Computation of Three-Loop Five-Point Feynman Integrals

Base on:

Dmitry Chicherin, Johannes Henn, Yongqun Xu, Shun-Qing Zhang, Yang Zhang [2512.17881]
Dmitry Chicherin, Yu Wu, Zihao Wu, Yongqun Xu, Shun-Qing Zhang, Yang Zhang [2512.17330]

and

Yuanche Liu, Antonela Matijašić, Julian Miczajka, Yingxuan Xu, Yongqun Xu, Yang Zhang [2411.18697]
(Phys. Rev. D 112, 016021)

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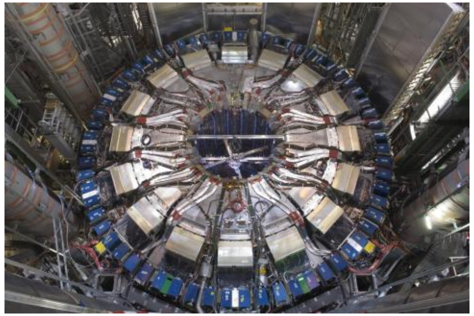
中国科学技术大学

中国物理学会高能物理分会第十二届全国会员代表大会暨学术年会

July 16, 2026

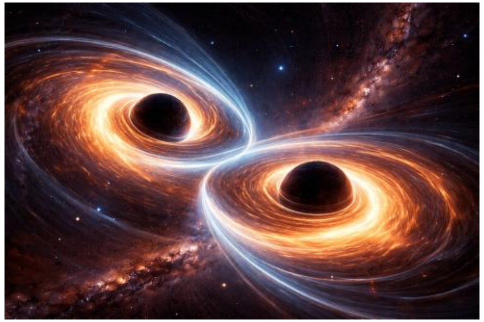
Background: Feynman Integrals — the basic building block of perturbative Quantum Field Theory



Collider
Phenomenology

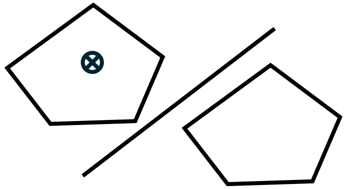
Scattering Amplitudes
Precision Physics

Future Collider:
HL-LHC CEPC,
FCC …

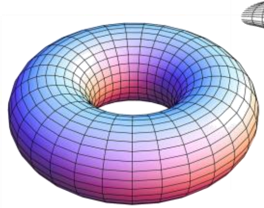


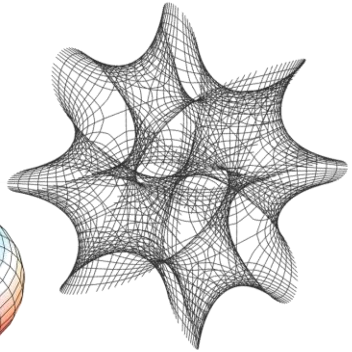
Gravitational Wave Physics

Understanding Structure of QFT



Feynman Integrals

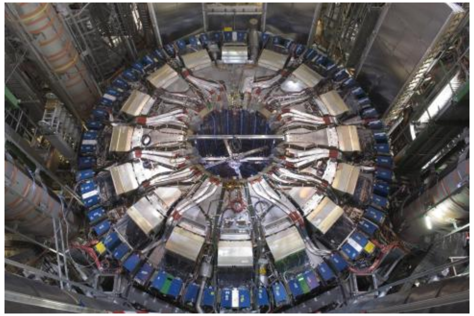




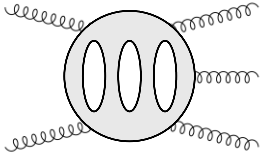
Mathematical Interest

[Henn'13] [Arkani-Hamed, Henn, Trnka '21] [Bree, Gasparotto, Matijašić, Mazloumi, Melnichenko, Pögel, Teschke, Wang, Weinzierl, Wu, Xu '25] [Driesse, Jakobsen, Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch '24] [Gehrmann, Henn, Lo Presti '18] [Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia '18] [Abreu, Page, Zeng '18] and many more..

Motivation: Massless Planar Three-Loop Five-Point Feynman Integrals In Dimensional Regularization

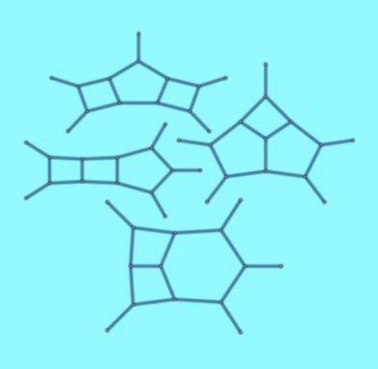


Collider Phenomenology



$\mathcal{A}_5^{(3)}(1,2,3,4,5)$

$$I_{\nu_1,\ldots,\nu_{18}}(\mathcal{N}) = e^{3\varepsilon\gamma_E}\int\prod_{j=1}^{3}\frac{\mathrm{d}^{4-2\varepsilon}\ell_j}{(i\pi)^{2-\varepsilon}}\frac{\mathcal{N}(\ell,k)}{\prod_{k=1}^{18}D_k^{\nu_k}}$$



[Liu, Matijašić, Miczajka, Xu, Xu, Zhang '24]
[Chicherin, Wu, Wu, Xu, Zhang, Zhang '25]

Ingredients for Scattering Amplitudes with three massless final states

$pp \to \gamma\gamma\gamma$ $\quad$ $pp \to \gamma jj$

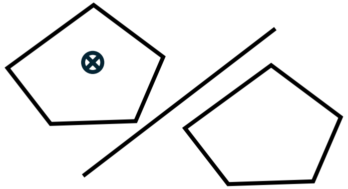
$pp \to \gamma\gamma j$ $\quad$ $pp \to jjj$

To reach 1% in LHC
we need NNLO

Prelude of various masses configuration e.g.

$pp \to t\bar{b}j$ $\quad$ $pp \to t\bar{t}j$

Understanding Structure of QFT



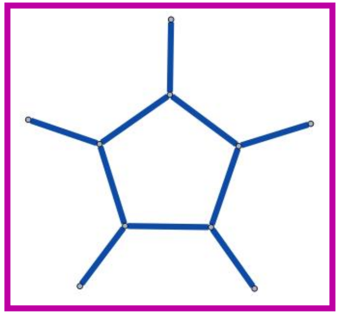
null Wilson loop with a Lagrangian insertion
(closely related to scattering amplitude)

[Chicherin, Henn, Xu, Zhang, Zhang '25]

History: Pentagon Feynman Integrals

Massless five-point integrals in dimensional regularization

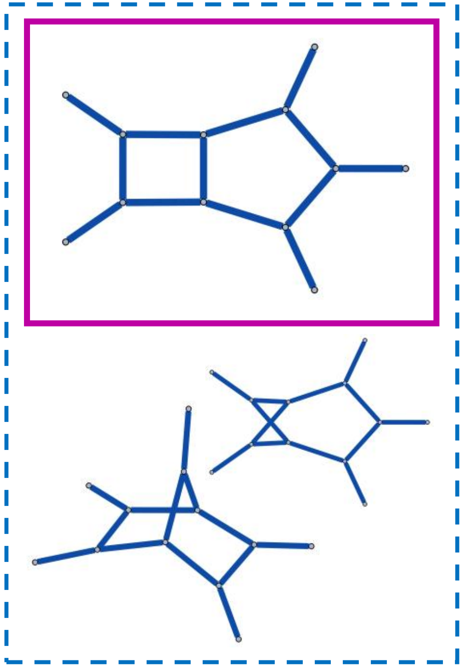
Planar Part



$s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

$\epsilon_5^2 = \mathrm{tr}(\gamma_5 \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4)^2$

[Bern, Dixon, Kosower ' 92]
[A. Kniehl, V. Tarasov '10]
[Gehrmann, Henn, Lo Presti '18]
[Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia '18]
[Abreu, Page, Zeng '18]
and so on…



Method of
Canonical Differential Equation
[Henn'13]

$$\mathrm{d}\mathbf{J} = \varepsilon \mathrm{d}\widetilde{A} \cdot \mathbf{J}$$

Function Space

↓ [Gehrmann, Henn, Lo Presti '18] [Chicherin, Sotnikov '20]

Scattering Amplitudes

[Abreu, Page, Pascual, Sotnikov '20]
[Chawdhry, Czakon, Mitov, Poncelet '21]
[Abreu, Febres-Cordero, Ita, Page, Sotnikov '21]
[Agarwal, Buccioni, von Manteuffel, Tancredi '21] $\times 2$
[A. Chawdhry, Czakon, Mitov, Poncelet '21]
[Badger, Brønnum-Hansen, Chicherin, Gehrmann, Bayu Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21]

↓

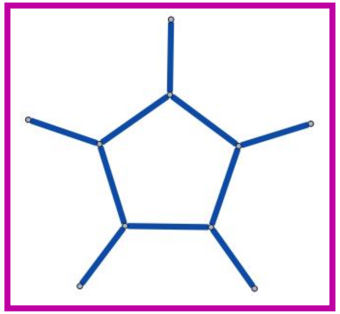
Cross Sections

[Kallweit, Sotnikov, Wiesemann '20]
[Chawdhry, Czakon, Mitov, Poncelet '20]
[Chawdhry, Czakon, Mitov, Poncelet '21]
[Badger, Gehrmann, Marcoli, Moodie '21]
[Czakon, Mitov, Poncelet '21]
[Badger, Czakon, Bayu Hartanto, Moodie, Peraro, Poncelet, Zoia '21]
[Chen, Gehrmann, Glover, Huss, Marcoli '22]

History: Pentagon Feynman Integrals

Massless five-point integrals in dimensional regularization

Planar Part



$$s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$$
$$\epsilon_5^2 = \mathrm{tr}(\gamma_5 \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4)^2$$

[Bern, Dixon, Kosower ' 92]
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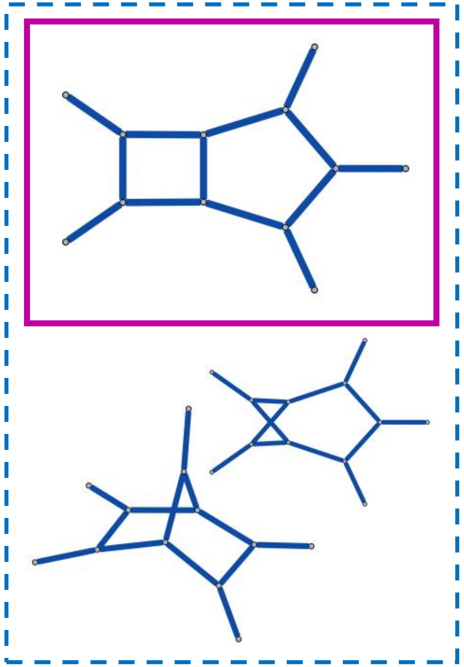
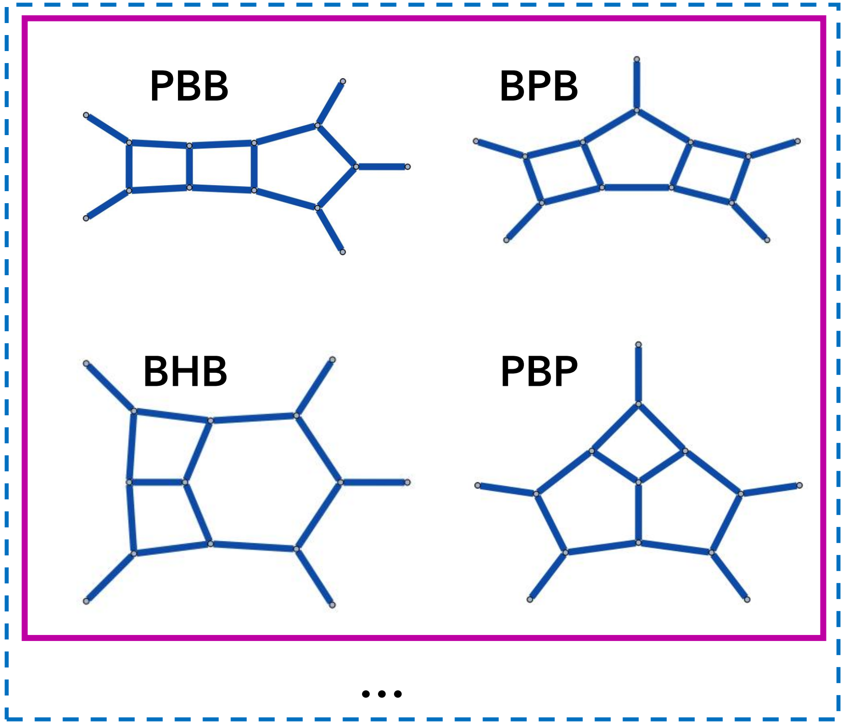
and so on…

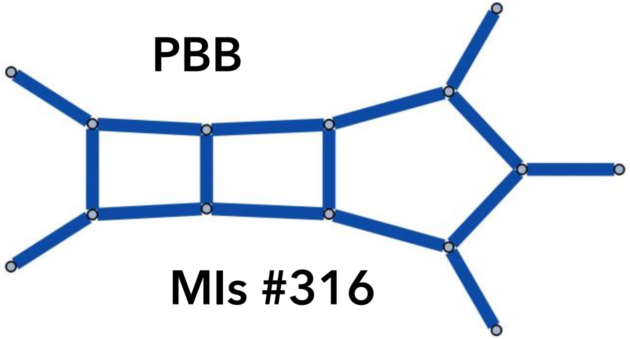


Method of
Canonical Differential Equation
[Henn'13] $\mathrm{d}\mathbf{J} = \varepsilon \mathrm{d}\widetilde{A} \cdot \mathbf{J}$



[Liu, Matijašić, Miczajka, Xu, Xu, Zhang '24]
[Chicherin, Wu, Wu, Xu, Zhang, Zhang '25]

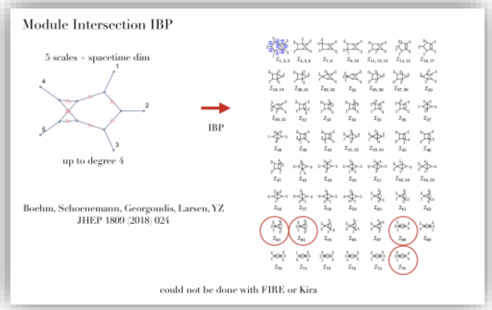
Integration-By-Parts (IBP) Reduction The current bottleneck



[Liu, Matijašić, Miczajka, Xu, Xu, Zhang '24]

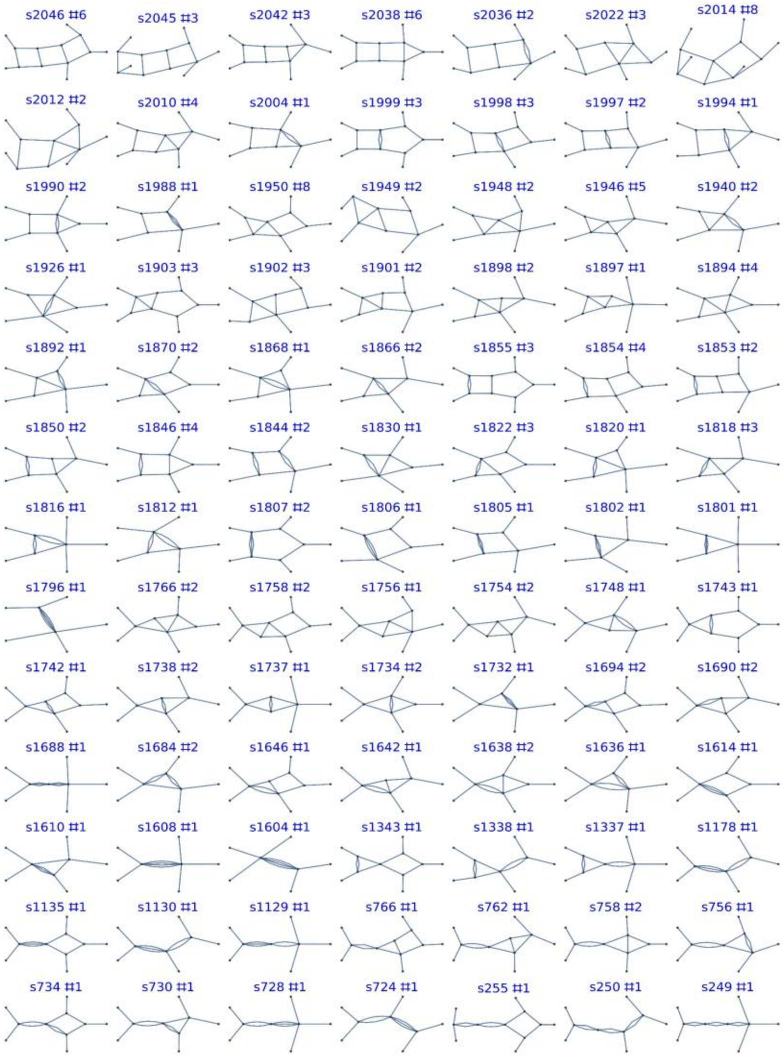
NeatIBP

~85,000 IBPs



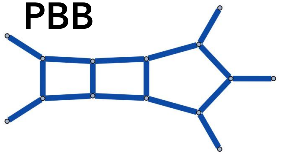
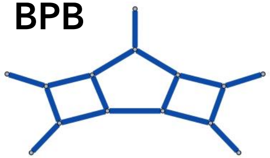
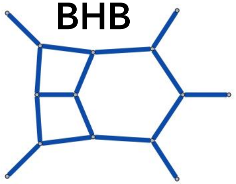
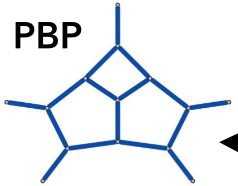
Similar Stories at Two Loop. Picture from Yang Zhang

👉See also: talk by Bo Feng
talk by Zihao Wu



Could not be done with FIRE or Kira(2.0)

Algebraic Geometry Accelerating IBP

Family	PBB 	BPB 	BHB 	PBP 
Master Integrals	316	367	431	734
Five-Point Integrals	120	164	169	342
# IBPs	~85,000	~183,000	Spanning Cuts	

Degree 4 /
Degree 3 + Dot 1

PRD Editor's Suggestion
[Liu, Matijašić, Miczajka, Xu, Xu, Zhang '24]

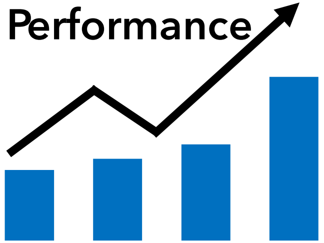
[Chicherin, Wu, Wu, Xu, Zhang, Zhang '25]

👉See also:
talk by Zihao Wu

NeatIBP
Powered by 1.1.0.3
[Wu, Böhm, Ma, Xu, Zhang '23]
[Wu, Böhm, Ma, Usovitsch, Xu, Zhang '25]

Loop ++
Leg ++
Scale ++

Performance

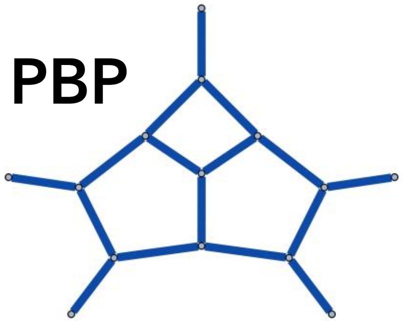


PBP: IBP Reduction & Differential Equation

Optimized Differential Operator $\frac{\partial}{\partial s_{ij}} = \beta_{ij}^{k\mu} \frac{\partial}{\partial p_k^\mu} \sim \frac{1}{D_i} + \text{Numerator}$ [Abreu, Page, Zeng '18]

$i \in \{1, 7, 9\}$

PBP



734 master integrals

Top sector MIs #19

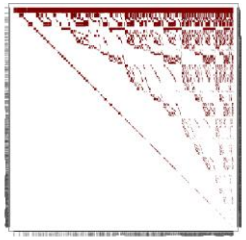
Reduce ~ 6800 targets in ~ 2 days with 4×50 CPUs

by Spanning Cuts

Degree 4 / Degree 3 + Dot 1

Semi-Analytically Reduction,
keep one variable at one time

Four dimensionless variables: $x_i = \left\{ \frac{s_{12}}{s_{45}}, \frac{s_{23}}{s_{45}}, \frac{s_{34}}{s_{45}}, \frac{s_{15}}{s_{45}} \right\}$



$\left(n_1, \frac{s_{23}}{s_{45}}, n_3, n_5\right)$

734×734

NeatIBP

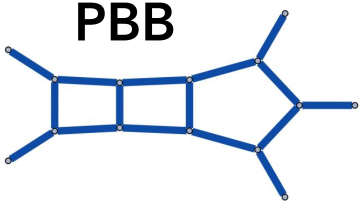
Powered by 1.1.0.3

[Wu, Böhm, Ma, Xu, Zhang '23]
[Wu, Böhm, Ma, Usovitsch, Xu, Zhang '25]

Pure Integrals: Leading Singularities Analysis

Sample expressions..

PBB

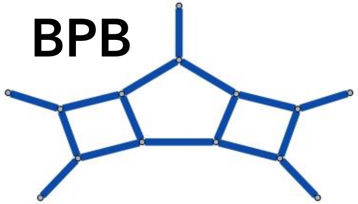


$$\mathcal{N}_1 = s_{12}s_{23}s_{45}^2(\ell_1+k_5)^2$$

$$\mathcal{N}_3 = \frac{s_{45}^2}{\epsilon_5}G(\ell_1,k_1,k_2,k_3,k_4)$$

$$\mathcal{N}_4 = \frac{s_{45}^2}{\epsilon_5}G\begin{pmatrix}\ell_1,k_1,k_2,k_3,k_4\\ \ell_2,k_1,k_2,k_3,k_4\end{pmatrix}$$

BPB

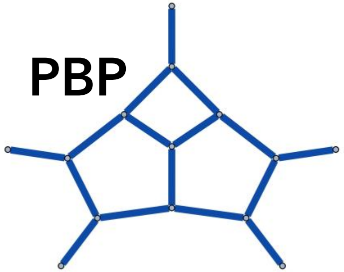


$$\mathcal{N}_1 = s_{12}s_{34}s_{45}^2(\ell_2-k_1)^2$$

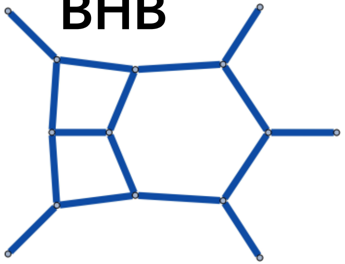
$$\mathcal{N}_5 = \frac{s_{45}s_{12}}{\epsilon_5}G(\ell_2,k_1,k_2,k_3,k_4)$$

$$\mathcal{N}_6 = \frac{s_{45}s_{12}}{\epsilon_5}G\begin{pmatrix}\ell_1,k_1,k_2,k_3,k_4\\ \ell_2,k_1,k_2,k_3,k_4\end{pmatrix}$$

PBP



BHB



$$\begin{aligned}\mathcal{N}_3 = &-2\frac{s_{12}s_{15}s_{23}s_{34}s_{45}}{\epsilon_5}s_{15}\ell_2^2\\
&+\frac{s_{23}s_{34}\left(s_{12}s_{15}+s_{45}s_{15}-s_{12}s_{23}+s_{23}s_{34}-s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^4\\
&+\frac{s_{34}s_{45}\left(s_{12}s_{15}-s_{45}s_{15}+s_{12}s_{23}-s_{23}s_{34}+s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^2\left(\ell_2-k_1\right)^2\\
&-\frac{s_{15}s_{45}\left(s_{12}s_{15}-s_{45}s_{15}-s_{12}s_{23}-s_{23}s_{34}+s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^2\left(\ell_2-k_1-k_2\right)^2\\
&+\frac{s_{12}s_{15}\left(s_{12}s_{15}-s_{45}s_{15}-s_{12}s_{23}+s_{23}s_{34}+s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^2\left(\ell_2-k_1-k_2-k_3\right)^2\\
&-\frac{s_{12}s_{23}\left(s_{12}s_{15}-s_{45}s_{15}-s_{12}s_{23}+s_{23}s_{34}-s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^2\left(\ell_2-k_1-k_2-k_3-k_4\right)^2\end{aligned}$$

$$\mathcal{N}_6 = \frac{\varepsilon}{1+2\varepsilon}\frac{s_{12}-s_{45}}{\epsilon_5}G\begin{pmatrix}\ell_1,\ell_2,k_1,k_2,k_3,k_4\\ \ell_1,\ell_2,k_1,k_2,k_3,k_4\end{pmatrix}$$

By leading singularities analysis in d-dimensional Baikov Rep.

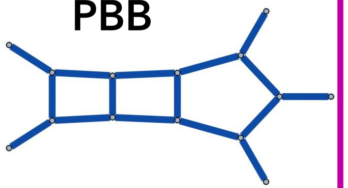
3L4P+1 off-shell leg from
[Di Vita, and Mastrolia, and Schubert, and Yundin '14]
[Henn, Lim, Bobadila'23]

DlogBasis [Henn, Mistlberger, Smirnov, Wasser '20]

Many thanks to my incredible collaborators!

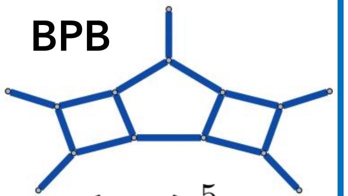
Canonical Differential Equation & Alphabet

PBB



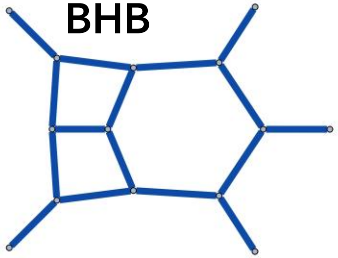
$\mathbb{A}_{\text{pl}}^{\text{2-loop}}$

BPB

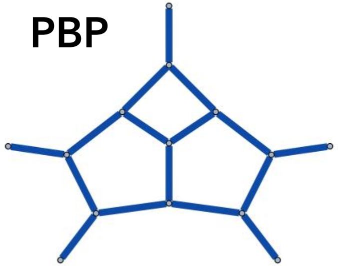


$\left\{\widehat{W}_i\right\}_{i=1}^{5}$

BHB



PBP



$\left\{\widetilde{W}_i\right\}_{i=16}^{20} \cup \left\{\widetilde{W}_i\right\}_{i=41}^{55} \cup \left\{\widetilde{W}_i\right\}_{i=76}^{80}$

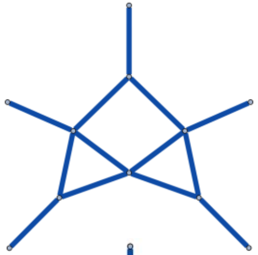
56 letters = 26 at two-loop +30 more at three-loop

Conjectured in [Chicherin, Henn, Trnka, Zhang '24]

Letters: DNA of Feynman integrals

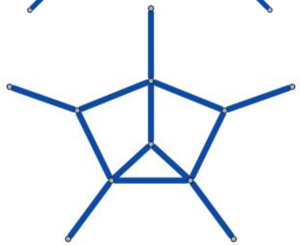
$$\mathrm{d}\mathbf{J} = \varepsilon \sum_{i=1}^{56} \mathbf{a}_i \mathrm{dlog} W_i \cdot \mathbf{J}$$

Originate from two topologies



$\left\{\widehat{W}_i\right\}_{i=1}^{5}$

$\widehat{W}_1 = s_{23}s_{34} - s_{34}s_{45} + s_{45}s_{15}$



New Square Root!

$\left\{\sqrt{\Delta_4^{(i)}}\right\}_{i=1}^{5}$

$$\begin{aligned}\Delta_4^{(1)} = & s_{15}^2 s_{12}^2 + s_{23}^2 s_{12}^2 - 2 s_{15} s_{23} s_{12}^2 - 2 s_{23}^2 s_{34} s_{12} \\ & + 2 s_{15} s_{23} s_{34} s_{12} + 2 s_{15} s_{34} s_{45} s_{12} + 2 s_{23} s_{34} s_{45} s_{12} \\ & + s_{23}^2 s_{34}^2 + s_{34}^2 s_{45}^2 - 2 s_{23} s_{34}^2 s_{45}\end{aligned}$$

Can be found by `Baikovletter` [Jiang, Liu, Xu, Yang '24]

How to solve this equation?

Determine Boundary Values

by absence of spurious singularities

Only a small subset of general solutions are "physical"

$$\begin{cases} \mathbf{I}^{(n)}(x) = \mathbf{I}^{(n)}(x_0) + \int_{x_0}^{x} \mathrm{d}\widetilde{\mathbf{A}} \cdot \mathbf{I}^{(n-1)} \\ \mathrm{Res} \dfrac{\mathrm{d}\widetilde{\mathbf{A}}}{\mathrm{d}x} \cdot \mathbf{I}^{(n)} \Big|_{y_i} = 0 \end{cases} \Rightarrow$$

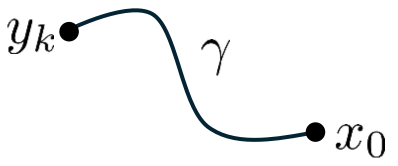
General Solutions & Free of Spurious Poles

One trivial integral

$$\mathbf{I}^{(n)}(x_0) = \mathbf{c}\mathbf{I}_1^{(n)}(x_0) + \mathbf{d} \cdot \int_\gamma \mathrm{d}\widetilde{\mathbf{A}} \cdot \mathbf{I}^{(n-1)}$$

1D path rationalizing all square roots:

$$\gamma : s_{12} = \frac{-t^2}{(1-t^2)^2}$$



$\rightarrow$ boundary values up to weight-six written in special values of GPL functions

[Gehrmann, Henn, Lo Presti '18] e.t.c

Spurious Singularities:

- Collinear Singularities $s_{35} = 0$
- Lower Dimension sub-space $G(p_1, p_2, p_3, p_4) = 0$
- Symmetric Point in Euclidean Region

$$x_0 = y_0 = \{-1, -1, -1, -1, -1\}$$

Boundary value is done
How about functions?

Solution of Three-Loop Five-Point Feynman Integrals

Solve 3L5P Canonical Differential Equation by:

6,000+ CPU hours by pySecDec

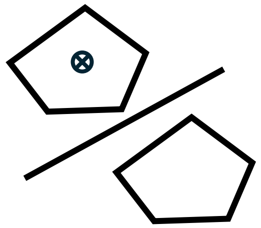
- DiffExp [Hidding '20]
- CHESS (Chebyshev-Lobatto pseudo-spectral) in One Minute [Liu, Zhang '26]
- Pentagon Functions (minimal functions space for 3L5P integrals) Ongoing, Stay Tune!!

PBB Family was cross-checked by AMFlow2.0 ! [Huang, Liu, Ma '26]

Solved by "Symbol" up to finite order weight-six

$$[W_{i_1}, W_{i_2} \cdots, W_{i_n}] = \int \mathrm{d}\log W_{i_n} [W_{i_1}, W_{i_2} \cdots, W_{i_{n-1}}]$$

only 2220 independent weight-six symbols $\ll 56^6$



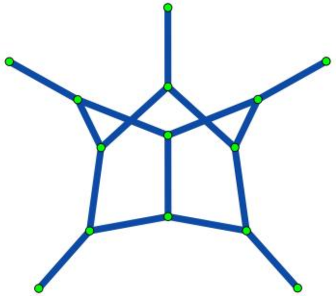
Bootstrapping Physical Quantities

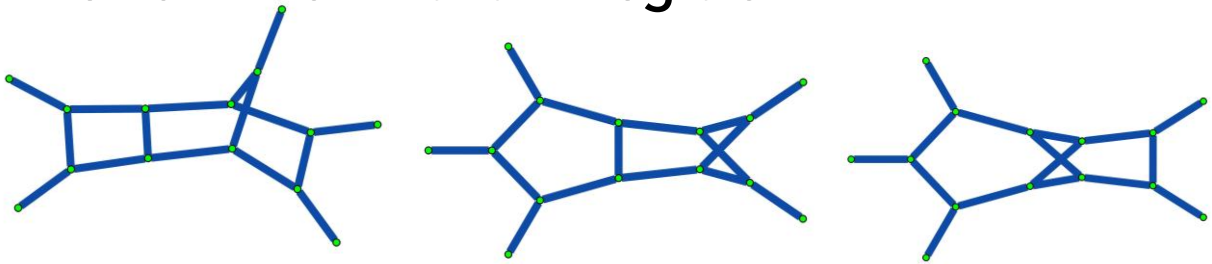
e.g. Pentagonal Wilson loop with Lagrangian insertion [Chicherin, Henn, Xu, Zhang, Zhang '25]
3L5P Scattering Amplitudes?

See also: talk by Yang Zhang

Into the Future

- **Toward Three-Loop Five-Point Non-Planar integrals**





And more non-planar topologies…

➡ Can we calculate these integrals? What is the non-planar alphabet letters?

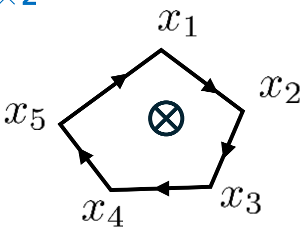
- **Three-Loop Five-Particles Scattering Amplitudes**

e.g. 3L5P all plus amplitudes in pure Yang-Mills, conjectured duality to $F_5^{(2)}$

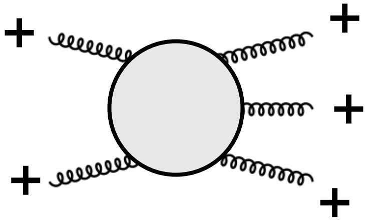
[Chicherin, Henn '22]$\times$2

$$\langle W[x_1, x_2, x_3, x_4, x_5] \mathscr{L}(x_0) \rangle$$
$$x_0 \to \infty$$

N=4 Super Yang Mills



?

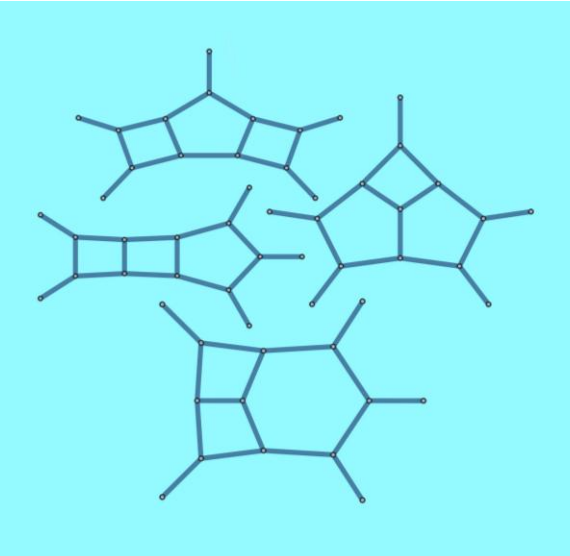


$$\mathcal{A}^{(3)}_{\mathrm{YM},5}(1^+, 2^+, 3^+, 4^+, 5^+)$$

pure Yang Mills

- And many more other directions…

Conclusion / Take-home messages



- Integration-By-Part Reduction
- Canonical Basis & Differential Equation
- Boundary Values
- Evaluating 3L5P Feynman Integrals
- Applications and Future Plans

Thank You!

Backup

Solved by Function basis: Pentagon Functions

- **Weight-1 to Weight-3 in Euclidean region,** same as Two-Loop Penta-Box

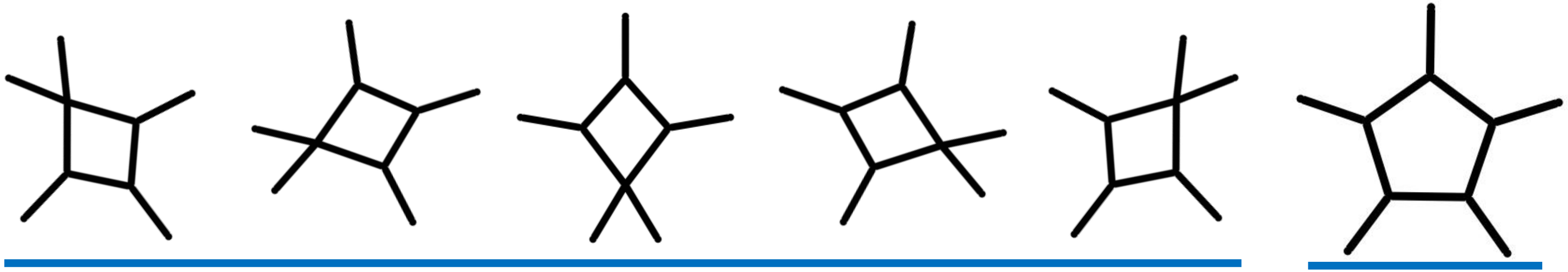
Length-5 Orbit: $f_{1,1}^{(1)} = \log(-s_{12}),\ f_{2,1}^{(1)} = \mathrm{Li}_2\left(1 - \frac{s_{45}}{s_{12}}\right),\ f_{3,1}^{(1)} = \mathrm{Li}_3\left(1 - \frac{s_{45}}{s_{12}}\right),\ f_{3,2}^{(1)},\ f_{3,3}^{(1)}$

+ cyclic permutation

Length-1 Orbit: Cyclic permutation singlet $f_{3,4}$

[Gehrmann, Henn, Lo Pres5 '18]

11 Master Integrals = 5 $\times$ bubbles +



$f_{3,i}^{(j)}$

$f_{3,4}$

Solved by Function basis: Pentagon Functions

- **Weight-1 to Weight-3 in Euclidean region,** same as Two-Loop Penta-Box

Length-5 Orbit: $f_{1,1}^{(1)} = \log(-s_{12}),\ f_{2,1}^{(1)} = \mathrm{Li}_2\left(1 - \frac{s_{45}}{s_{12}}\right),\ f_{3,1}^{(1)} = \mathrm{Li}_3\left(1 - \frac{s_{45}}{s_{12}}\right),\ f_{3,2}^{(1)},\ f_{3,3}^{(1)}$

+ cyclic permutation

Length-1 Orbit: Cyclic permutation singlet $f_{3,4}$

[Gehrmann, Henn, Lo Pres5 '18]

$$f_{3,4} = \mathrm{Li}_3\left(..\sqrt{..}\right) + \mathrm{Li}_2\left(..\sqrt{..}\right)\log(..) + \zeta_2 \log(..\sqrt{..}) + \zeta_3$$

Can we do this Calculus?

$$\int \frac{\log(1-x)\log x}{(x-2)\sqrt{1+4x}}\mathrm{d}x = ?$$

How to put a $\sqrt{\cdots}$ into Li_3

- Function Type ?
- Admissible Argument ?

Novel explicit function representation for $f_{3,4}$

Simplest genuine pentagon function, a curial ingredient

Used symbol techniques $\mathrm{SB}(\mathrm{Li}_n(z)) = -(1-z) \otimes \underbrace{z \otimes \ldots \otimes z}_{n-1}$

Only $\log, \mathrm{Li}_2, \mathrm{Li}_3$ are needed

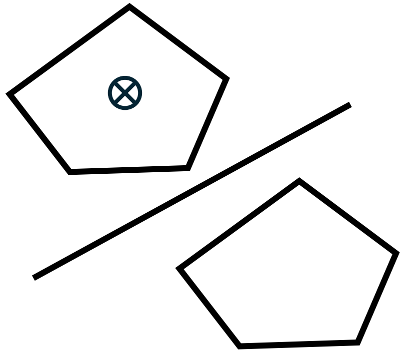
How about higher transcendental weight?

explicit functions become infeasible…

Pentagonal Wilson Loop with Lagrangian Insertion at Three-Loop

$\mathcal{N} = 4$ Super Yang-Mills in planar limit

$$F_5 = \frac{\langle W[x_1, x_2, x_3, x_4, x_5]\mathscr{L}(x_0)\rangle}{\langle W[x_1, x_2, x_3, x_4, x_5]\rangle} =$$



$$W[x_1, \cdots, x_n] = \mathrm{tr_F P} \exp\left(\mathrm{i} g_{\mathrm{YM}} t^a \oint A^a_\mu \mathrm{d}x^\mu\right)$$

n-cusp null Wilson loop with a Lagrangian insertion normalized by Wilson loop without Insertion

Symbol Bootstrap & Direct Computation

Theoretical Importance / Simpler Structure

closely related to scattering amplitudes

- Null Wilson Loop ~ amplitudes in N=4 SYM
- Natural in AdS/CFT
- Duality with all-plus pure Yang-Mills amplitudes ?
- Positivity and geometry
- …

[Alday, Maldacena '07] [Drummond, Korchemsky, Sokatchev '07]
[Arkani-Hamed, Henn, Trnka '21] [Chicherin, Henn '22] [Chicherin, Henn '22]
[Chicherin, Henn, Trnka, Zhang '24] [Brown, Henn, Mazzucchelli, Trnka '24] and so on.

- Free of IR divergence
- Uniformly Transcendental of 2L
- Exact dual conformal symmetry
- All Loop Leading Singularity
- All Loop Integrand from Amplituhedron
- …

👉 See also: talk by Dmitry Chicherin at Amplitudes 2022, Prague

➡ A good application of our Three-Loop Five-Point integrals

Review of perturbative results

Wilson loop with one Lagrangian insertion at weak coupling $F_n = \sum_{L \geq 1} (g^2)^{L+1} F_n^{(L)}$

<table>
<tr><th></th><th>Four-Point</th><th>Five-Point</th><th>Six-Point</th></tr>
<tr><td>Tree-Level</td><td>[Alday, Buchbinder, Tseytlin ‘11]</td><td colspan="2" rowspan="2">[Chicherin, Henn ‘22]
n-point Tree-Level and One-Loop</td></tr>
<tr><td>One-Loop</td><td>[Alday, Heslop, Sikorowski ‘12]</td></tr>
<tr><td>Two-Loop</td><td>[Alday, Henn, Sikorowski ‘13]</td><td>[Chicherin, Henn ‘22]</td><td>[Carrôlo, Chicherin, Henn Yang, Zhang ‘25]</td></tr>
<tr><td>Three-Loop</td><td>[Henn, Korchemsky, Mistlberger ‘ 19]</td><td>[Chicherin, Henn, Xu, Zhang, Zhang ‘25] $F_5^{(3)}$</td><td></td></tr>
</table>

See also
[Alday, Buchbinder, Tseytlin ‘11] Four-Point @ Strong Coupling
[Arkani-Hamed, Henn, Trnka ‘21] Four-Point Negative geometries expansion

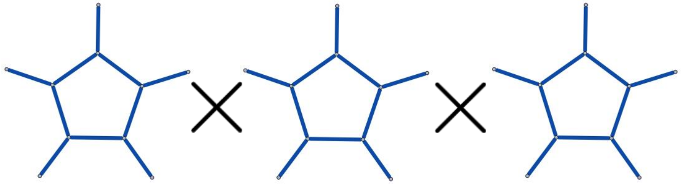
We are now here

Direct Computation of $F_5^{(3)}$

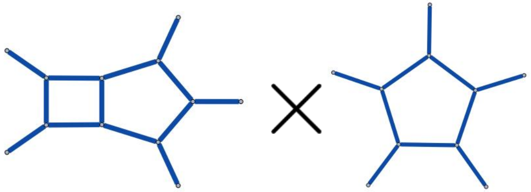
Step1: Separating Targets from integrand

$\mathscr{D}_1 = (\ell_1 - k_1)^2$	$\mathscr{D}_6 = (\ell_2 - k_1)^2$	$\mathscr{D}_{11} = (\ell_3 - k_1)^2$
$\mathscr{D}_2 = (\ell_1 - k_1 - k_2)^2$	$\mathscr{D}_7 = (\ell_2 - k_1 - k_2)^2$	$\mathscr{D}_{12} = (\ell_3 - k_1 - k_2)^2$
$\mathscr{D}_3 = (\ell_1 - k_1 - k_2 - k_3)^2$	$\mathscr{D}_8 = (\ell_2 - k_1 - k_2 - k_3)^2$	$\mathscr{D}_{13} = (\ell_3 - k_1 - k_2 - k_3)^2$
$\mathscr{D}_4 = (\ell_1 - k_5)^2$	$\mathscr{D}_9 = (\ell_2 - k_5)^2$	$\mathscr{D}_{14} = (\ell_3 - k_5)^2$
$\mathscr{D}_5 = \ell_1^2$	$\mathscr{D}_{10} = \ell_2^2$	$\mathscr{D}_{15} = \ell_3^2$
$\mathscr{D}_{16} = (\ell_1 - \ell_2)^2$	$\mathscr{D}_{17} = (\ell_1 - \ell_3)^2$	$\mathscr{D}_{18} = (\ell_2 - \ell_3)^2$

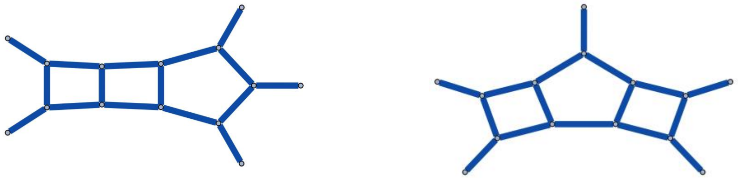
Loop momenta coupling



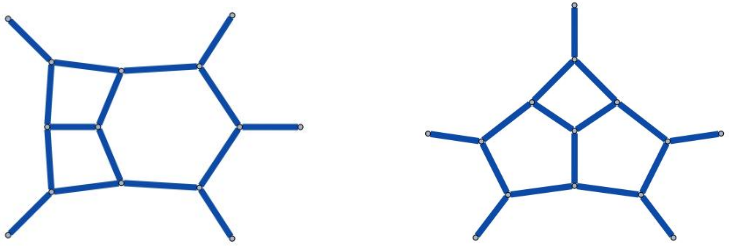
Free of $\mathscr{D}_{16}, \mathscr{D}_{17}, \mathscr{D}_{18}$



One of $\mathscr{D}_{16}, \mathscr{D}_{17}, \mathscr{D}_{18}$



Two of $\mathscr{D}_{16}, \mathscr{D}_{17}, \mathscr{D}_{18}$



All of $\mathscr{D}_{16}, \mathscr{D}_{17}, \mathscr{D}_{18}$

with all cyclic permutation C_5

Direct Computation of $F_5^{(3)}$

Step 2: Reduce target integrals into pure basis by NeatIBP

Step 3: Assembling with numerator ⇨ Results $F_5^{(3)} = r_0 g_0^{(3)} + \sum_{i=1}^{5} r_i g_i^{(3)}$

Leading Singularities:

$$r_0 = \mathrm{tr}_-((p_1+p_2)(p_2+p_3)(p_3+p_4)(p_4+p_5))$$

$$r_i = \frac{s_{i+2,i+3}}{s_{i+1,i+4}} \mathrm{tr}_-(p_{i+1}p_{i+2}p_{i+3}p_{i+4})$$

Weight–Six Symbol:

$$g_i^{(3)} = \sum_{i_1,\ldots,i_6} t_{i_1,\ldots,i_6} W_{i_1} \otimes \ldots \otimes W_{i_6}$$

involves 1,373,140 terms…

Intermediate expression involves 3,107,425 terms

- ☑ Consistent with Symbol bootstrap
- ☑ Infra Red finite: All ε poles vanished
- ☑ Respect soft and collinear limits
- ☑ Free of new square roots $\left\{\sqrt{\Delta_4^{(i)}}\right\}_{i=1}^{5}$
- ☑ Involves totally 30 letters

Duality to four-loop five-point all plus pure Yang-Mills amplitudes?

[Chicherin, Henn '22] $\times 2$
[Wu, Böhm, Ma, Xu, Zhang '23]
[Chicherin, Henn, Trnka, Zhang '24]
[Brown, Henn, Mazzucchelli, Trnka '24]
[Wu, Böhm, Ma, Usovitsch, Xu, Zhang '25]

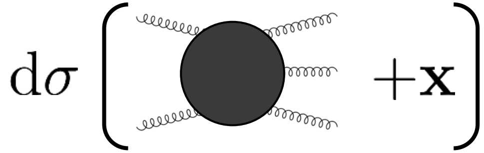
Letter	v notation	momentum notation	cylic
W_1	v_1	$2p_1 \cdot p_2$	$+$ cyclic (4)
W_6	$v_3 + v_4$	$2p_4 \cdot (p_3 + p_5)$	$+$ cyclic (4)
W_{11}	$v_1 - v_4$	$2p_3 \cdot (p_4 + p_5)$	$+$ cyclic (4)
W_{16}	$v_4 - v_1 - v_2$	$2p_1 \cdot p_3$	$+$ cyclic (4)
W_{21}	$v_3 + v_4 - v_1 - v_2$	$2p_3 \cdot (p_1 + p_4)$	$+$ cyclic (4)
W_{26}	$\frac{v_1 v_2 - v_2 v_3 + v_3 v_4 - v_1 v_5 - v_4 v_5 - \sqrt{\Delta}}{v_1 v_2 - v_2 v_3 + v_3 v_4 - v_1 v_5 - v_4 v_5 + \sqrt{\Delta}}$	$\frac{\mathrm{tr}[(1-\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2]}{\mathrm{tr}[(1+\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2]}$	$+$ cyclic (4)
W_{31}	$\sqrt{\Delta}$	$\mathrm{tr}[\gamma_5\not{p}_1\not{p}_2\not{p}_3\not{p}_4]$	

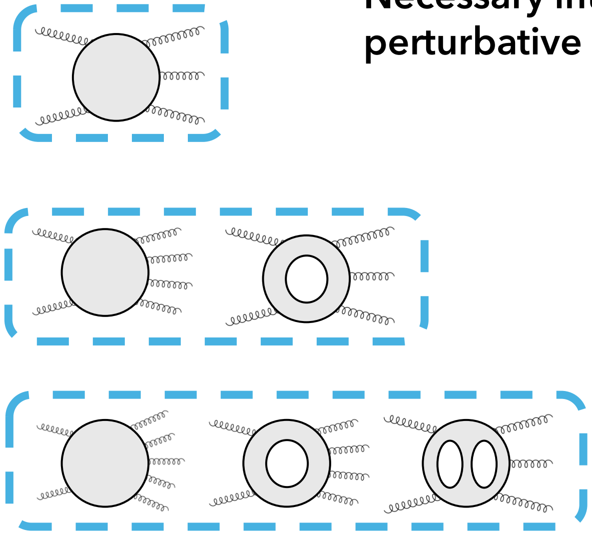
Five-particle letter alphabet at Two-Loop
[Gehrmann, Henn, Lo Pres5 '18]

Letter Alphabets are DNA of Feynman Integrals

Toward the Five-particle cross sections at NNNLO

Necessary integrals for each perturbative order

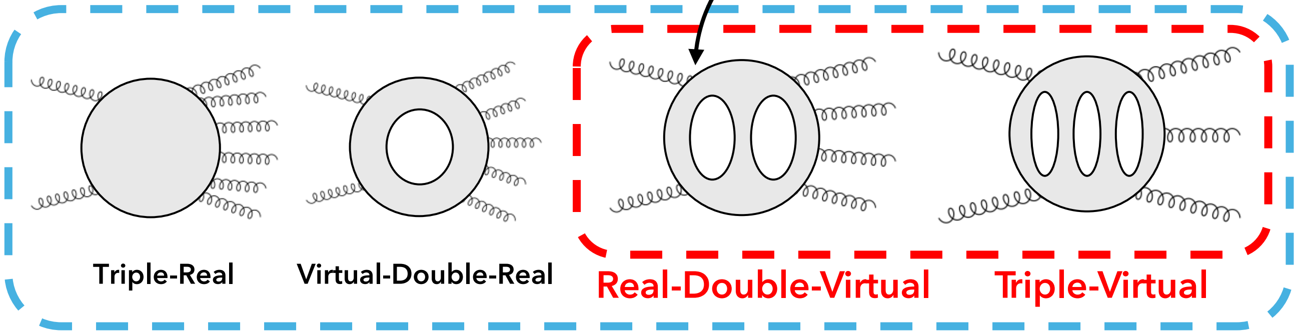




Legs \ Loops	1	2	3	4
4	☑	Most	Some	Some
5	☑	Some	3L5P	
6	☑	2L6P		
7	☑			

Frontier

Two-loop six-point computation



$\mathcal{O}(\alpha_s^7)$