Four-loop non-singlet splitting functions in QCD

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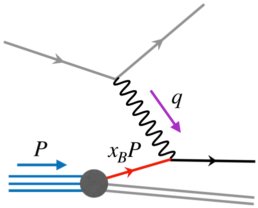
Parton densities and splitting functions



- Bjorken variable

$$x_B = \frac{-q^2}{2P \cdot q}$$

- Factorization

$$\sigma \sim \sum_a f_{a|N}(x_B) \otimes \hat{\sigma}_a(x_B)$$

- Splitting functions govern the DGLAP evolutions of PDFs

$$f_{i|N}(Q_0) \to f_{i|N}(Q) : \frac{df_{i|N}}{d\ln Q} = 2\sum_k P_{ik} \otimes f_{k|N}, \quad P_{ik} = a_s P_{ik}^{(0)} + a_s^2 P_{ik}^{(1)} + a_s^3 P_{ik}^{(2)} + a_s^4 P_{ik}^{(3)}$$

Mellin transform: $\gamma_{ij}(n) = -\int_0^1 dx\, x^{n-1} P_{ij}(x)$

Non-singlet & singlet splitting functions

- Non-singlet case
  - Flavor non-singlet

$$\frac{dT_i^{\pm}}{d\ln\mu} = 2P_{\text{ns}}^{\pm} \otimes T_i^{\pm}, \quad i = 3, 8, \cdots n_f^2 - 1$$

$$T_3^{\pm} = u^{\pm} - d^{\pm},\, T_8^{\pm} = u^{\pm} + d^{\pm} - 2s^{\pm}, \cdots, \quad q^{\pm} = q \pm \bar{q}$$

  - Valence non-singlet

$$\frac{dq^V}{d\ln\mu} = 2P_{\text{ns}}^V \otimes q^V, \quad P_{\text{ns}}^V = P_{\text{ns}}^- + P_{\text{ns}}^s, \quad q^V = \sum_{k=1}^{n_f} q_k^-$$

- Singlet case

$$\frac{d}{d\ln\mu}\begin{pmatrix} \Sigma \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix}, \quad \Sigma = \sum_{k=1}^{n_f} q_k^+$$

Formulation of DGLAP evolution equation

- Before existence of QCD (derive equation in QED and pseudoscalar Yukawa theory)

Deep inelastic e p scattering in perturbation theory #1

V.N. Gribov (St. Petersburg, INP), L.N. Lipatov (St. Petersburg, INP) (1972)

Published in: *Sov.J.Nucl.Phys.* 15 (1972) 438-450, *Yad.Fiz.* 15 (1972) 781-807

cite claim reference search 5,688 citations

- Preprint March 1977

Asymptotic Freedom in Parton Language #1

Guido Altarelli (Ecole Normale Superieure), G. Parisi (IHES, Bures-sur-Yvette) (Mar, 1977)

Published in: *Nucl.Phys.B* 126 (1977) 298-318

DOI cite claim reference search 8,846 citations

- Submit to journal in April 1977

Calculation of the Structure Functions for Deep Inelastic Scattering and e+ e- Annihilation by Perturbation Theory in Quantum Chromodynamics. #3

Yuri L. Dokshitzer (St. Petersburg, INP) (1977)

Published in: *Sov.Phys.JETP* 46 (1977) 641-653, *Zh.Eksp.Teor.Fiz.* 73 (1977) 1216-1240

cite claim reference search 90 citations

Motivation of $P_{ij}^{(3)}$: required by high-precision physics

- (HL-)LHC will produce precise data
- Several $\hat{\sigma}$ are available at N3LO $\rightarrow$ N3LO PDF $\leftarrow P_{ij}^{(3)}$
- Why properly matched PDFs matter?
  - Using an N^{3}LO $\hat{\sigma}$ with NNLO PDFs is inconsistent – how large is the error?

$$\Delta_{\mathrm{NNLO}}^{\mathrm{exact}} = \left| \frac{\sigma_{\mathrm{aN^3LO-PDF}}^{\mathrm{N^3LO}} - \sigma_{\mathrm{NNLO-PDF}}^{\mathrm{N^3LO}}}{\sigma_{\mathrm{aN^3LO-PDF}}^{\mathrm{N^3LO}}} \right|, \quad \Delta_{\mathrm{NNLO}}^{\mathrm{est}} = \frac{1}{2} \left| \frac{\sigma_{\mathrm{NNLO-PDF}}^{\mathrm{NNLO}} - \sigma_{\mathrm{NLO-PDF}}^{\mathrm{NNLO}}}{\sigma_{\mathrm{NNLO-PDF}}^{\mathrm{NNLO}}} \right|$$

  - Two aN^{3}LO PDF sets exist (MSHT20, NNPDF4.0) based on approximated $P_{ij}^{(3)}$

channel	NNLO	aN^{3}LO	shift
$gg \rightarrow H$	46.5 pb	44.9 pb	-3.3%
H VBF	4.35 pb	4.45 pb	$+2.3\%$
HV assoc.	$\sim$0.86 pb	$\sim$0.86 pb	$< 1\%$

$\Delta_{\mathrm{NNLO}}^{\mathrm{est}} = 0.9\%$ vs. 3.3% actual ($gg \rightarrow H$)

  - Naive estimate from the NLO$\rightarrow$NNLO step badly underestimates the true shift

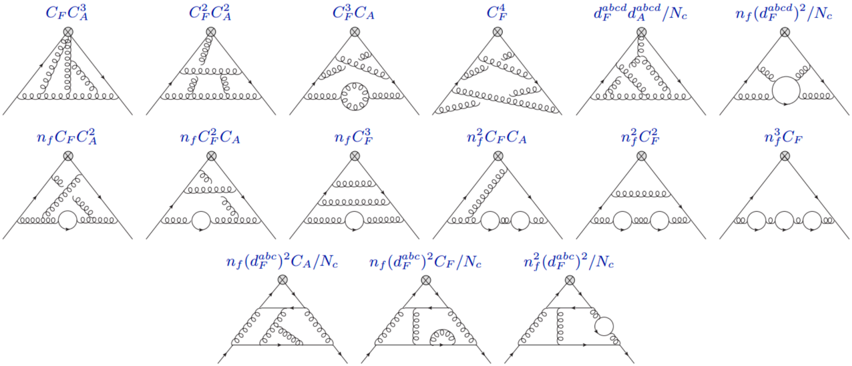
Why do we need exact $P_{ij}^{(3)}$?

- Eliminate any remaining uncertainties, promote aN3LO PDF $\to$ N3LO PDF
- Small-x divergent terms are difficult to predict (from BFKL)
- Especially $P_{gq}^{(3)}$ has more unknown small-x divergent terms
  - NNPDF assumes a Casimir-scaling relation linking $P_{gq}^{(3)}$ to $P_{gg}^{(3)}$; MSHT does not
  - Result: $\sim 2\sigma$ disagreement in $P_{gq}^{(3)} \Rightarrow$ gluon PDF differs by 3–4% for $10^{-3} \lesssim x \lesssim 10^{-1}$
  - Exact $P_{gq}^{(3)}$ is necessary to resolve the MSHT/NNPDF tension
- Theoretical interest:
  - To test the generalized Gribov-Lipatov reciprocity [Dokshitzer, Marchesini,Salam,06';Basso, Korchemsky, 06'; Mitov,Moch,Vogt,06'; Chen, TZY, Zhu, Zhu, 20']

$$2\gamma^S(n,\mu) = 2\gamma^T(n + 2\gamma^S(n,\mu),\mu)$$

  - To study new analytic structures

Sample Feynman diagrams



- We follow Wilson's OPE framework, compute operator matrix elements with an operator insertion: $O_q = \bar{\psi} \not{\Delta} (\Delta \cdot D)^{n-1} \psi$; $\mathcal{O}(16k)$ diagrams at four-loop order

Remarks from a competitor group

- Within one month, another group writes a paper to study our results

Properties and implications of the four-loop non-singlet splitting functions in QCD #2

S. Moch (Hamburg U., Inst. Theor. Phys. II), A. Vogt (Hamburg U., Inst. Theor. Phys. II and U. Liverpool (main)) (May 5, 2026)

e-Print: 2605.03889 [hep-ph]

Abstract

We have studied the recently completed analytic all-N expressions for the four-loop anomalous dimensions corresponding to the next-to-next-to-next-to-leading order splitting functions for the non-singlet quark distribution in perturbative QCD. The results agree with fixed-N values beyond those published so far. Their structural consistency with theoretical requirements is established.

1 Introduction

Recently, the penultimate milestone has been reached of research projects spanning more than a decade: the exact expressions have been completed of the fourth-order contributions to the splitting functions for the scale dependence (evolution) of the non-singlet quark distributions of hadrons [1].

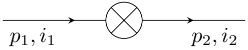
Acknowledgements

We compliment the authors of ref. [1] on their achievement.

[1] T. Gehrmann, A. von Manteuffel, V. Sotnikov and T. Yang, *The four-loop non-singlet splitting functions in QCD*, arXiv:2604.09534

Tracing parameter to retain all-n dependence

- Non-standard terms appearing in the Feynman rules
- Example: Feynman rules for O_q at lowest order



$$\to \not{\Delta}(\Delta \cdot p_1)^{n-1}$$

- Retain all-n dependence?
  - Sum non-standard terms into linear propagators by a tracing parameter t[Ablinger,Bluemlein,Hasselhuhn,Klein,Schneiderm,Wissbrock,12'],

$$(\Delta \cdot p)^{n-1} \to \sum_{n=1}^{\infty} t^n (\Delta \cdot p)^{n-1} = \frac{t}{1 - t\Delta \cdot p}, \quad \sum_{j=0}^{n-3} (\Delta \cdot p_1)^{n-3-j} (\Delta \cdot p_2)^j \to \cdots$$

  - IBP reduction with respect to linear propagator
  - Parameter-t space $\to$ n-space by re-expanding t to higher orders

Technical aspects

- Raw integrand
  - Raw integrand generated by QGRAF,FORM
  - 549 top topologies with 13 denominators
  - 3GB integrand with 3M integrals
- IBP reduction ($\mathcal{O}(6000)$ masters)[Chetyrkin, Tkachov 81', Laporta 00']
  - Finite field sampling $+$ function reconstruction[Manteuffel, Schabinger 14'; Peraro 15']
  - Custom optimization of IBP equation generation/selection[Driesse, Jakobsen, Mogul, Plefka, Sauer, Usovitsch 24'; Guan, Liu, Ma, Wu 24'; Bern, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng 24'; von Hippel, Wilhelm 25'; Song, TZY, Cao, Luo, Zhu 25'; Lange, Usovitsch, Wu 25']
  - Divide big system into smaller systems by spanning cuts/sectors[Larsen, Zhang 15'; Guan, Liu, Ma, Wu,24']
- Differential equation
  - Derive DE[Kotikov 91'; Gehrmann,Remiddi,99'] with respect to t

Master solution from series expansion of DE

- DE with respect to the tracing parameter t:

$$d\vec{I}(t,\epsilon)/dt = A(t,\epsilon)\vec{I}(t,\epsilon)$$

- Largest DE from a top topology: $\mathcal{O}(2000)$ masters, $\mathcal{O}(700\mathrm{MB})$ size
- Boundary condition: 28 four-loop self-energy master integrals in the $t \to 0$ limit [Baikov, Chetyrkin 10'; Lee, Smirnov, Smirnov 11']
  - Example: $\int \frac{1}{(1-t\Delta\cdot l_1)(1-t\Delta\cdot l_2)} \times \frac{1}{l_1^2(l_1-l_2)^2\cdots} \to \int \frac{1}{l_1^2(l_1-l_2)^2\cdots}$
- Derive recursion relations for each master

$$I(t,\epsilon) = \sum_{m=m_{\mathsf{min}}}^{m_{\mathsf{max}}} \sum_{n=0}^{\infty} a_{m,n}\epsilon^m t^n, \quad a_{m,n} = (\#1)a_{m,n-1} + (\#2)a_{m-1,n} + \cdots$$

- Compute each master to $\mathcal{O}(t^{4000})$ and enough ϵ order

Reconstruct A.D. from fixed moments

- Derive recursion relations for OME to obtain $\mathcal{O}(4000)$ moments
- Renormalization: $\langle q(p)|O_{\mathsf{ns}}|q(p)\rangle^{\mathsf{R}} = Z_\psi Z_{\mathsf{ns}} \langle q(p)|O_{\mathsf{ns}}|q(p)\rangle^{\mathsf{B}}$
- $Z_{\mathsf{ns}}^{(4)} = \frac{1}{4\epsilon}\gamma_{\mathsf{ns}}^{(3)} + \frac{1}{\epsilon^2}\left(-\frac{1}{4}\beta_0\gamma_{\mathsf{ns}}^{(2)} + \cdots\right) + \frac{1}{\epsilon^3}(\cdots) + \frac{1}{\epsilon^4}(\cdots)$
- Generic ansatz:

$$\gamma_{\mathsf{ns}}^{(3)}(n) = \sum_a \sum_{k=0}^{7} \sum_{w=0}^{7-k} c_{akw} D_a^k S_w(n)\,, \qquad D_a^k = (n+a)^{-k}$$

$S_w(n)$ is HSs of weight w

We choose: $a = \{0,1\}$ for $\gamma_{\mathsf{ns}}^{(3)\pm}$, $\{0,1,-1,2,-2\}$ for $\gamma_{\mathsf{ns}}^{(3)\,V}$

- Successfully reconstructed $\gamma_{\mathsf{ns}}^{(3)}$ with all-n dependence using $\mathcal{O}(4000)$ moments

Analytic results for $\gamma_{\mathrm{ns}}^{(3)}$

- Analytic results with all-n dependence via HSs to weight 7

$$\text{Example: } S_{-6,1}(n) = \sum_{\tau_2=1}^{n} \frac{(-1)^{\tau_2}}{\tau_2^6} \sum_{\tau_1=1}^{\tau_2} \frac{1}{\tau_1^1}$$

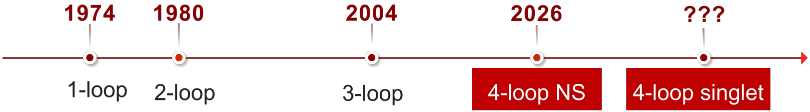
- Non-n_f terms for $\gamma_{\mathrm{ns}}^{(3)\pm}$, n_f terms for $\gamma_{\mathrm{ns}}^{(3)s}$ *New*

```
((1 + (-1)^n)*color[ca^3*cf]*(373793/648 + 2662/(27*n^4) - 50006/(81*n^3) +
   146482/(81*n^2) - 283147/(81*n) - 2662/(27*(1 + n)^4) +
   50006/(81*(1 + n)^3) - 146482/(81*(1 + n)^2) + 283147/(81*(1 + n)) -
   320*S[-7, n] + (-1600/3 - 832/(3*n) + 832/(3*(1 + n)))*S[-6, n] +
   (-2728/9 + 1000/(3*n^2) - 6920/(9*n) + 1000/(3*(1 + n)^2) +
     6920/(9*(1 + n)))*S[-5, n] + (2728/9 - 1000/(3*n^2) + 6920/(9*n) -
     1000/(3*(1 + n)^2) - 6920/(9*(1 + n)))*S[5, n] +
   (1600/3 + 832/(3*n) - 832/(3*(1 + n)))*S[6, n] + 320*S[7, n] +
   1216*S[-6, 1, n] - (4160*S[-5, -2, n])/3 +
   (1456/3 + 3808/(3*n) - 3808/(3*(1 + n)))*S[-5, 1, n] +
   (2048*S[-5, 2, n])/3 + 384*S[-4, -3, n] + (-368 - 1152/n + 1152/(1 + n))*
    S[-4, -2, n] + (-18320/9 + 6112/(3*n^2) - 1824/n + 6112/(3*(1 + n)^2) +
```

- Agree with partially known results
  - n_f terms for $\gamma_{\mathrm{ns}}^{(3)\pm}$ [Gehrmann,Manteuffel,Sotnikov,TZY,24';Kniehl, Moch, Velizhanin, Vogt 25']
  - $n \leq 16$ for $\gamma_{\mathrm{ns}}^{(3)}$ [Moch,Ruijl,Ueda,Vermaseren,Vogt,17']

Summary

- Computer algebra is heavily used, private code are used to tackle the bottleneck
- $\mathcal{O}(10\text{TB})$ intermediate data $\to$ $\mathcal{O}(100\text{KB})$ final analytic results, a reduction of 10^8
- Complete, exact results for $P_{\text{ns}}^{(3)}$ for the first time
- Results in the small-x limit
  - New quartic color structure $(d_F^{abcd} d_A^{abcd}/N_c)$ starts to contribute at a_s^4, deviation with predictions from resummation[Davies,Kom,Moch,Vogt,22']
- Timeline of splitting functions in QCD



  - Singlet: first step $\gamma_{qq} = \gamma_{\text{ns}}^{(3)+} + \gamma_{\text{ps}}^{(3)}$, result done, in preparation; $\gamma_{qg}^{(3)}, \gamma_{gq}^{(3)}, \gamma_{gg}^{(3)}$ in progress
- Our results will be used by MSHT/NNPDF/CTEQ in their next global fittings