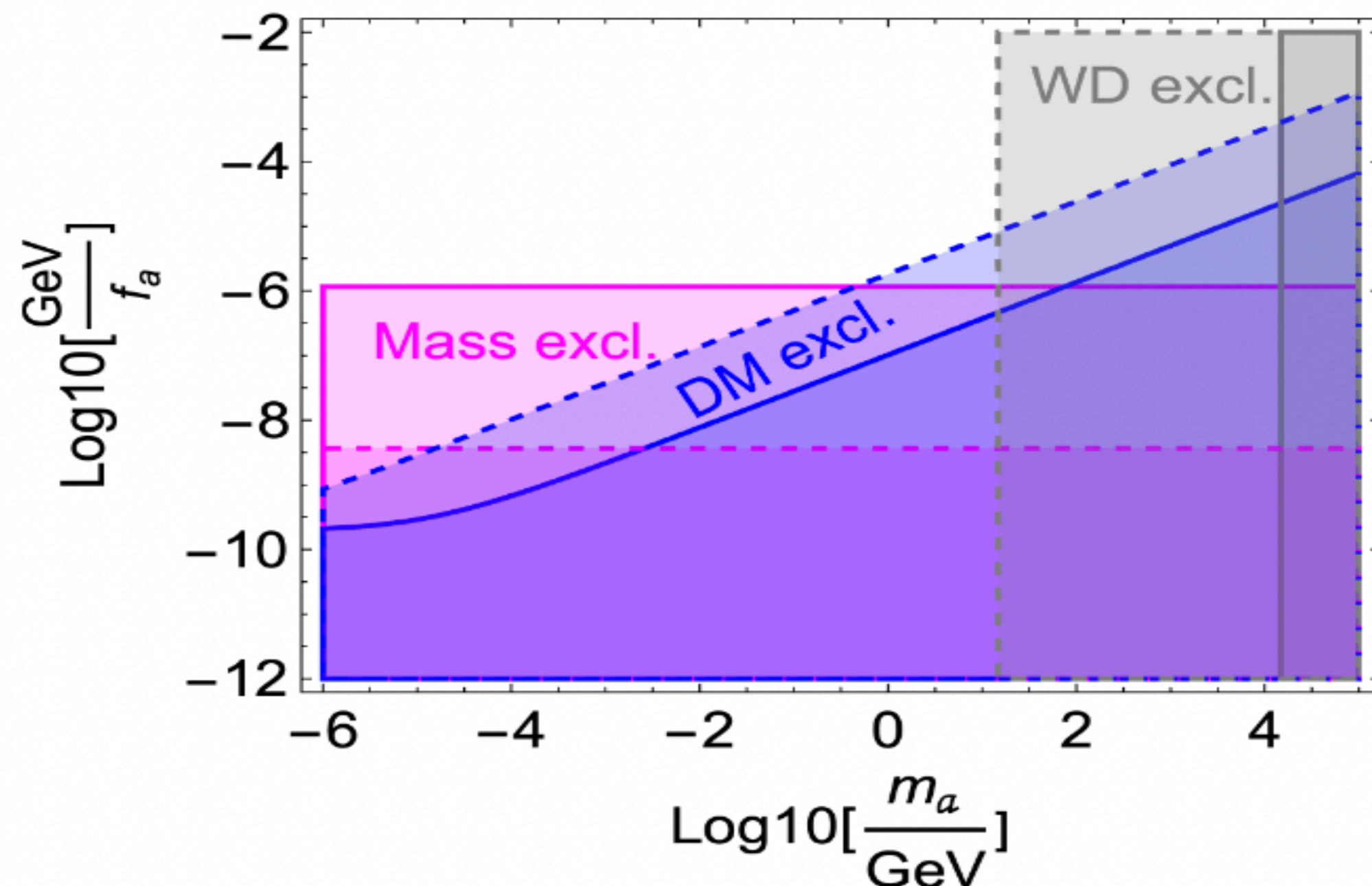
$$\Omega_a h^2 = \frac{\rho_a(t_0)}{\rho_{0,c}/h^2} = 1.024 \times 10^{-19} \left(\frac{8}{3}\right)^{3/(2p)} \left(\frac{2p-1}{3-2p}\right) C_d^{3/(2p)-1} \left(\frac{m}{\text{GeV}}\right)^{3/p-3/2} \left(\frac{f_a}{\text{GeV}}\right)^{4-3/p} \Xi^{1-3/(2p)} \left(\frac{\text{csc}(\pi/N_{\text{DW}})}{N_{\text{DW}}^2}\right)^{3/p-2}$$

Misalignment mechanism:

$$(\Omega_{a,0}(t_0)h^2 \simeq 4.63 \times 10^{-3} (f_{a,\text{new}}/10^{10}\text{GeV})^{1.19})$$

Axion DM from pure axion string:

$$\Omega_{a,\text{st}}(t_0)h^2 = 2.0 \times (f_a / 10^{12}\text{GeV})^{1.19}$$



WD excl.:

$$t_{\text{dec}} < t_{\text{WD}} \quad t_{\text{WD}} = \frac{3}{16\pi G \sigma_{\text{wall}}}$$

Mass excl.:

bias term dominating over the QCD instanton effect in the contributions to axion mass term

$$\Xi < 2 \times 10^{-45} N_{\text{DW}}^{-2} \left(\frac{10^{10}\text{GeV}}{f_a}\right)^4$$

DM excl.

$$\Omega_a(t_0)h^2 \leq 0.12$$

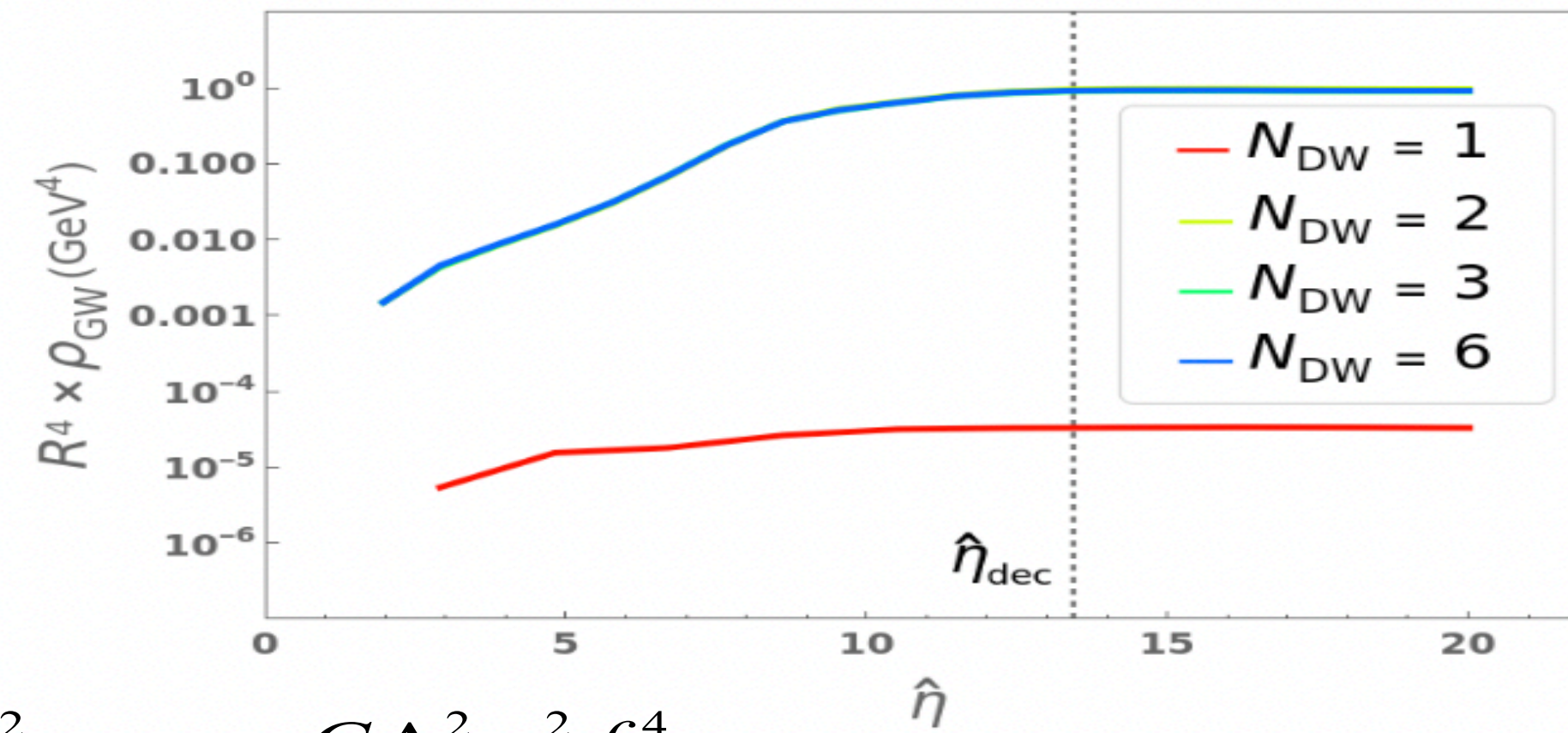
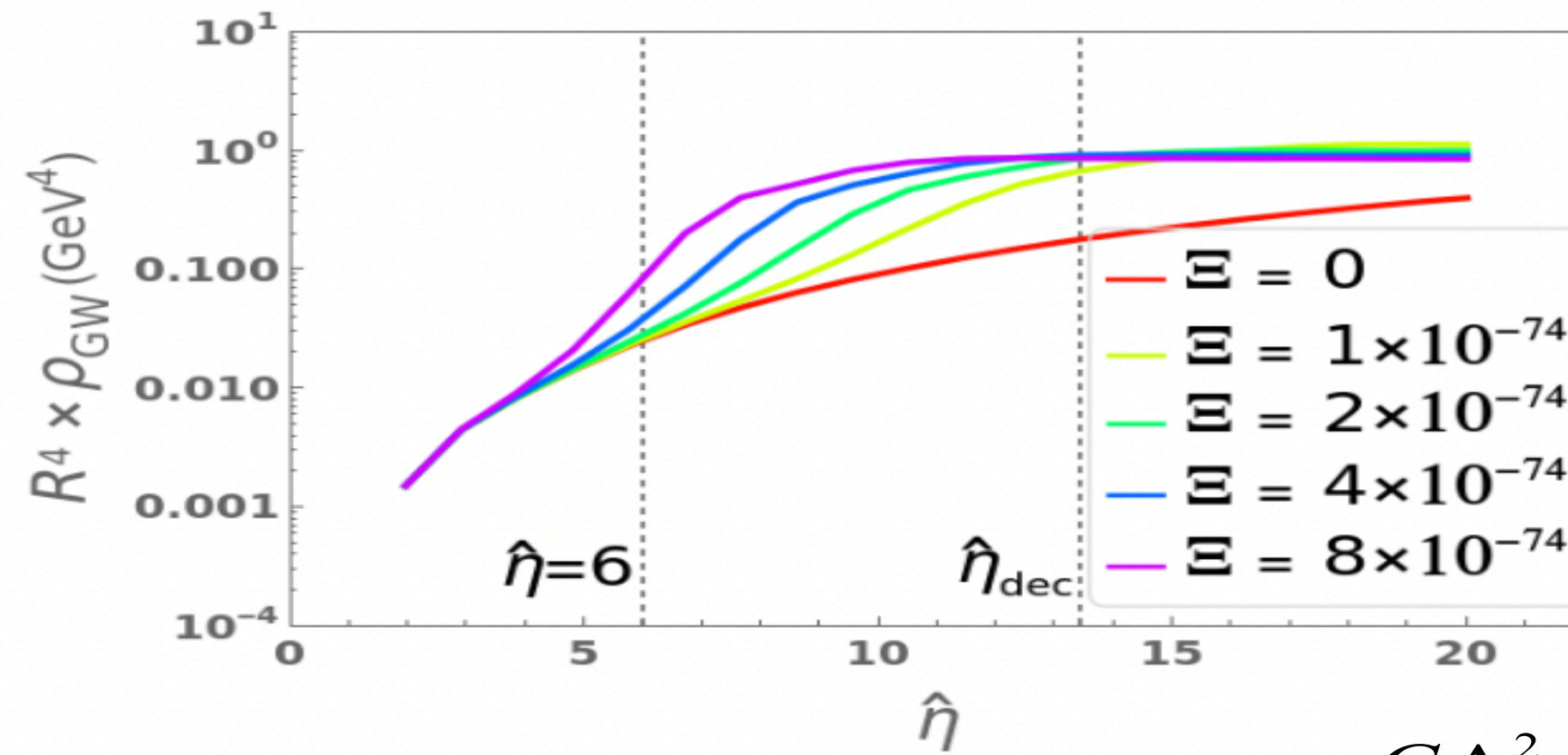
► String-wall GW

$$\ddot{h}_{ij} + 3\frac{\dot{R}}{R}\dot{h}_{ij} - \frac{\nabla^2}{R^2}h_{ij} = \frac{16\pi G}{R^2}\Pi_{ij}^{\text{TT}}$$

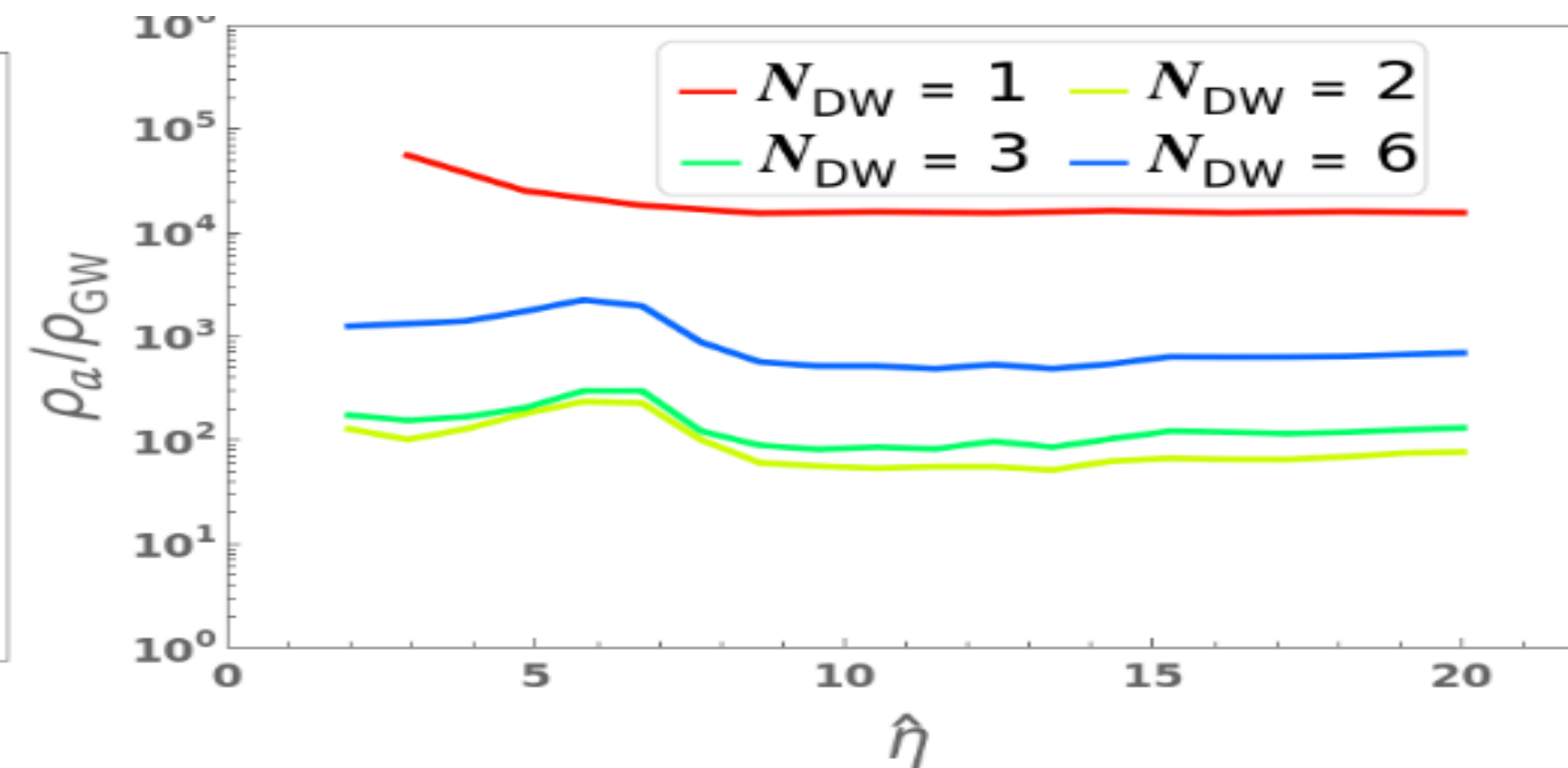
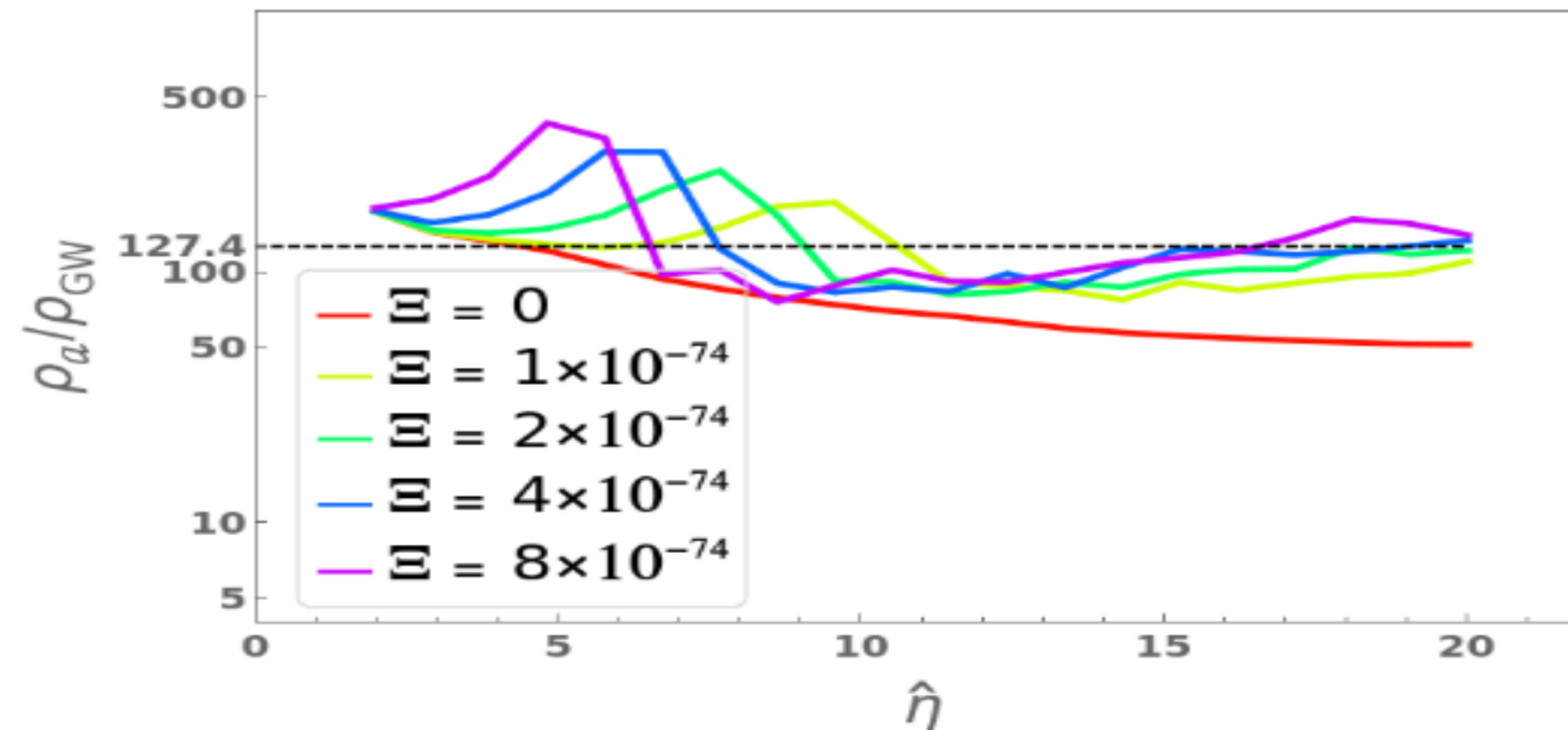
$$\rho_{\text{GW}}(t) = t_{00} = \frac{1}{32\pi G}\langle\partial_\mu h_{ij}^{\text{TT}}(\mathbf{x}, t)\partial_\nu h^{\text{TT},ij}(\mathbf{x}, t)\rangle|_{\mu=\nu=0} = \frac{1}{32\pi G}\langle\dot{h}_{ij}(\mathbf{x}, t)\dot{h}_{ij}(\mathbf{x}, t)\rangle$$

$$\Pi_{ij} \equiv T_{ij} - pg_{ij} = T_{ij} - \frac{\delta_{ij}}{3}\sum_l T_{ll}$$

$$= (\partial_i\phi_1\partial_j\phi_1 + \partial_i\phi_2\partial_j\phi_2) - \frac{\delta_{ij}}{3}\sum_l(\partial_l\phi_1\partial_l\phi_1 + \partial_l\phi_2\partial_l\phi_2)$$

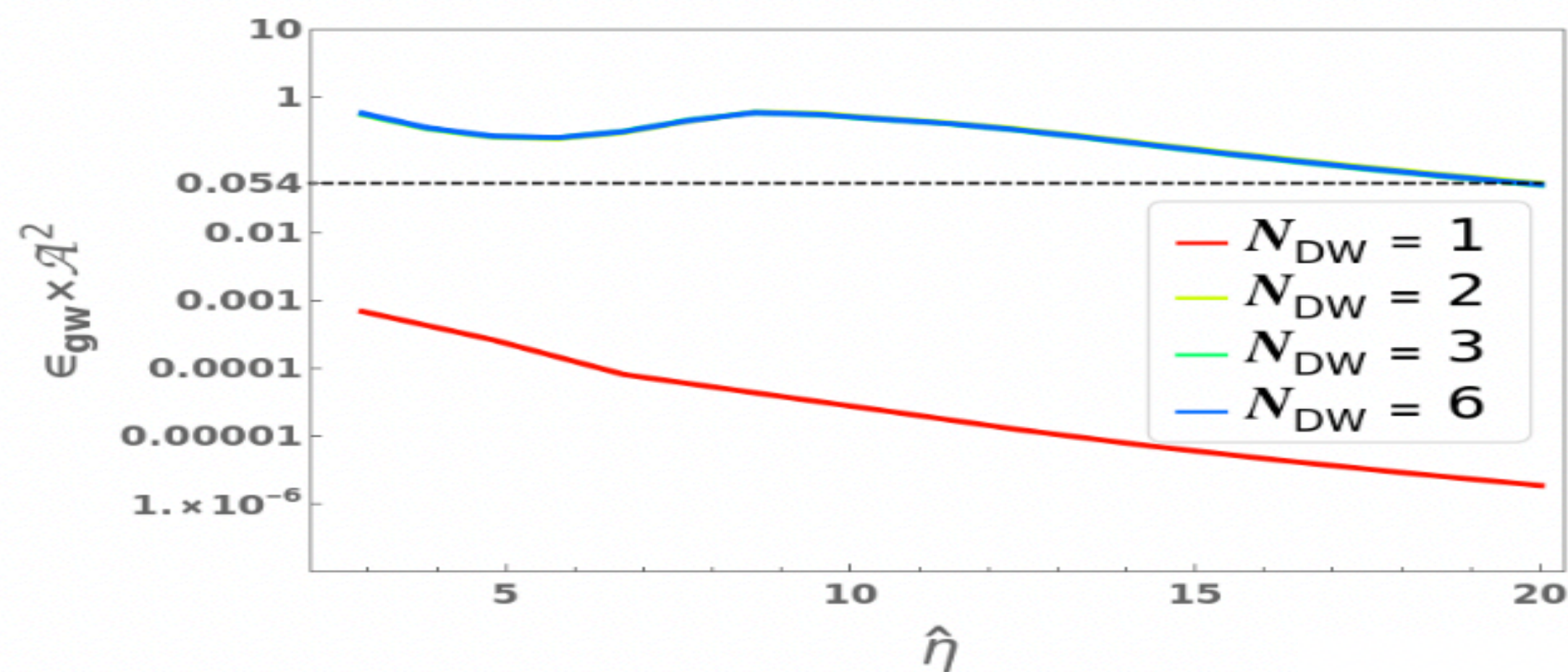


$$\rho_{\text{gw}} = \varepsilon_{\text{gw}} G A^2 \sigma_{\text{wall}}^2 \propto \varepsilon_{\text{gw}} G A^2 m^2 f_a^4$$

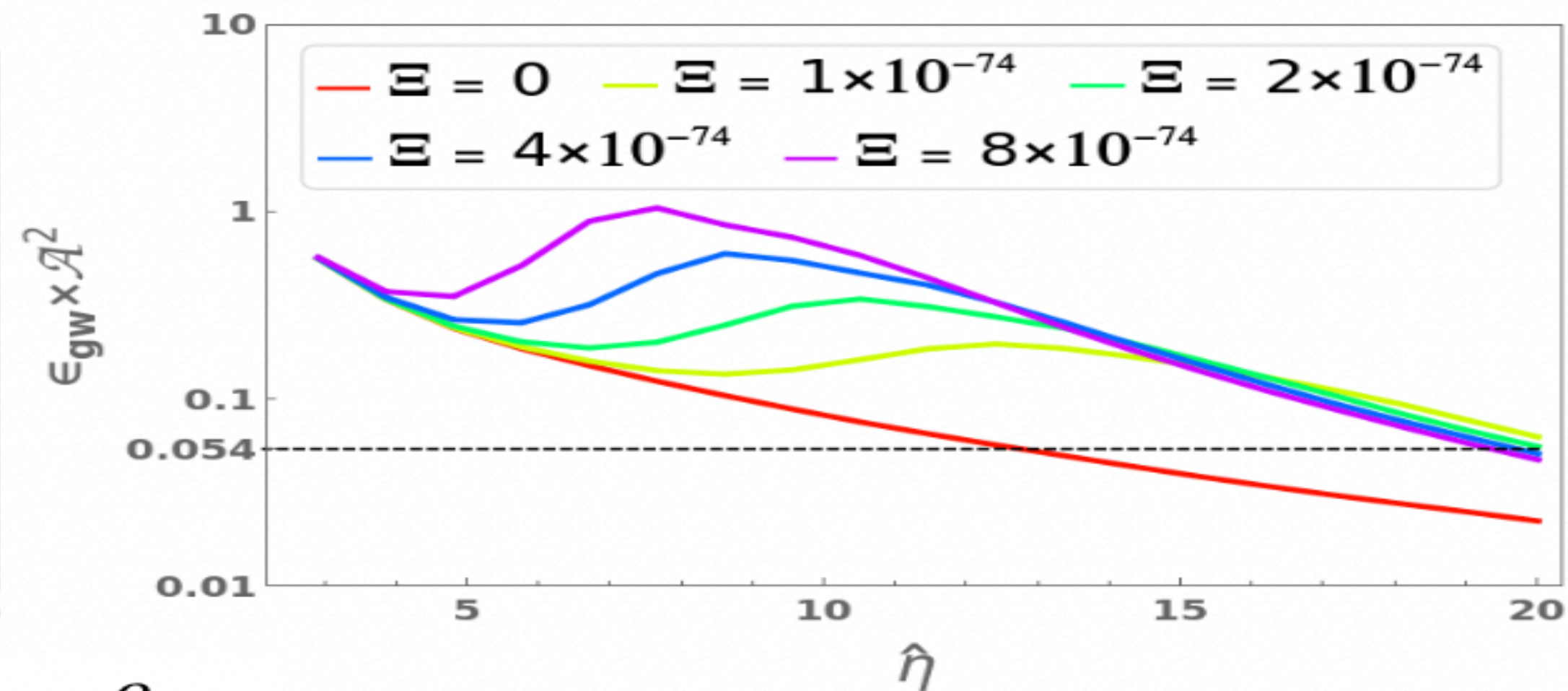


axion string-wall networks mainly decay to axion

► String-wall GW

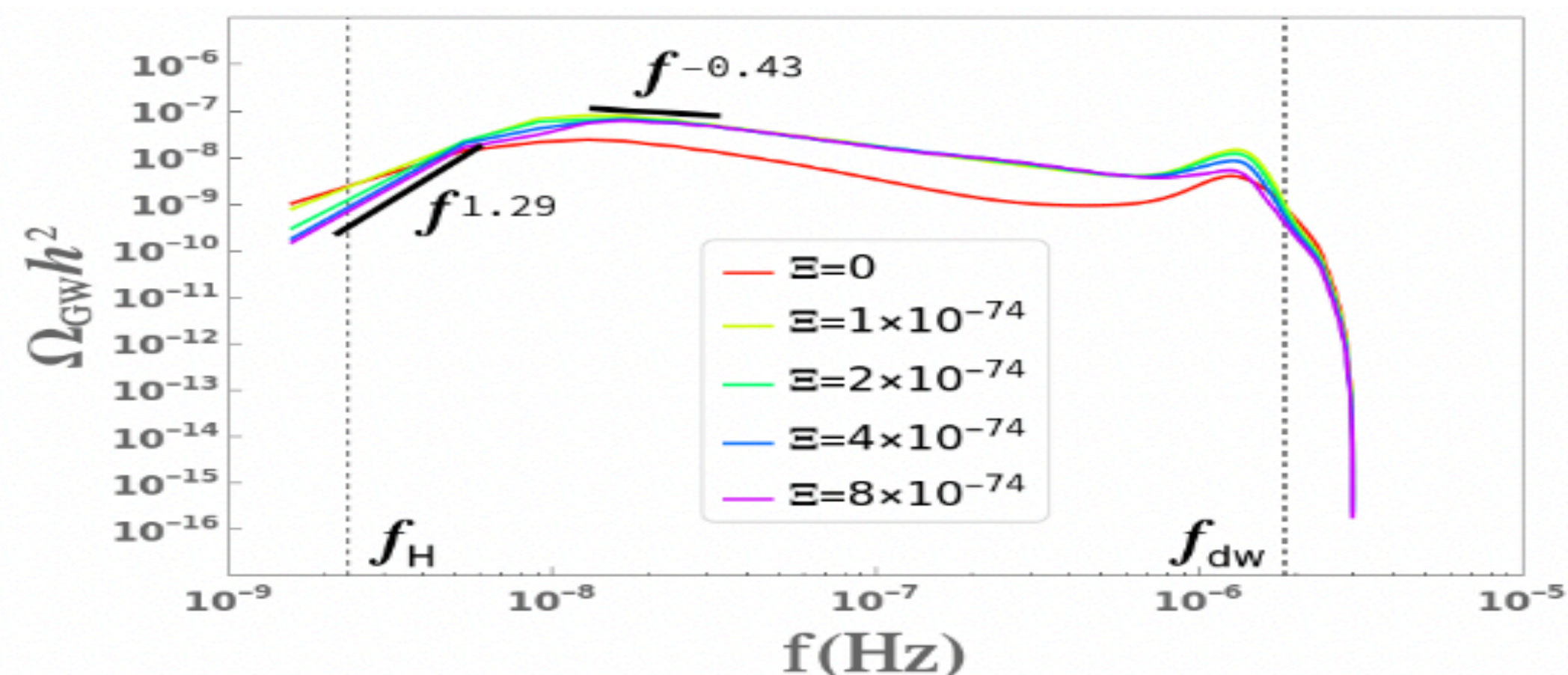


Efficiency parameter

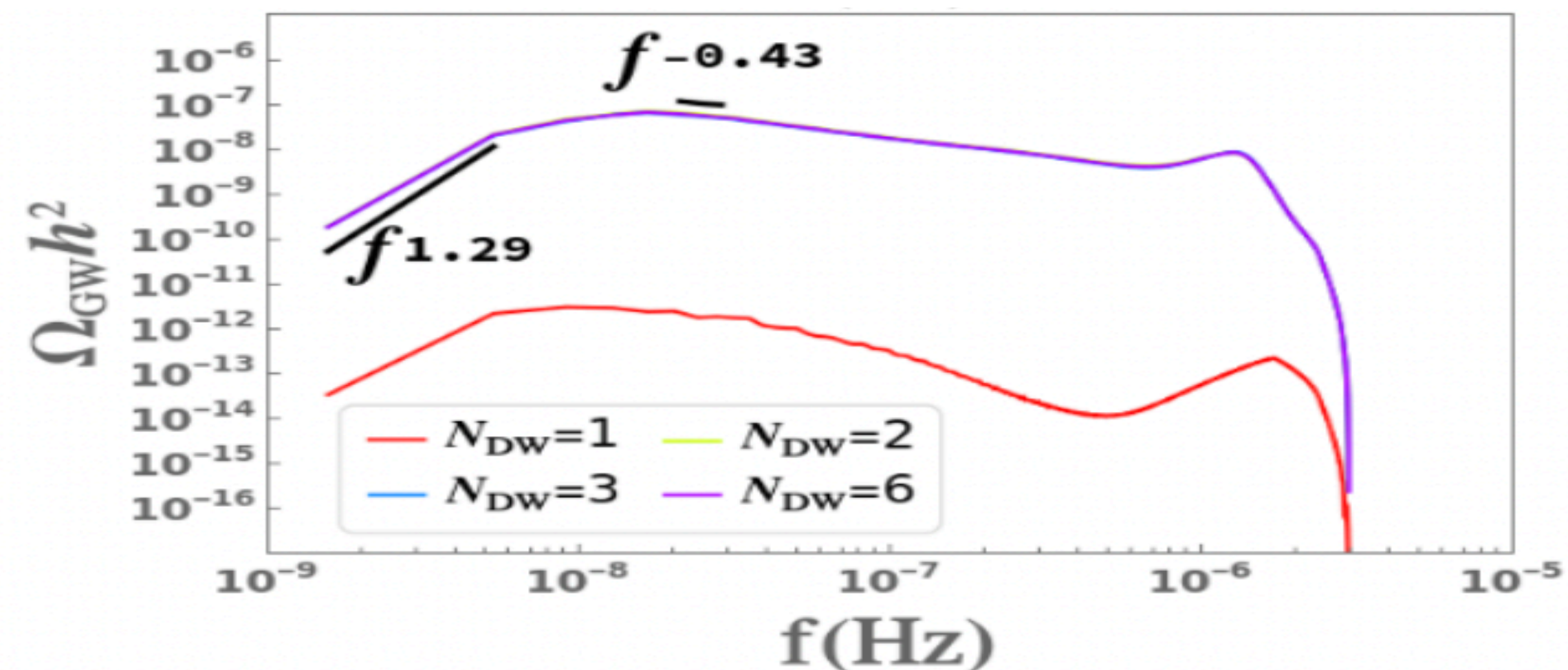


$$\epsilon_{\text{gw}} \mathcal{A}^2 = \frac{\rho_{\text{gw}}}{G\sigma_{\text{wall}}^2} \sim \text{Const.}$$

$$\rho_{\text{gw}} \propto Gm^2 f_a^4$$



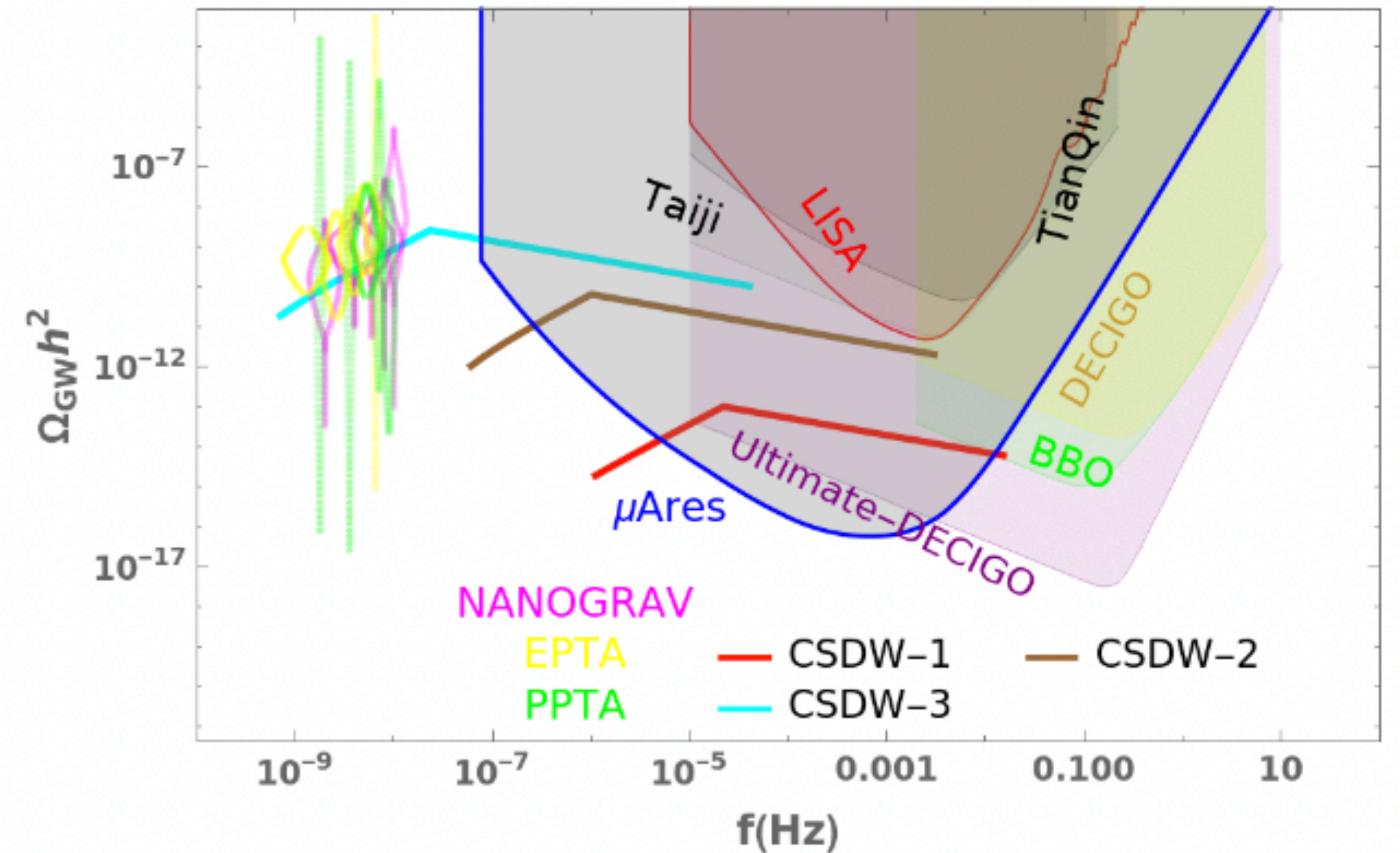
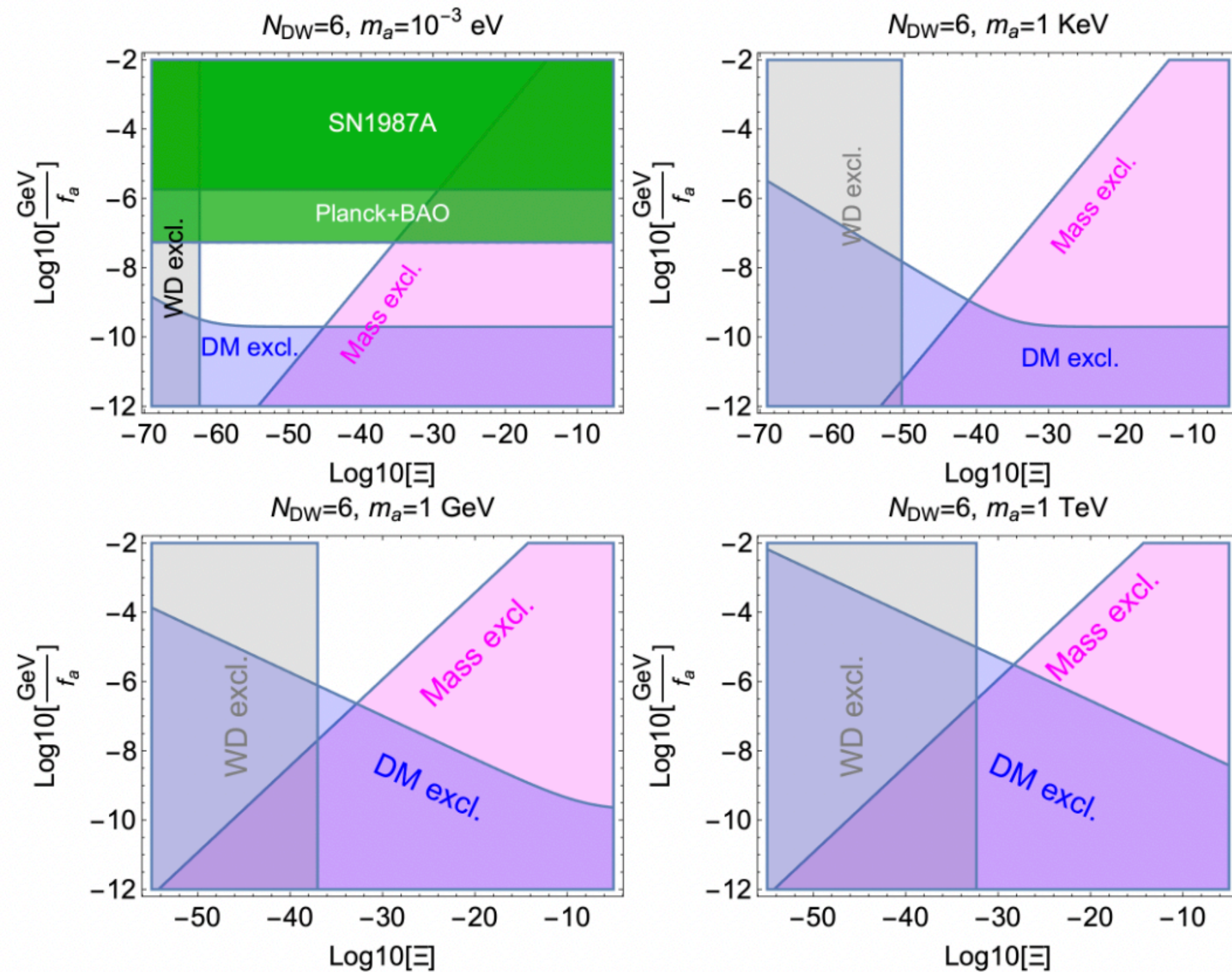
$$\begin{aligned} \Omega_{\text{GW}} &\equiv \frac{1}{\rho_c} \frac{d\rho_{\text{GW}}}{d\ln k} \\ &= \frac{1}{32\pi G} \frac{1}{(2\pi)^3 V} \int d\Omega |\mathbf{k}|^3 |\dot{h}_{ij}(\mathbf{k}, t)|^2 \end{aligned}$$



$$\begin{aligned} \Omega_{\text{GW},0} h^2 &= \frac{h^2}{\rho_{c,0}} \frac{d\rho_{\text{GW},0}}{d\ln k_{p,0}} \\ &= \Omega_{\text{rad},0} h^2 \left(\frac{g_0}{g_e} \right)^{1/3} \left\{ \frac{1}{\rho_{c,e}} \frac{d\rho_{\text{GW},e}}{d\ln k_{co,e}} \right\} \end{aligned}$$

► Axion DM and GW from String-wall

Our research provides the possibility of searching for axion models or constraining model parameters through GW detection experiments



Simultaneously explanation of DM and GWs with axion string-wall networks,

only for $N_{DW} > 1$ of ALP scenarios.

For $N_{DW} = 1$, the GW energy density appears undetectable for QCD axions and axion-like particles.

► Axion DM with PMF

- The action and potential are

$$S = \int d^4x \sqrt{-g} \left[-\partial_\mu \phi \partial^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\phi) - \alpha \frac{\phi}{v} F_{\mu\nu} \tilde{F}^{\mu\nu} \right]$$

$$m_a(T)^2 = \frac{\alpha_a \Lambda^4}{f_a^2 (T/\Lambda)^{6.68}} \quad (T > 100 \text{ MeV}, f_a = \frac{v}{N_{DW}})$$

$$V(\phi) = \frac{m_a(T)^2 v^2}{N_{DW}^2} \left(1 - \cos \left(N_{DW} \frac{\phi}{v} \right) \right)$$

$$V_{eff}(\phi) = V(\phi) + \alpha \frac{\phi}{v} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

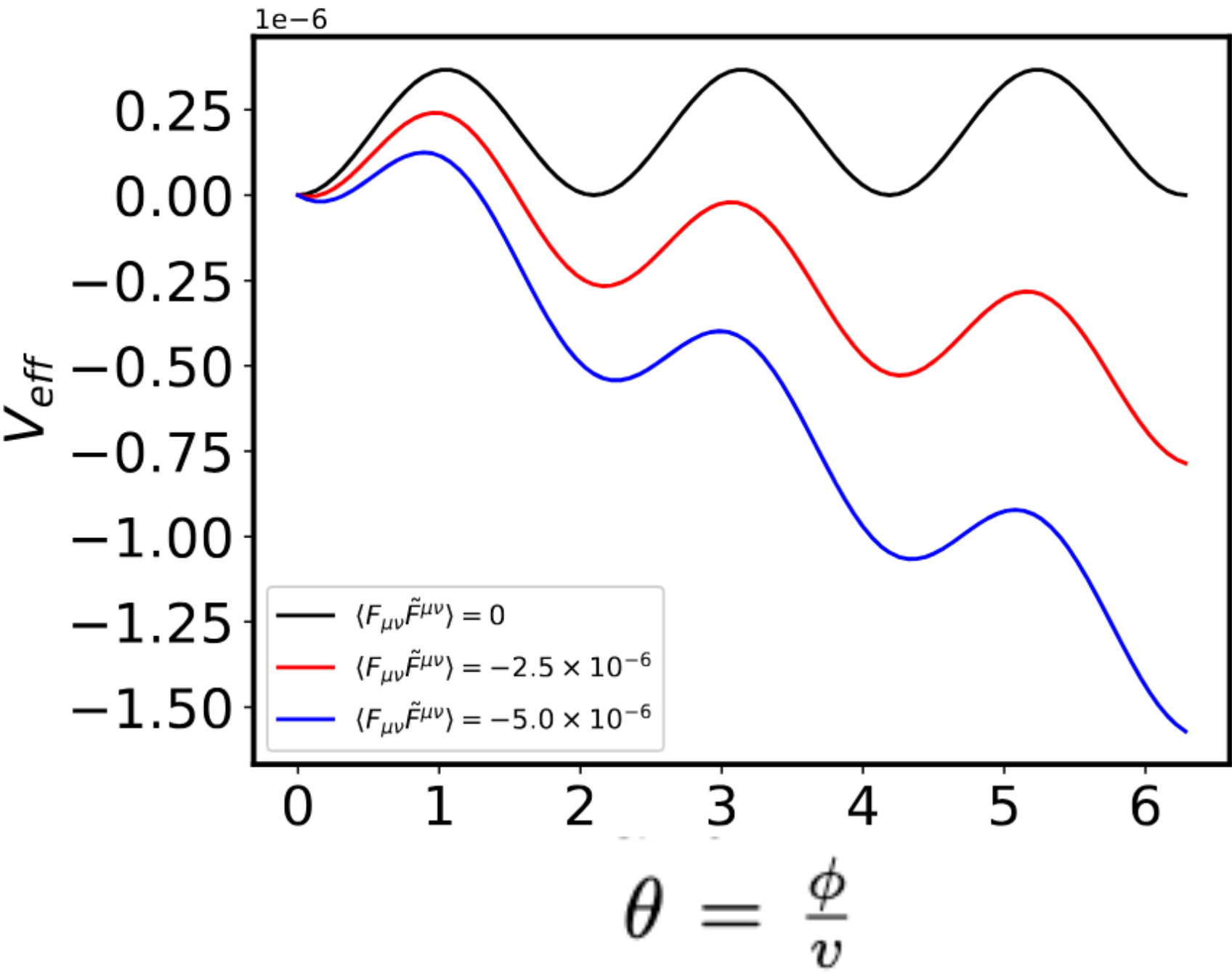
- The EOMs are

$$\begin{aligned} \phi'' &= \partial_i \partial_i \phi - 2\mathcal{H}\phi' - a^2 \frac{dV}{d\phi} - a^2 \frac{\alpha}{4v} F_{\mu\nu} \tilde{F}^{\mu\nu} \\ E'_i &= -\epsilon_{ijk} \partial_j B_k - \frac{\alpha}{v} \phi' B_i + \frac{\alpha}{v} \epsilon_{ijk} \partial_j \phi E_k \\ \partial_i A'_i &= -\frac{\alpha}{v} \partial_i \phi \epsilon_{ijk} \partial_j A_k \end{aligned}$$

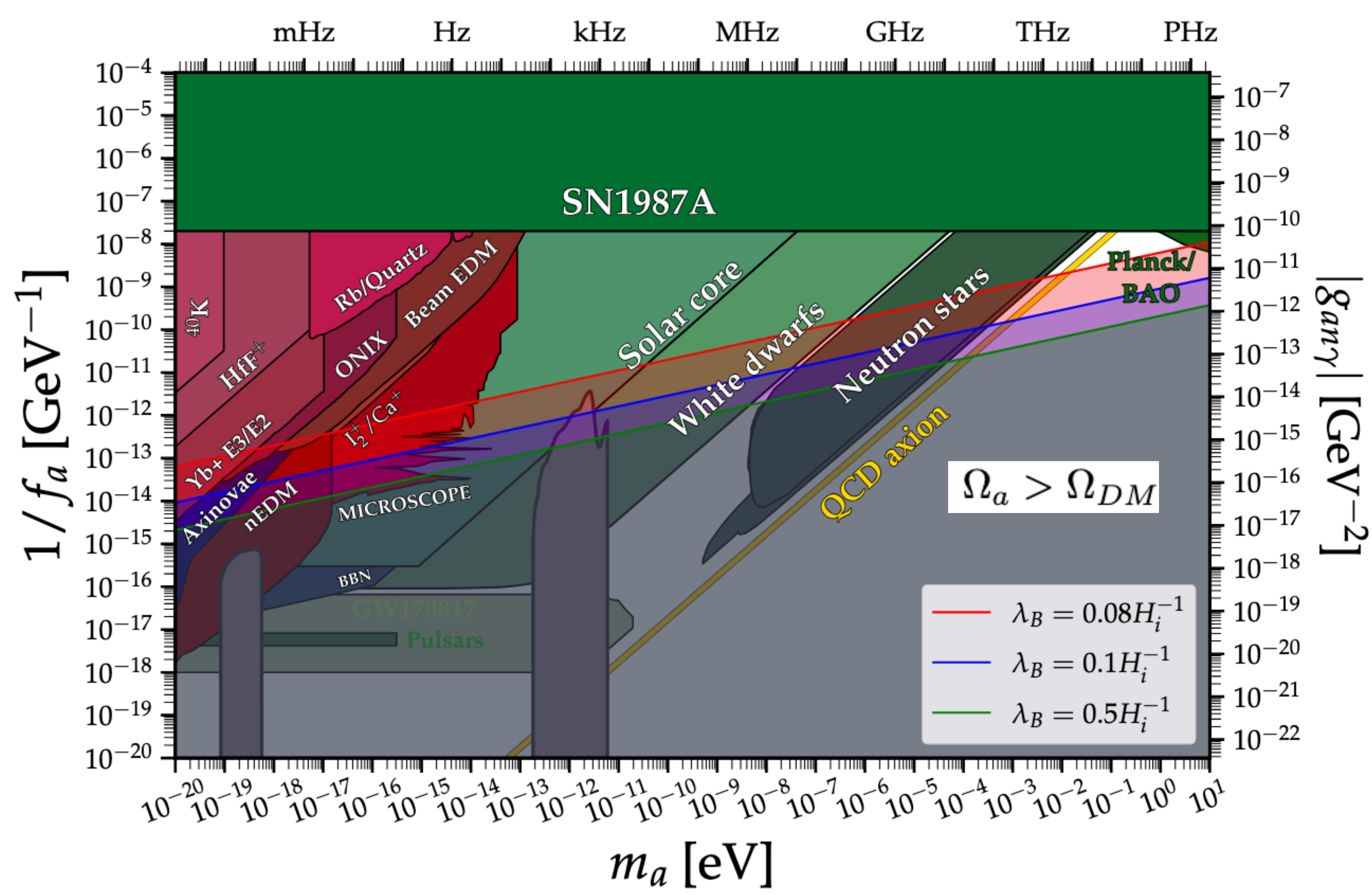
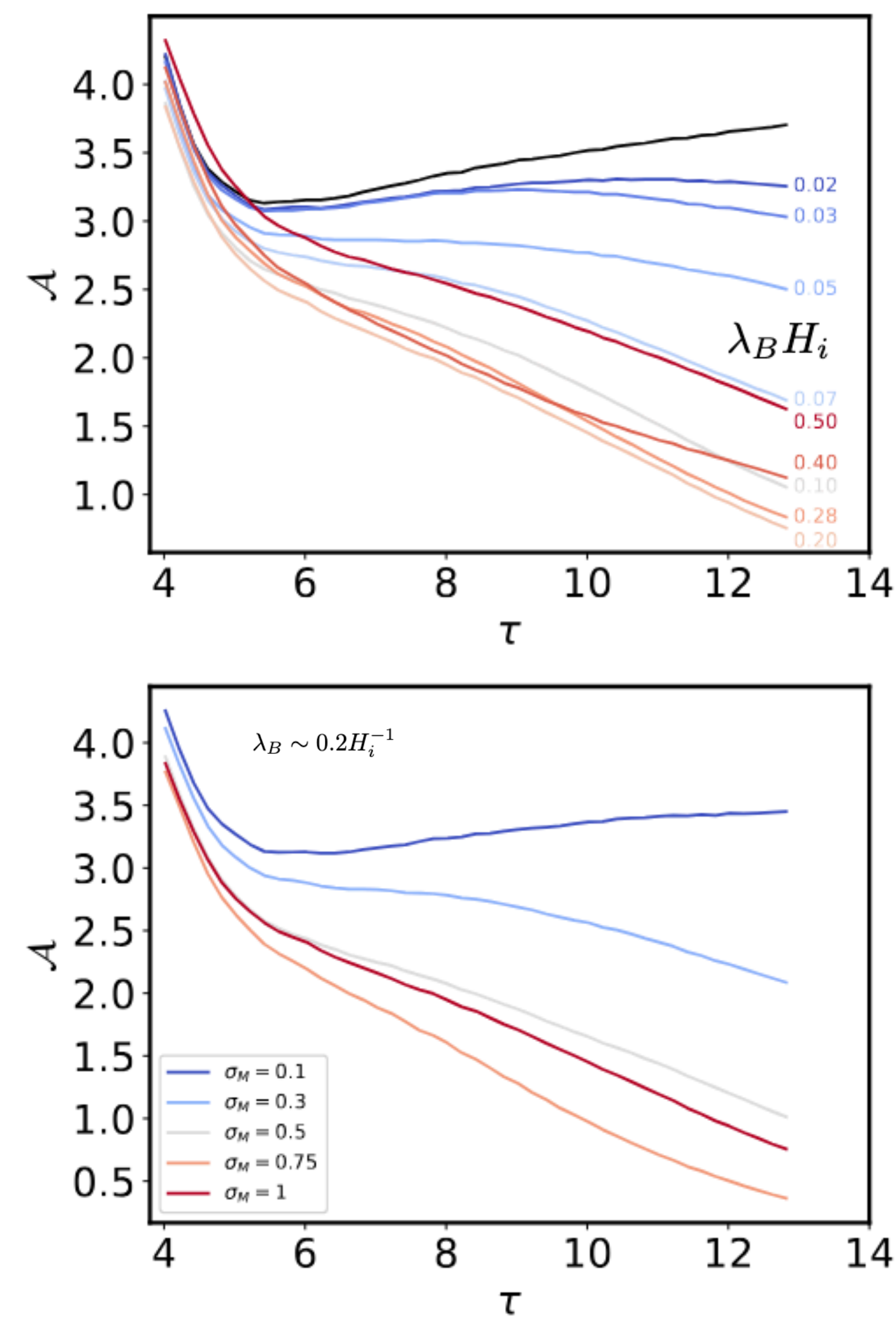
$$E_i = \frac{A'_i}{a^2}, \quad B_i = \frac{1}{a^2} \epsilon_{ijk} \partial_j A_k \quad F_{\mu\nu} \tilde{F}^{\mu\nu} = 4 \sum_i E_i B_i \text{ is Chern-Simons (CS)}$$

magnetic fields $B_i(\vec{k}) = B_{ini} \Theta(k - k_{UV}) (\delta_{ij} - \hat{k}_i \hat{k}_j - i \sigma_M \epsilon_{ijl} \hat{k}_l) g_j(\vec{k}) k^n$

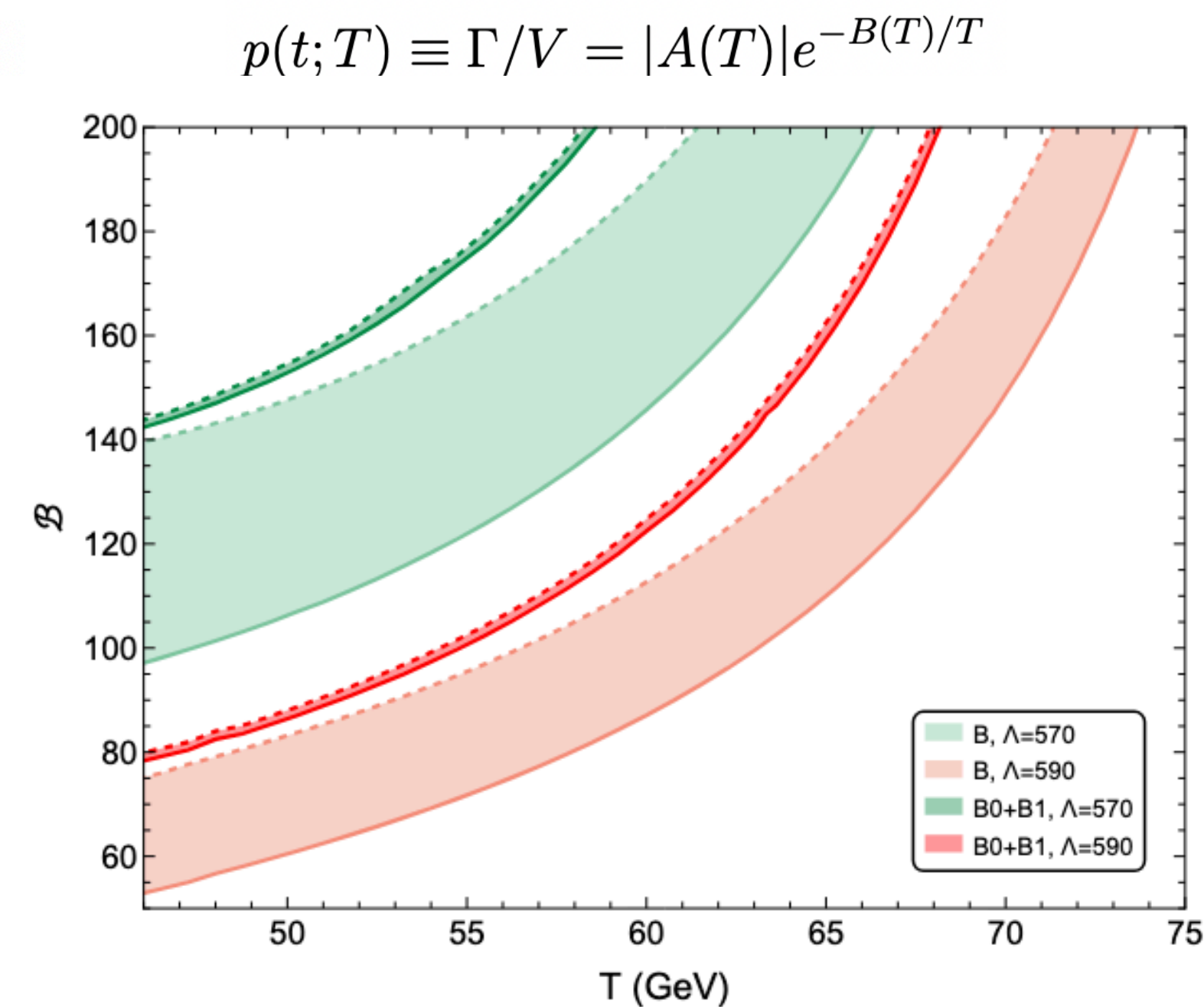
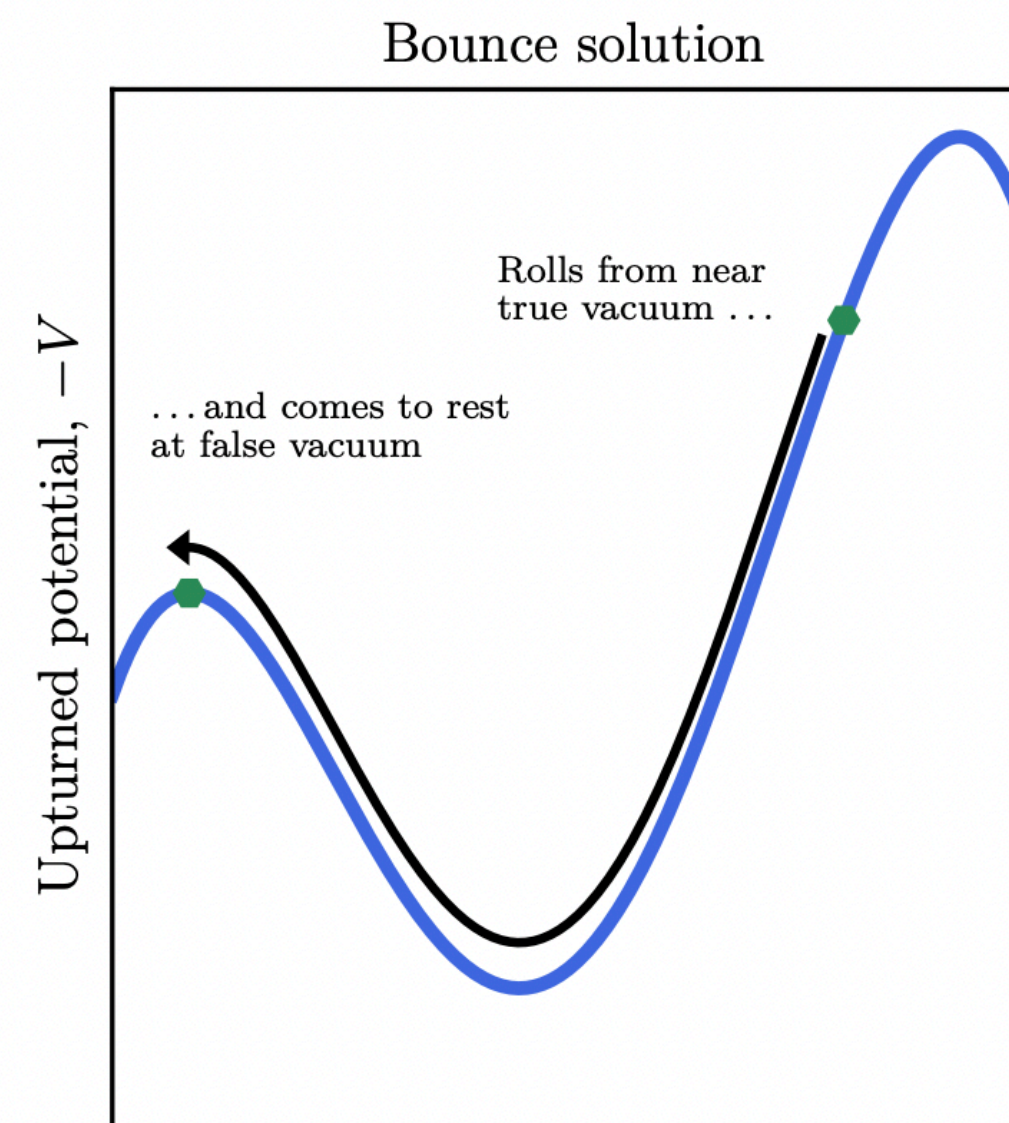
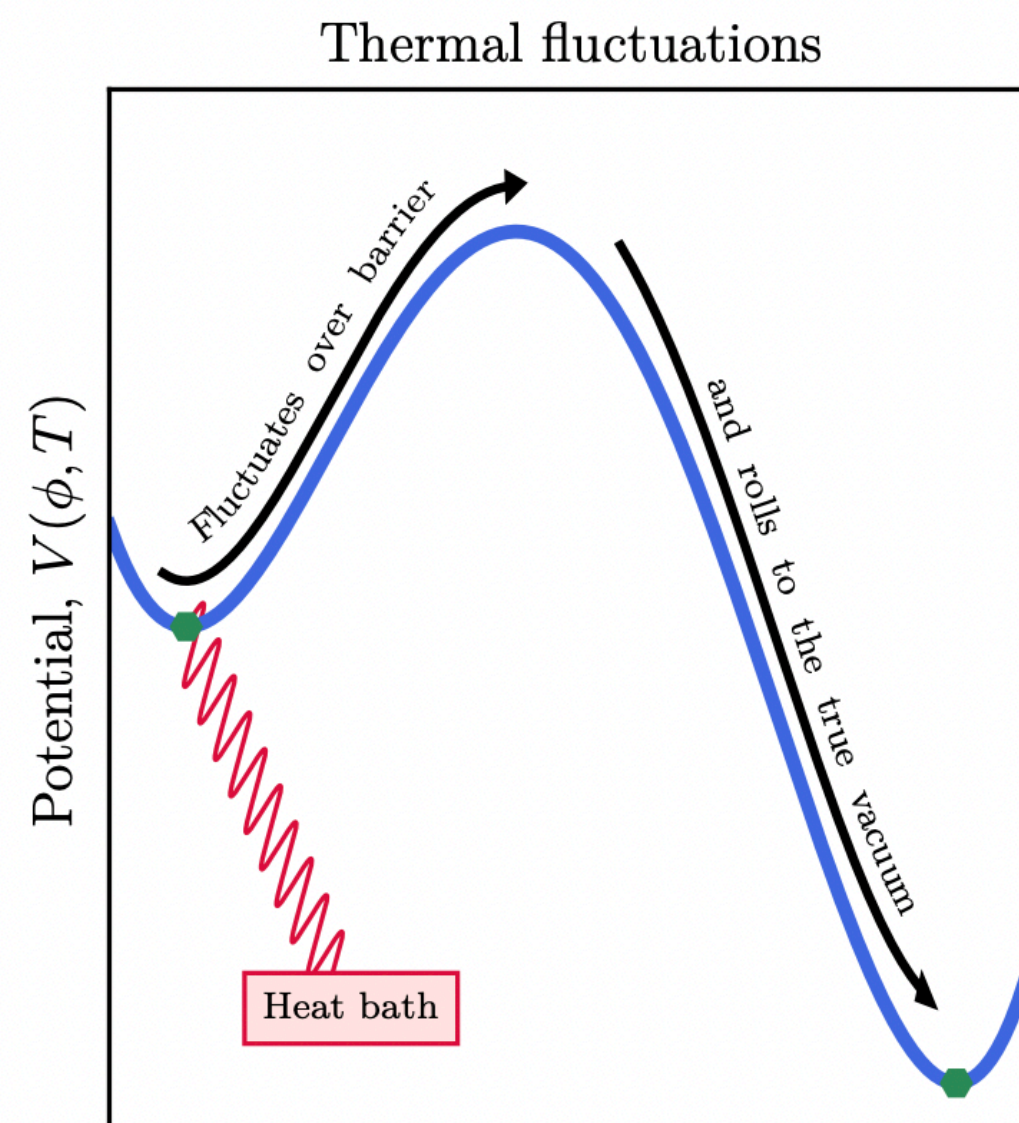
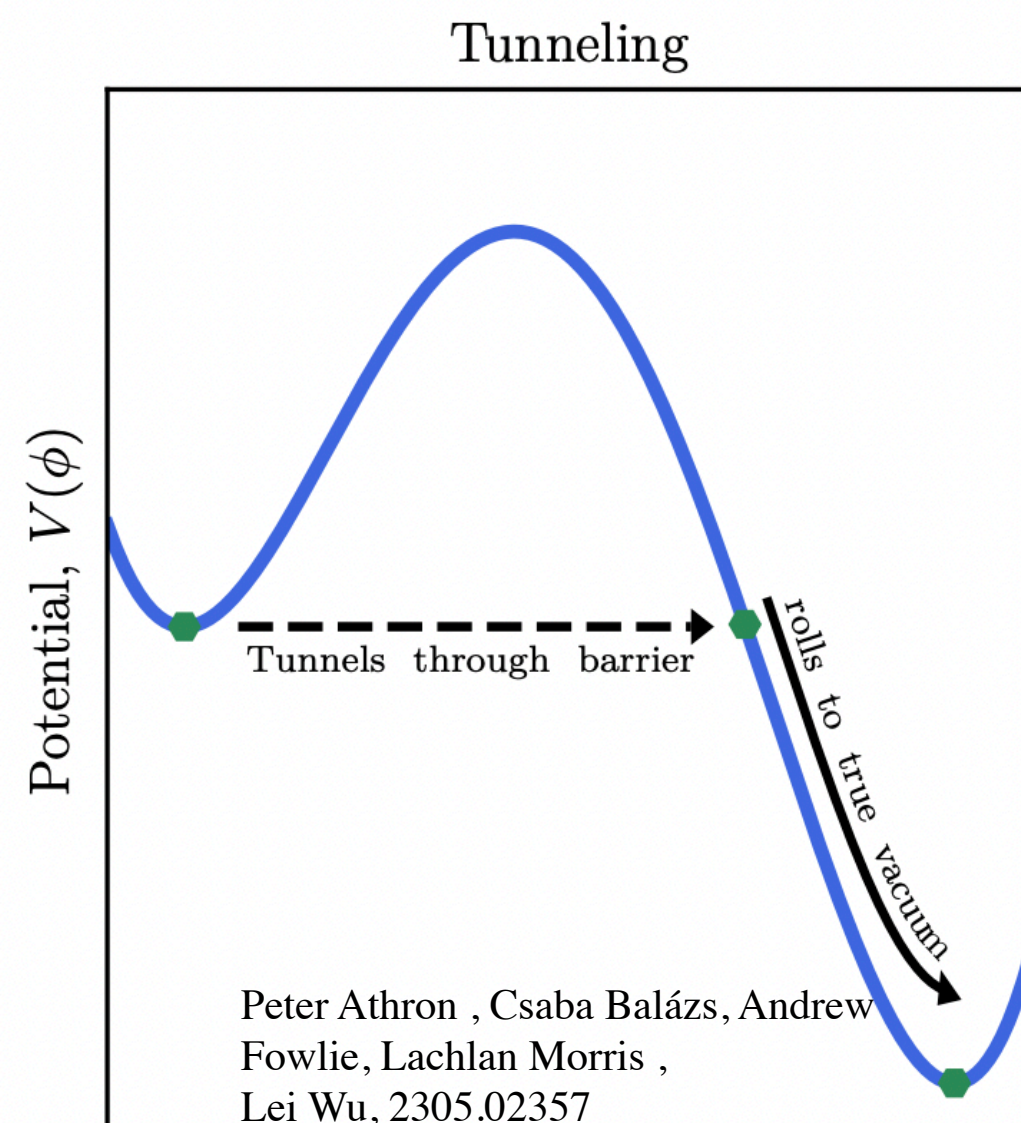
$g_j(\vec{k})$ is Gauss random field and $\sigma_M \in [-1, 1]$ determines the helicity of magnetic field. n is the spectrum index. (2409.16124)



► Axion DM with PMF



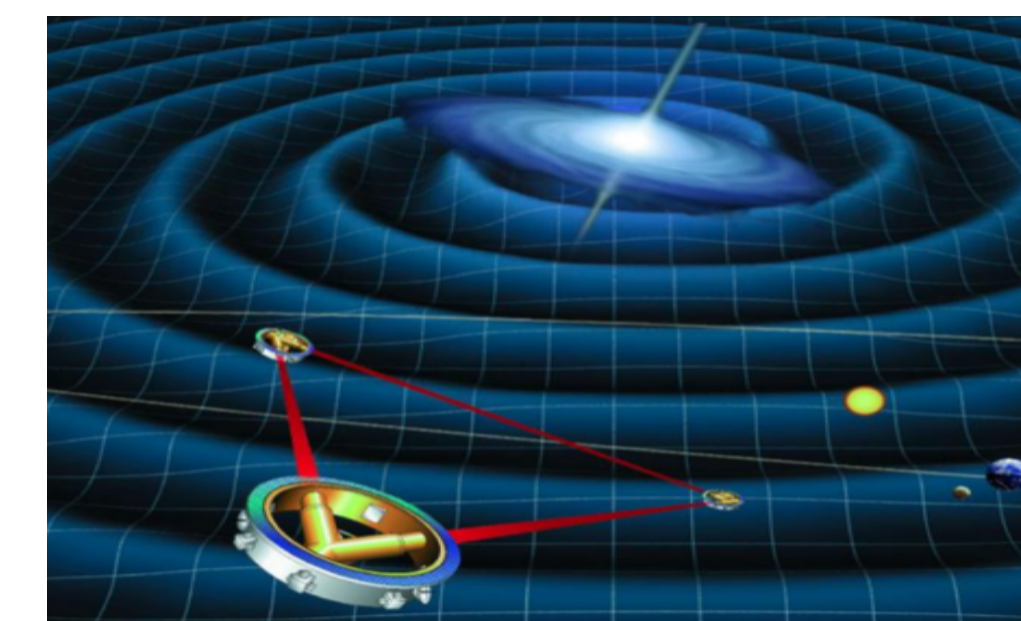
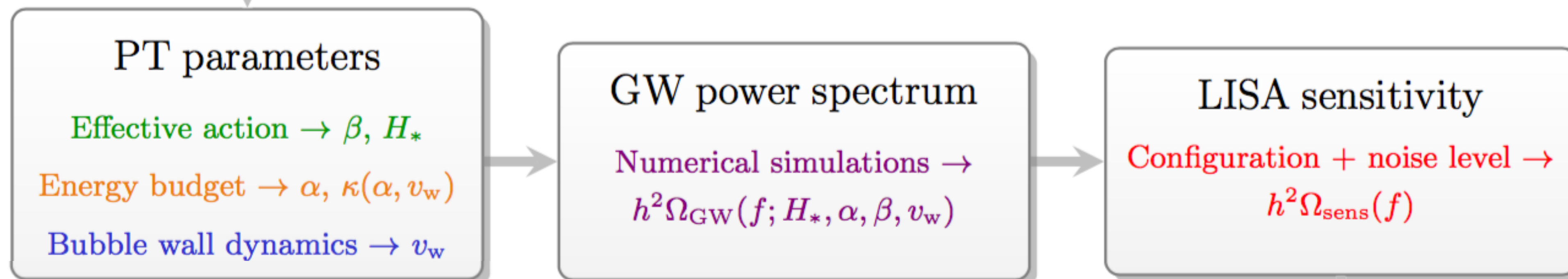
► Vacuum decay and FOPT



Jie Liu, Renhui Qin, Ligong Bian,
e-Print: 2512.05565

Thermal EFT+MC lattice simulation

Classical lattice simulation



► 1+1D simulation-across Quantum-Thermal crossover

The decay probability is formulated as a phase-space average over field configurations, with the false-vacuum basin specified by an indicator functional

$$P_{\text{FV}}(t) = \frac{1}{Z} \int D\phi \frac{D\Pi}{2\pi} \Theta_{\text{FV}}[\phi] W[\phi, \Pi; t].$$
$$W[\phi, \Pi; t] = \int D\varphi e^{-\frac{i}{\hbar} \int dx \Pi \varphi} \left\langle \phi + \frac{\varphi}{2} \left| \hat{\rho}(t) \right| \phi - \frac{\varphi}{2} \right\rangle.$$

Consider initial density matrix operator: $\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}$. The interacting thermal state is approximated by a self-consistent Gaussian quasi-free state.

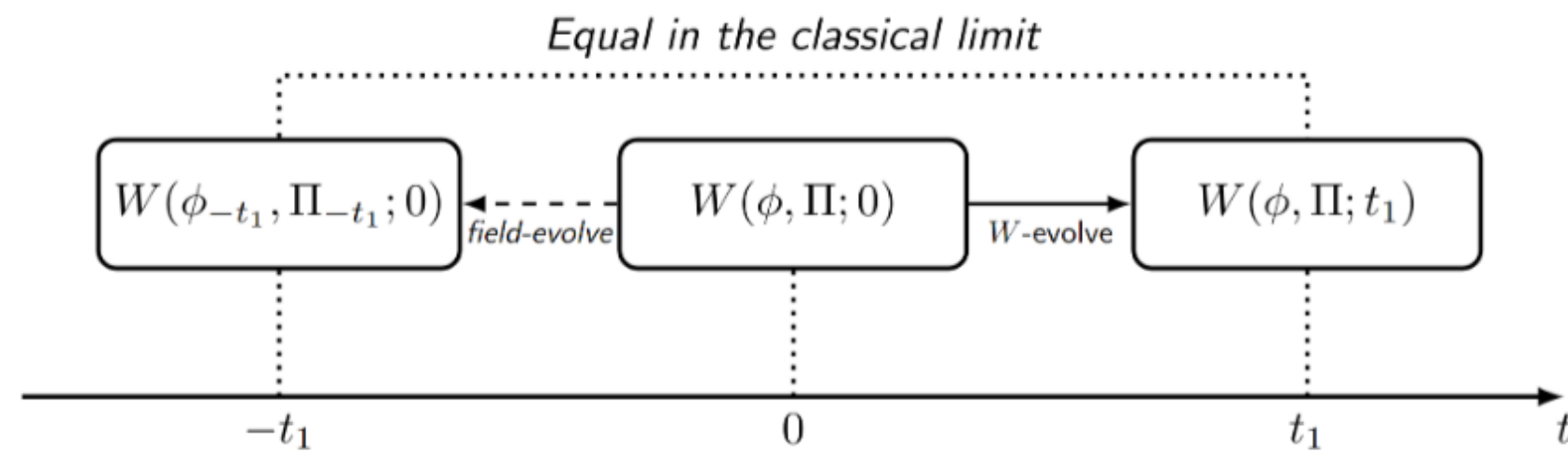
Mode-by-Mode Positive Wigner Sampling:

$$W_k(\eta_k, \xi_k) = \exp \left[-\tanh \frac{\beta \omega_k}{2} \left(\frac{|\xi_k|^2}{\omega_k} + \omega_k |\eta_k|^2 \right) \right], \quad \omega_k = \sqrt{k^2 + M_H^2}.$$
$$\langle |\eta_k|^2 \rangle = \frac{1}{2\omega_k} \coth \frac{\beta \omega_k}{2}, \quad \langle |\xi_k|^2 \rangle = \frac{\omega_k}{2} \coth \frac{\beta \omega_k}{2}.$$

This positive Hartree-Gaussian Wigner functional allows direct Monte Carlo sampling of initial field and momentum modes, including both zero-point and thermal fluctuations.

► 1+1D simulation-across Quantum-Thermal crossover

The initial quantum-thermal ensemble is evolved by classical Hamiltonian trajectories after neglecting higher-order $\hbar^2$ terms in the Wigner equation.



$$\hat{L} = \int d^3x \left(\frac{1}{2} \partial^\mu \hat{\phi} \partial_\mu \hat{\phi} - V(\hat{\phi}) \right) \quad \hat{H} = \int d^3x \hat{\mathcal{H}} = \int d^3x \left[\frac{\hat{\Pi}^2}{2} + \frac{(\nabla \hat{\phi})^2}{2} + V(\hat{\phi}) \right]$$

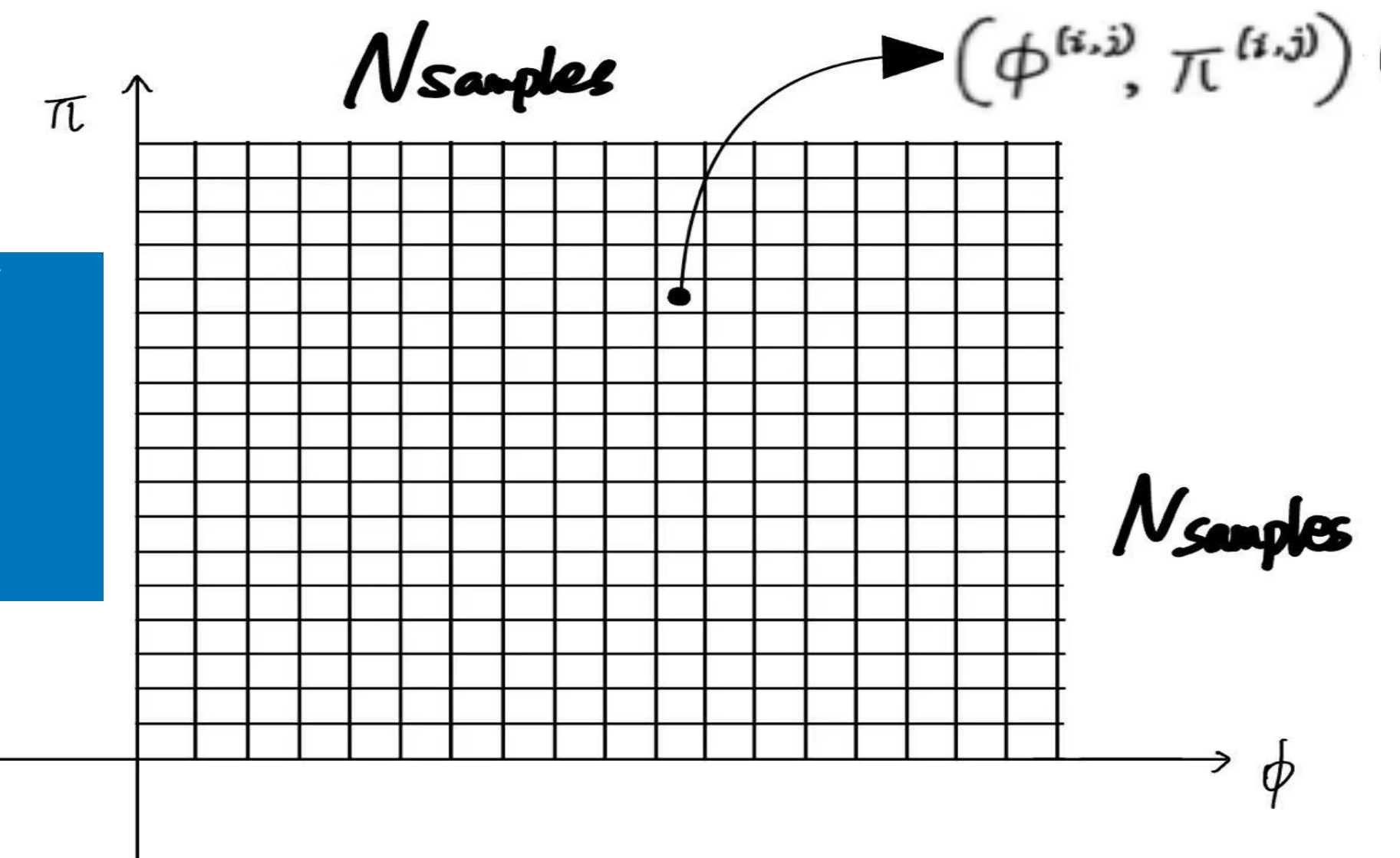
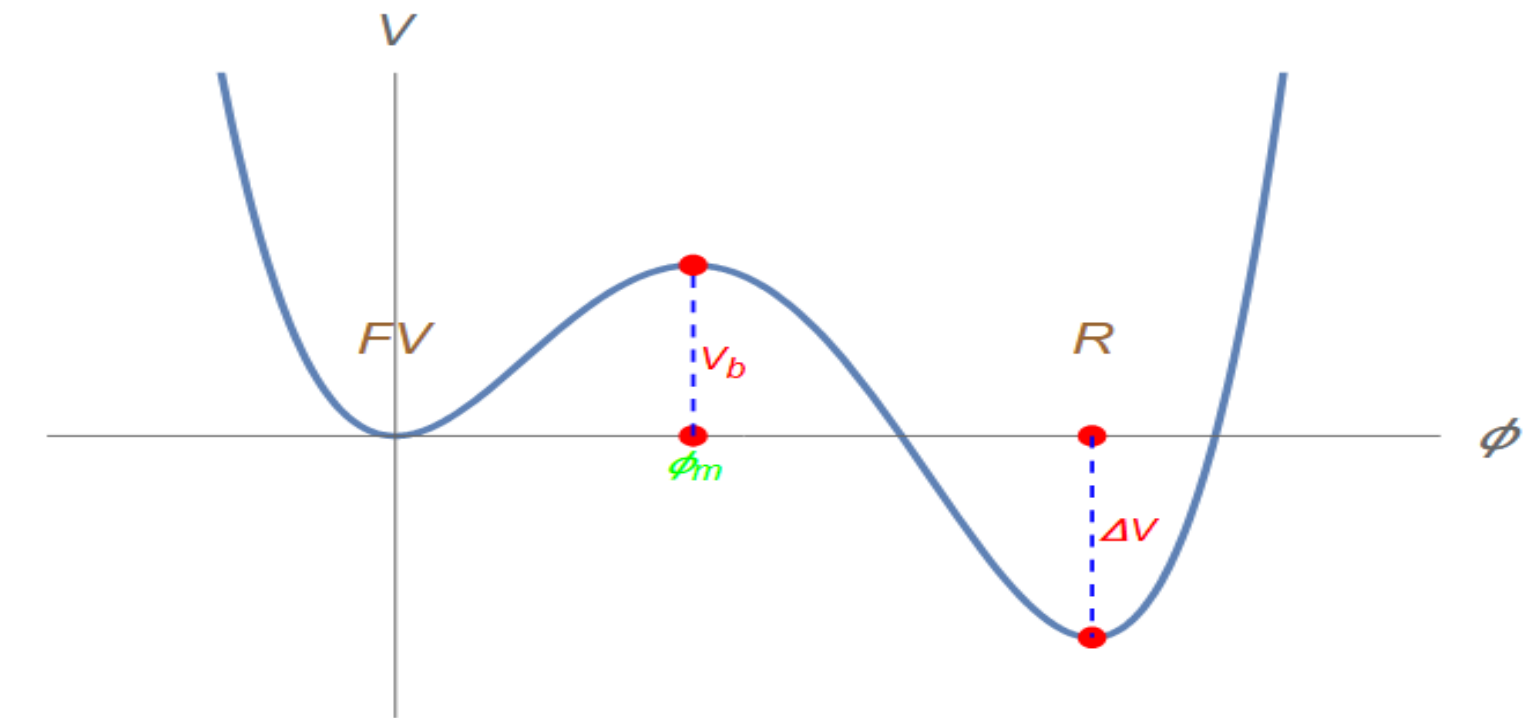
$$i\hbar \frac{\partial}{\partial t} \hat{\rho}(t) = [\hat{H}, \hat{\rho}(t)] \quad \frac{\partial}{\partial t} W[\phi, \pi; t] = -2H \frac{1}{i\hbar} \sin\left(\frac{i\hbar}{2} \Lambda\right) W[\phi, \pi; t]$$

$$\Lambda = \overleftarrow{\frac{\delta}{\delta \Pi}} \overrightarrow{\frac{\delta}{\delta \phi}} - \overleftarrow{\frac{\delta}{\delta \Phi}} \overrightarrow{\frac{\delta}{\delta \Pi}} \text{ is the Poisson bracket operator}$$

Ignore $O(\hbar^2)$

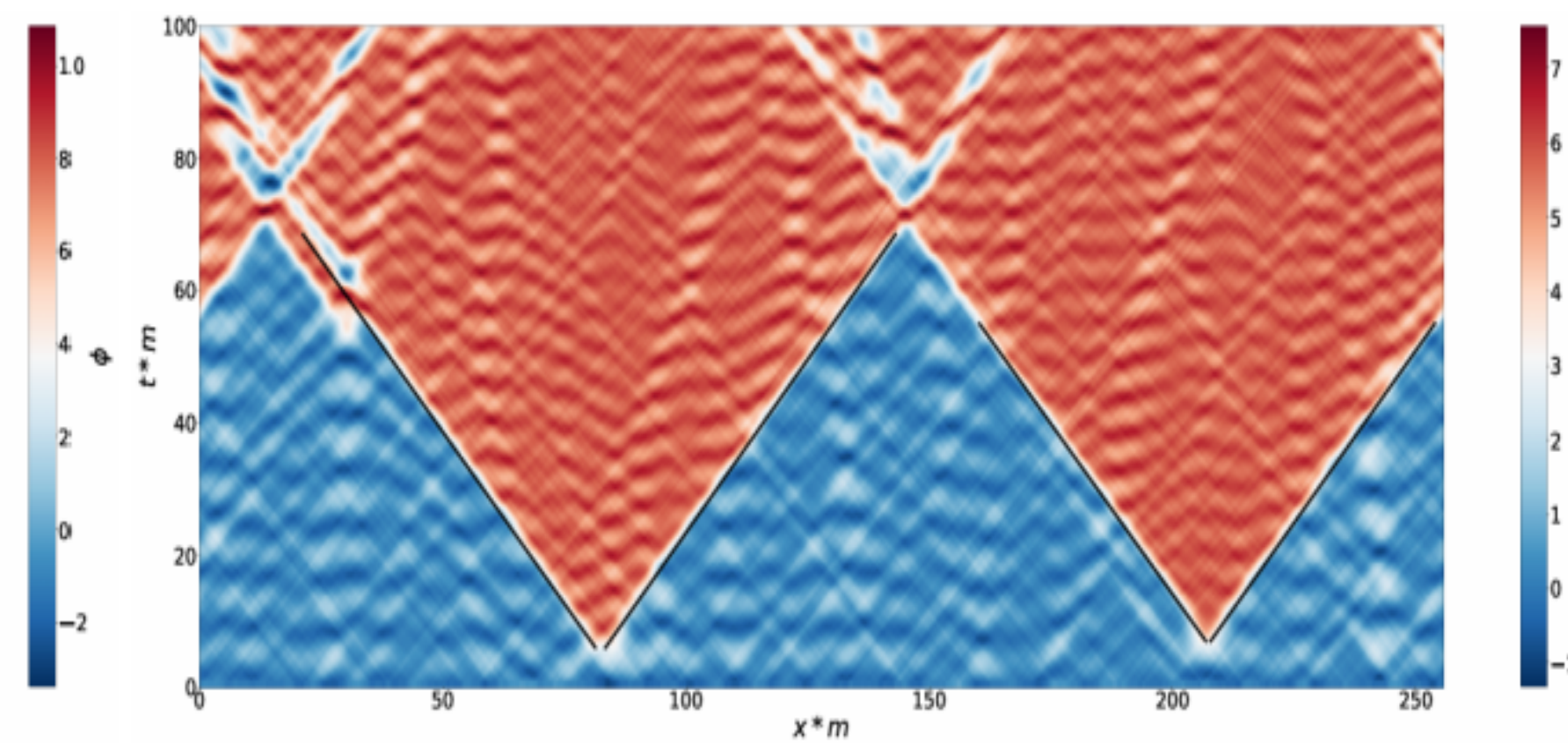
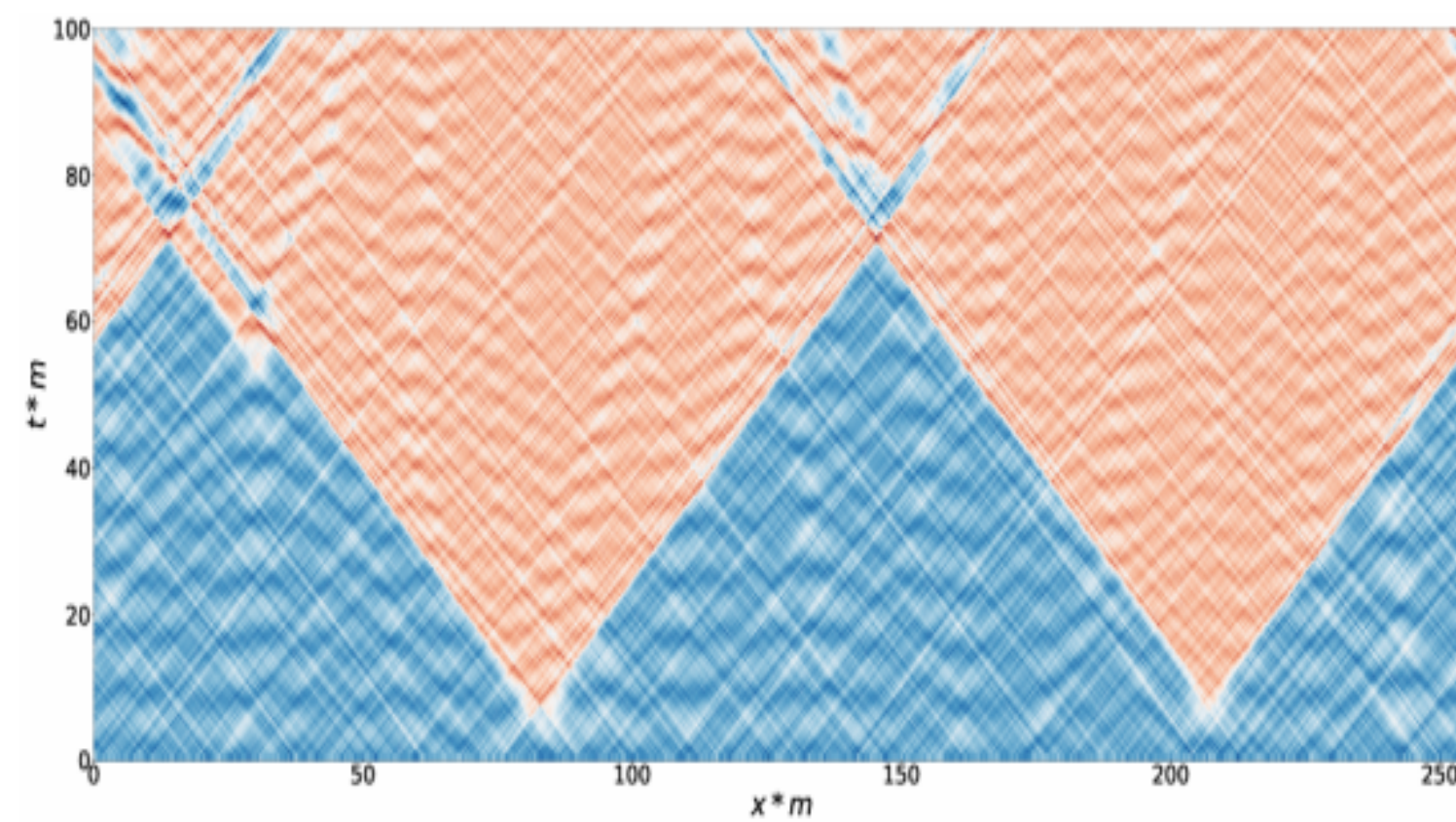
$$\left[\frac{\partial}{\partial t} + \int d^3x \left(\frac{\delta H}{\delta \Pi} \frac{\delta}{\delta \phi} - \frac{\delta H}{\delta \phi} \frac{\delta}{\delta \Pi} \right) \right] W[\phi, \Pi; t] = 0$$

$$\begin{cases} \frac{\delta H}{\delta \Pi} = \frac{d\phi}{dt} = \Pi \\ -\frac{\delta H}{\delta \phi} = \frac{d\Pi}{dt} = \nabla^2 \phi - \frac{\delta V(\phi)}{\delta \phi} \end{cases} \quad W[\phi, \Pi; t] = W[\phi-t, \Pi-t; 0]$$

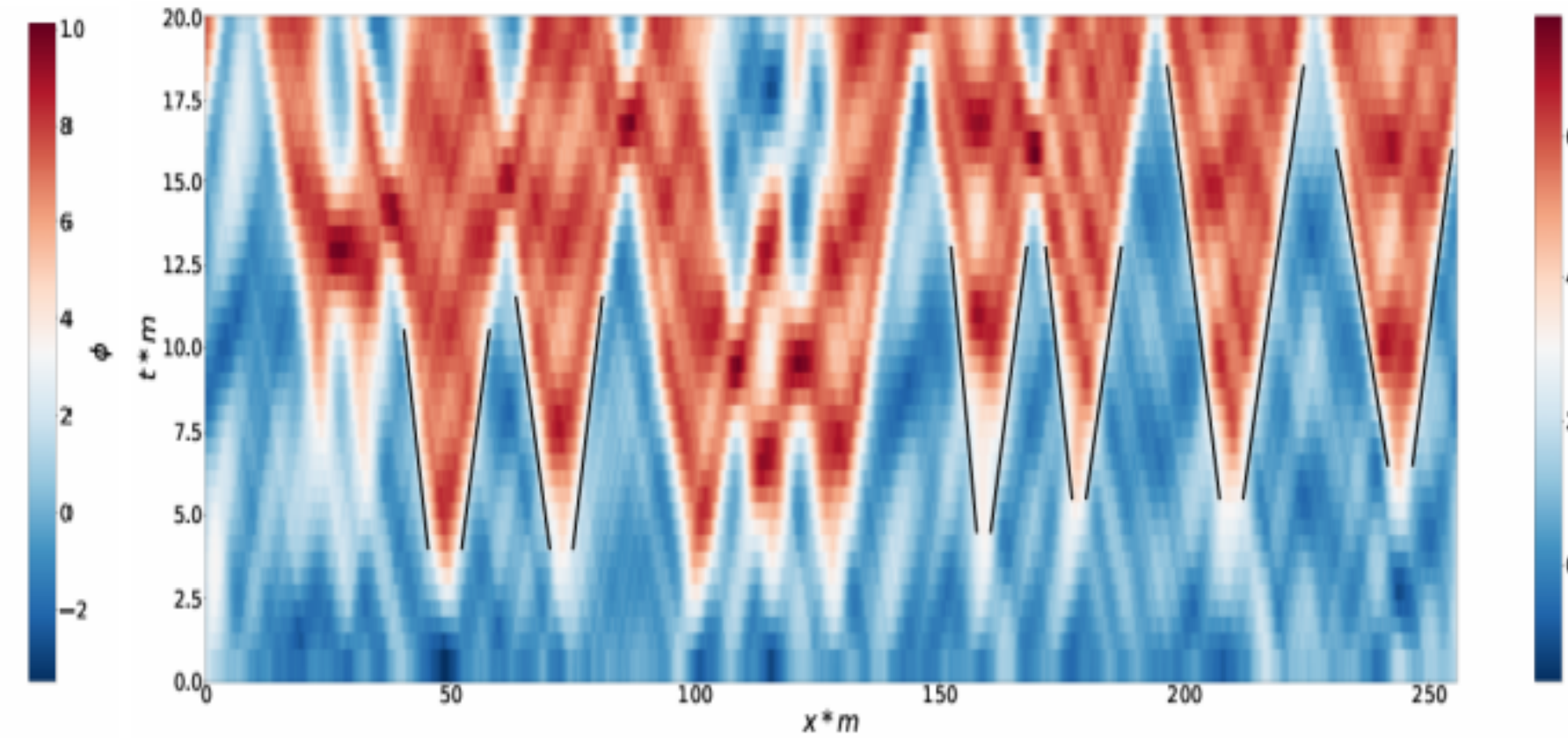
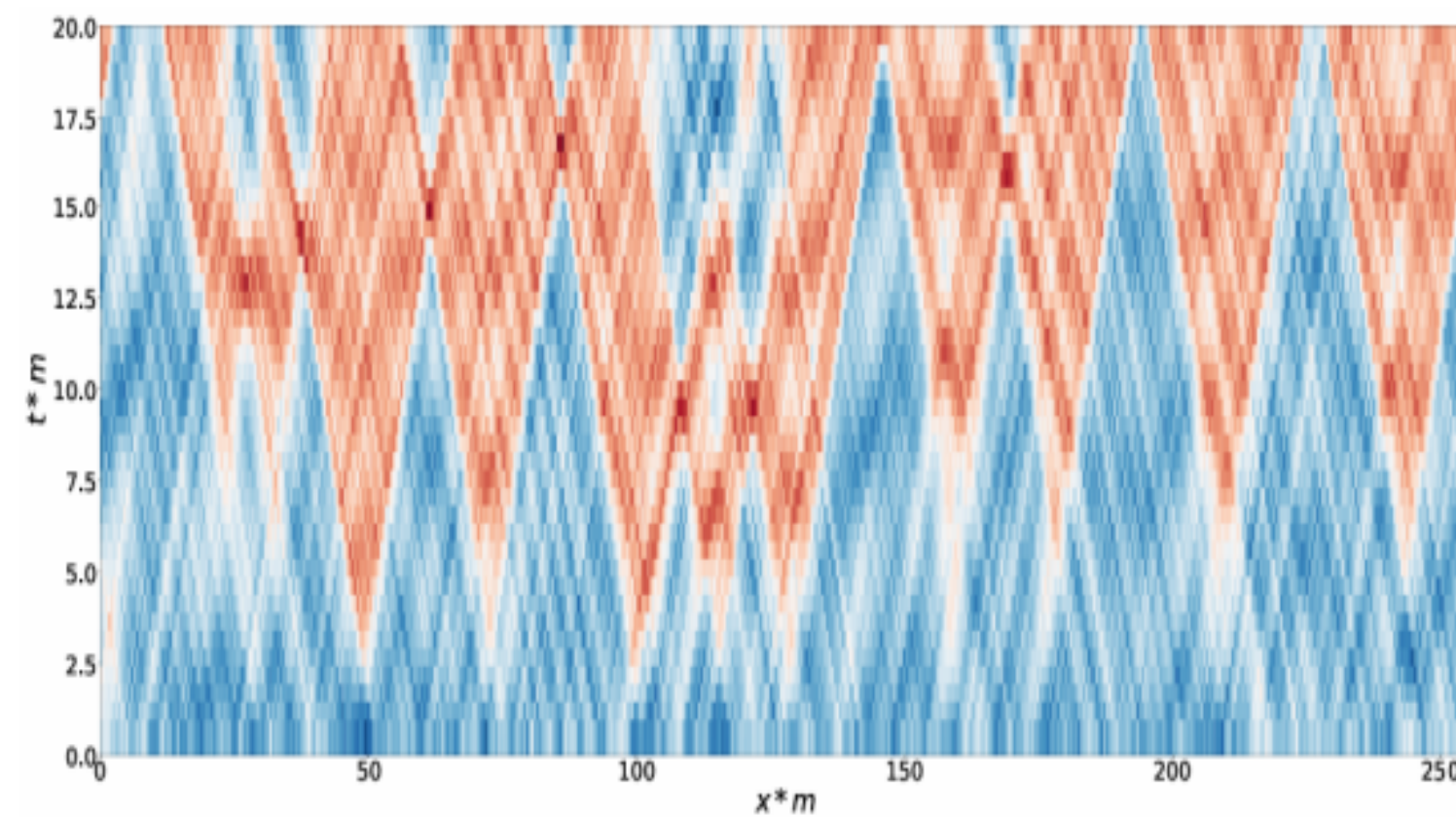


Nsamples = 10^5
 $\Delta x = 0.25$
 $\Delta t = 0.01$
 $N_x = 1024$

Real-Time Dynamics: Low vs High Temperature

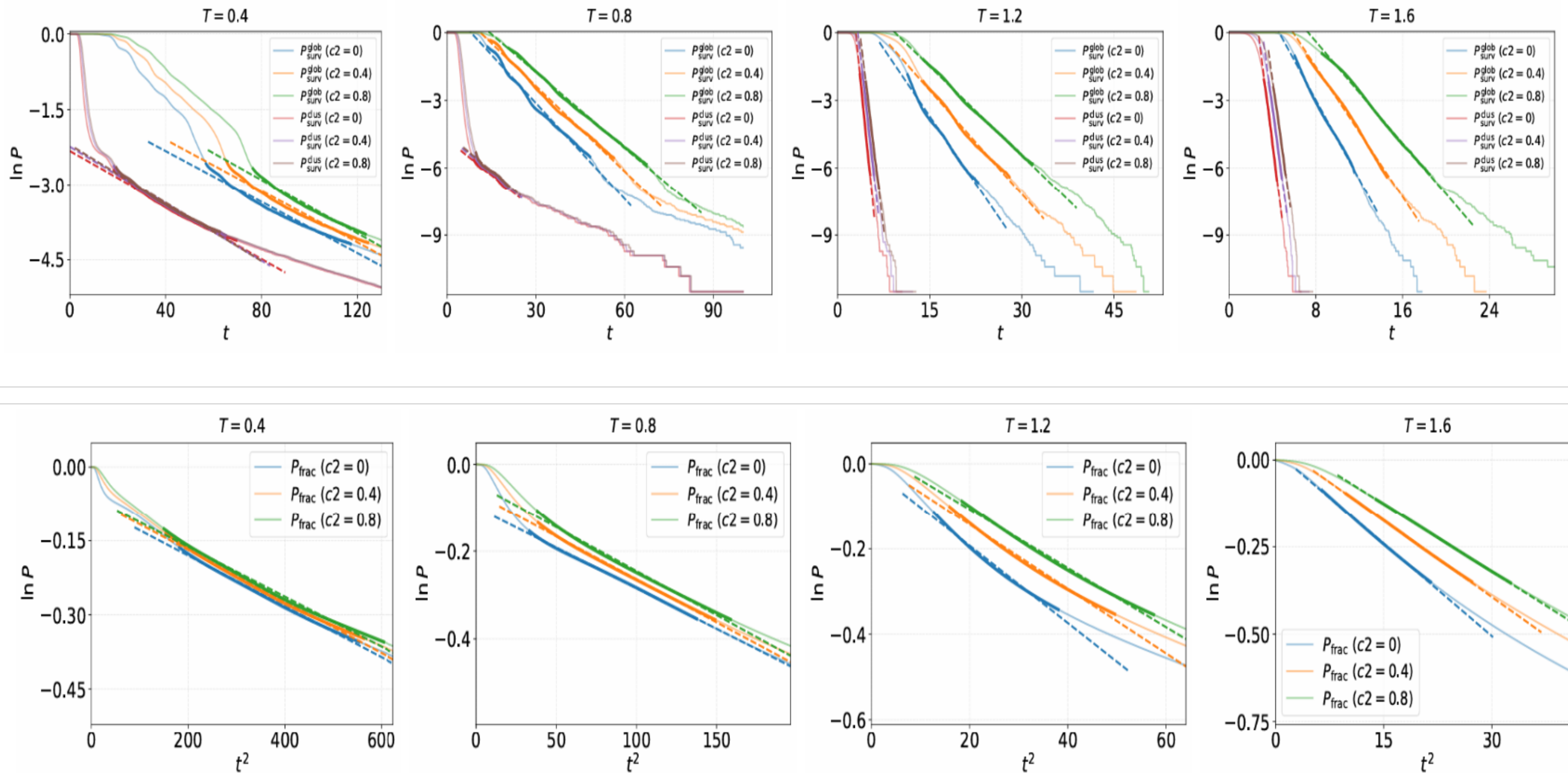


Low Tem:
Dilute kink–antikink-like domains; partial reflection or return can occur.

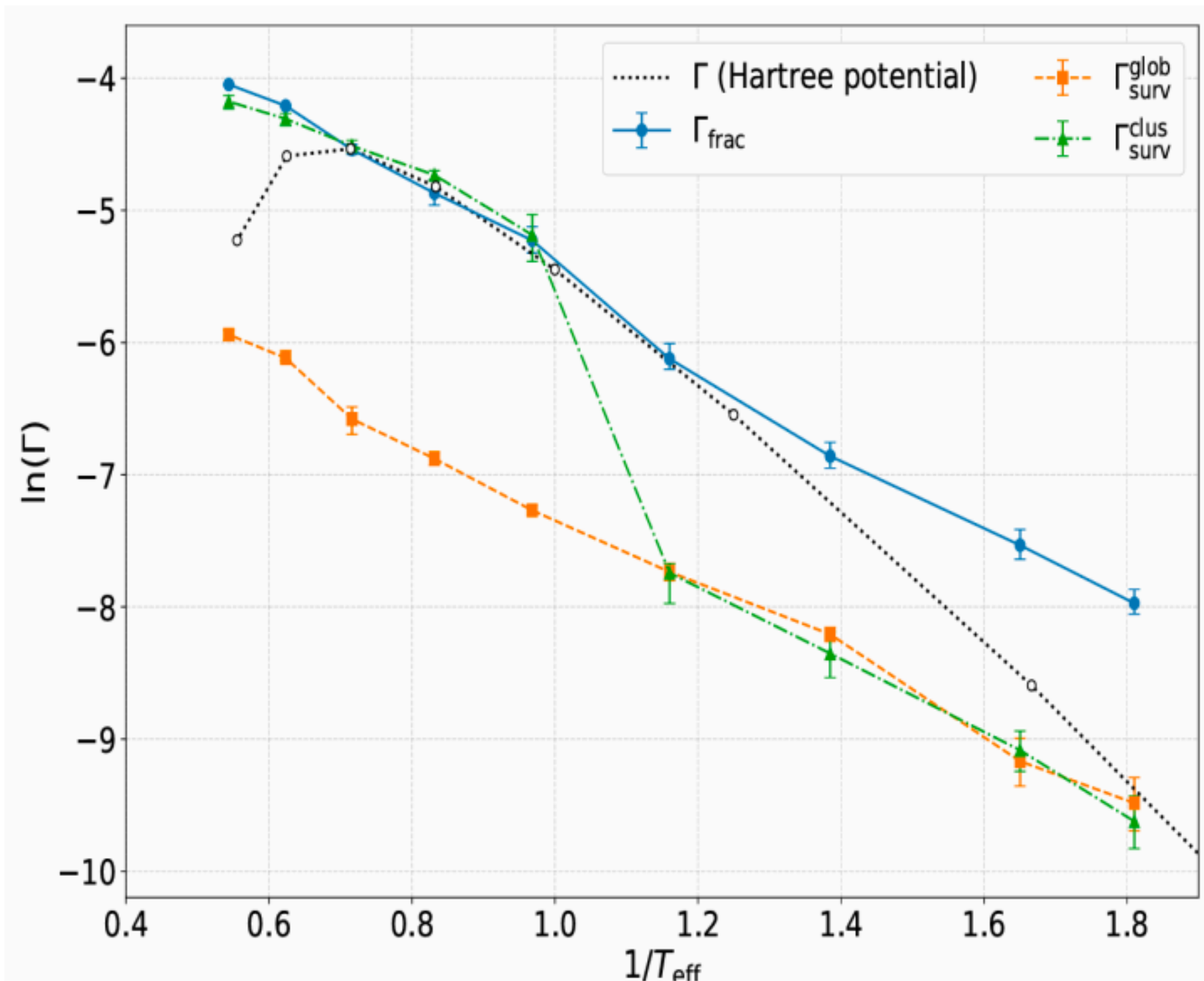


High Tem:
Multiple local transformed regions appear and grow rapidly; dynamics is closer to nucleation-and-growth.

► 1+1D simulation-across Quantum-Thermal crossover



► 1+1D simulation-across Quantum-Thermal crossover



Low Tem:

Events are dilute.

Global and connected-cluster identify similar events.

Fraction observable can be distorted by transient conversion and kink–antikink reflection.

High Tem:

Many local seeds.

Connected-cluster detects persistent domains.

Global average waits until a large part of the box converts.

Therefore global survival underestimates the instantaneous nucleation rate.

► Lattice electroweak theory-classical simulation

$\Phi(t, \mathbf{x})$: Higgs field doublet defined on sites;

$U_i(t, \mathbf{x})$ and $V_i(t, \mathbf{x})$: SU(2) and U(1) link fields, defined on the link between the neighboring sites $\mathbf{x}$ and $\mathbf{x} + \mathbf{i}$, $\Phi(t, \mathbf{x})$, $U_i(t, \mathbf{x})$ and $V_i(t, \mathbf{x})$ are defined at time steps $t + \Delta t$, $t + 2\Delta t$, ... ;

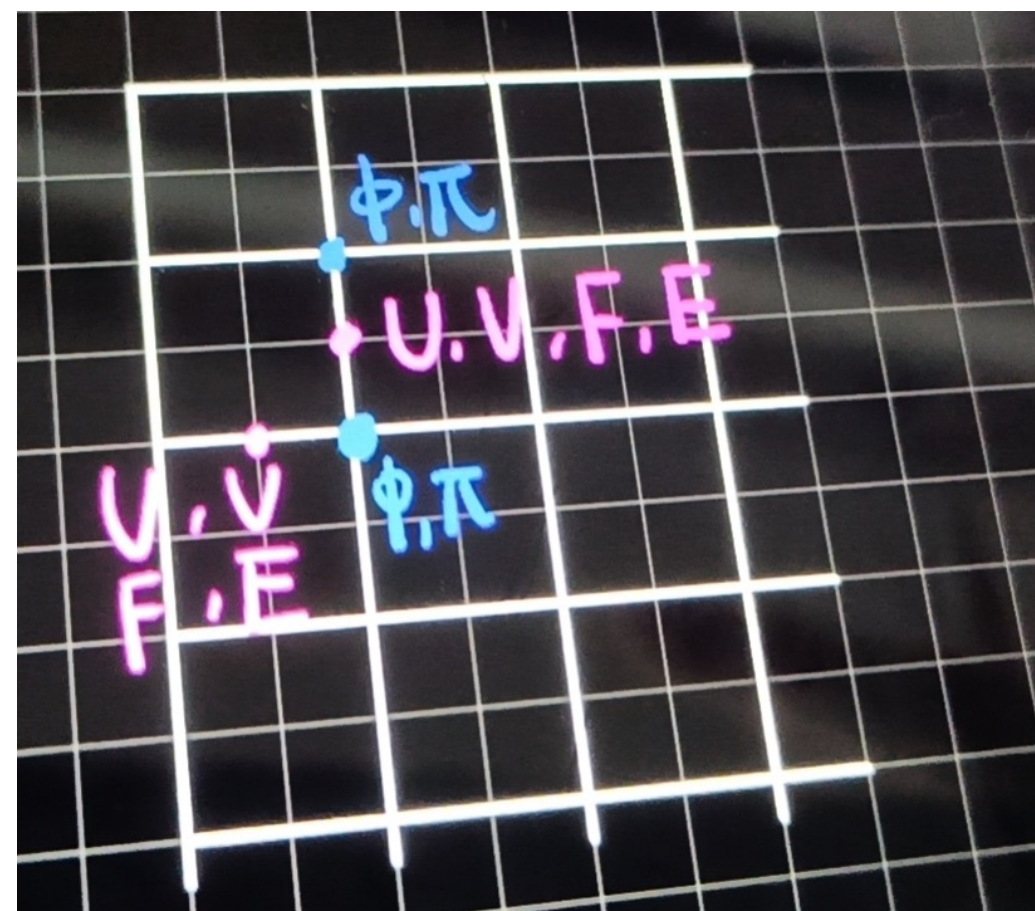
Conjugate momentum fields: $\Pi(t+\Delta t/2, \mathbf{x})$, $F(t+\Delta t/2, \mathbf{x})$ and $E(t+\Delta t/2, \mathbf{x})$, are defined at time steps $t + \Delta t/2$, $t + 3\Delta t/2$.

$$U_i(t, x) = \exp \left(-\frac{i}{2} g \Delta x \sigma^a W_i^a \right)$$

$$U_0(t, x) = \exp \left(-\frac{i}{2} g \Delta t \sigma^a W_0^a \right)$$

$$V_i(t, x) = \exp \left(-\frac{i}{2} g \Delta x B_i \right)$$

$$V_0(t, x) = \exp \left(-\frac{i}{2} g \Delta t B_0 \right).$$



$$D_i \Phi = \frac{1}{\Delta x} [U_i(t, x) V_i(t, x) \Phi(t, x + i) - \Phi(t, x)]$$

$$D_0 \Phi = \frac{1}{\Delta t} [U_0(t, x) V_0(t, x) \Phi(t + \Delta t, x) - \Phi(t, x)].$$

$$\Phi(t + \Delta t, x) = \Phi(t, x) + \Delta t \Pi(t + \Delta t/2, x)$$

$$V_i(t + \Delta t, x) = \frac{1}{2} g' \Delta x \Delta t E_i(t + \Delta t/2, x) V_i(t, x)$$

$$U_i(t + \Delta t, x) = g \Delta x \Delta t F_i(t + \Delta t/2, x) U_i(t, x),$$

Temporal gauge
 $U_0(t, \mathbf{x}) = I_2, V_0(t, \mathbf{x}) = 1$

leapfrog

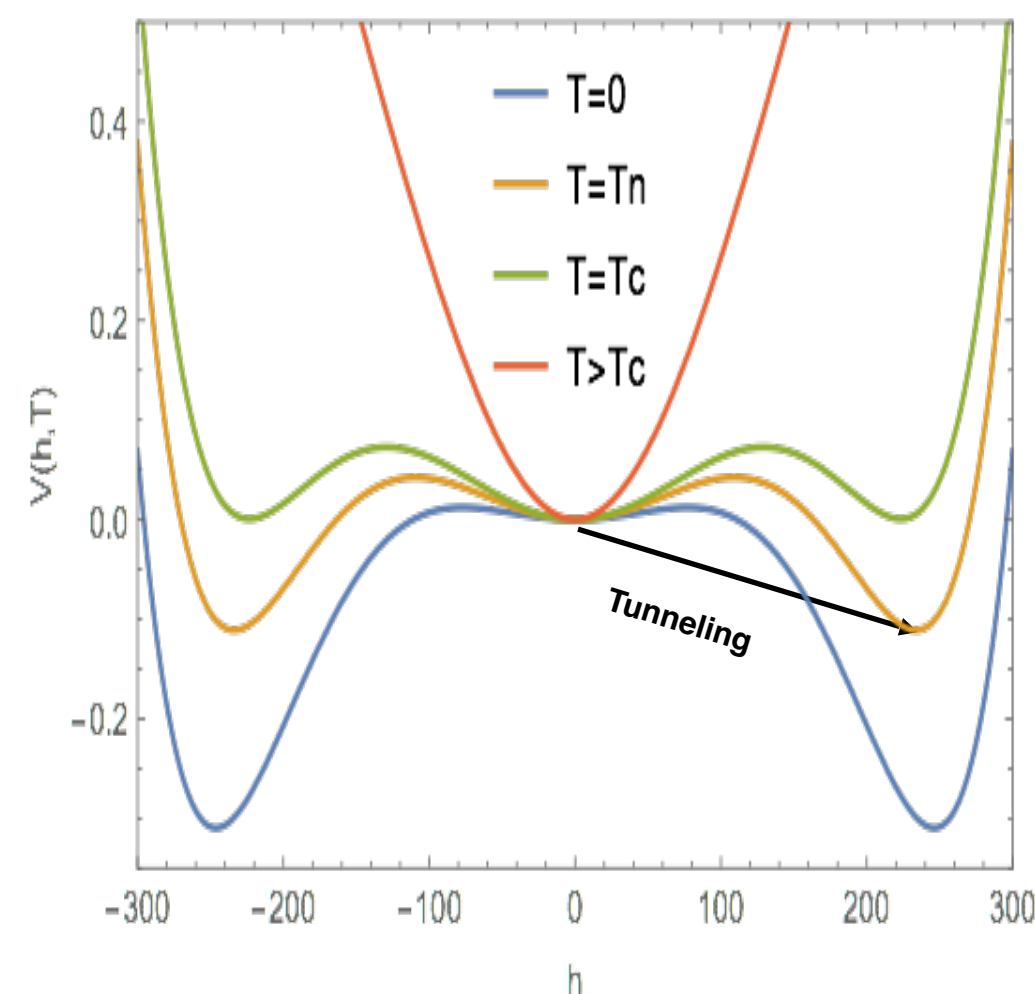
Field basis equation of motion

$$\begin{aligned}\partial_0^2 \Phi &= D_i D_i \Phi - \frac{dV(\Phi)}{d\Phi}, \\ \partial_0^2 B_i &= -\partial_j B_{ij} + g' \text{Im}[\Phi^\dagger D_i \Phi], \\ \partial_0^2 W_i^a &= -\partial_k W_{ik}^a - g \epsilon^{abc} W_k^b W_{ik}^c + g \text{Im}[\Phi^\dagger \sigma^a D_i \Phi], \\ \partial_0 \partial_j B_j - g' \text{Im}[\Phi^\dagger \partial_0 \Phi] &= 0, \\ \partial_0 \partial_j W_j^a + g \epsilon^{abc} W_j^b \partial_0 W_j^c - g \text{Im}[\Phi^\dagger \sigma^a \partial_0 \Phi] &= 0.\end{aligned}$$

Lattice implementation

$$\begin{aligned}\Pi(t + \Delta t/2, x) &= \Pi(t - \Delta t/2, x) + \Delta t \left\{ \frac{1}{\Delta x^2} \sum_i [U_i(t, x) V_i(t, x) \Phi(t, x + i) \right. \\ &\quad \left. - 2\Phi(t, x) + U_i^\dagger(t, x - i) V_i^\dagger(t, x - i) \Phi(t, x - i)] - \frac{\partial U}{\partial \Phi^\dagger} \right\} \\ \text{Im}[E_k(t + \Delta t/2, x)] &= \text{Im}[E_k(t - \Delta t/2, x)] + \Delta t \left\{ \frac{g'}{\Delta x} \text{Im}[\Phi^\dagger(t, x + k) U_k^\dagger(t, x) V_k^\dagger(t, x) \Phi(t, x)] \right. \\ &\quad \left. - \frac{2}{g' \Delta x^3} \sum_i \text{Im}[V_k(t, x) V_i(t, x + k) V_k^\dagger(t, x + i) V_i^\dagger(t, x) \right. \\ &\quad \left. + V_i(t, x - i) V_k(t, x) V_i^\dagger(t, x + k - i) V_k^\dagger(t, x - i)] \right\} \\ \text{Tr}[i\sigma^m F_k(t + \Delta t/2, x)] &= \text{Tr}[i\sigma^m F_k(t - \Delta t/2, x)] + \Delta t \left\{ \frac{g}{\Delta x} \text{Re}[\Phi^\dagger(t, x + k) U_k^\dagger(t, x) V_k^\dagger(t, x) i\sigma^m \Phi(t, x)] \right. \\ &\quad \left. - \frac{1}{g \Delta x^3} \sum_i \text{Tr}[i\sigma^m U_k(t, x) U_i(t, x + k) U_k^\dagger(t, x + i) U_i^\dagger(t, x) \right. \\ &\quad \left. + i\sigma^m U_k(t, x) U_i^\dagger(t, x + k - i) U_k^\dagger(t, x - i) U_i(t, x - i)] \right\},\end{aligned}$$

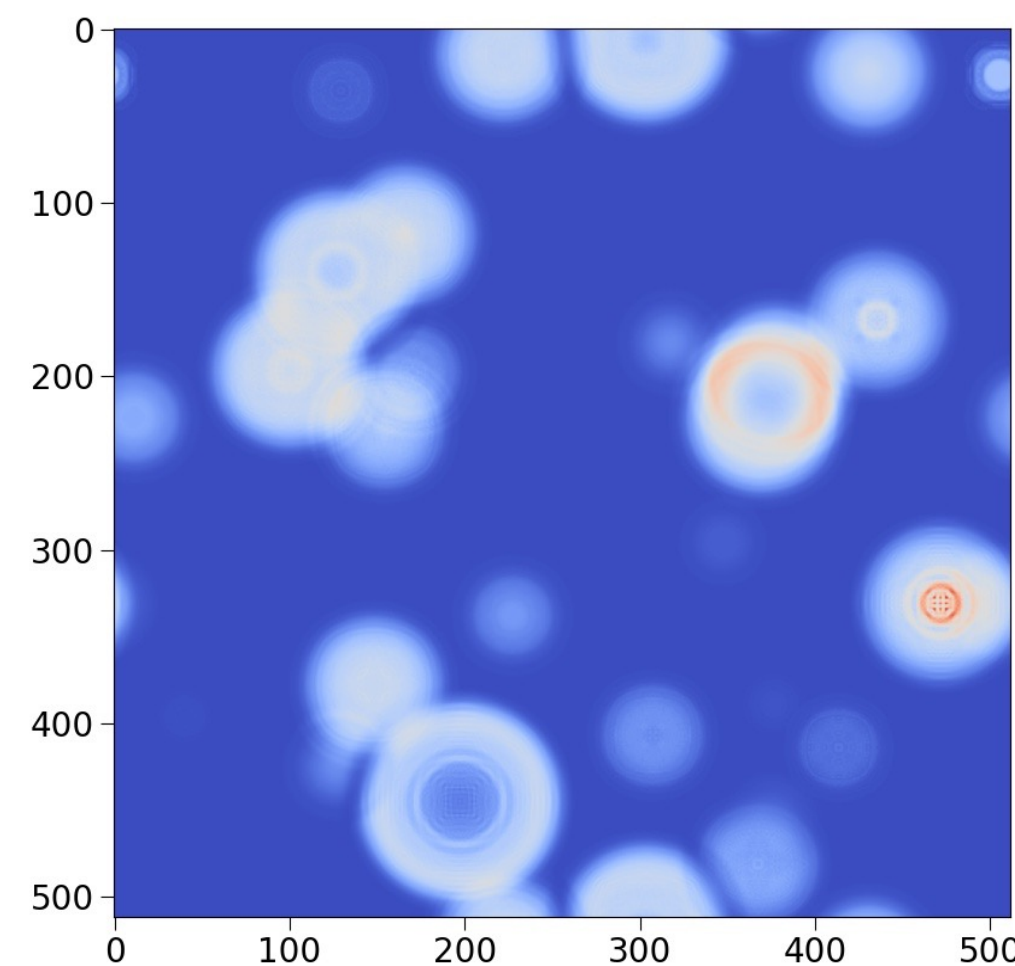
Finite-T Veff



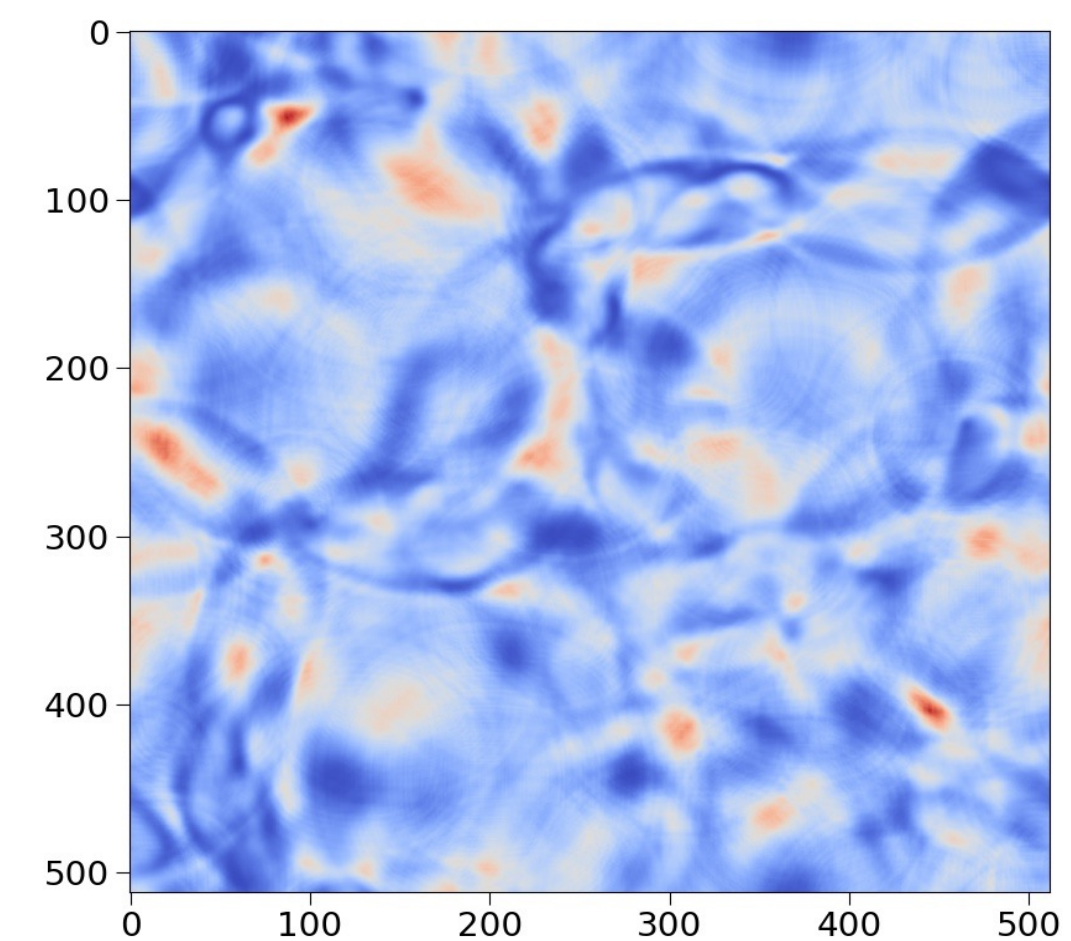
Finite-T calculation

temperature (T_*)
duration (β)
strength (α)

Nucleation



Expansion&Percolation



1st Phase transition + Magnetic field = Matter-antimatter asymmetry ?

Higgs doublet **U(1) gauge field strength**

$$\mathcal{L} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - \frac{1}{4} W_{\mu\nu}^a W^{a\mu\nu} - \frac{1}{4} Y_{\mu\nu} Y^{\mu\nu} - \frac{1}{2} Y_{\mu\nu}^{\text{ex}} Y^{\mu\nu} + \mathcal{V}(\Phi)$$

SU(2) gauge field strength **External U(1) gauge field strength**
Dose not change with time

$$D_\mu = \partial_\mu - ig \frac{\sigma^a}{2} W_\mu^a - ig' \frac{1}{2} (Y_\mu + Y_\mu^{\text{ex}})$$

is the covariant derivative.

$$\mathcal{V}(\Phi) = -\mu^2 (\Phi^\dagger \Phi - \eta^2) + A (\Phi^\dagger \Phi - \eta^2)^{3/2} + \lambda (\Phi^\dagger \Phi - \eta^2)^2$$

$$\partial_\mu j_B^\mu = N_g \left[\frac{g^2}{16\pi^2} \text{Tr} (W_{\mu\nu} \tilde{W}^{\mu\nu}) - \frac{g'^2}{32\pi^2} Y_{\mu\nu} \tilde{Y}^{\mu\nu} \right]$$

Adler, PR 177, 2426 (1969)
Bell, Jackiw, NCA 60, 47 (1969)
't Hooft, PRL 39, 8 (1976)
't Hooft, PRD 14, 3432 (1976)

$$\Delta n_B = n_B(t) - n_B(0) = N_g \frac{\Delta N_{CS}(t)}{V} = N_g \frac{N_{CS}(t) - N_{CS}(0)}{V}$$

Number of families of fermions $N_g = 3$

Chern-Simons Number density:

$$\frac{N_{CS}(t)}{V} = \frac{1}{V} \frac{1}{32\pi^2} \epsilon^{ijk} \int d^3x \left[-g'^2 (Y_i + Y_i^{\text{ex}})(Y_{jk} + Y_{jk}^{\text{ex}}) + g^2 \left(W_i^a W_{jk}^a - \frac{g}{3} \epsilon^{abc} W_i^a W_j^b W_k^c \right) \right]$$

Helicity density:

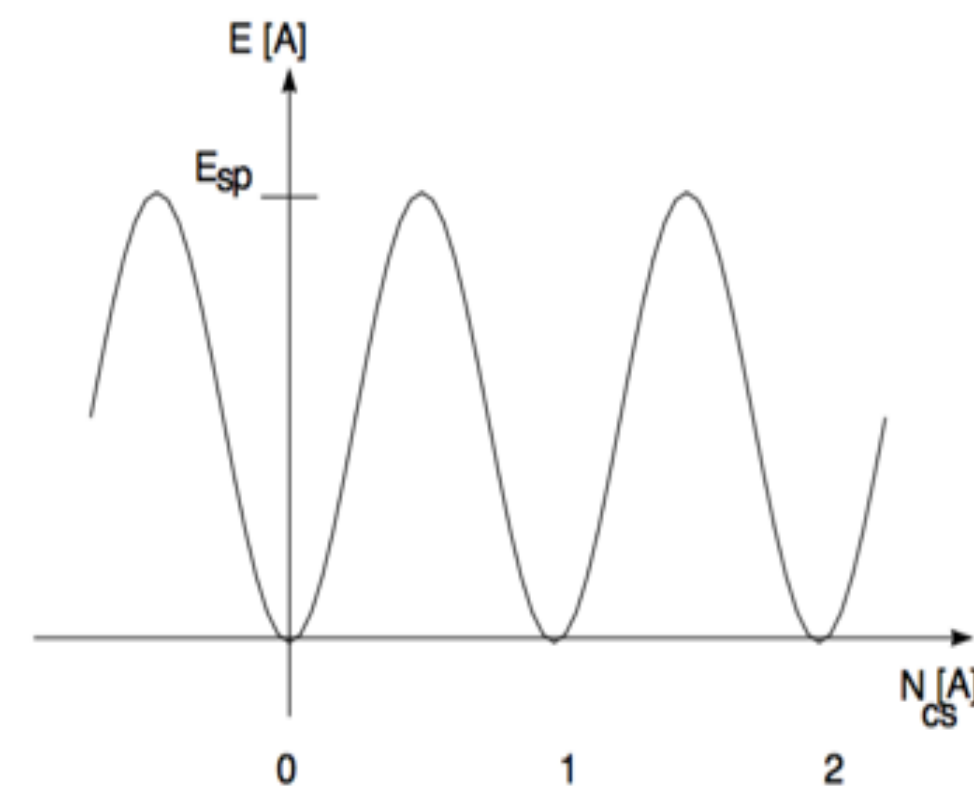
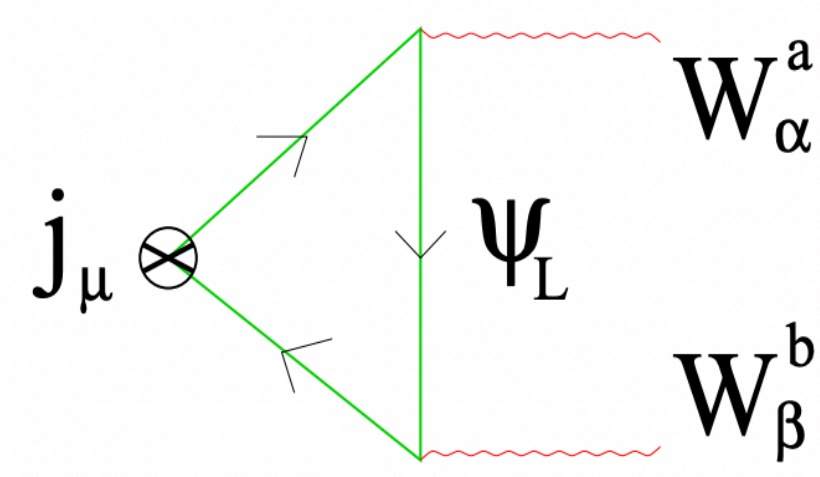
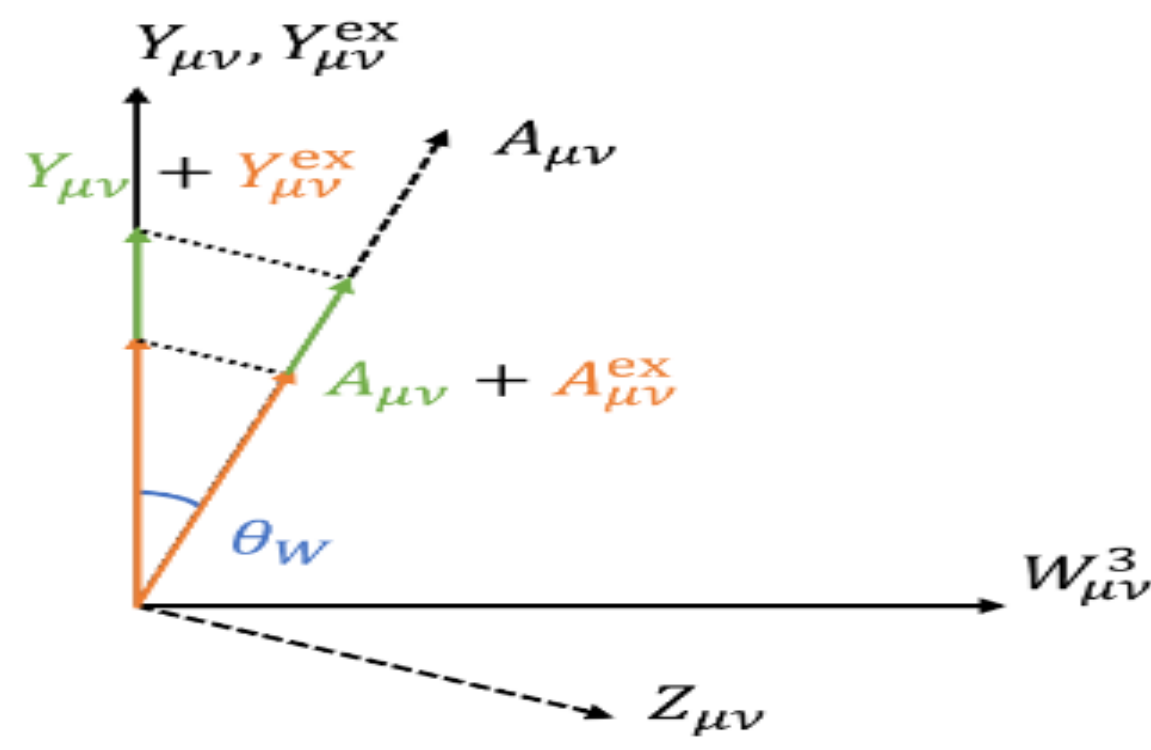
$$h_Y = \frac{H_Y}{V} = \frac{1}{V} \epsilon^{ijk} \int d^3x (Y_i + Y_i^{\text{ex}})(Y_{jk} + Y_{jk}^{\text{ex}}) = \frac{1}{V} \int d^3x (Y + Y^{\text{ex}}) \cdot (B_Y + B_Y^{\text{ex}})$$

Net Baryon density:

$$\eta_B(t) = \frac{n_B}{s} = \frac{45 N_g}{2\pi^2 g_{*S} T^3} \frac{\Delta N_{CS}(t)}{V}$$

$$\text{SU}(2)_L \times \text{U}(1)_Y \xrightarrow[\text{symmetry breaking}]{\text{Spontaneous}} \text{U}(1)_{\text{em}}$$

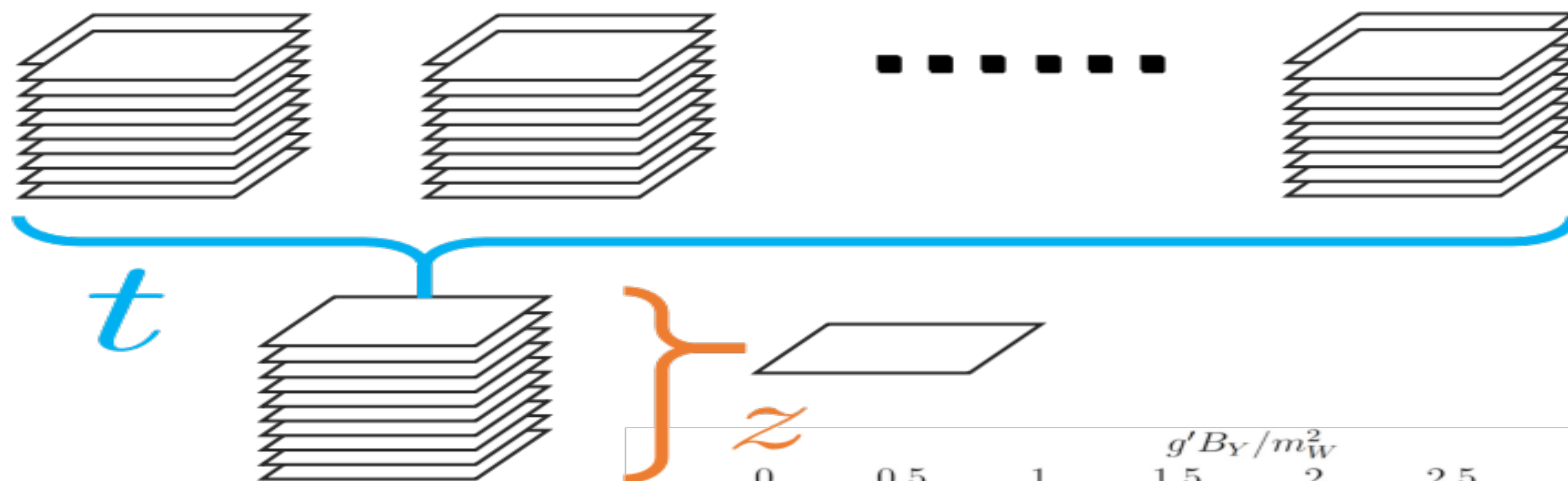
$\Phi, W_\mu^a, Y_\mu \rightarrow A_\mu, H, Z_\mu, W_\mu^\pm$



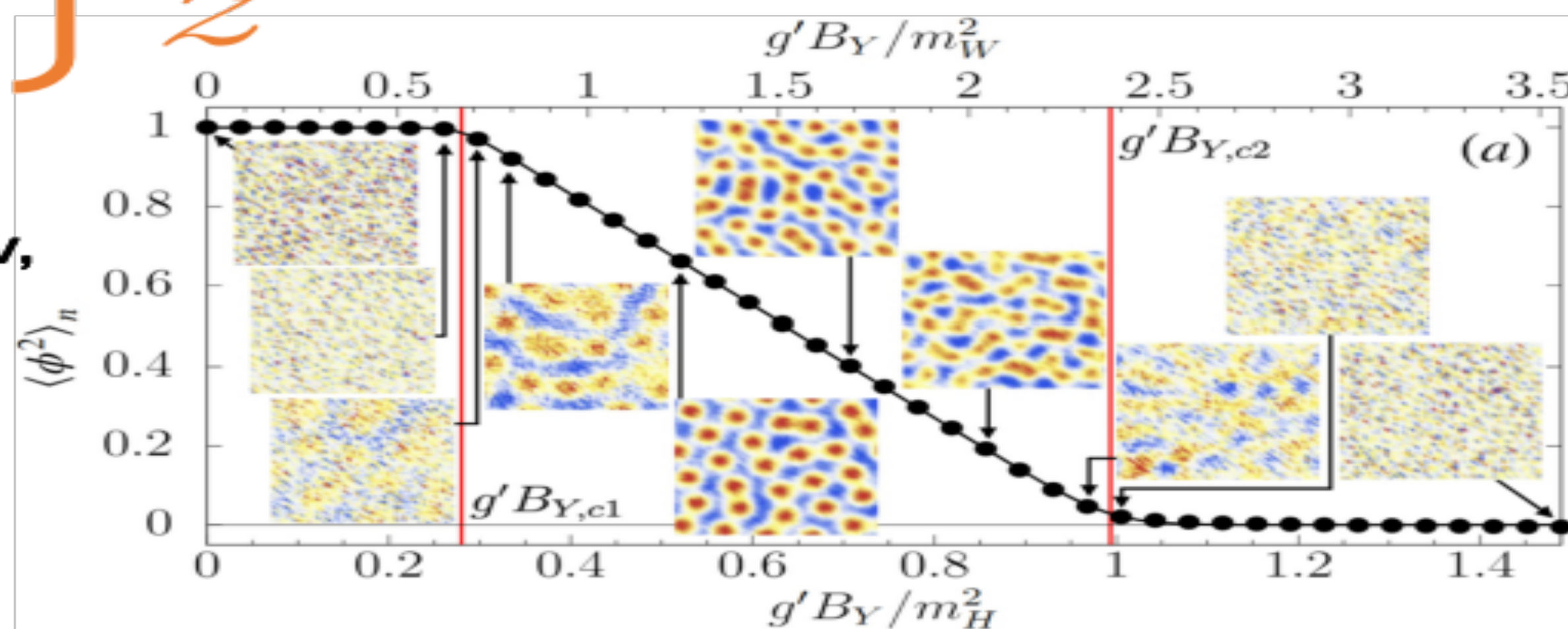
► Bubble collision and PMF

When $\frac{g' B_Y}{m_W^2} > 1$, the Higgs field will form a hexagonally arranged vortex, which is called Ambjørn-Olesen condense.

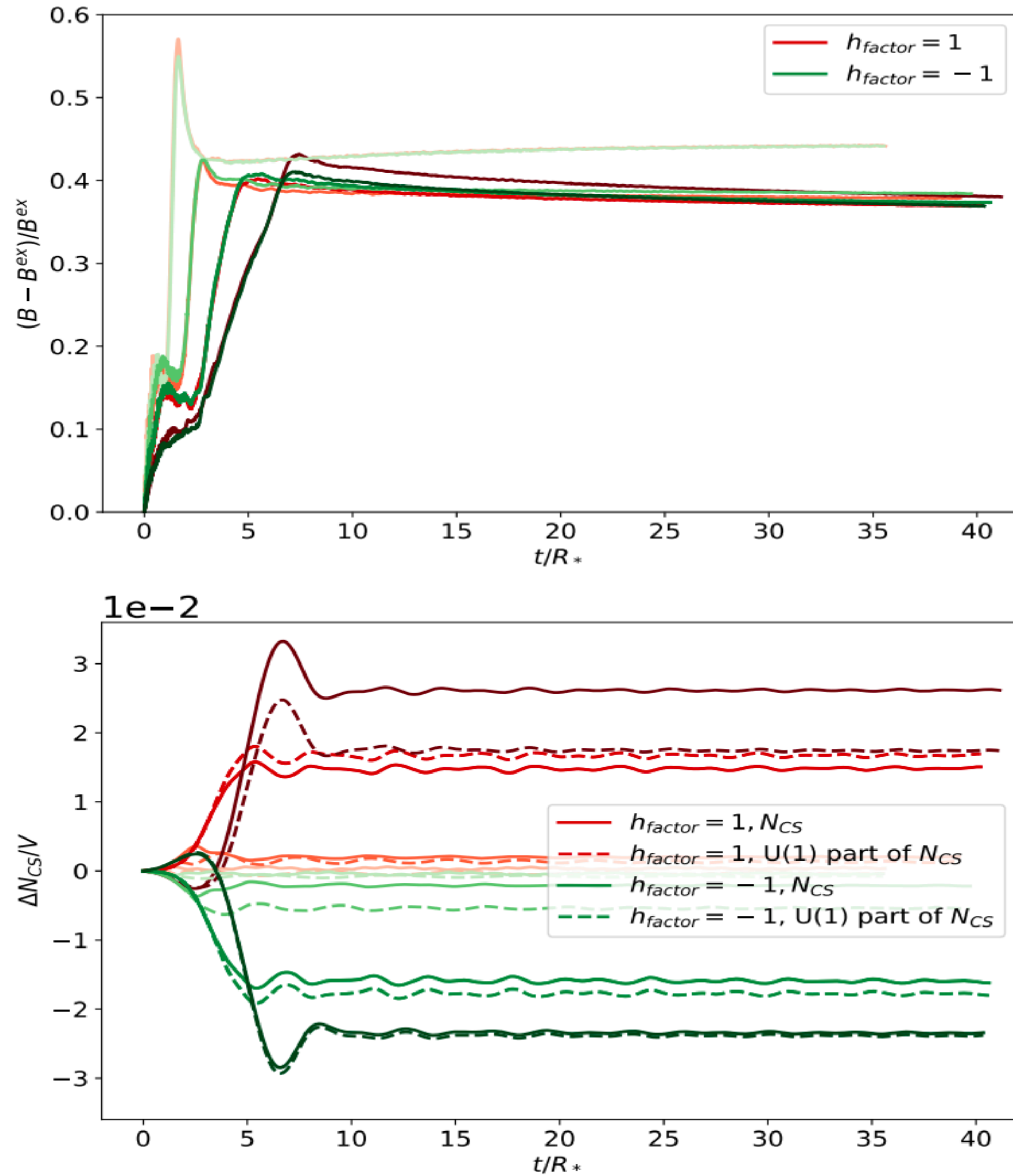
Salam, Strathdee, NPB 90, 203 (1975)
Linde, PLB 62, 435 (1976)
Nielsen, Olesen, NPB 144, 376 (1978)
Ambjørn, PLB 220, 659 (1989)



Chernodub, Goy, Molochkov,
PRL 130 (2023)

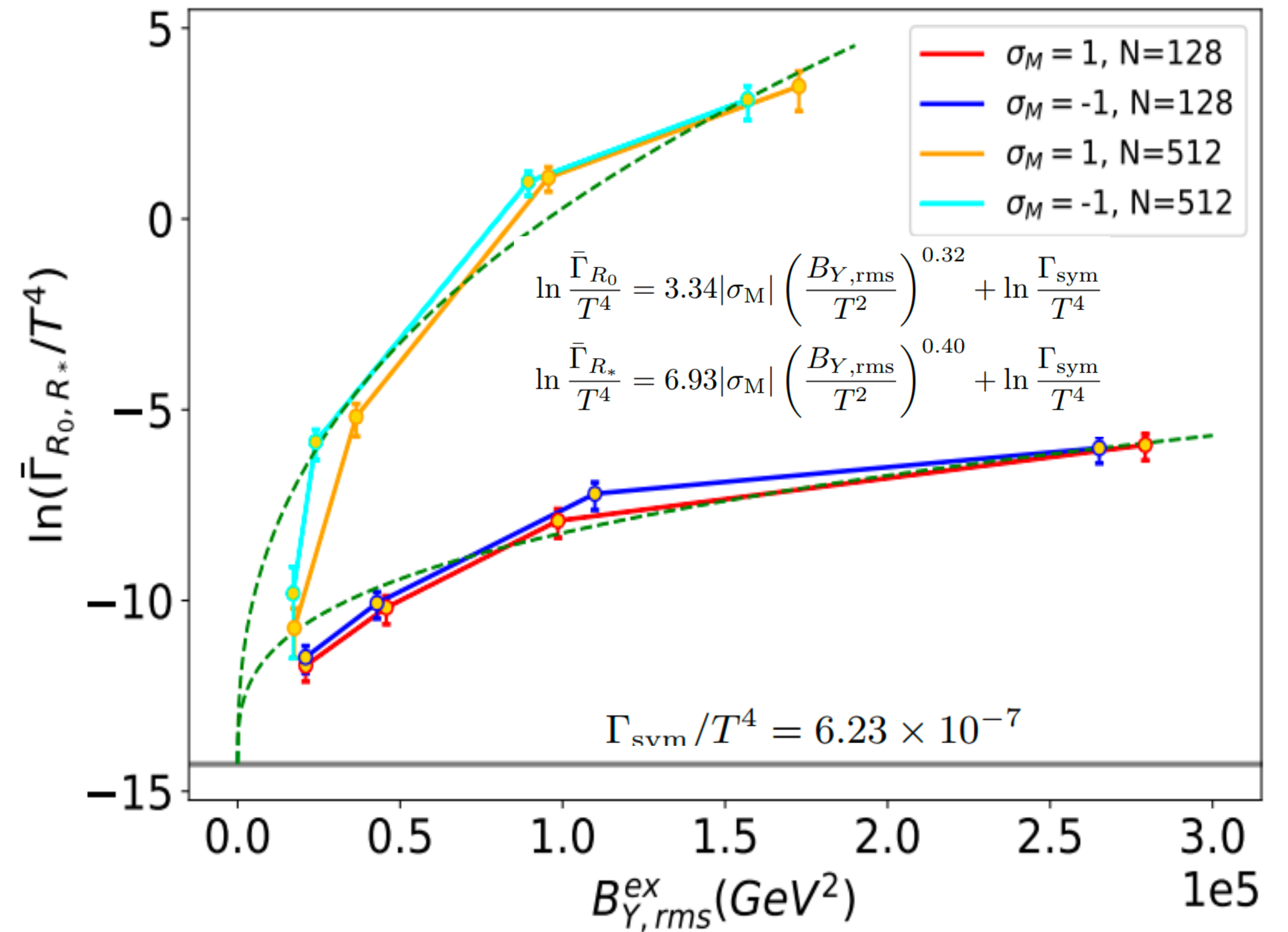


► Bubble collision and PMF

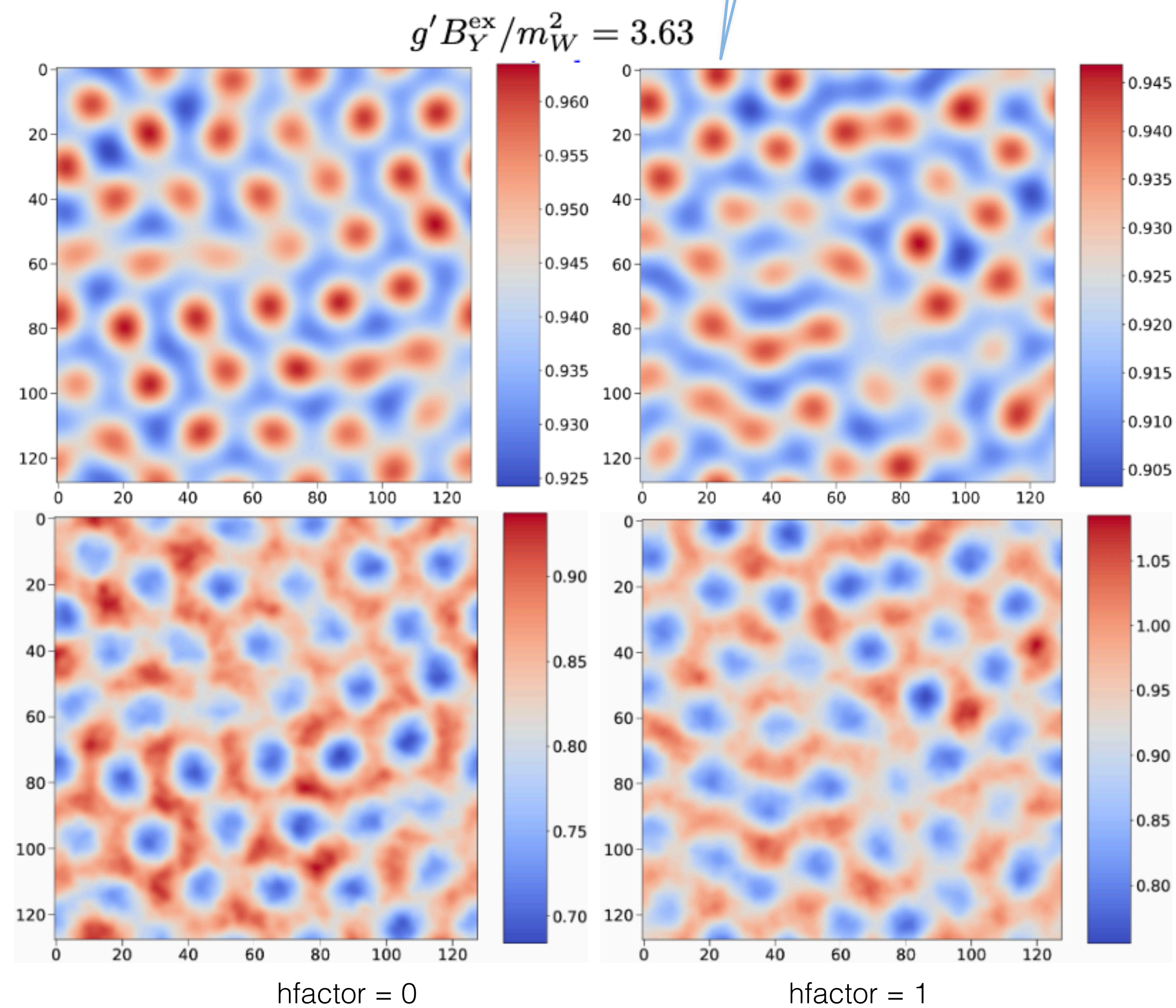
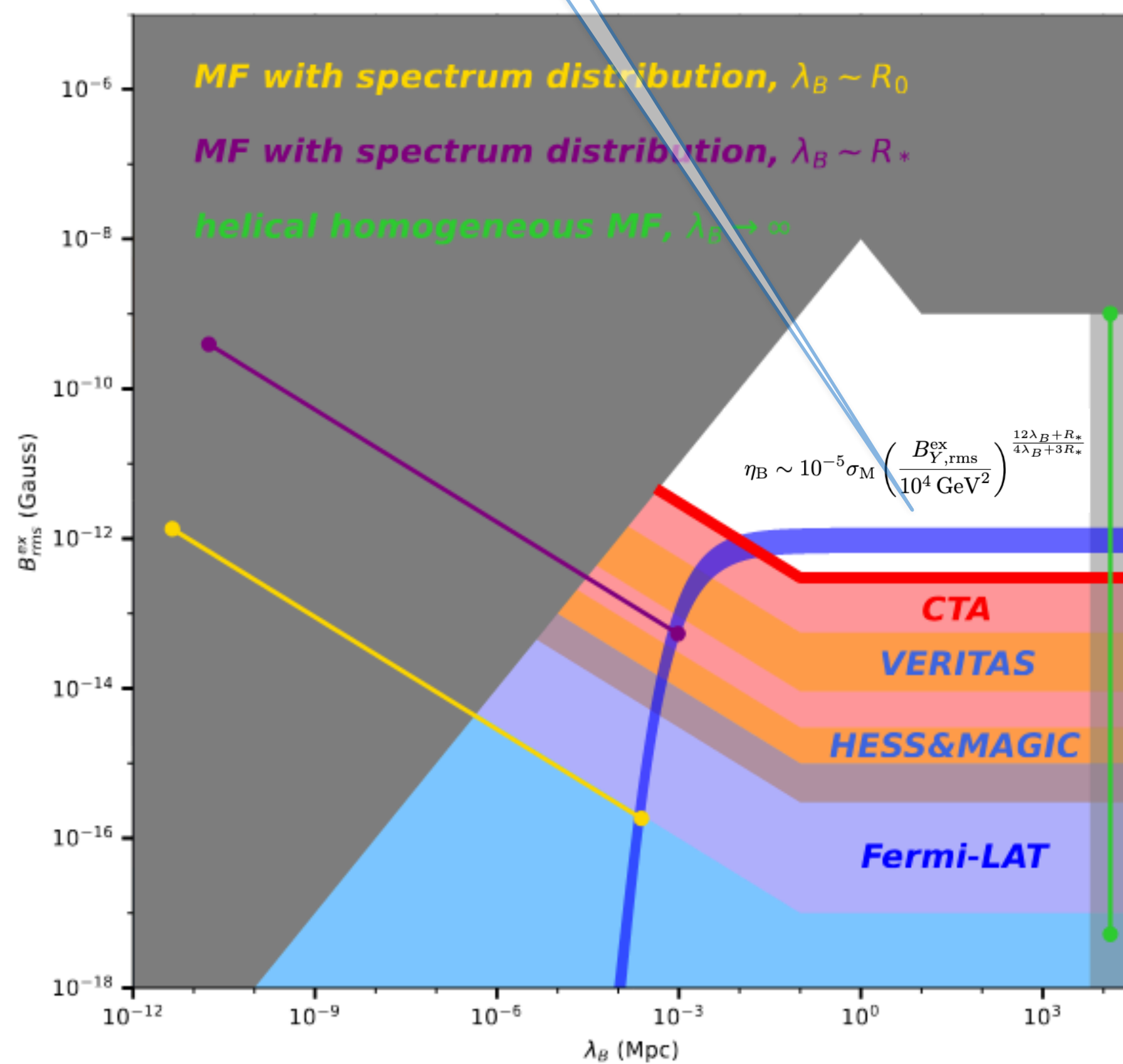
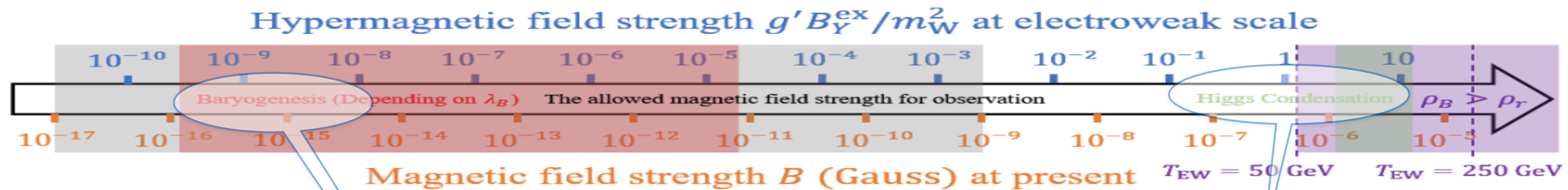


Sphaleron rate

$$\Gamma(t) = \frac{1}{V} \frac{d}{dt} [\langle (\Delta N_{\text{CS}}(t))^2 \rangle - \langle \Delta N_{\text{CS}}(t) \rangle^2]$$



► Bubble collision and PMF



❖ Summary

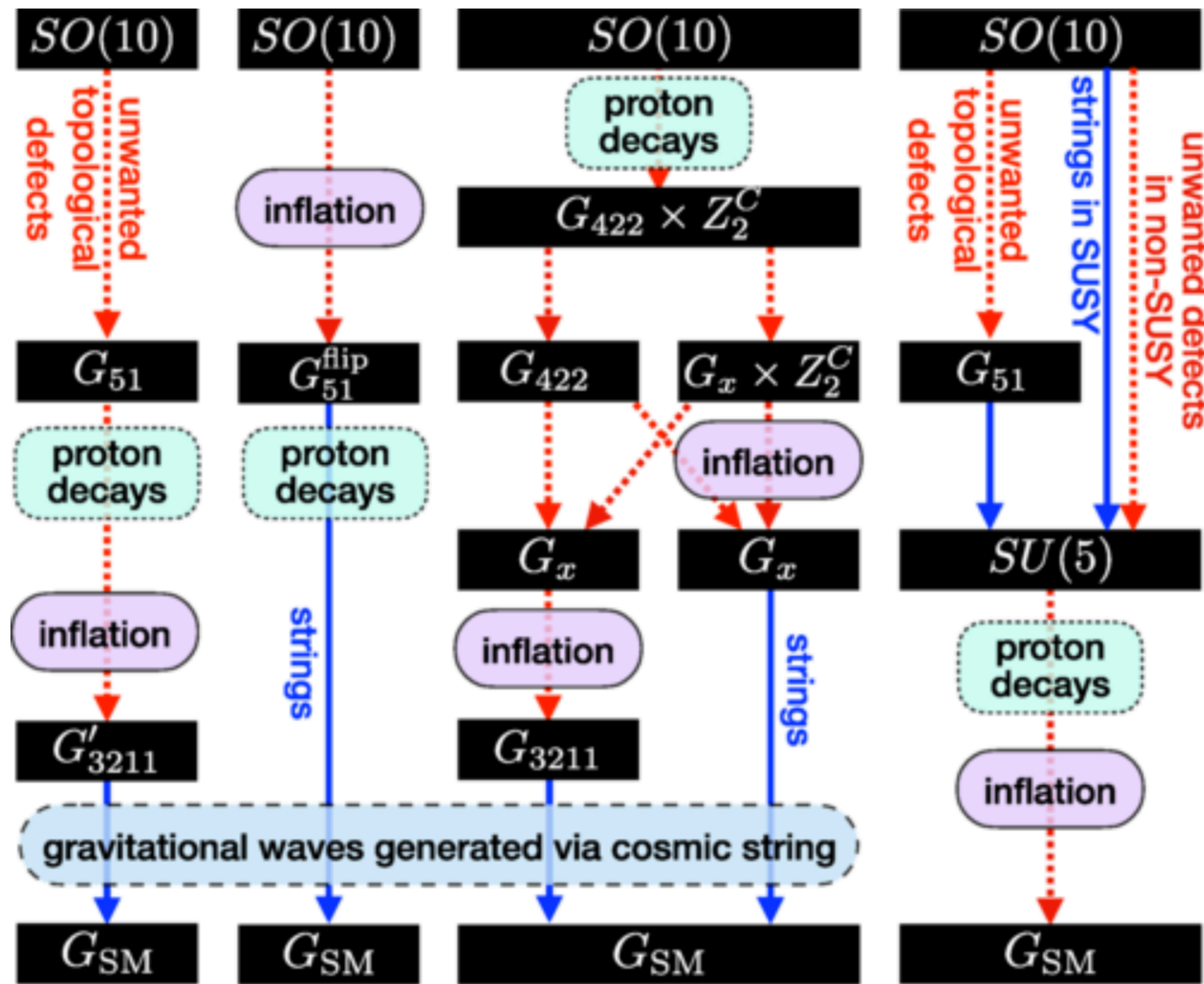
- **GW of ALP particles in the DFSZ case can be probed by GW detectors**
- **GW of Axion/ALP dark matter scenario in KSVZ cannot be probed**
- **DW Decay through PMF**
- **Direct measurement of the vacuum decay rate at finite temperature**
- **Bubble dynamics prefer PMF and GW production from EW lattice**

❖ Future

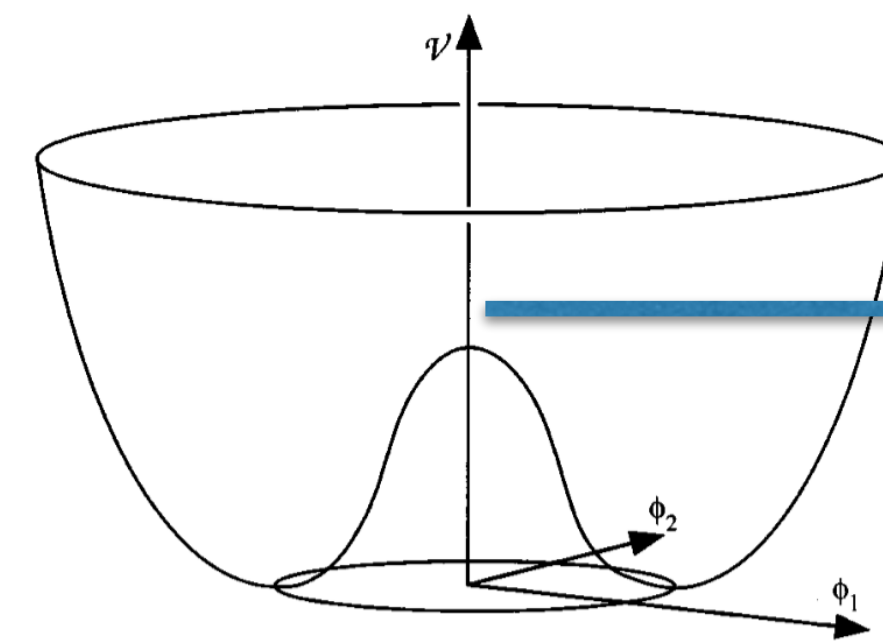
- **Dark matter and PT**
- **Topological defects: Magnetic monopoles, cosmic strings, domain walls**
- **Magnetic fields, baryogenesis, and Axion dynamics**

谢谢！

► Cosmic string

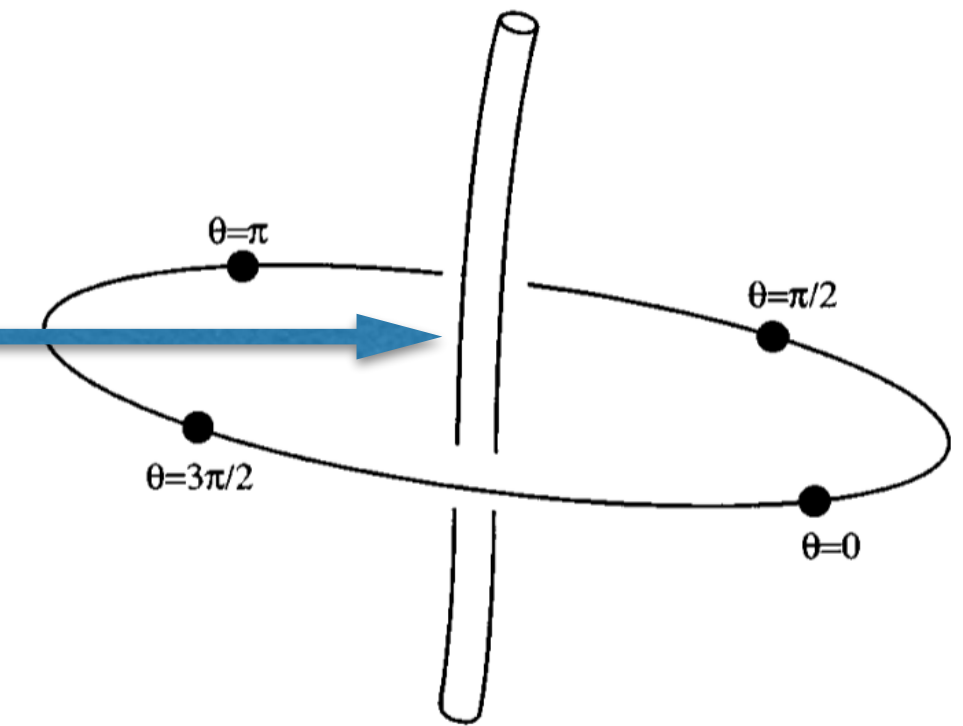


SSB of the U(1)

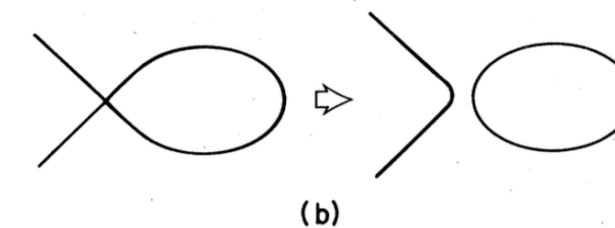
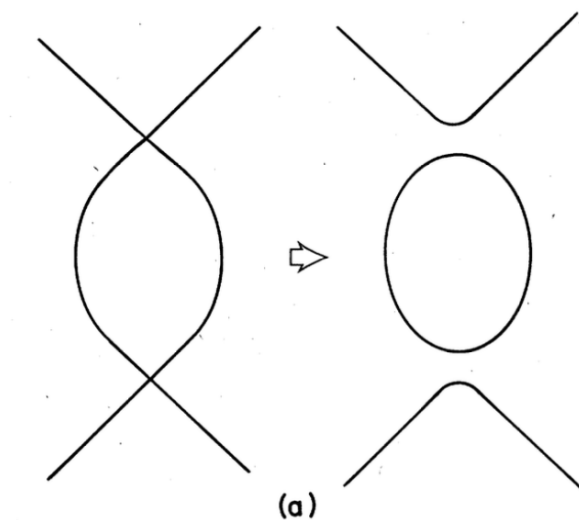


T. W. B. Kibble

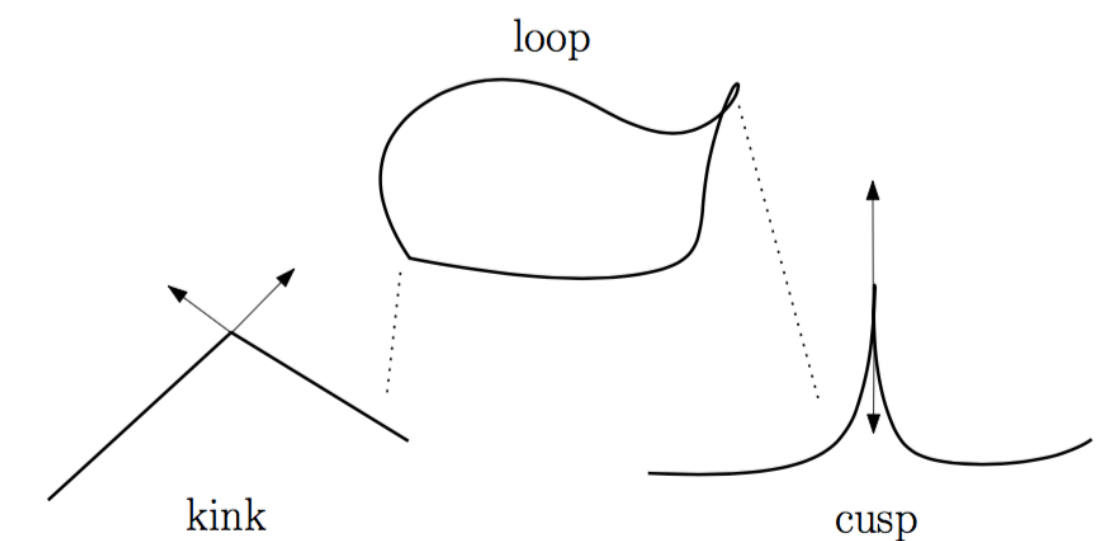
Cosmic string



Loops

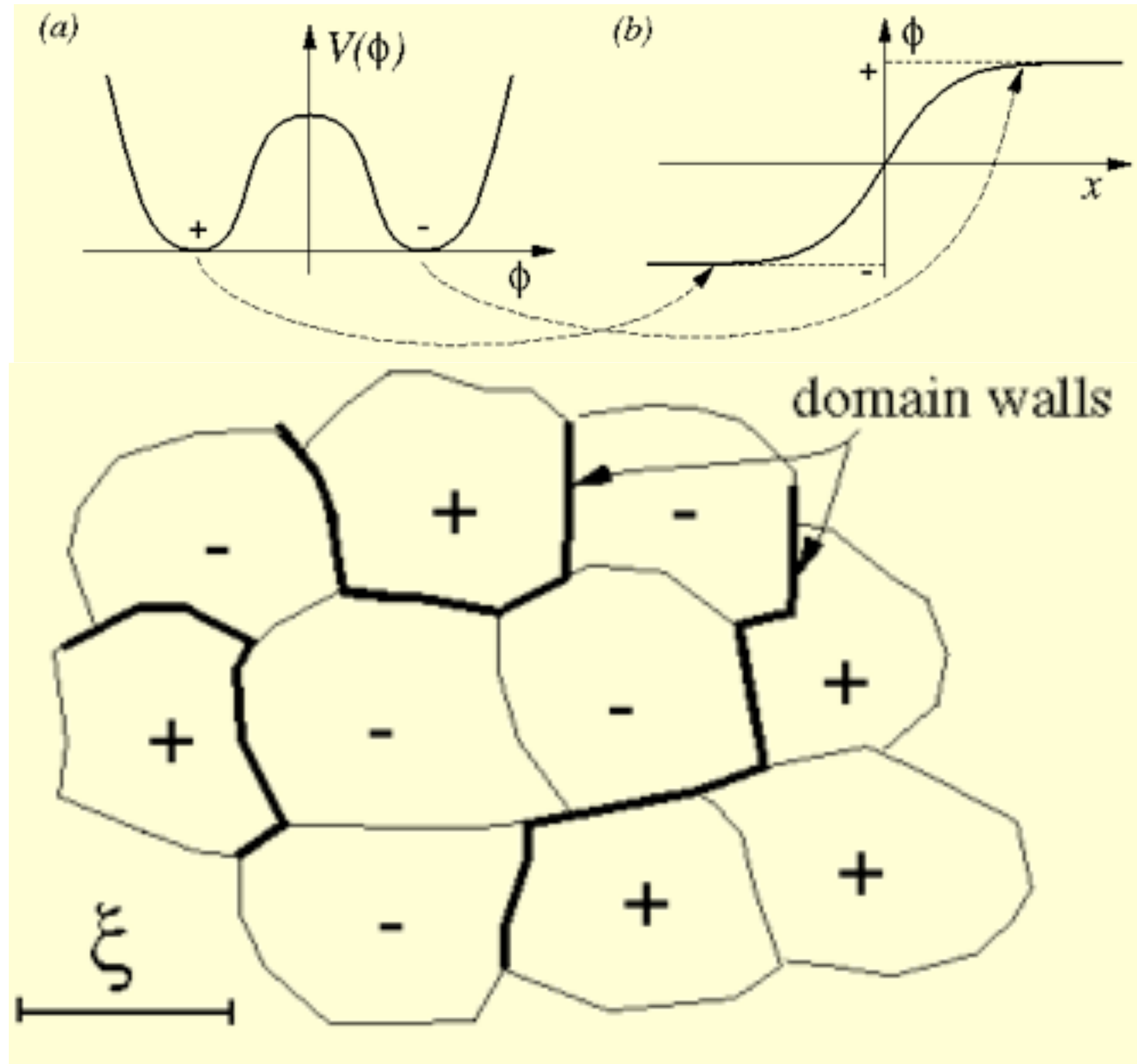


GW radiations

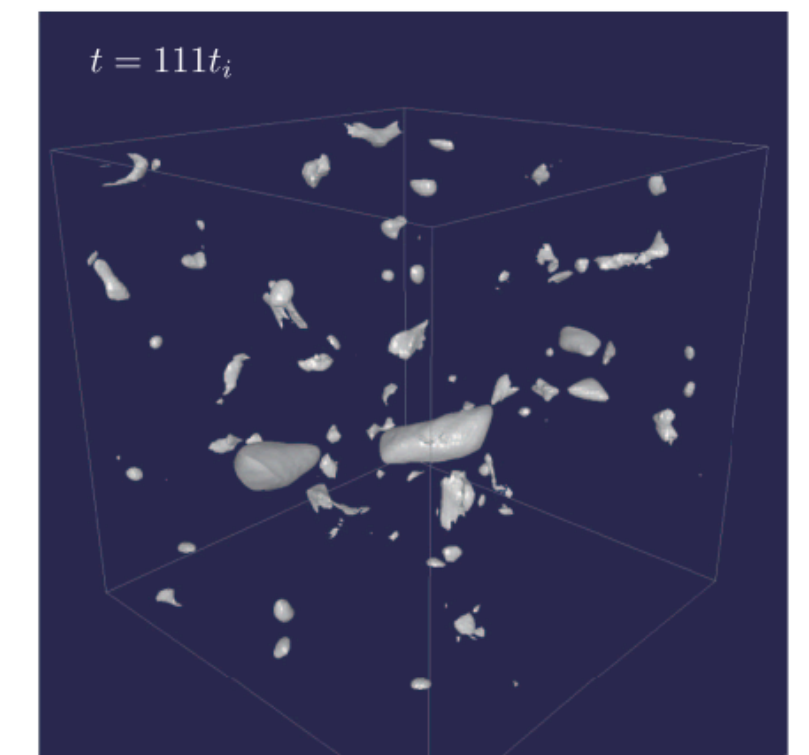
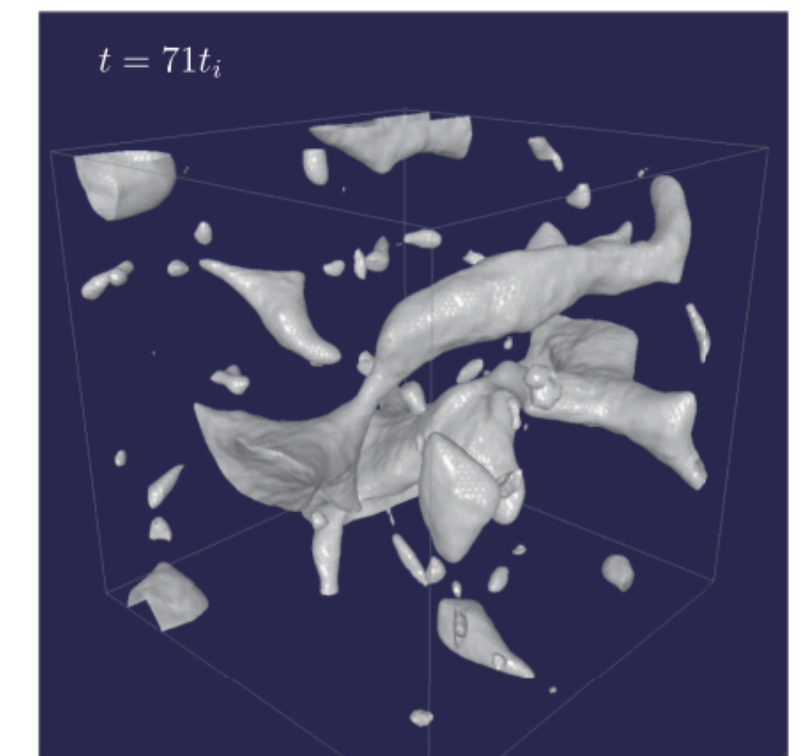
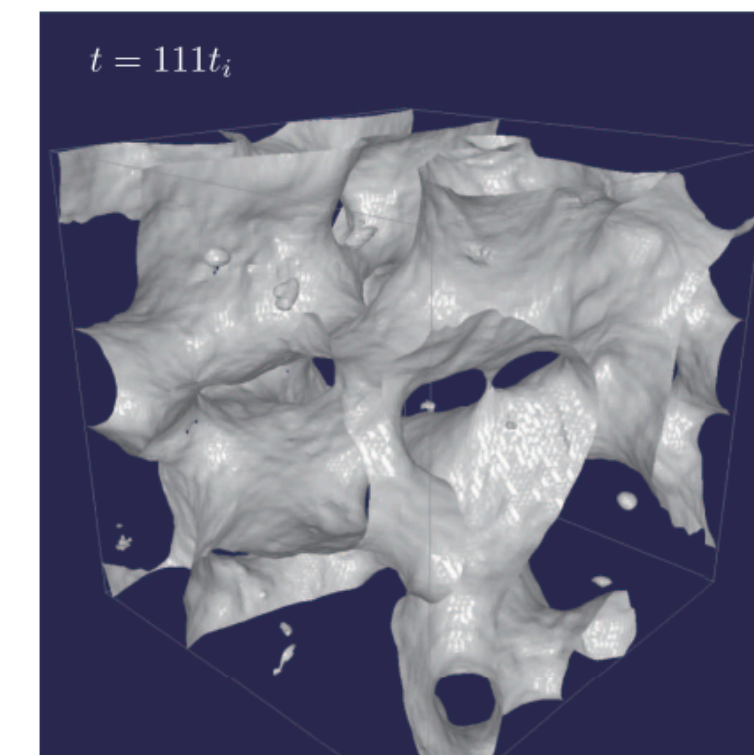
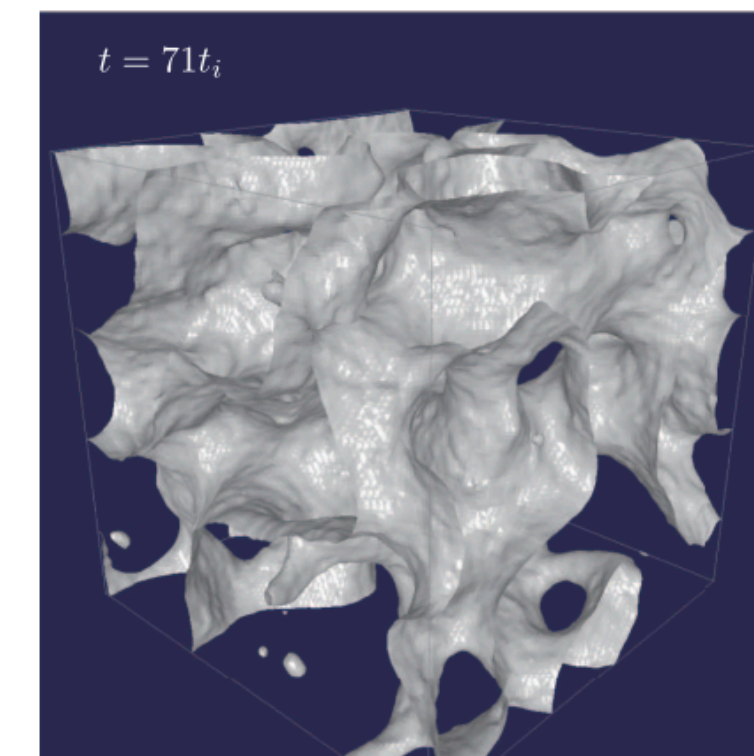
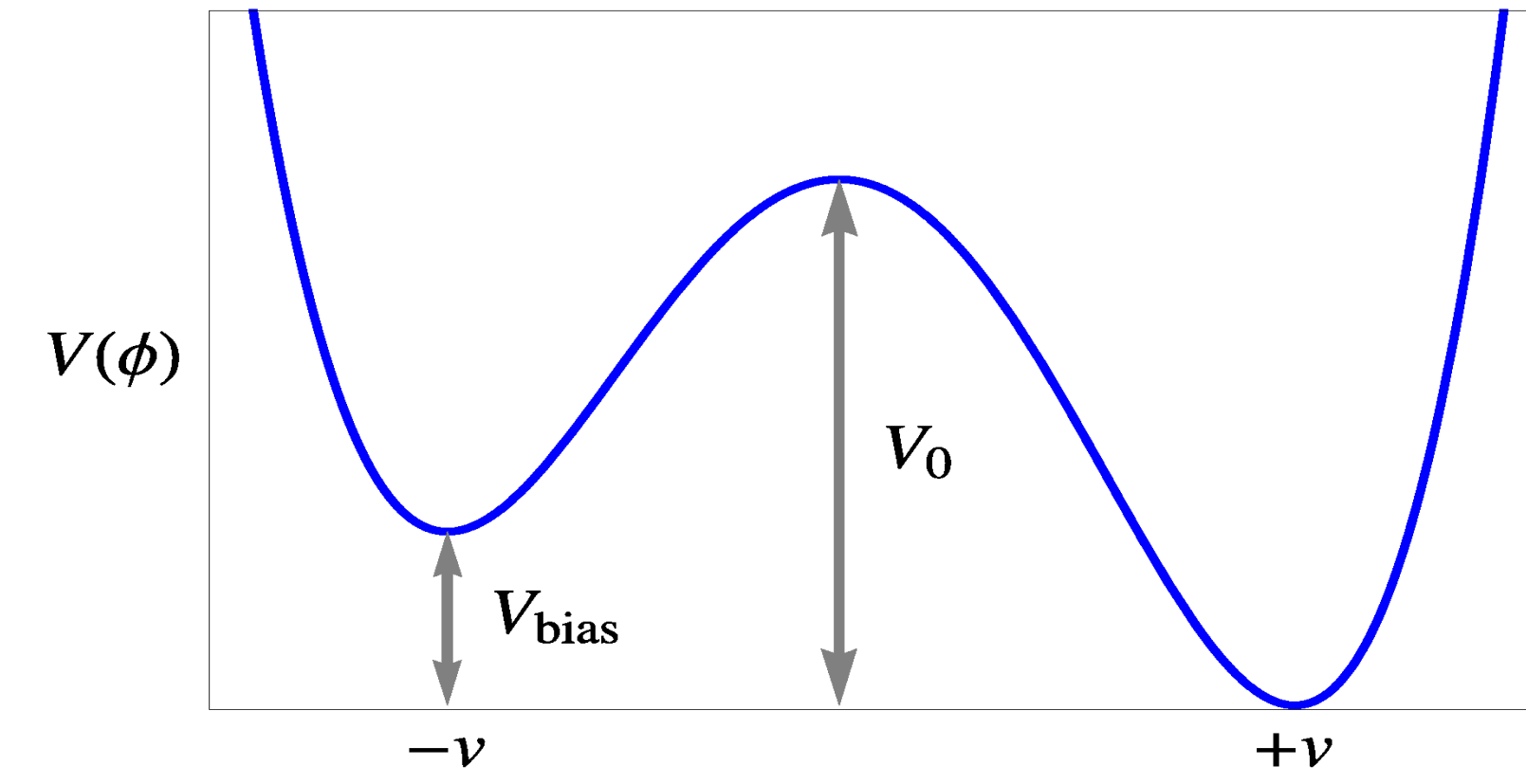


Yann Gouttenoire *et al* JCAP07(2020)032

Domain wall



Kibble mechanism



Indicator Functional: How to Define a field configuration Decay?

$$P_{\text{FV}}(t) = \frac{1}{Z} \int D\phi \frac{D\Pi}{2\pi} \Theta_{\text{FV}}[\phi] W[\phi, \Pi; t] \xrightarrow{\text{Monte Carlo average}} P_{\text{FV}}(t) \simeq \frac{1}{N_s} \sum_{i=1}^{N_s} \Theta_{\text{FV}}[\phi_t^{(i)}].$$

The main ambiguity is not the Monte Carlo average itself, but how Θ_{FV} is implemented for an extended field configuration. Different choices define different real-time observables.

- Global survival:

$$\bar{\phi}_\ell(t) > \phi_{\text{th}}. \quad \text{Whole-volume average crossing.}$$

- Connected-cluster survival:

$$\phi_\ell(x, t) > \phi_{\text{th}} \quad \text{over a connected interval with } L > L_c, \Delta t > \tau_{\text{hold}}. \quad \text{Persistent supercritical domain.}$$

- False-vacuum fraction:

$$f_{\text{FV}}^{(i)}(t) = \frac{1}{L} \int dx \Theta(\phi_{\text{th}} - \phi_\ell^{(i)}(x, t)),$$

$$P_{\text{frac}}(t) = \frac{1}{N_s} \sum_i f_{\text{FV}}^{(i)}(t).$$

Global and connected-cluster criteria are sample-level survival observables, while P_{frac} measures spatial conversion.

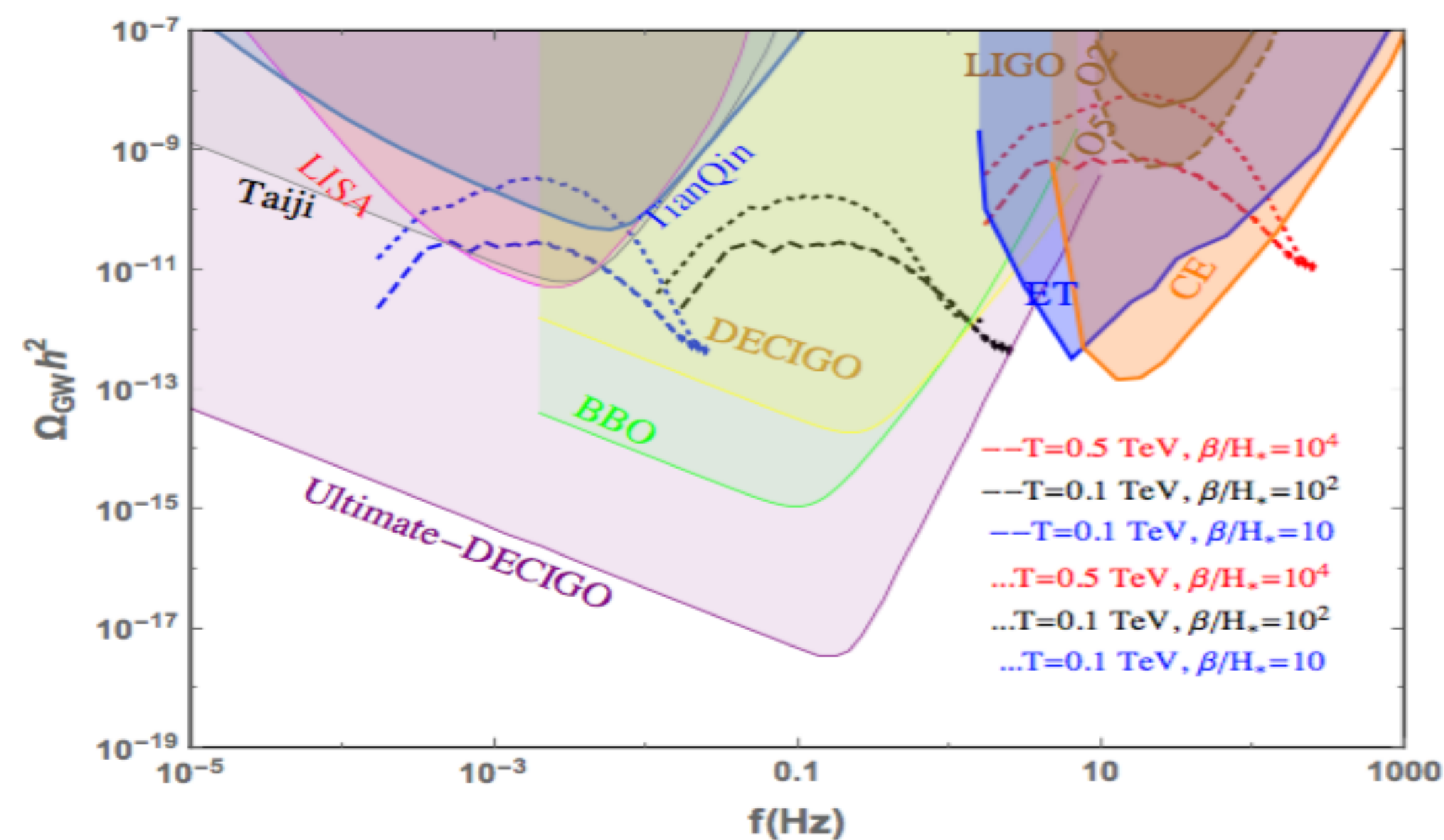
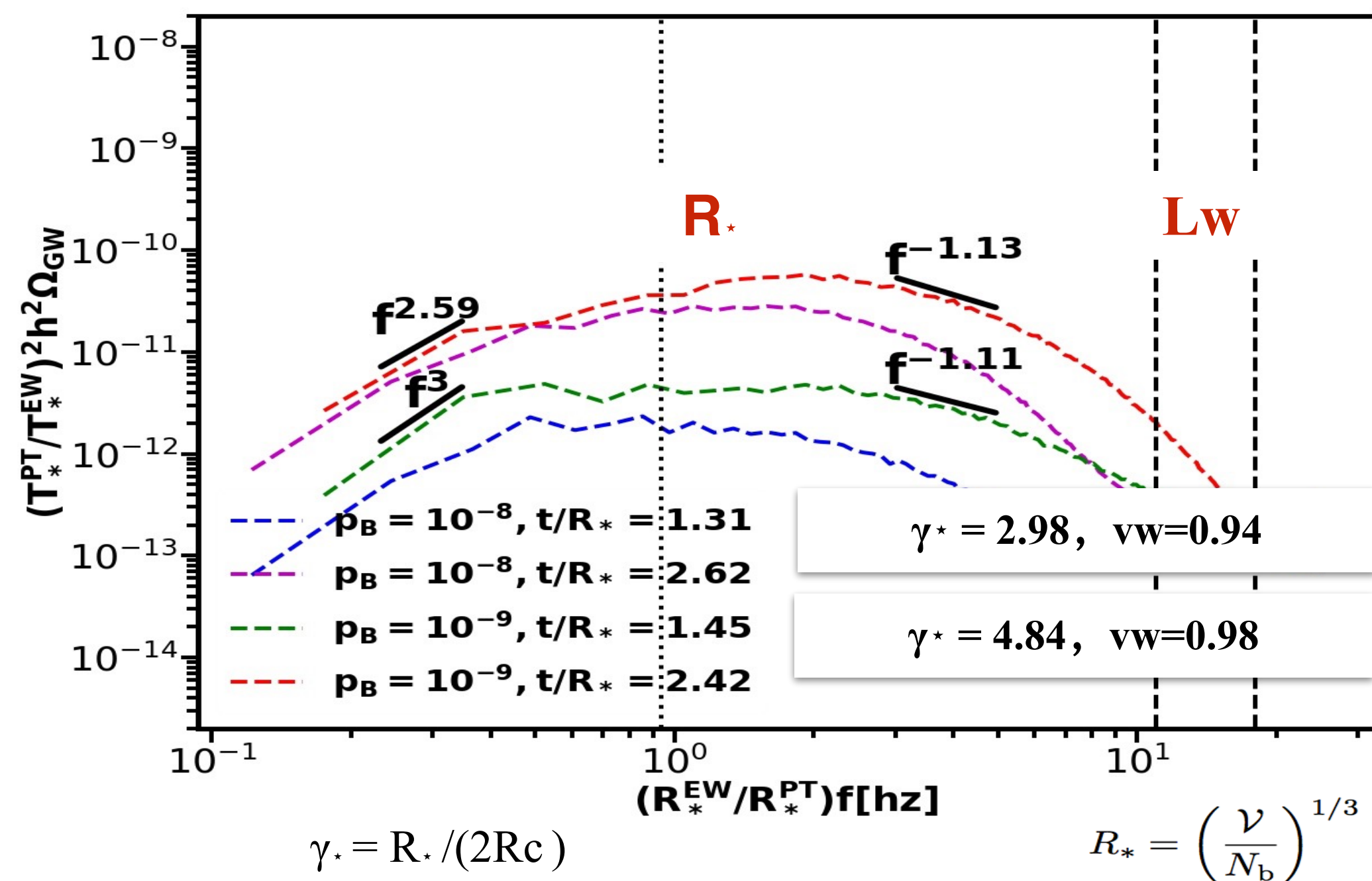
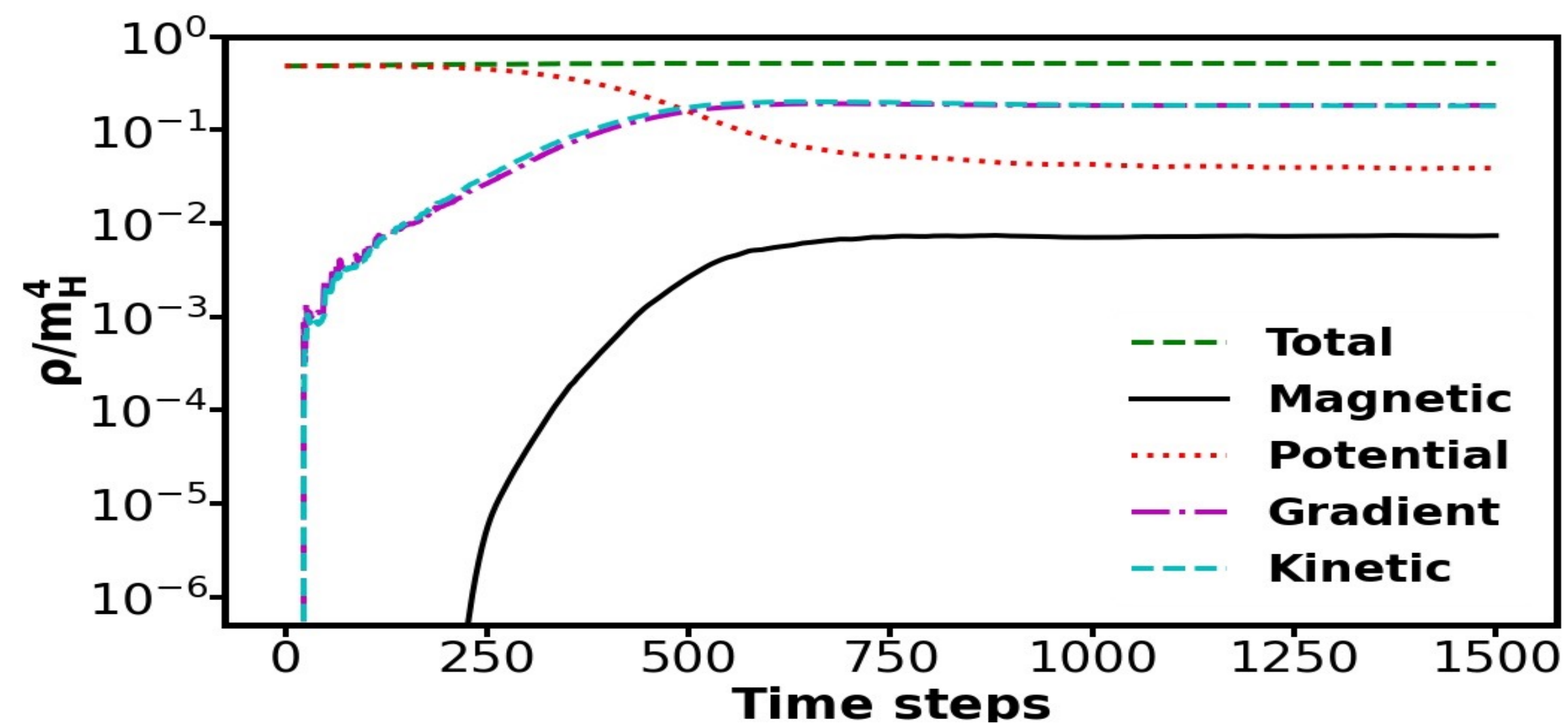
► Bubble collision and GWs generation

$$\ddot{h}_{ij} - \nabla^2 h_{ij} = 16\pi G T_{ij}^{TT}$$

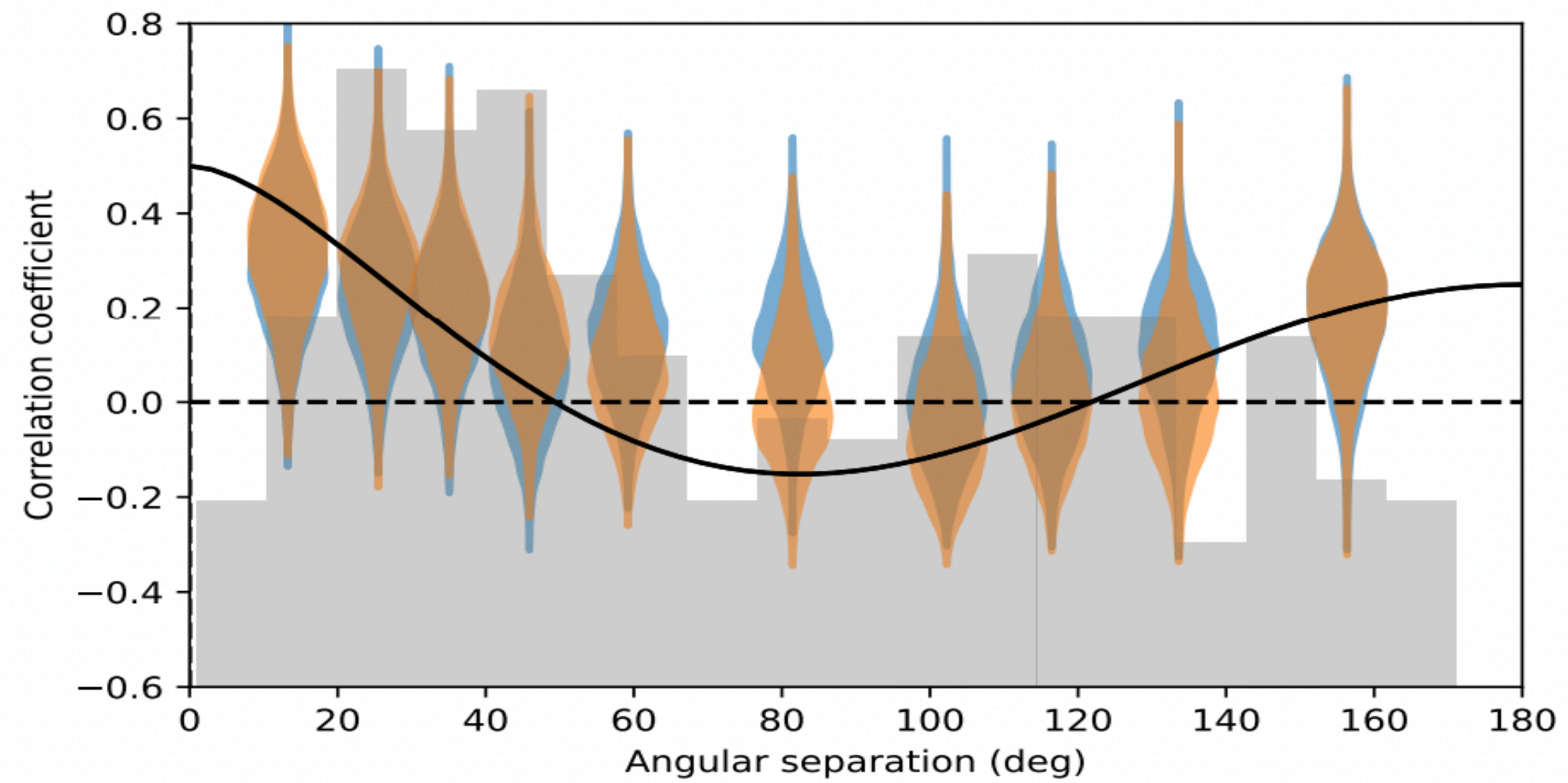
$$T_{\mu\nu} = \partial_\mu \Phi^\dagger \partial_\nu \Phi - g_{\mu\nu} \frac{1}{2} \text{Re}[(\partial_i \Phi^\dagger \partial^i \Phi)^2]$$

$$\langle \dot{h}_{ij}^{TT}(\mathbf{k}, t) \dot{h}_{ij}^{TT}(\mathbf{k}', t) \rangle = P_h(\mathbf{k}, t) (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}')$$

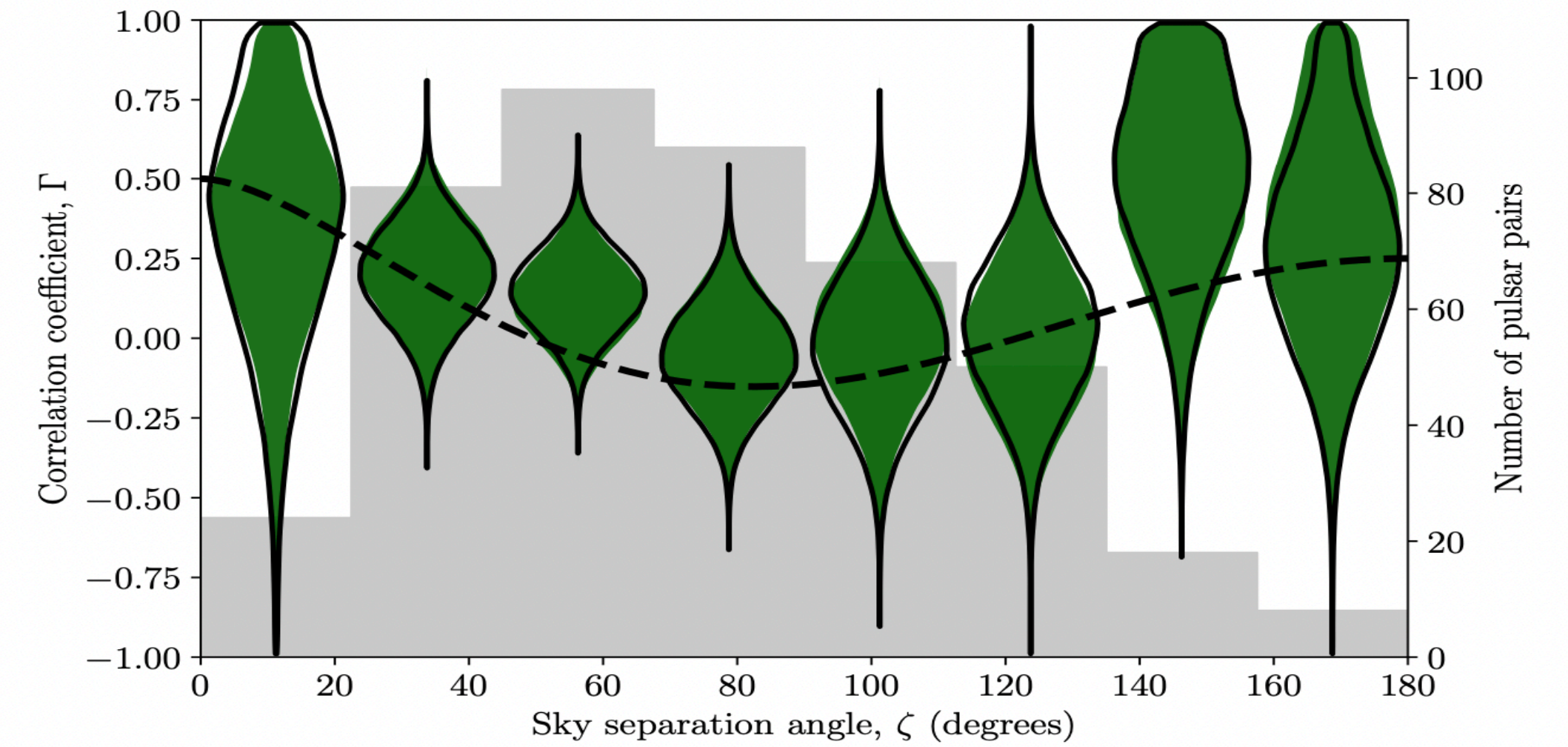
$$\frac{d\Omega_{\text{gw}}}{d\ln(k)} = \frac{1}{32\pi G \rho_c} \frac{k^3}{2\pi^2} P_h(\mathbf{k}, t)$$



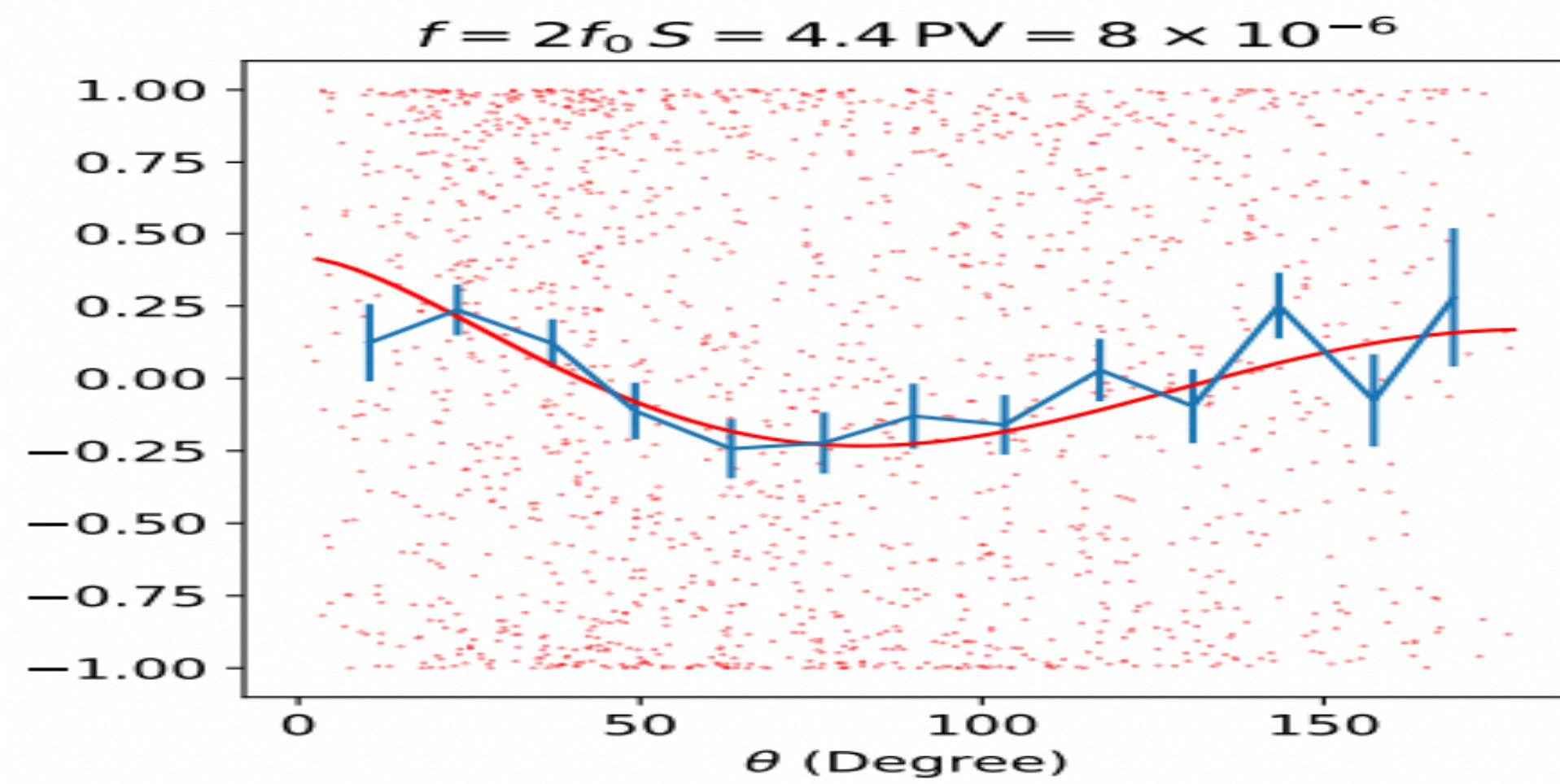
New dataset from PTAs



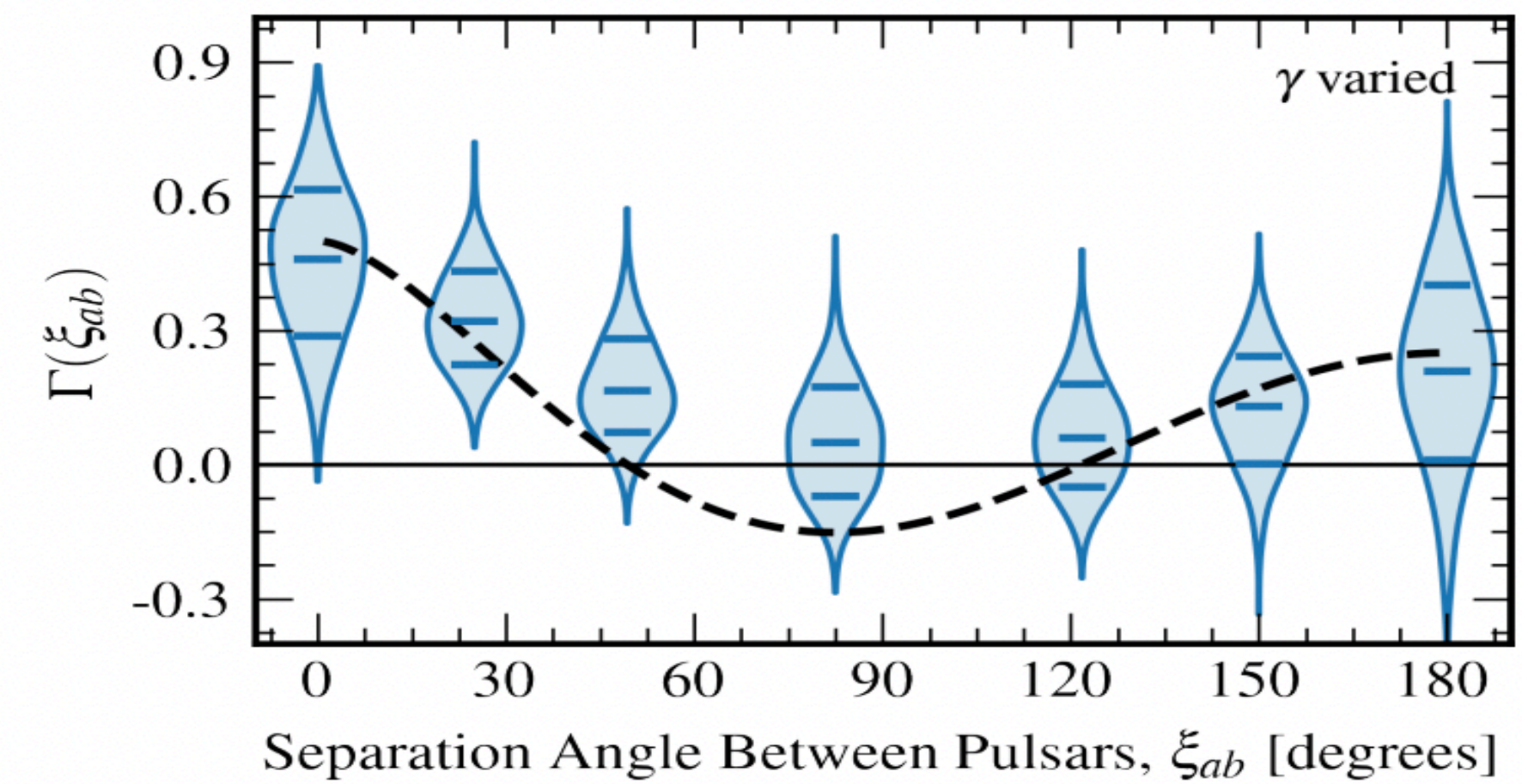
EPTA,2306.16214



PPTA,2306.16215

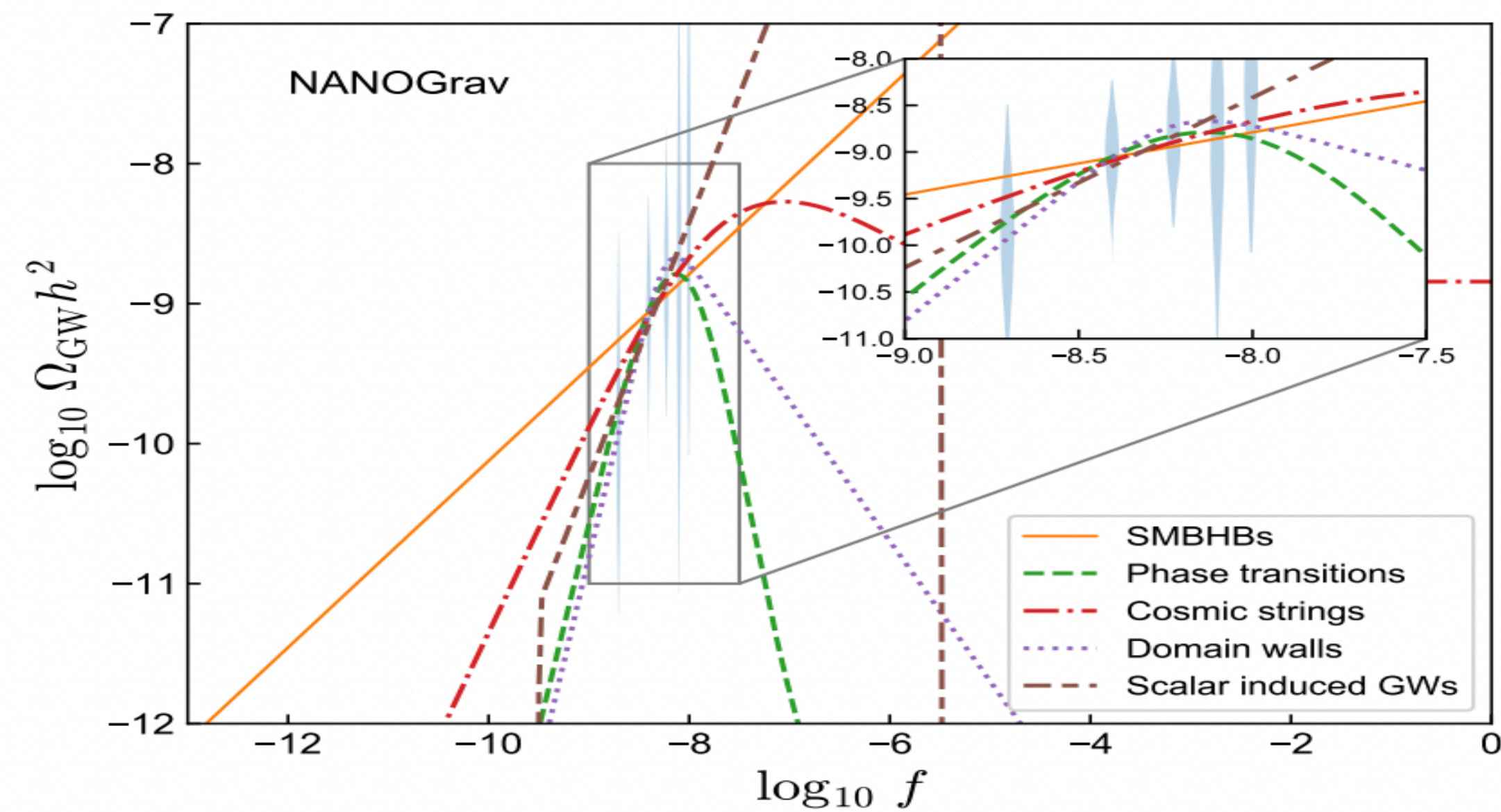
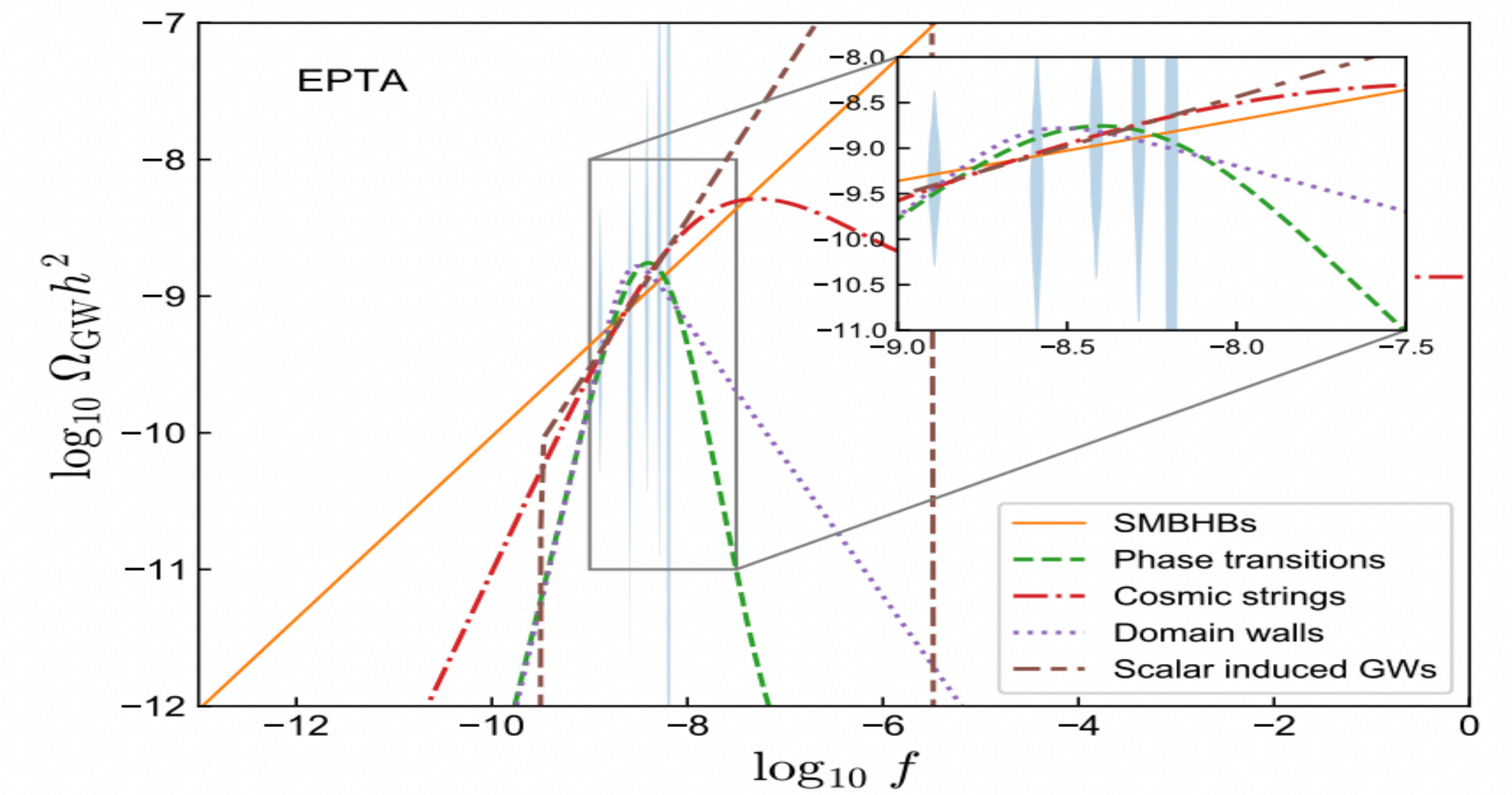
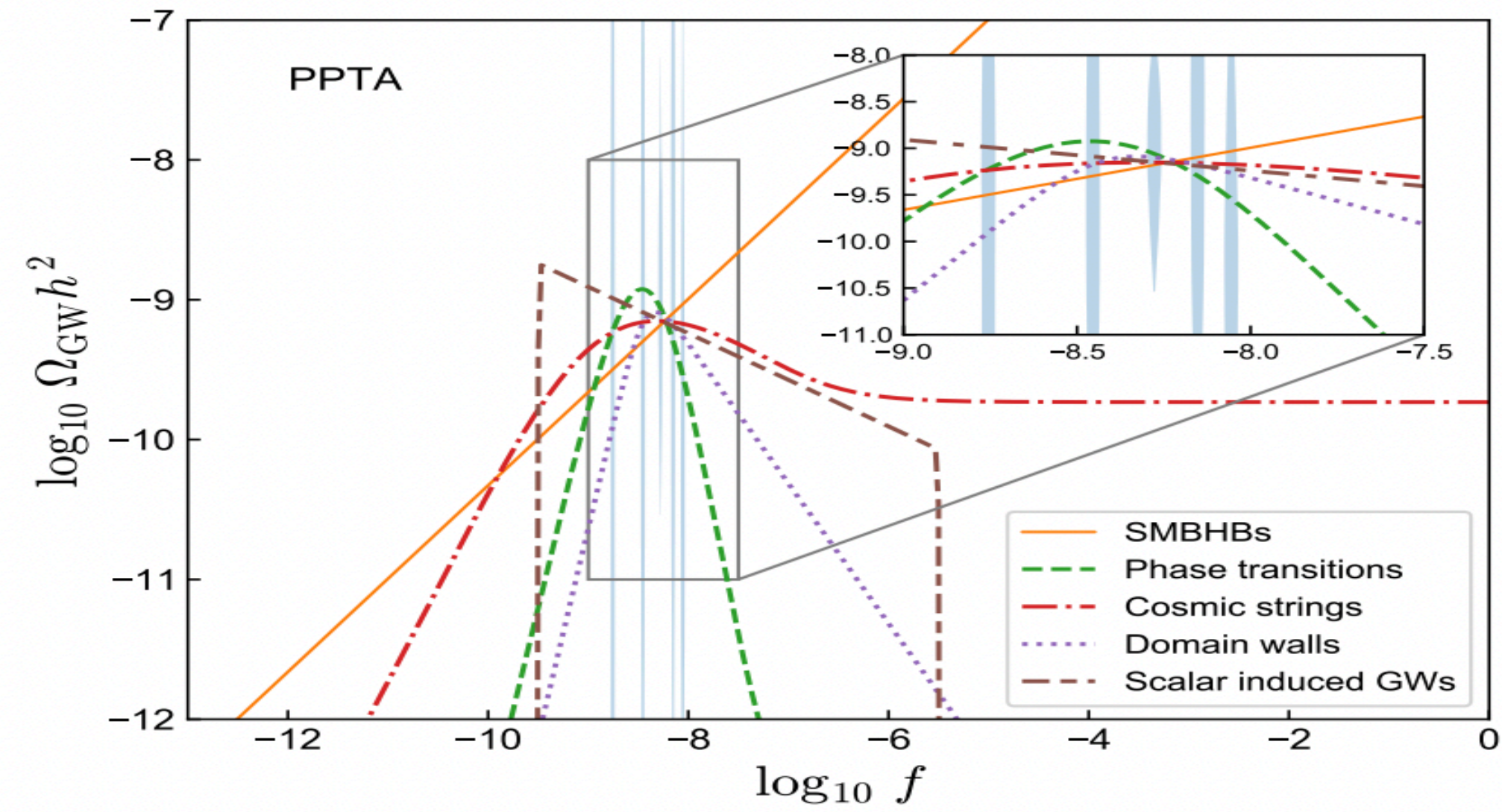


CPTA ,2306.16216



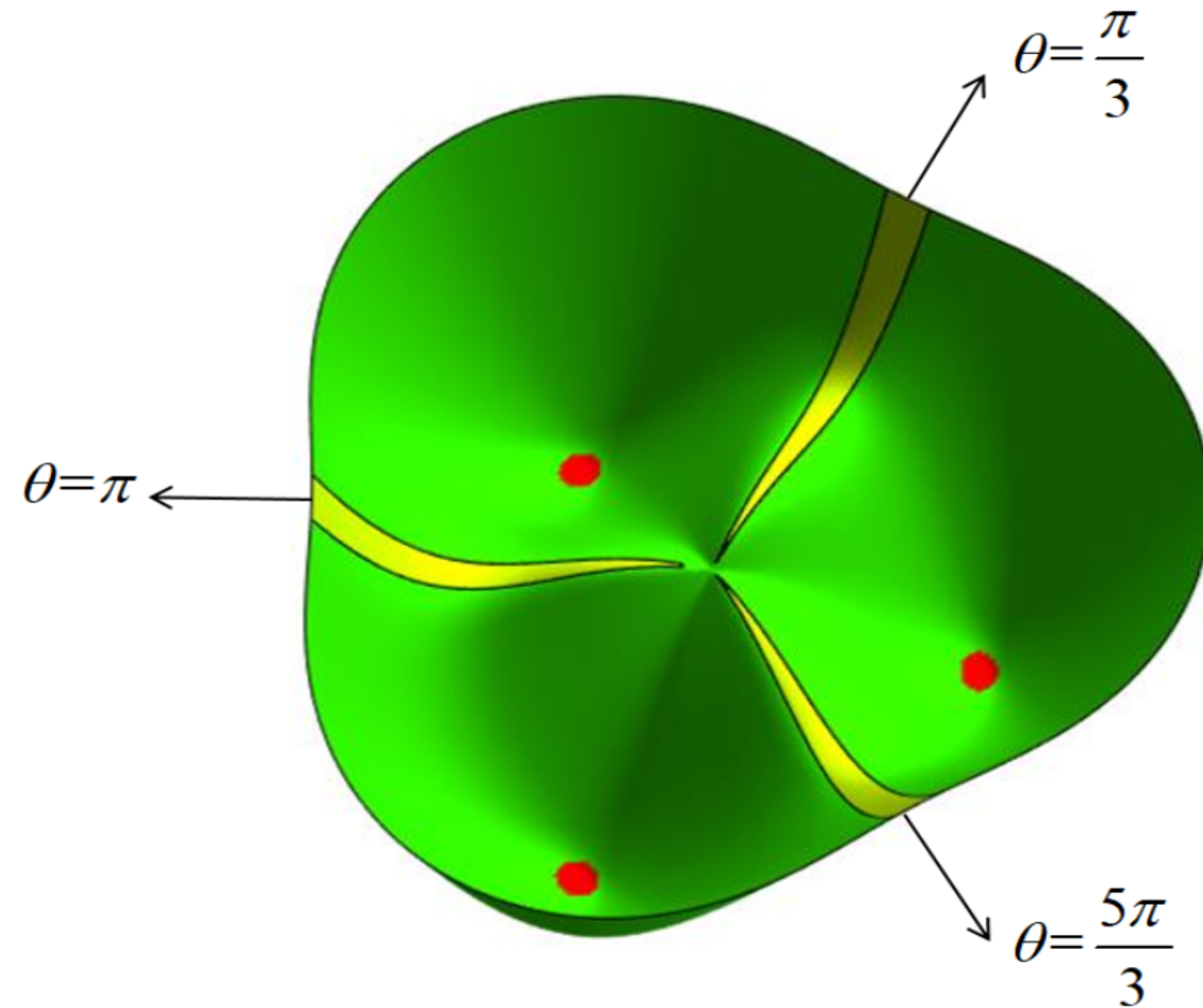
NANOGrav,2306.16213

Gravitational wave sources for Pulsar Timing Arrays



► Domain wall's identification

Top view of the shape of potential energy



Comoving area density

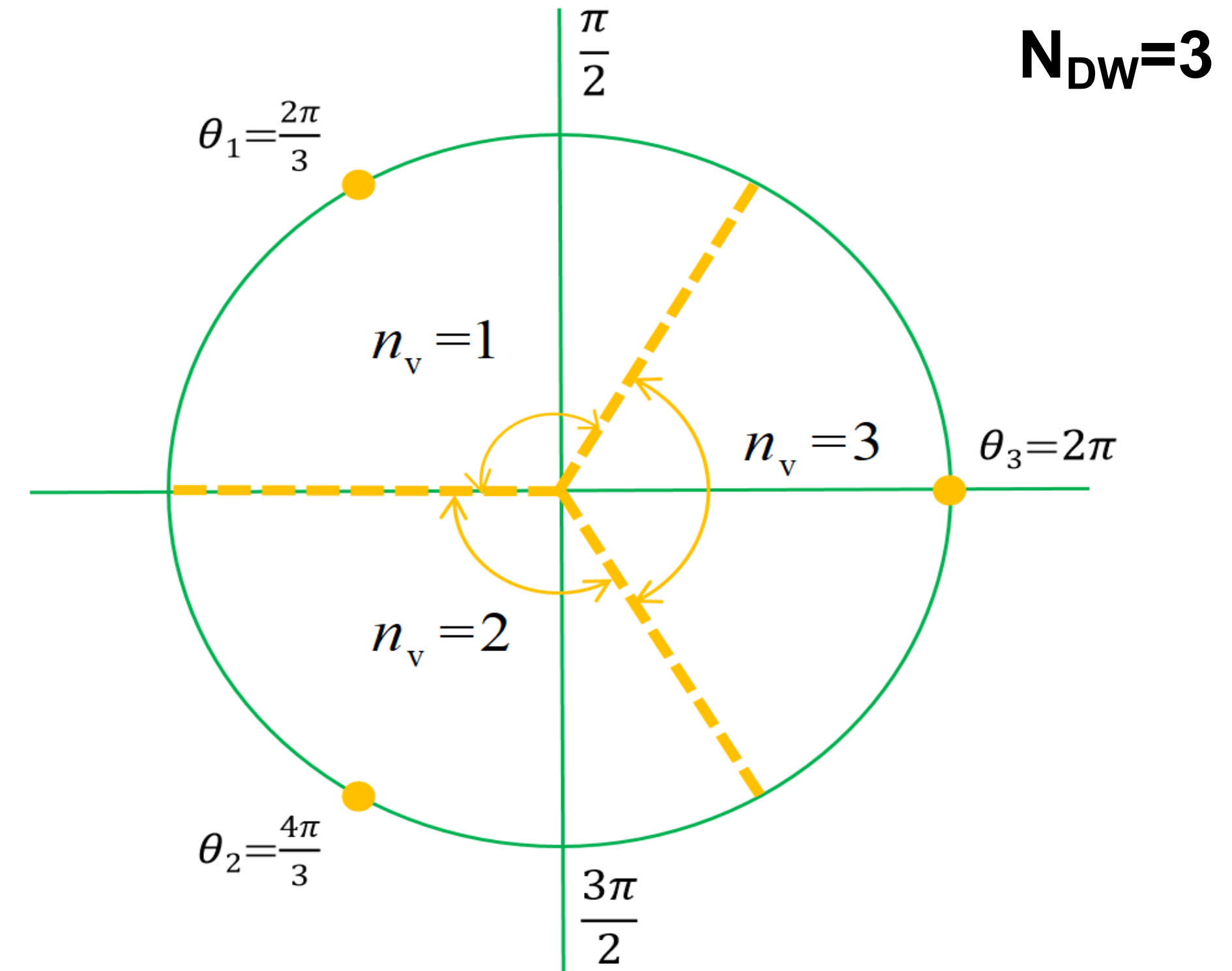
$$A/V = C \sum_{\text{links}} \delta \frac{|\nabla \theta|}{|\theta_{,x}| + |\theta_{,y}| + |\theta_{,z}|}$$

$\theta_{,i}$ ($i = x, y, z$) : spatial derivatives of the dimensionless axion field $\theta(x)$

Area parameter A of DW (scaling parameter of DW)

$$A = \Delta A \sum_{\text{links}} \delta \frac{|\nabla \theta|}{|\theta_{,x}| + |\theta_{,y}| + |\theta_{,z}|}$$

The distribution of fields in phase space

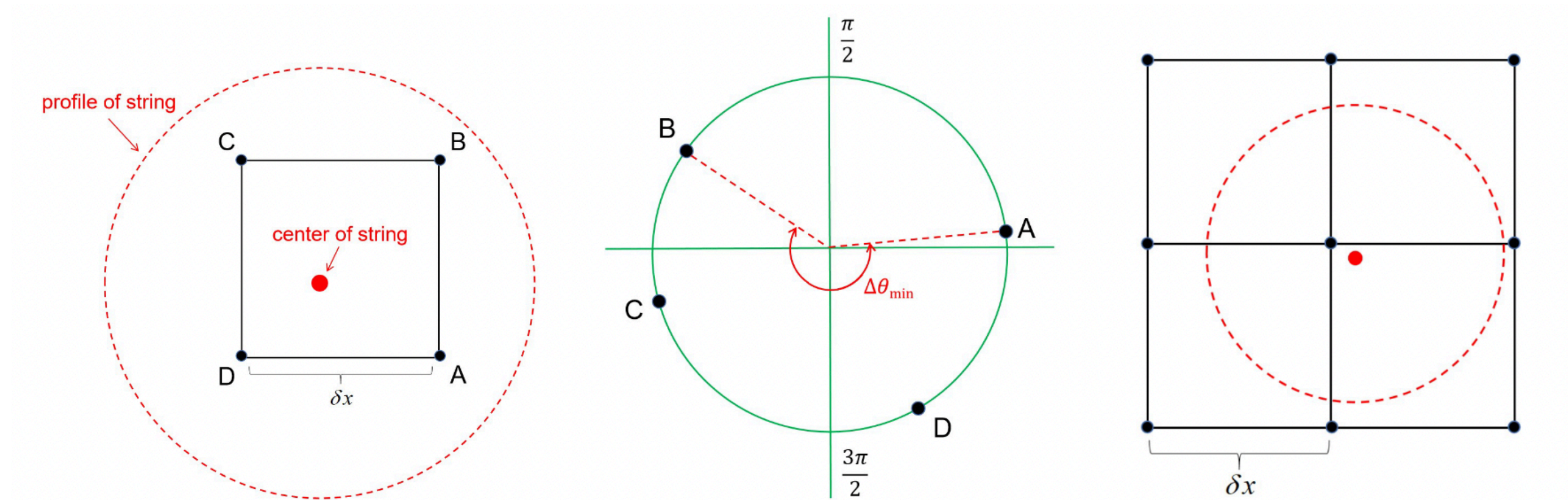


$$\xi_{\text{dw}} \equiv \mathcal{A} = \frac{\rho_{\text{wall}}}{\sigma_{\text{wall}}} t, \quad \text{with } \rho_{\text{wall}} = \frac{\sigma_{\text{wall}} A}{R(t)V}$$

$\Delta A = (\delta x)^2$ is the comoving area of one grid surface

► Identification of Cosmic string

String penetrates the square loop if the minimum phase range which contains the four points is greater than π and the phase changes continuously



For a specific square loop, assuming that the minimum phase at four points is $\theta_{\min}$

- (1) $\theta_{\min} < \pi$.
- (2) There exists at least one phase at another point minus $\theta_{\min}$ is greater than π .
- (3) There exists at least one phase at another point minus $\theta_{\min}$ is smaller than π .
- (4) Denote the phase closest to π in all phases greater than π as θ_a , and denote the phase closest to π in all phases smaller than π as θ_b , it is required to meet $\theta_a - \theta_b < \pi$.
- (5) Calculate the difference between the phases at each of two adjacent points in a counterclockwise direction, the multiplication of the four differences is required to be negative.

Bubble dynamics of FOPT

From QFT

$$p(t; T) \equiv \Gamma/V = |A(T)|e^{-B(T)/T}$$

Tunneling

$$A(T) = T \left(\frac{S_3[\bar{\phi}(T)]}{2\pi T} \right)^{\frac{3}{2}} \left(\frac{\det'[-\nabla^2 + V''(\bar{\phi})]}{\det[-\nabla^2 + V''(\phi_f)]} \right)^{-\frac{1}{2}}$$

Fluctuation

$$A(T) = \frac{\sqrt{-\lambda_-}}{2\pi} \left(\frac{S_3[\bar{\phi}(T)]}{2\pi T} \right)^{\frac{3}{2}} \left(\frac{\det'[-\nabla^2 + V''(\bar{\phi})]}{\det[-\nabla^2 + V''(\phi_f)]} \right)^{-\frac{1}{2}} \\ = \frac{1}{2\pi} \left(\frac{S_3[\bar{\phi}(T)]}{2\pi T} \right)^{\frac{3}{2}} \left(\frac{\det^+[-\nabla^2 + V''(\bar{\phi})]}{\det[-\nabla^2 + V''(\phi_f)]} \right)^{-\frac{1}{2}},$$

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False vacuum fraction considering bubbles expansion

Bubble Volume

$$h(t) = \exp \left[- \int_{t_c}^t dt' \frac{4\pi}{3} v_w^3 (t-t')^3 \frac{\Gamma(t')}{\mathcal{V}} \right]$$

$$\frac{\Gamma}{\mathcal{V}} = \frac{\Gamma_f}{\mathcal{V}} e^{\beta(t-t_f)}$$

$$\beta \equiv \left. \frac{d}{dt} \log \left(\frac{\Gamma(t)}{\mathcal{V}} \right) \right|_{t=t_f}$$

$$-\log h(t) \simeq \int_{t_c}^t dt' \frac{4\pi}{3} v_w^3 (t-t')^3 \frac{\Gamma_f}{\mathcal{V}} e^{\beta(t'-t_f)} \\ = \frac{4\pi}{3} v_w^3 \frac{\Gamma_f}{\mathcal{V}} \frac{3!}{\beta^4} e^{\beta(t-t_f)}.$$

$$h(t_f) = 1/e$$

$$8\pi \frac{v_w^3}{\beta^4} \frac{\Gamma_f}{\mathcal{V}} = 1$$

$$h(t) = \exp \left[- e^{\beta(t-t_f)} \right]$$

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