# Techniques of multi-loop Feynman integral computation and their applications

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# Contents of this talk

Background

- Why multi-loop Feynman integrals
- Review of Methods

Introduction to NeatIBP package

- The algorithm
- Examples and applications

The spanning cuts consistency problem and one of its answer (new)

Summary

Advances in HEP

**High precision Higgs physics,**
**New physics beyond standard model,**
**…**

High-precision experimental measurements

High-precision theoretical predictions


(picture from the internet)

LHC ……

In the future…

HL-LHC

CEPC

FCC ……

…

Phase space integration,
IR subtraction,
…

High-order Amplitudes

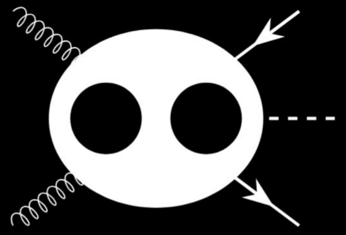
perturbation

QFT

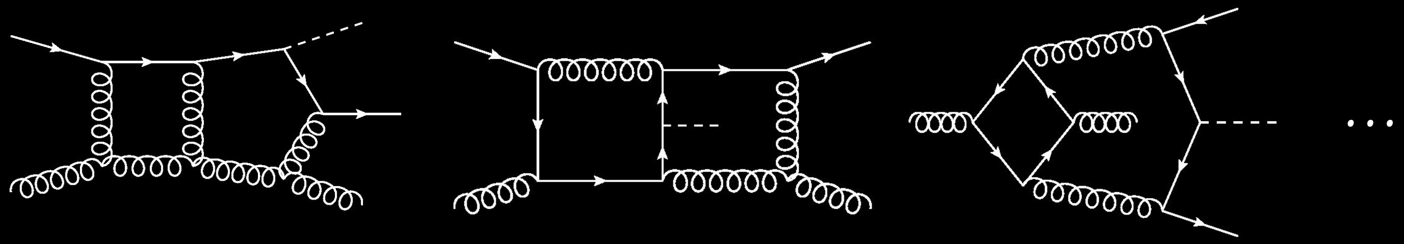
*Computing fixed-order amplitudes for scattering processes remains a key obstacle to producing precise predictions for the LHC and HL-LHC. – Les Houches Wish List 2023*

# Amplitudes from multi-loop Feynman integrals

***Amplitudes*** *(in high order expansion)*



$\sum$ ***Multi-loop Feynman diagrams***



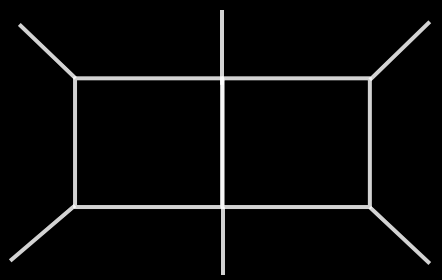
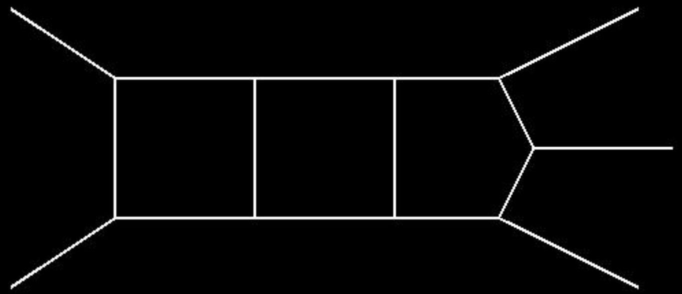
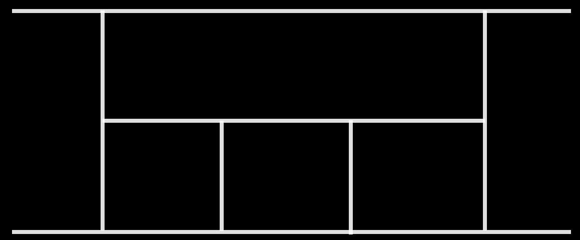
$\sum$ ***Multi-loop Feynman integrals***

$$\int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{\mathcal{N}}{D_1^{\delta_1} \cdots D_m^{\delta_m}}.$$

*Multi-loop Feynman integrals are key objects to evaluate high-order amplitudes.*

# Frontier of Feynman integral computations

| | One-loop | Two-loop | Three-loop | $\geqslant$ Four-loop |
|---|---|---|---|---|
| Four-point | Known | Most known, with some hard problems unknown | Partially known | Frontier |
| Five-point | Known | Partially known | Frontier | Unknown |
| $\geqslant$ Six-point | Known | Frontier | Unknown | Unknown |

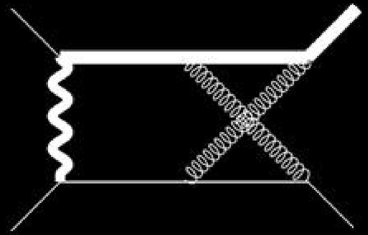




*Feynman integral computations is one of the bottlenecks for plenty of computations in precision frontier.*
*Its breakthrough relies highly on the development of techniques*

# Review of Methods

# Workflow of analytic computation of Feynman integrals

Feynman diagrams



Feynman rules

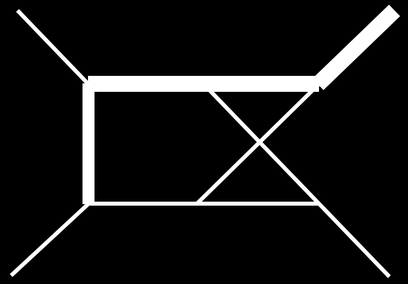
Feynman integrals

$$\int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{\mathcal{N}}{D_1^{\delta_1} \cdots D_m^{\delta_m}}.$$

Tensor reduction

Scalar integrals

$$\int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}}$$



Integration-by-parts (IBP) reduction

Master Integrals

A linear independent basis of the scalar integrals

Evaluation methods:

(canonical) differential equations, method of region…

AMFlow, pySecDec, Fiesta,…

Analytic functions / Numerical results

# Integration by parts (IBP) reduction

Feynman integral family

$$I[\alpha_1,\cdots,\alpha_n]=\int\frac{\mathrm{d}^Dl_1}{i\pi^{D/2}}\cdots\frac{\mathrm{d}^Dl_L}{i\pi^{D/2}}\frac{1}{D_1^{\alpha_1}\cdots D_n^{\alpha_n}}$$

IBP relations

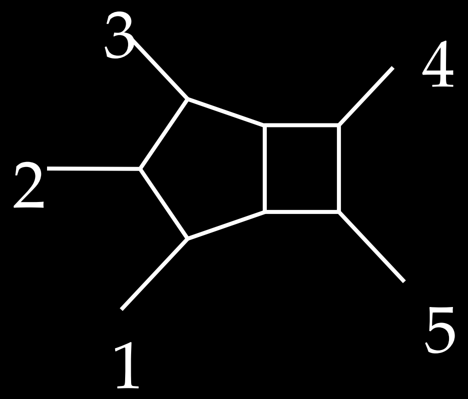
$$0=\int\frac{\mathrm{d}^Dl_1}{i\pi^{D/2}}\cdots\frac{\mathrm{d}^Dl_L}{i\pi^{D/2}}\frac{\partial}{\partial l_k^\mu}\frac{v^\mu}{D_1^{\alpha_1}\cdots D_n^{\alpha_n}}\qquad 0=\sum_{\{\alpha_i\}}c^i_{\alpha_1\cdots\alpha_n}I[\alpha_1,\cdots,\alpha_n]$$

IBP reduction

Linear system of IBP relations $\xrightarrow{\text{Gaussian Elimination}}$ $I[\alpha_1,\cdots,\alpha_n]=\sum_{i=1}^{N}\underbrace{c_i}_{\text{Coefficients}}\overbrace{I_i}^{\text{Master integrals}}$

# Integration by parts reduction

## Problem : High computation cost for reduction




Target integrals with degree-5 numerators
Quantity: 2483
# of IBP (FIRE6): **11207942**

Gaussian elimination of a HUGE matrix with Analytic elements

$$\begin{bmatrix} s_{12} + s_{23} & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

**10^7 × 10^7**

We cannot finish the reduction analytically in a tolerable time

**Our goal:**
**Compute of high-order amplitudes that are needed for high energy physics**

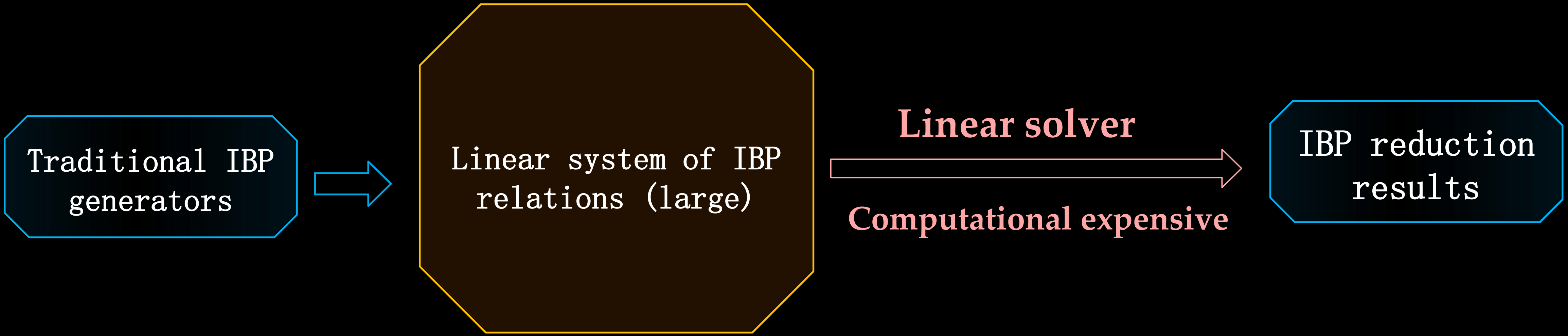
**How:**

1. Developments of new methods and computation tools to solve the bottlenecks in multi-loop Feynman integrals

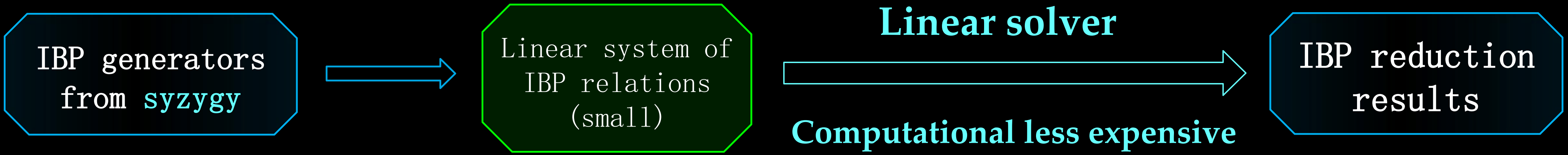
2. Applying the new methods and computation tools to advance the computations

# Introduction to NeatIBP

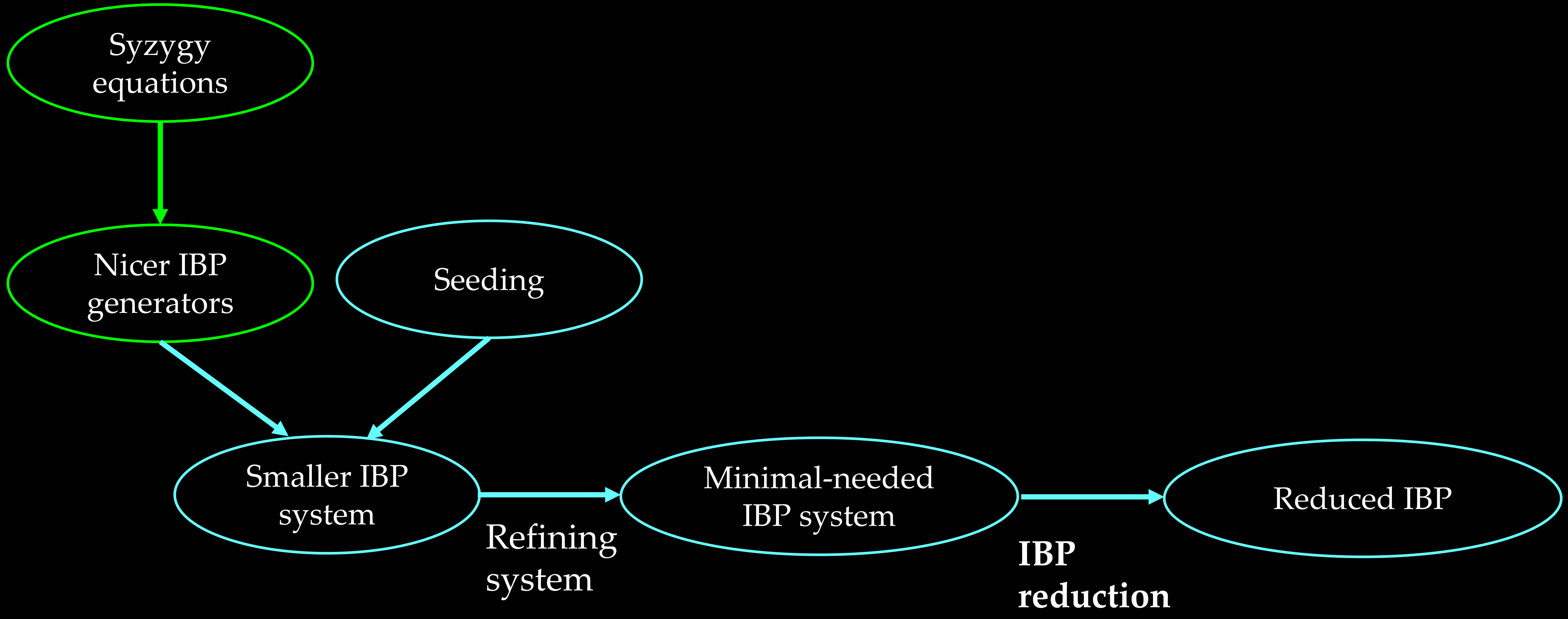
# Traditional IBP algorithm



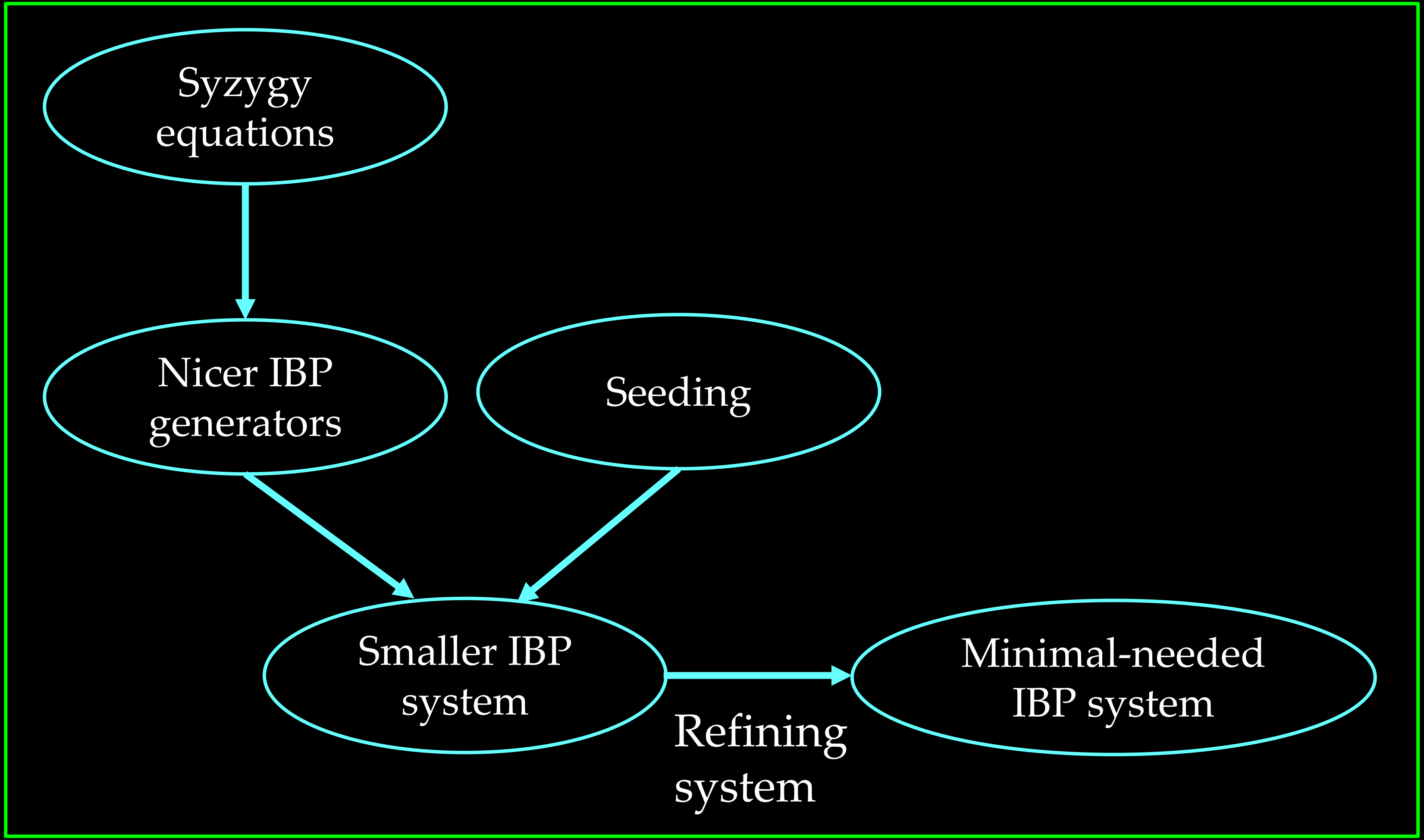

# NeatIBP:



# The work flow and program package NeatIBP



# The work flow and program package NeatIBP

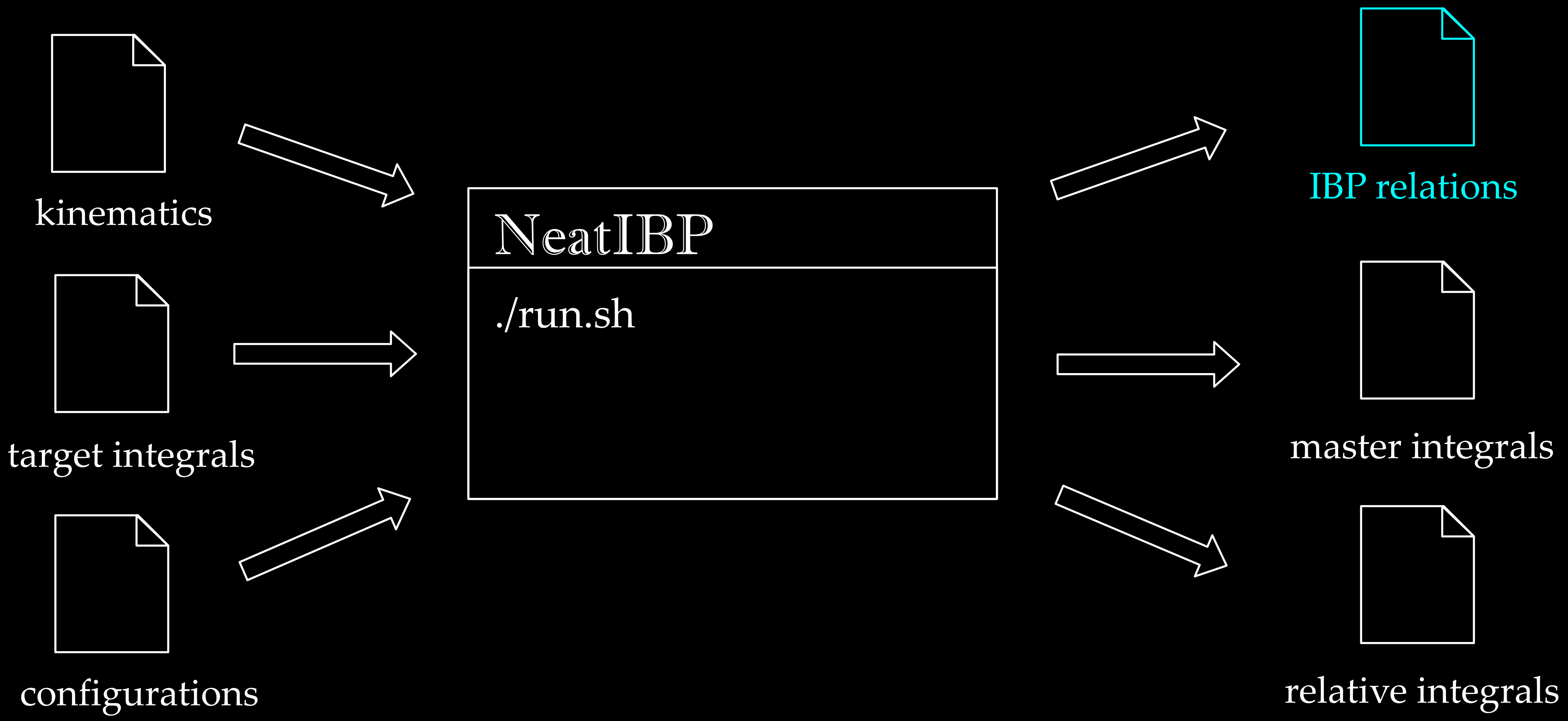



**NeatIBP**

Based on:
Mathematica
Singular

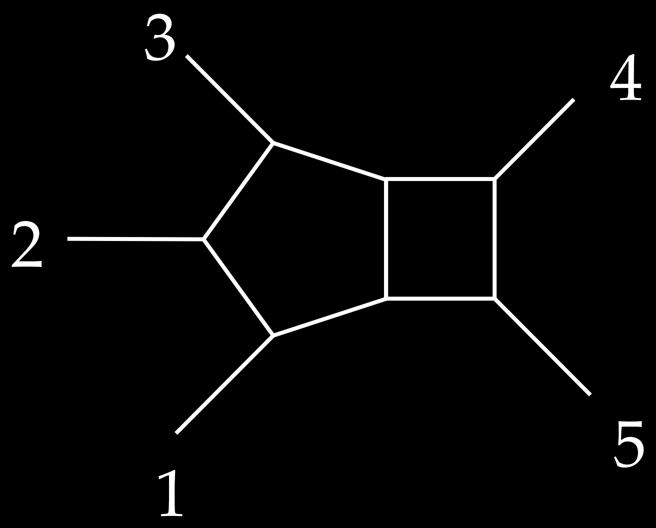
https://github.com/yzhphy/NeatIBP

# NeatIBP inputs & outputs



# Performance of NeatIBP

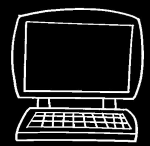
2L5P Example



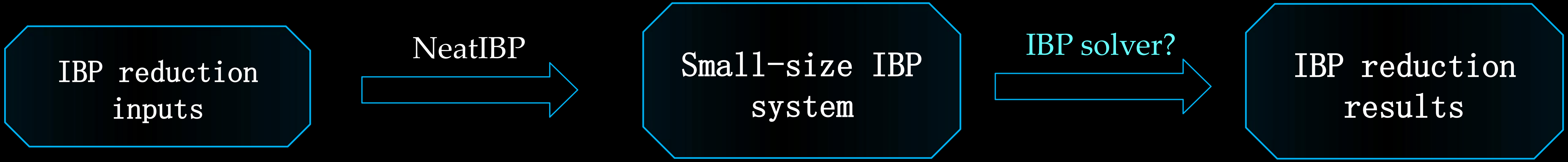

Target integrals:
Max numerator degree: 5
Max denominator power: 1
Quantity: 2483

# of IBP (FIRE6): **11207942**

# of IBP (NeatIBP): **14120**

Time used: 27m at  CPU threads: 20
RAM: 128GB

# NeatIBP 1.1 new feature: kira interface

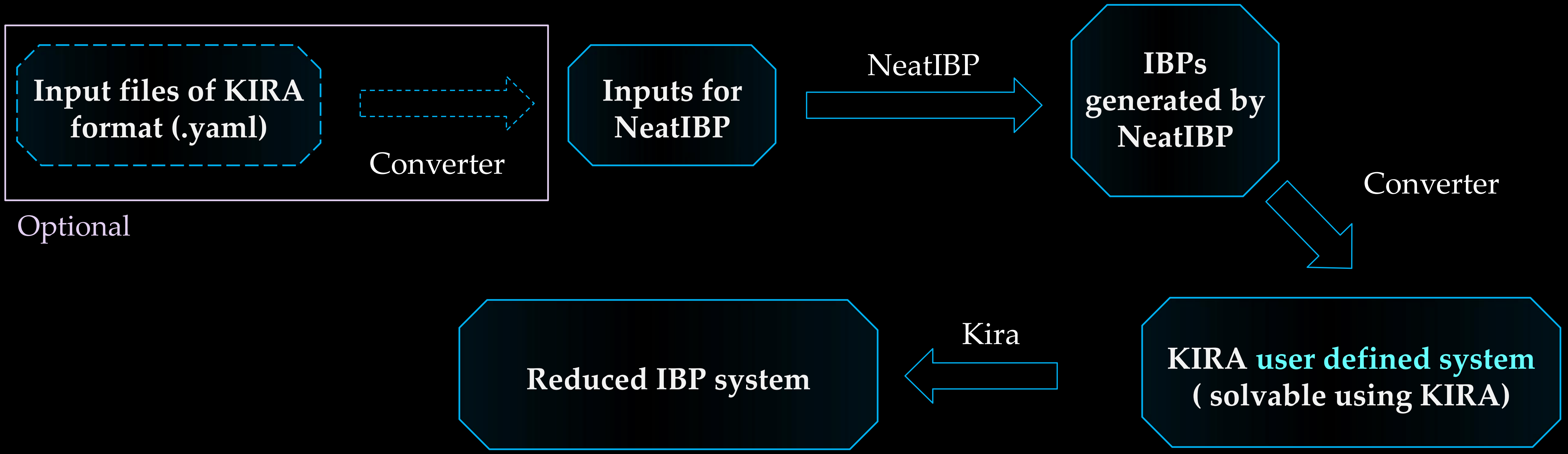



KIRA Comput.Phys.Commun. 230 (2018) 99-112, Comput.Phys.Commun. 266 (2021) 108024, etc.
as a popular integral reduction software, is a very nice IBP solver for NeatIBP output.

Kira allows user to feed in linear systems of IBP relations (called user defined system) and then reduce them .

The Kira+NeatIBP interface is included in NeatIBP 1.1

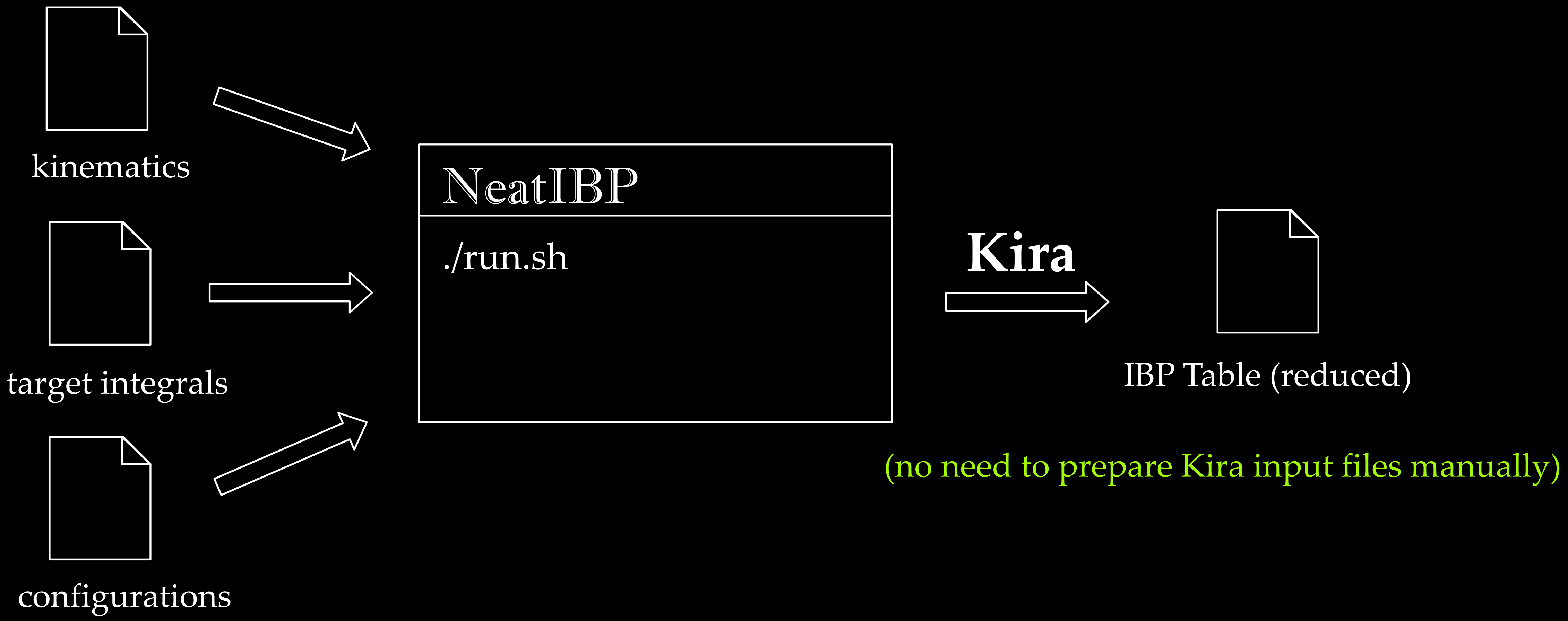
# NeatIBP 1.1 new feature: kira interface




In NeatIBP 1.1, to use Kira to reduce the IBP generated by NeatIBP, add the following commands into NeatIBP “config.txt”

```
PerformIBPReduction=True;
IBPReductionMethod=“Kira”;
KiraCommand=“/some/path/kira”
FermatPath=“/some/path/fer64”
```

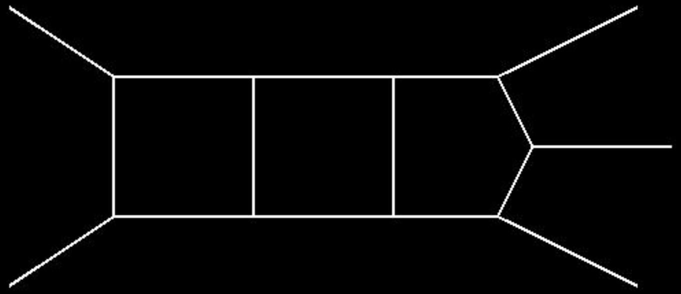
# I/O of NeatIBP with Kira interface



# NeatIBP Applications

# Applications of NeatIBP

## A three-loop five-point planar diagram



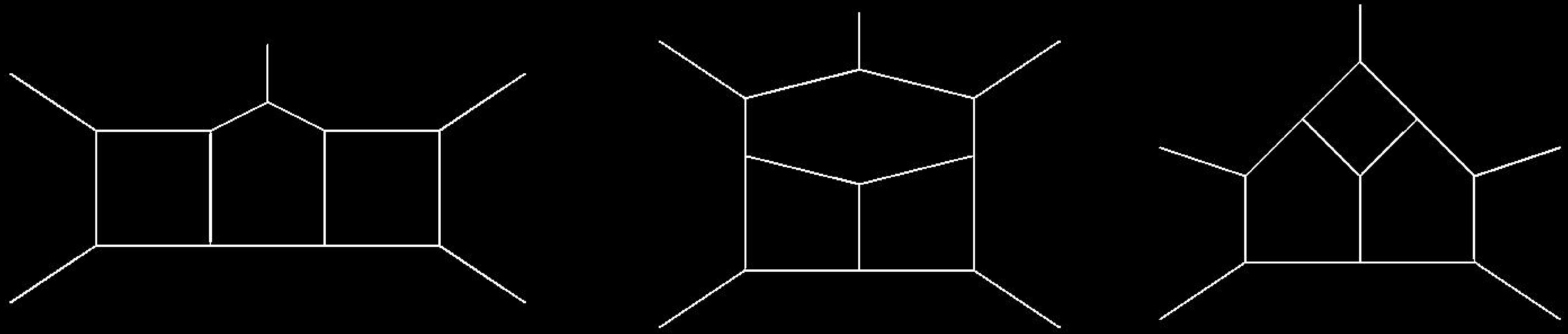
Yuanche Liu, Antonela Matijašić, Julian Miczajka, Yingxuan Xu, Yongqun Xu, Yang Zhang, Phys.Rev.D 112 (2025) 1, 016021

*"On a laptop, **NeatIBP** finds about 85000 IBP identities in full kinematics dependence within several hours. These relations are sufficient to derive the differential equation. As a comparison, standard IBP tools would generate three orders of magnitude more IBP relations, which make the reduction computation impossible."*

| Traditional (Laporta) IBPs | NeatIBP |
|---|---|
| ~ 1 0000 0000 IBPs | ~ 85000 IBPs |
| IBP reduction impossible | IBP reduction doable |

# Applications of NeatIBP

## More three-loop five-point planar diagrams



Dmitry Chicherin, Yu Wu, Zihao Wu, Yongqun Xu, Shun-Qing Zhang, Yang Zhang, 2512.17330

With the spanning cuts method developed in the latest version of NeatIBP (1.1), there are break throughs on the above diagrams.

# Applications of NeatIBP

## $W\gamma\ \gamma$ production at LHC

### Cross section measurements

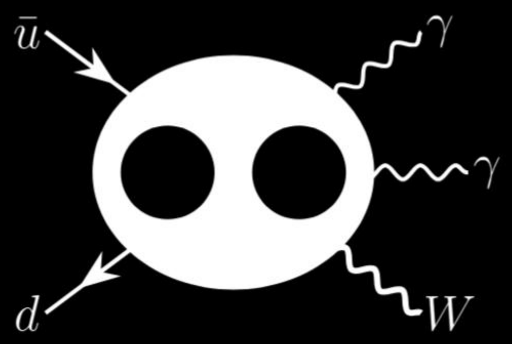
**ATLAS**

Phys. Rev. Lett. 115 (2015) 031802
Phys. Lett. B 848 (2024) 138400

**CMS**

JHEP 10 (2017) 072
JHEP 10 (2021) 174

### Latest amplitude progress: NNLO correction in QCD




Simon Badger, Heribertus Bayu Hartanto, ZW,
Yang Zhang and Simone Zoia, *JHEP* 03 (2025) 066

# Applications of NeatIBP

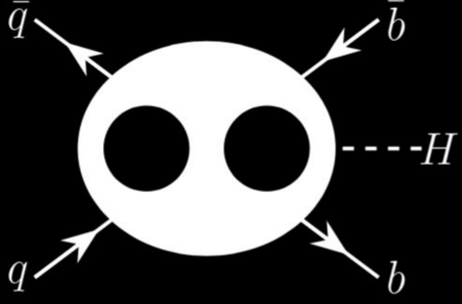
## $b\bar{b}H$ production at LHC

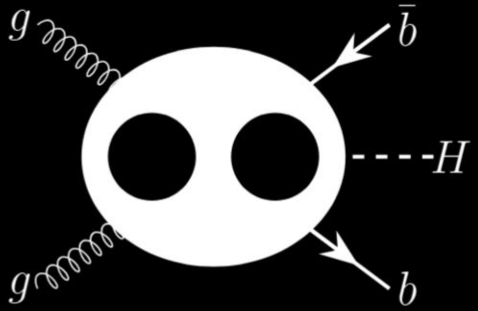
### Cross section measurements

**CMS**

Phys. Lett. B 860 (2025) 139173

### Latest amplitude progress: two-loop five-point amplitudes in 5FS

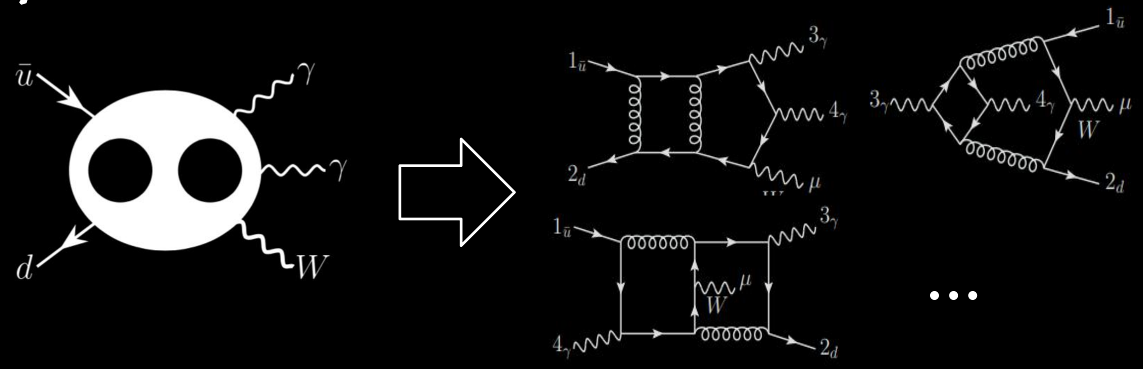


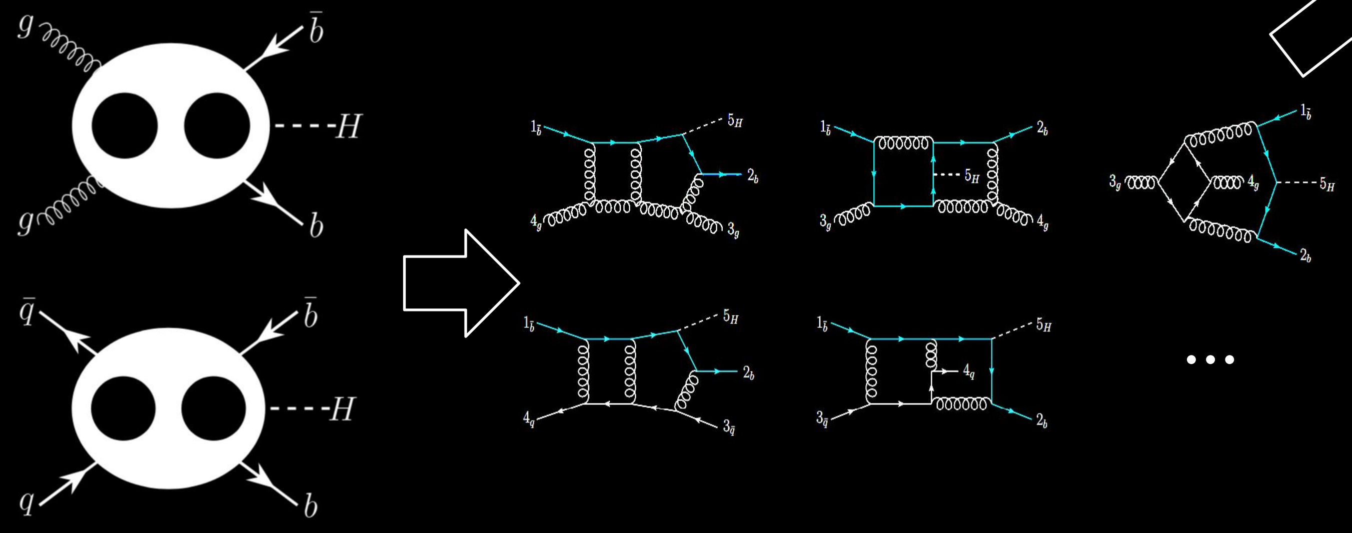

Simon Badger, Heribertus Bayu Hartanto, Rene Poncelet, ZW, Yang Zhang and Simone Zoia, *JHEP* 12 (2025) 221

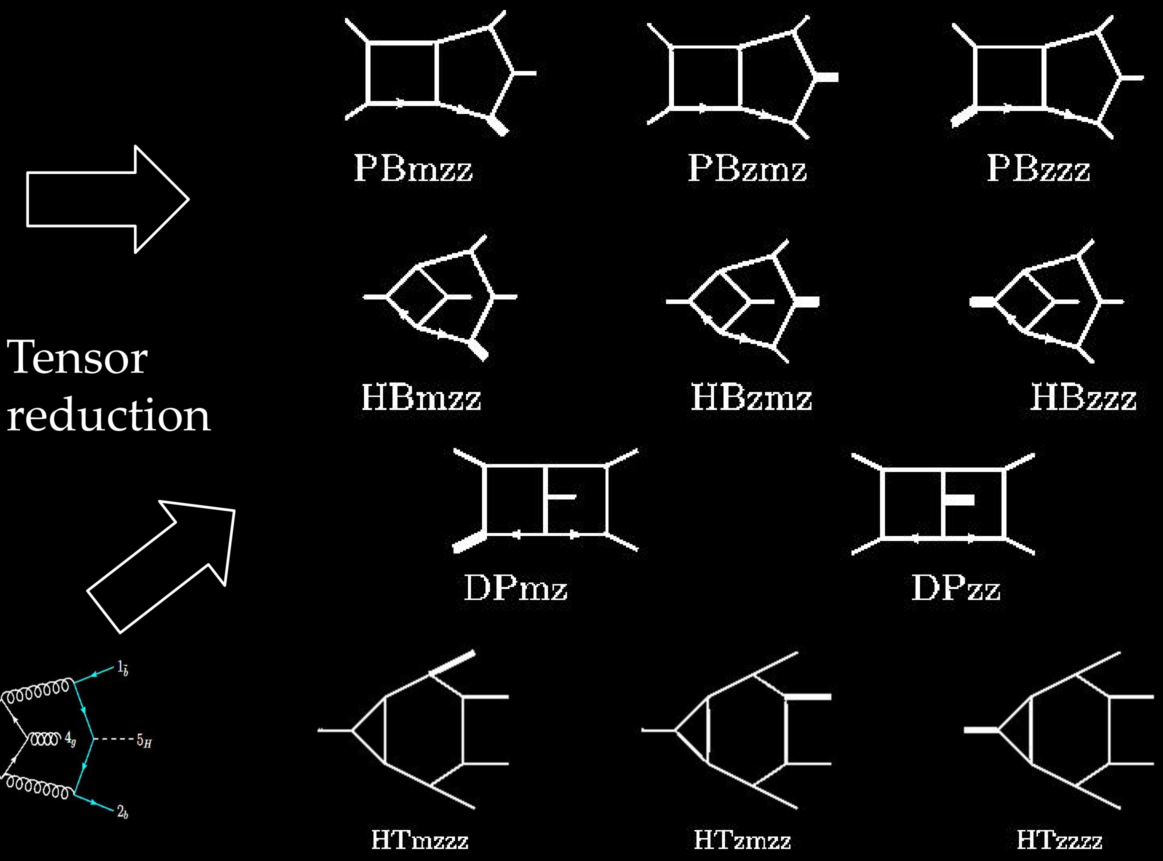
# Scalar integral families

## $W\gamma\gamma$

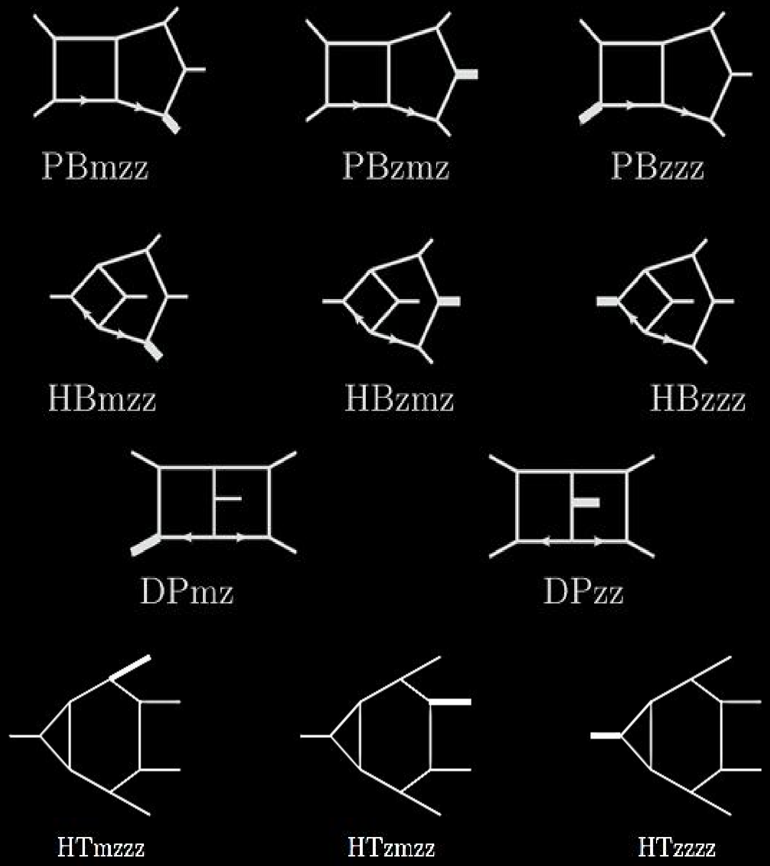



## $b\bar{b}H$



# Workflow and performance of the tools

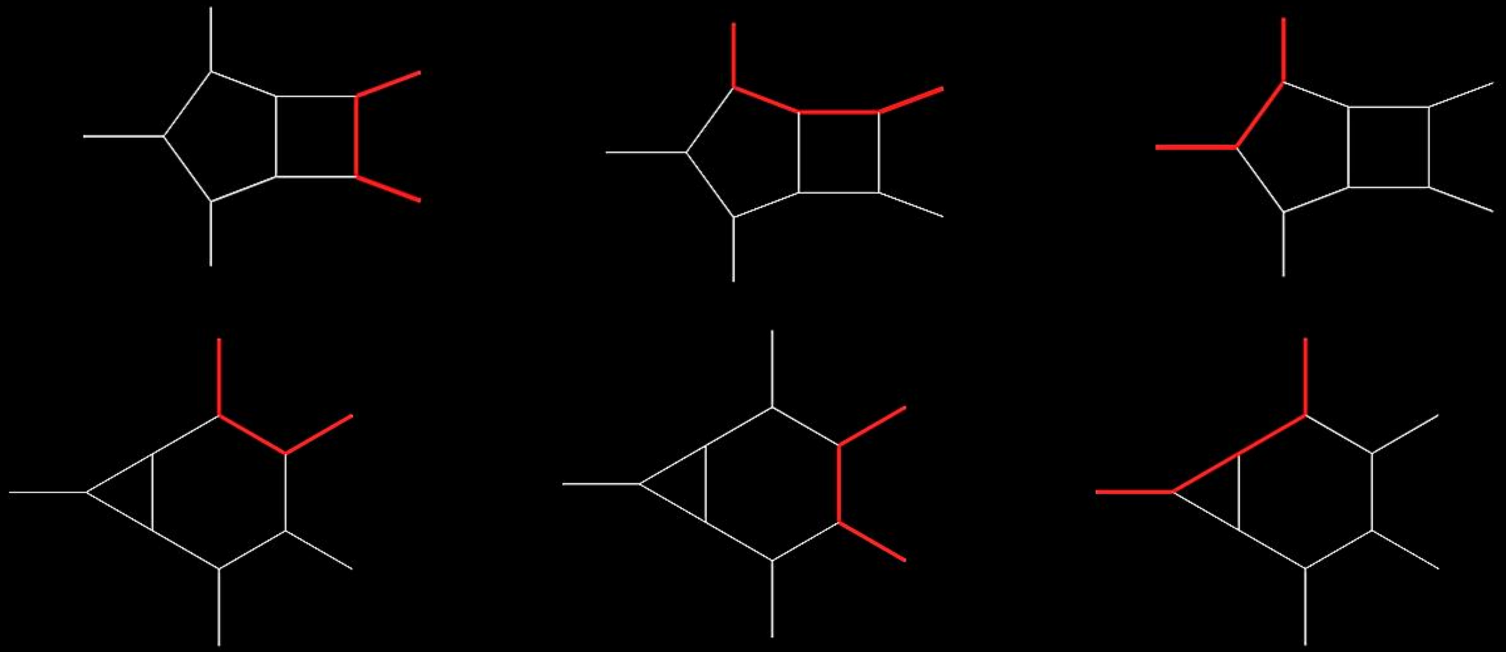



IBP systems from NeatIBP

| family | deg. | # IBPs | # integrals | IBP disk size | running time |
|---|---|---|---|---|---|
| DPmz | 5 | 26673 | 27432 | 71.4 MB | 7h5m |
| DPzz | 5 | 61777 | 63880 | 375.8 MB | 21h54m* |
| HBmzz | 5 | 15428 | 15916 | 17.0 MB | 5h38m |
| HBzmz | 5 | 10953 | 11289 | 13.4 MB | 5h45m |
| HBzzz | 5 | 21126 | 21766 | 38.0 MB | 9h32m |
| PBmzz | 5 | 10224 | 10329 | 7.8 MB | 2h23m |
| PBzmz | 5 | 11610 | 11791 | 6.5 MB | 2h50m |
| PBzzz | 5 | 8592 | 8752 | 5.8 MB | 2h50m |
| HTmzzz | 4 | 3120 | 3176 | 1.5 MB | 1h12m |
| HTzmzz | 4 | 6594 | 6650 | 2.5 MB | 1h31m |
| HTzzzz | 4 | 4680 | 4631 | 4.0 MB | 2h31m |

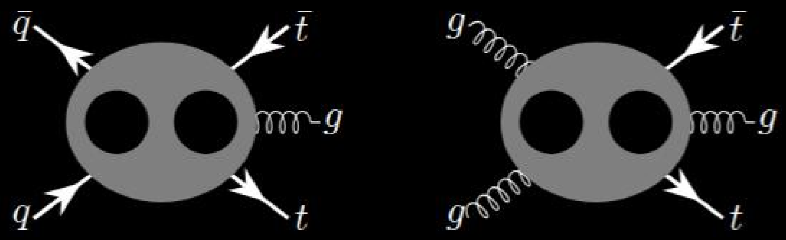
# Applications of NeatIBP

Two-loop five-point diagrams with massive propagators



Applications:

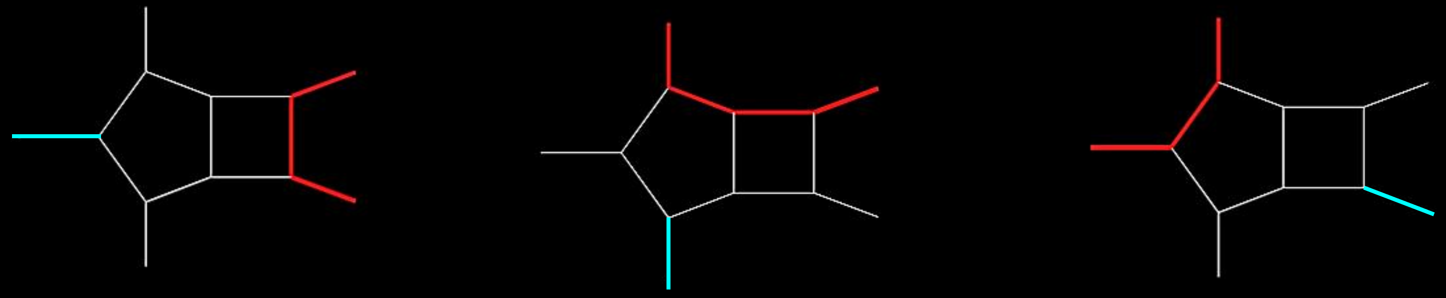
$t\,\bar{t}\,j$ production




JHEP 07 (2024) 073
SciPost Phys. 19 (2025) 6, 165
JHEP 03 (2025) 070
2511.11424
2411.10856

# Applications of NeatIBP

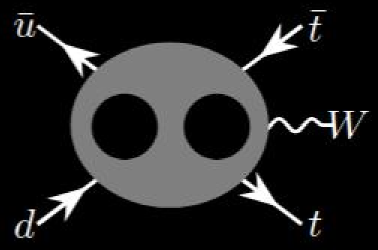
Two-loop five-point diagrams with massive propagators



Applications:

$t\,\bar{t}\;\;W$ production

JHEP 07 (2025) 001



# Applications of NeatIBP (1/2)

| Studies that used NeatIBP | Reference | Date |
|---|---|---|
| All-order structure of static gravitational interactions and the seventh post-Newtonian potential | arXiv: 2604.14134 | 2026.4.15 |
| Two-loop all-plus helicity amplitudes for self-dual Higgs boson with gluons | JHEP 03 (2026) 011 | 2026.3.2 |
| Three-loop helicity amplitudes of four-lepton scattering in QED | arXiv: 2602.10807 | 2026.2.11 |
| Differential equations for energy correlators in any angle | JHEP 02 (2026) 025 | 2026.2.2 |
| Algorithmic studies of IBP Equations | arXiv: 2512.05923 | 2025.12.5 |
| Three-loop five-point planar Feynman diagram | arXiv: 2512.17330 | 2025.12.19 |
| Nucleon mass in covariant baryon chiral perturbation theory at leading two-loop order | PoS HADRON2025 (2026) 014 | 2025.12.23 |
| Three-loop pentagonal Wilson loop with Lagrangian insertion | arXiv: 2512.17881 | 2025.12.19 |
| Six-loop gravitational interactions at the sixth post-Newtonian order | arXiv: 2512.19498 | 2025.12.22 |
| One-loop amplitudes for $t\bar{t}j$ and $t\bar{t}\gamma$ productions at the LHC through $O(\epsilon^2)$ | SciPost Phys. 19 (2025) 6, 165 | 2025.12.24 |
| Double virtual QCD corrections to $t\,\bar{t}\,j$ production at the LHC | arXiv: 2511.11424 | 2025.11.14 |
| $J/\psi$ polarization and $p_T$ distribution in $c\,\bar{c}$ associated hadroproduction | Phys.Rev.D 112 (2025) 7, L071501 | 2025.10.1 |

# Applications of NeatIBP (2/2)

| Studies that used NeatIBP | Reference | Date |
|---|---|---|
| Two-loop Feynman integrals for leading color $W\,t\,\bar{t}$ production | JHEP 07 (2025) 001 | 2025.7.1 |
| Two-loop QCD helicity amplitudes for $g\,g \to g\,t\,\bar{t}$ at leading color | JHEP 03 (2025) 070 | 2025.3.11 |
| Full-color double-virtual amplitudes for $q\,\bar{q} \to b\,\bar{b}\,H$ | JHEP 03 (2025) 066 | 2025.3.11 |
| Three-loop five-point pentagon-box-box Feynman diagram | Phys.Rev.D 112 (2025) 1, 016021 | 2024.11.27 |
| Two-loop QCD corrections for $p\,p \to t\,\bar{t}\,j$ | arXiv: 2411.10856 | 2024.11.16 |
| Two-loop amplitudes for $W\,\gamma\,\gamma$ production at LHC | JHEP 12 (2025) 221 | 2024.12.30 |
| NLO corrections to $J/\Psi\;c\,\bar{c}$ photoproduction | Phys.Rev.D 110 (2024) 9, 094047 | 2024.11.11 |
| Two-loop five-point two-mass planar integrals | JHEP 10 (2024) 167 | 2024.10.23 |
| Two-loop integrals for $t\,\bar{t}\,j$ production at hadron colliders in the leading color approximation | JHEP 07 (2024) 073 | 2024.7.9 |

# Advanced feature of NeatIBP: the spanning cuts method

# NeatIBP 1.1 new feature: the spanning cuts method

Generalized unitarity cuts in Baikov representation

$$I_{\alpha_1,\cdots,\alpha_n}\big|_{\mathcal{C}-\mathrm{cut}} \propto \oint_0 \prod_{i\in\mathcal{C}} \mathrm{d}z_i \int \prod_{i\notin\mathcal{C}} \mathrm{d}z_i P^{\alpha} \frac{1}{z_1^{\alpha_1}\cdots z_n^{\alpha_n}}$$

# NeatIBP 1.1 new feature: the spanning cuts method

Generalized unitarity cuts in Baikov representation

$$I_{\alpha_1,\cdots,\alpha_n}|_{\mathcal{C}-\text{cut}} \propto \oint_0 \prod_{i\in\mathcal{C}} \mathrm{d}z_i \int \prod_{i\notin\mathcal{C}} \mathrm{d}z_i P^{\alpha} \frac{1}{z_1^{\alpha_1}\cdots z_n^{\alpha_n}}$$

Cuts change the integrals but preserve IBP relations (conjecture)

$$\sum_i c_i I_i = 0 \longrightarrow \sum_i c_i (I_i|_{\mathcal{C}-\text{cut}}) = 0$$

Cuts increases the number of *zero sectors* in a family

$$\mathcal{C} \not\subseteq S \Rightarrow I|_{\mathcal{C}-\text{cut}} = 0, \forall I \in \text{Sector } S$$

## NeatIBP 1.1 new feature: the spanning cuts method

A spanning cuts $\{\mathcal{C}_i\}$ is defined so that for all nonzero sector $S$ in a family, $\exists \mathcal{C}_i$, such that $C_i \subseteq S$

```
SpanningCutsMode=True;
```

Step 1: Run NeatIBP on each cut

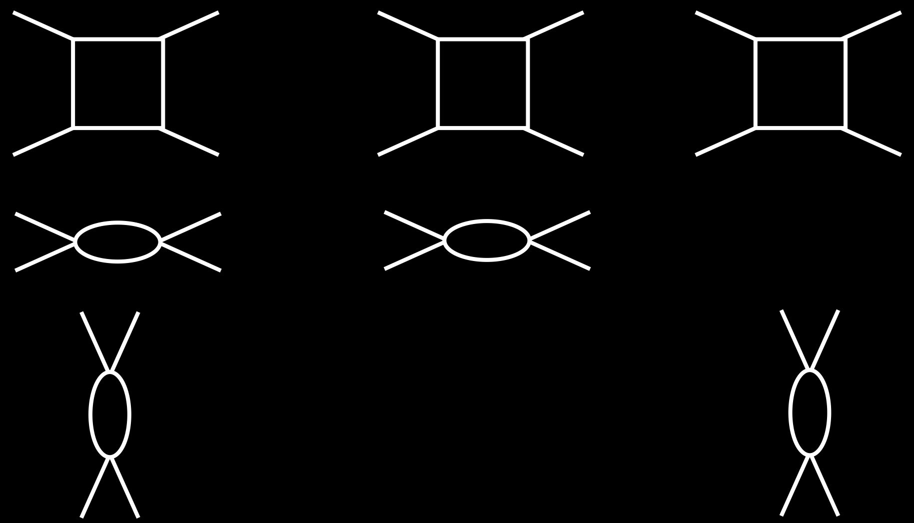
Current version of NeatIBP avoids cutting multiple propagators. Thus, for integrals such that $\alpha_i = 1, i \in \mathcal{C}$, $\mathcal{C}$-cut means

$$P \to P|_{z_i \to 0, i \in \mathcal{C}}$$

Step 2: Reduce the IBP system on each cut  Kira

Step 3: Merge the reduced IBP system of all cuts

{1,3} - cut {2,4} - cut



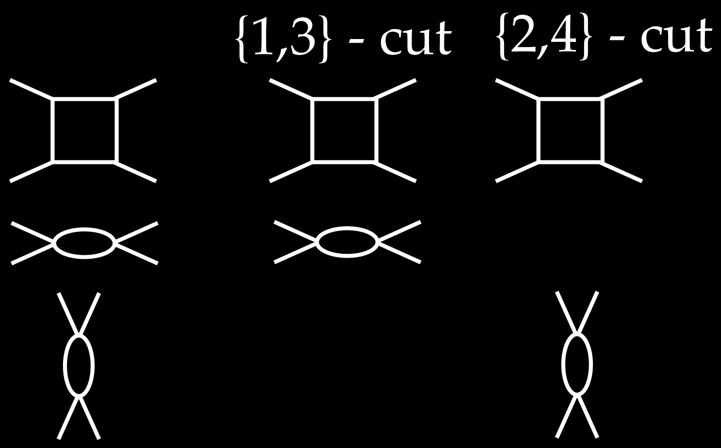
Benefit:

1. Split a difficult problem into simpler pieces
2. Parallelizable

# Merging the spanning cut systems

$$I|_{\{1,3\}\text{-cut}} = \sum_i c^{1234}_{\{1,3\},i} I^{1234}_i|_{\{1,3\}\text{-cut}} + \sum_i c^{13}_{\{1,3\},i} I^{13}_i|_{\{1,3\}\text{-cut}}$$

$$I|_{\{2,4\}\text{-cut}} = \sum_i c^{1234}_{\{2,4\},i} I^{1234}_i|_{\{2,4\}\text{-cut}} + \sum_i c^{24}_{\{2,4\},i} I^{24}_i|_{\{2,4\}\text{-cut}}$$




merge

$$I = \sum_i c^{1234}_i I^{1234}_i + \sum_i c^{13}_i I^{13}_i + \sum_i c^{24}_i I^{24}_i$$

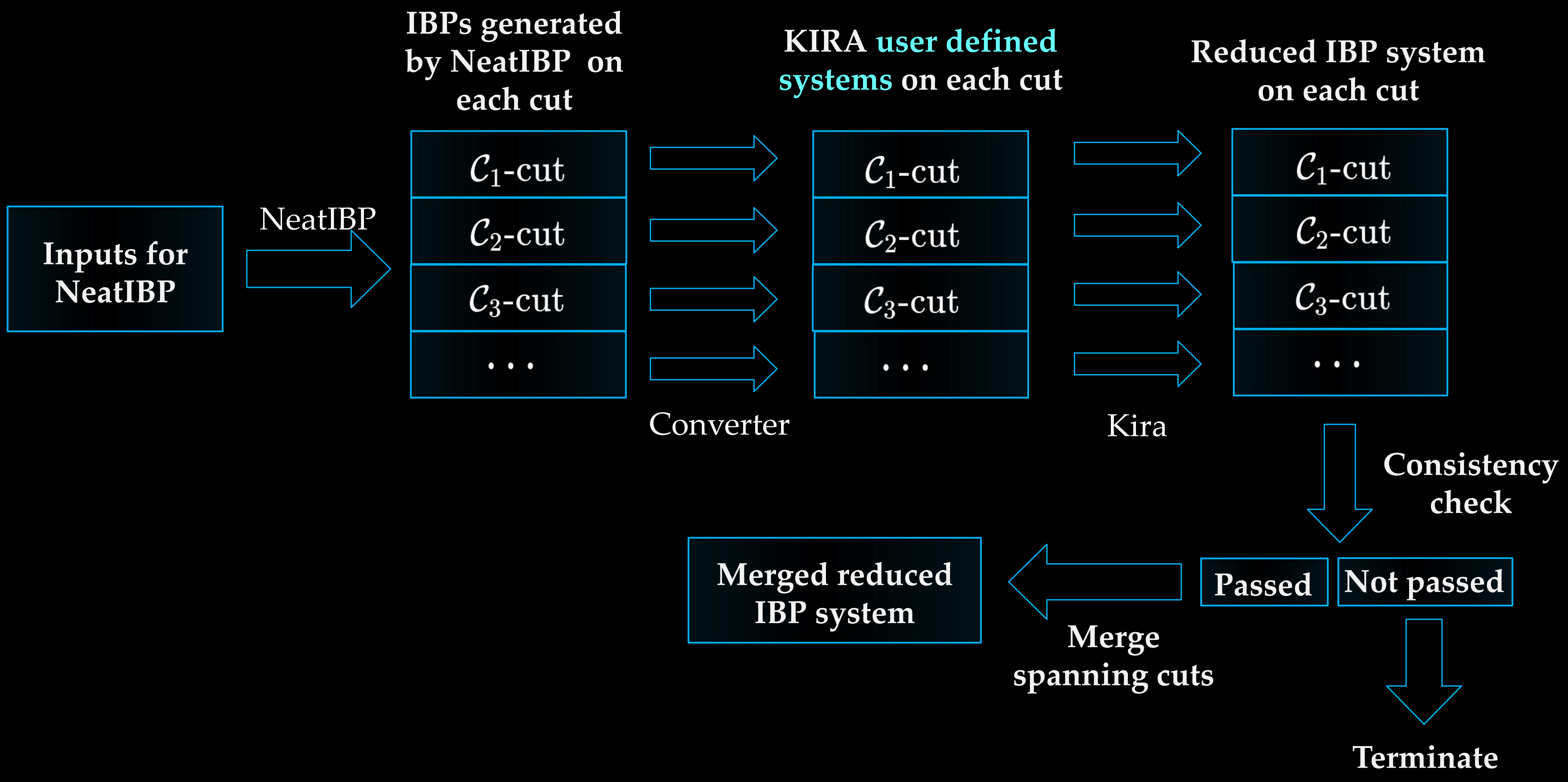
Currently, NeatIBP uses a "direct" merging strategy:

$$\begin{cases} c^{13}_i = c^{13}_{\{1,3\},i} \\ c^{24}_i = c^{24}_{\{2,4\},i} \\ c^{1234}_i = \boxed{c^{1234}_{\{1,3\},i} = c^{1234}_{\{2,4\},i}} \end{cases}$$

Consistency condition

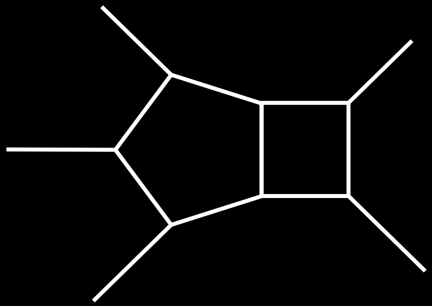
NeatIBP checks consistency conditions before merging the spanning cut systems. This step cannot be omitted because the condition is NOT always guaranteed according to our observation.

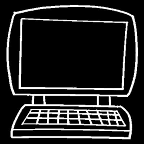
# The Kira+NeatIBP interface (spanning cuts mode)



# A baby example of Kira+NeatIBP performance (with spanning cuts)

Two-loop five-point, with numerator degree 3





CPU threads: 20
RAM: 128GB

Running NeatIBP + Kira in spanning cuts mode:

1. NeatIBP: Generates equations on 10 spanning cuts, using 5m
2. Number of equations: vary from 130 ~ 624, on different cuts
3. Kira running time: 10m (including converting, checking consistency, reducing and merging, in parallel mode)

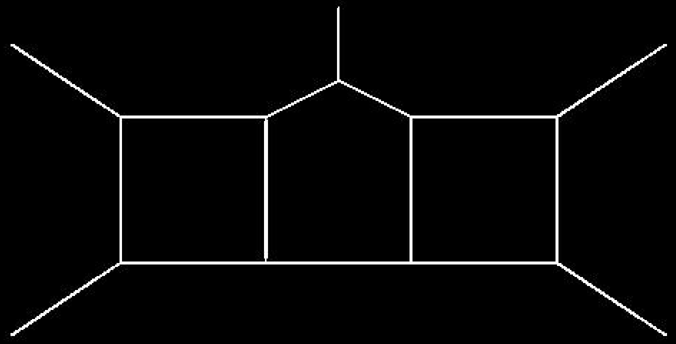
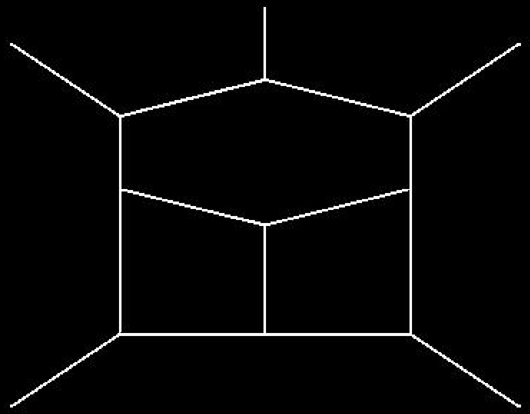
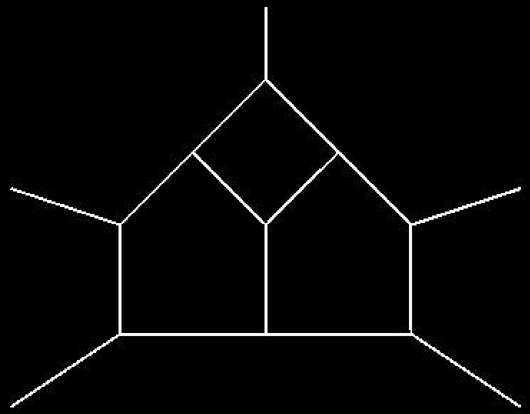
Running NeatIBP + Kira in normal mode:

1. NeatIBP: Generates 1358 equations spanning cuts, using 16m
2. Kira reduction time used: 27m

# The spanning cuts consistency problem

## Three-loop five-point planar diagrams

Dmitry Chicherin, Yu Wu, Zihao Wu, Yongqun Xu, Shun-Qing Zhang, Yang Zhang, 2512.17330

Rely on spanning cuts method, the consistency conditions holds

The importance of the spanning cuts method + its consistency problem = The problem must be solved

# An answer to the consistency problem

# Previous studies

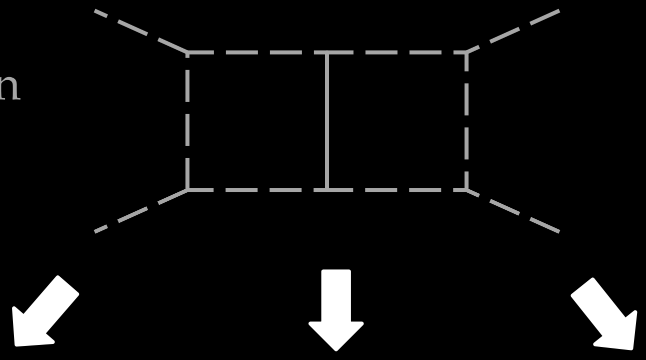
Observation: Magic relation spoils the spanning cuts method

Xin Guan, Xiao Liu, Yan-Qing Ma, Wen-Hao Wu, Comput.Phys.Commun. 310 (2025) 109538

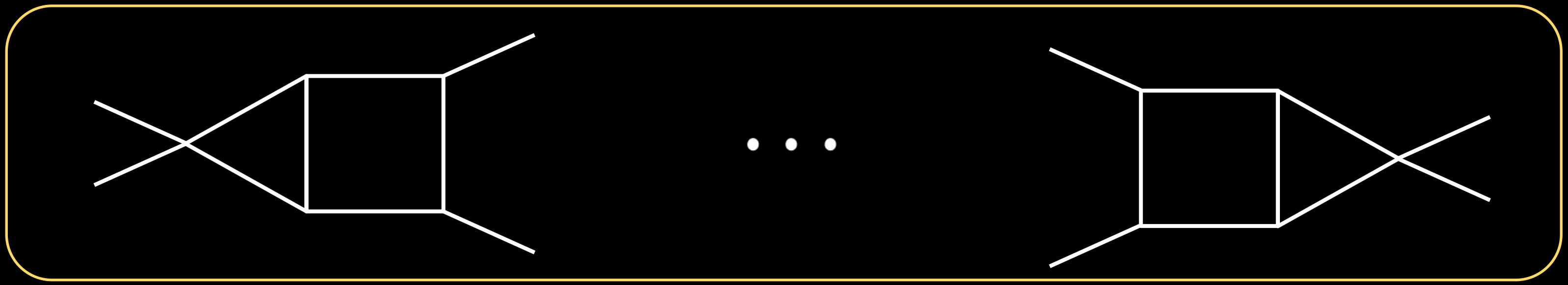
Giulio Crisanti, Hjalte Frellesvig, Andrzej Pokraka, Sid Smith, 2605.29789 (Magic relation leads to high dimensional critical variety)

Magic relations:

IBP relation generated in a sector



Contains integrals only in its subsectors



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## Question: why?

Cuts change the integrals but preserve IBP relations (conjecture)

Magic relations are IBP relations

$\Big\} \xrightarrow{?}$ Magic relations may be broken under cuts

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## Question: why?

Not always true! ZW, Yingxuan Xu, 2606.02485

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$\Big\} \longrightarrow$ Magic relations may be broken under cuts

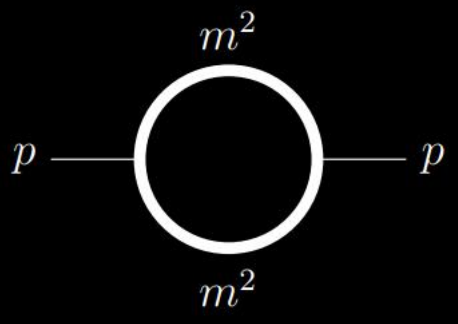
# A first look into the diagrams with inconsistency

ZW, Yingxuan Xu, 2606.02485

Key point:

Generalized unitarity cuts some times breaks the IBP relations, if hidden terms (see later) are erased

An example:




$$P_1 = l^2 - m^2,\ P_2 = (l+p)^2 - m^2. \qquad p^2 = 0.$$

$$I_{\alpha_1,\alpha_2} = \int \frac{\mathrm{d}^D l}{i\pi^{\frac{D}{2}}} \frac{1}{P_1^{\alpha_1} P_2^{\alpha_2}}.$$

One of the IBP relations:

$$0 = \int \frac{\mathrm{d}^D l}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial l^\mu} \frac{p^\mu}{P_1 P_2} = (I_{0,2} - I_{2,0})$$

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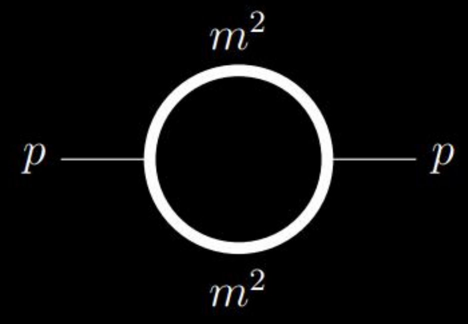


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One of the IBP relations:

Vanishes under {1}-cut

$$0 = \int \frac{\mathrm{d}^D l}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial l^\mu} \frac{p^\mu}{P_1 P_2} = (\cancel{I_{0,2}} - \underline{I_{2,0}})$$

=0 under {1}-cut?

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=0 under {1}-cut? No!

$$I_{2,0}\big|_{\{1\}\text{-cut}} = -2(m^2)^{\frac{D-4}{2}}\Gamma(2-\frac{D}{2}) \neq 0$$

The underlying reason: the hidden terms proportional to Feynman prescription parameters in the IBP relations were naively erased

The propagators with Feynman prescription parameters

$$D_k = P_k - i\eta_k, \quad \bar{D}_k = P_k + i\eta_k$$

{k}-cut:

$$D_k \to \ 2\pi i\delta(P_k) = \lim_{\eta_k \to 0^+} (\frac{1}{D_k} - \frac{1}{\bar{D}_k})$$

The IBP with Feynman prescription parameters

Finite

$$0 = \int \frac{\mathrm{d}^D l}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial l^\mu} \frac{p^\mu}{D_1 D_2} = (\mathcal{I}_{0,2} - \mathcal{I}_{2,0} - \underbrace{i(\eta_2 - \eta_1)\boxed{(\mathcal{I}_{1,2} + \mathcal{I}_{2,1})}}_{\text{Hidden terms, vanishes}})$$

Hidden terms, vanishes

Taking $\eta_1 \to -\eta_1$ to apply {1}-cut

Divergent $\sim \dfrac{1}{\eta_1 + \eta_2}$

$$0 = \int \frac{\mathrm{d}^D l}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial l^\mu} \frac{p^\mu}{\bar{D}_1 D_2} = (\mathcal{I}_{0,2}|_{\{1\}\text{-mir}} - \mathcal{I}_{2,0}|_{\{1\}\text{-mir}} - \underbrace{i(\eta_2 + \eta_1)\boxed{(\mathcal{I}_{1,2}|_{\{1\}\text{-mir}} + \mathcal{I}_{2,1}|_{\{1\}\text{-mir}})}}_{\text{Hidden terms, finite}})$$

Hidden terms, finite

where

$$f(\eta_1, \cdots, \eta_n)|_{s\text{-mir}} := f(\eta_1, \cdots, \eta_n)|_{\eta_k \to -\eta_k,\ \text{for all } k \in s}$$

$$I_{2,0}|_{\{1\}\text{-cut}} = -2(m^2)^{\frac{D-4}{2}} \Gamma(2 - \frac{D}{2}) \neq 0 \xleftarrow{\text{explains}} \boxed{-2(m^2)^{\frac{D-4}{2}} \Gamma(2 - \frac{D}{2}) \neq 0}$$

## The first answer to the question

Oh no, the spanning cuts seems to break this IBP relation …

$$I_{0,2} - I_{2,0} = 0 \quad \text{but} \quad I_{0,2}|_{\{1\}\text{-cut}} - I_{2,0}|_{\{1\}\text{-cut}} \neq 0 \quad \times$$

Don't worry, this IBP relation still holds under the cut if you keep the hidden terms.

$$0 = (\mathcal{I}_{0,2} - \mathcal{I}_{2,0} \overbrace{\boxed{- i(\eta_2 - \eta_1)(\mathcal{I}_{1,2} + \mathcal{I}_{2,1})}}^{=0})$$

$$- \qquad 0 = (\mathcal{I}_{0,2}|_{\{1\}\text{-mir}} - \mathcal{I}_{2,0}|_{\{1\}\text{-mir}} - i(\eta_2 + \eta_1)(\mathcal{I}_{1,2}|_{\{1\}\text{-mir}} + \mathcal{I}_{2,1}|_{\{1\}\text{-mir}}))$$

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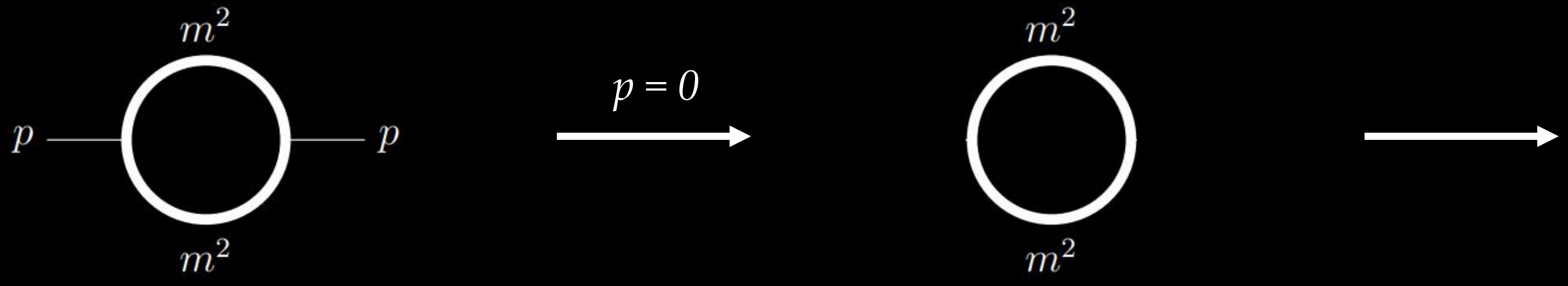
$$0 = I_{0,2}|_{\{1\}\text{-cut}} - I_{2,0}|_{\{1\}\text{-cut}} \underbrace{- i(\eta_2 + \eta_1)(\mathcal{I}_{1,2}|_{\{1\}\text{-mir}} + \mathcal{I}_{2,1}|_{\{1\}\text{-mir}}))}_{\text{Hidden terms}} \quad \checkmark$$

## Why singularity?

$$-i(\eta_2+\eta_1)\boxed{(\mathcal{I}_{1,2}|_{\{1\}\text{-mir}}+\mathcal{I}_{2,1}|_{\{1\}\text{-mir}})}\ \text{Divergent} \sim \frac{1}{\eta_1+\eta_2}$$

The external-massless bubble diagram is equivalent to a vacuum diagram




Hidden relation

$$\boxed{P_1 = P_2}$$

$$I_{\alpha_1,\alpha_2} = I_{\alpha_1+\alpha_2,0}$$

$$\lim_{\eta_1\to 0+}(\mathcal{I}_{\alpha_1,\alpha_2}|_{\{1\}\text{-cut}}) \propto \int \mathrm{d}^D l \frac{\delta^{(\alpha_1-1)}(P_1)}{(P_1-i\eta_2)^{\alpha_2}} = \int \mathrm{d}^D l \frac{\delta^{(\alpha_1-1)}(P_1)}{(P_1-i\eta_2)^{\alpha_2}} \propto \frac{1}{\eta_2^{\alpha_1+\alpha_2-1}}$$

## Hidden relations

In general, for a sector $S$, if the integral $I_{\alpha_1,\cdots,\alpha_n}$ on this sector is unchanged after we do the following replacement

$$\boxed{P_i = f(P_1,\cdots,P_{i-1},P_{i+1},\cdots,P_n)}\quad \text{Hidden relations}$$

in the integrand, where $f$ satisfies $f(0,0,\cdots,0) = 0$, cutting the propagators other than $P_i$ will introduce a divergence related to the Feynman prescriptions.

## Searching for linear hidden relations $P_i = \sum_{j=1, j\neq i}^{n} c_j P_j$

Feynman parameterization /
Lee-Pomeransky representation

$$\sum_{i=1}^{n} x_i P_i \longrightarrow \sum_{j=1, j\neq i}^{n} (x_j + c_j x_i) P_j \longrightarrow \Big(-\frac{\partial}{\partial x_i} + \sum_{j=1, j\neq i}^{n} c_j \frac{\partial}{\partial x_j}\Big) G = 0$$

Algorithm: Find the annihilating differential operator for LP polynomial:

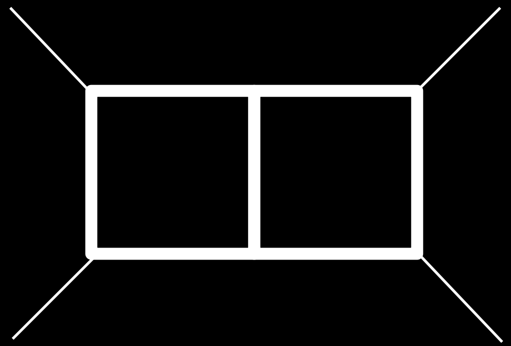
$$(\vec{v} \cdot \vec{\partial}) G = 0$$

It leads to hidden relation:

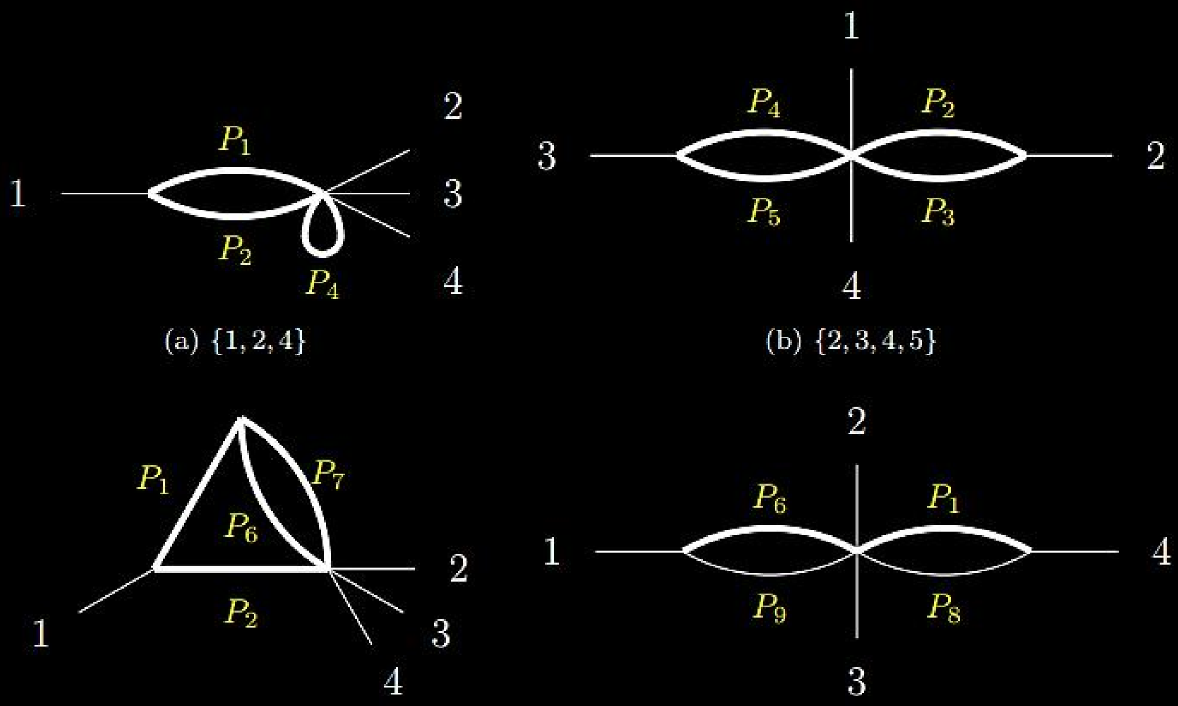
$$\vec{v} \cdot \vec{P} = 0$$

# Some examples of linear hidden relations

## Top sector: massive double box

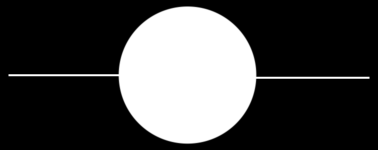


| sector | ann. diff. vector(s) | hidden relation(s) |
|---|---|---|
| {1,2,4} | $\partial_1 - \partial_2$ | $P_1 - P_2$ |
| {2,3,4,5} | $\partial_2 - \partial_3$, $\partial_5 - \partial_4$ | $P_2 - P_3$, $P_4 - P_5$ |
| {1,2,6,7} | $\partial_1 - \partial_2$ | $P_1 - P_2$ |
| {1,6,8,9} | $\partial_1 - \partial_6 - \partial_8 + \partial_9$ | $P_1 - P_6 - P_8 + P_9$ |



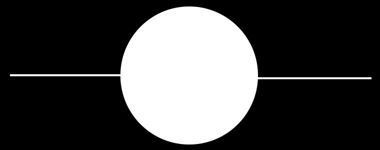

(a) $\{1, 2, 4\}$ (b) $\{2, 3, 4, 5\}$ (c) $\{1, 2, 6, 7\}$ (d) $\{1, 6, 8, 9\}$

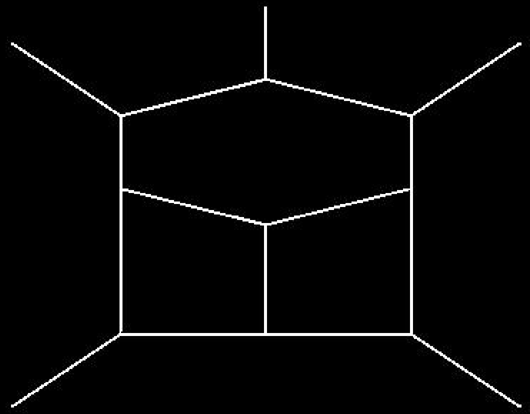
A typical feature: one external massless momentum with massive propagators



# The problematic feature

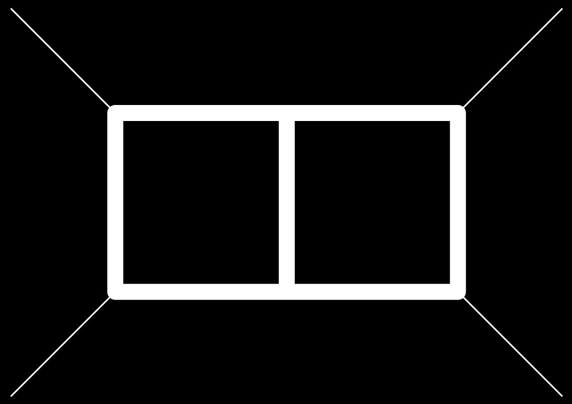
One external massless momentum with massive propagators





Safter: without internal massive propagators

(because the vacuum diagram vanishes)



Less safe: with internal massive propagators

# The problematic feature

One external massless momentum with massive propagators

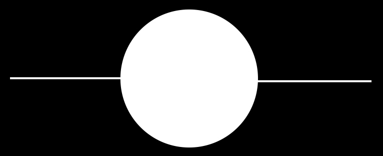
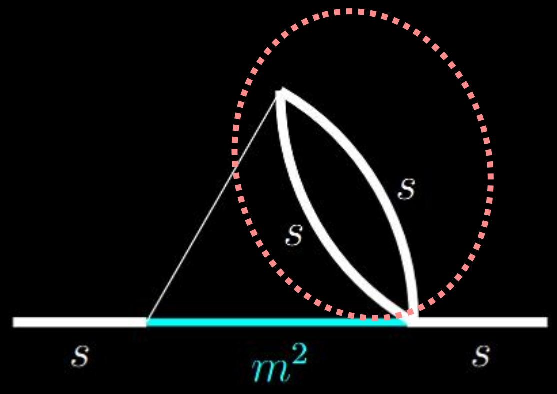
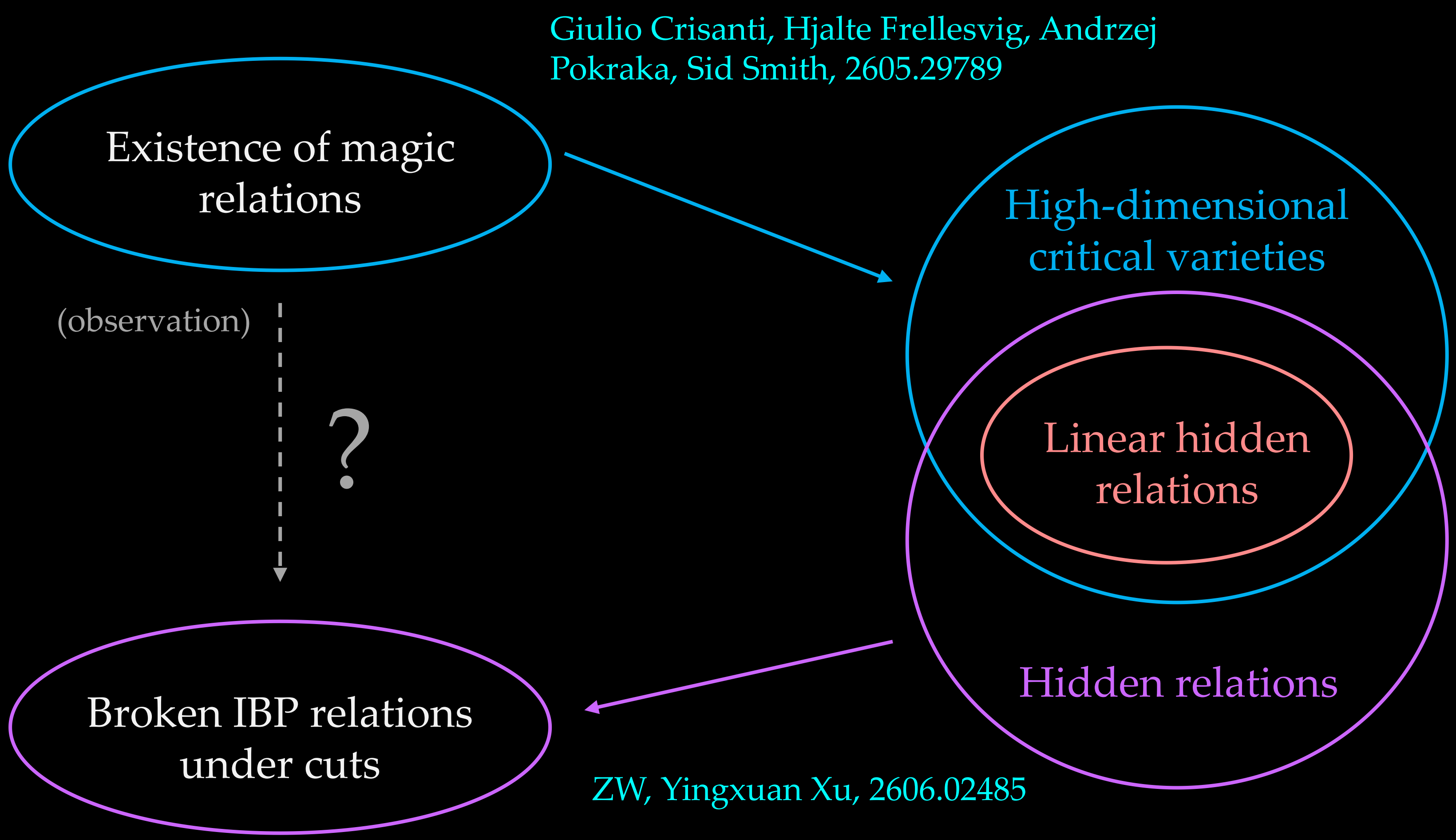


Diagram with such sub-structure, with consistency problem




No linear hidden relations found in this diagram

The underlying relation between hidden relations and magic relation and critical varieties…




The existence of linear hidden relation

$$(\vec{v}\cdot\vec{\partial})G = 0$$

indicates a higher dimensional (linear)critical variety

# Summary

**We aim at the computation of multi-loop Feynman integrals.**

For this purpose, we need to perform IBP reduction. This is a challenging work due to high reduction cost.

We introduced NeatIBP and its applications, aiming at breaking through the bottleneck problems concerning heavy IBP computation.

We introduced a study to the spanning cuts consistency problem.