Symbolic Reduction of Feynman Integrals via Generating Functions

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Contents

Motivation

- Scattering amplitude is one of the central objects in QFT. It connects the theoretical prediction with the experiment. Thus, its computation is extremely important.
- With the huge data collected by experiments, precise theoretical computation become necessary to match up with uncertainties from experiments
- Current frontiers are the computation of scattering amplitudes with **multiple legs** and **higher loops**.

Because the integrands from Feynman diagrams are not arbitrary, there are some common features in the expressions, for example, the same propagators. A very important consequence is that any multi-loop integral can be expanded by a small set of integrals as

$$I = \sum_i c_i I_i \tag{1}$$

where c_i is the rational function of external information, while I_i is called master integrals.

The procedure of the above decomposition is called the **reduction**

With the above simple but extremely powerful idea, the integration can be divided into two parallel parts:

- The computation of **master integrals**
- The computation of **reduction coefficients**

For higher loop computations, there are a lot of proposals to do the reduction. One of the very successful and commonly used methods is the **IBP method** (plus its various generalizations).

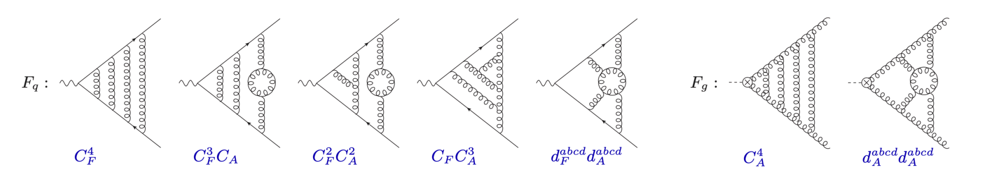
[Chetyrkin, Tkachov, 1981; Tkachov, 1981]

Bottleneck: IBP reduction

- However, it is very nontrivial to solve the IBP relations. A systematic and widely used approach is the Laporta Algorithm, i.e., solving a group of linear equations [Laporta, 2001]
- However, the size of equations increases factorially with # of loops and # of legs, requiring a lot times and resources. It becomes the **bottleneck** for a lot of cutting edge computations

Challenges

- There is one example: Four-loop form factors [Lee, Manteuffel, Schabinger, Smirnov, Smirnov,Steinhauser, 2022]



  - Takes 6 years to complete the calculation: 2016-2022
  - 5728 diagrams for F_q, and 43220 diagrams for F_g
  - Need to solve linear system involving up to 10^8 equations
  - $\mathcal{O}(40)$ finite field numbers + degree of D to $\mathcal{O}(300)$
  - Compressed IBP reduction tables consumes $\mathcal{O}(20\text{TB})$ on disk.

Developments

Facing this challenge, there has been a lot of work recently:

- Improvement of IBP methods:
  - Syzygy method: [Gluza, Kajda, and Kosower, 2010; Schabinger, 2011; Larsen, Zhang, 2015; Barakat, Brüser, Fieker, Huber and Piclum, 2022; Wu, Boehm, Ma, Xu, and Zhang, 2023; Smith, Zeng, 2025]
  - Finite field: [Kant, 2013; Manteuffel and Schabinger, 2014; Peraro, 2016; Klappert and Lange, 2019; Peraro, 2019; Magerya, 2022; Belitsky, Smirnov and Yakovlev, 2023; Chawdhry, 2023; Liu, 2023; Maier, 2024;]
  - Block triangular forms: [Guan, Liu, Ma and Wu, 2024]
  - Choices of seed integrals: [Lange, Usovitsch and Wu, 2025; Hippel and Wilhelm, 2025; Song, Yang, Cao, Luo and Zhu, 2025; Zeng, 2025; Berman, Charton, Luna, Wilhelm, Zeng, 2026; Shih, 2026]
  - Transverse integration identities: [Chestnov, Fontana, Peraro, 2024]
  - Geometric order relation: [The ϵ-collaboration: Bree, Gasparotto, Matijasic, Mazloumi, Melnichenko, Pögel, Teschke, Wang, Weinzierl, Wu and Xu, 2025, 2026]
  - Finding reduction rules: [Liu and Mitov, 2025; de la Cruz and Kosower, 2026; Gersdorff and Lessa, 2026;]

Developments

- Intersection theory: [Mastrolia and Mizera, 2018; Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi, Mizera, 2019; Frellesvig, Gasparotto, Mandal, Mastrolia, Mattiazzi, Mizera, 2019; Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi, Mizera, 2020; Chestnov, Frellesvig, Gasparotto, Mandal, Mastrolia, 2022; Brunello, Chestnov, Crisanti, Frellesvig, Mandal, Mastrolia, 2024]

Generating Function

Another improvement of the IBP method is to use the generating functions:

- The early use appeared in [Ablinger, Blümlein, Raab, Schneider and Wiβbrock, 2014; Kosower, 2018], where specific task of reduction has been carried out
- Systematic reduction for one-loop tensor-reduction is done in [Feng, 2022]
- Another exploration to derive reduction rules is given in [Guan, Li and Ma, 2023]

In this talk, we will present our recent work on finding **complete symbolic reduction rules** for general multi-loop integrals using the generating function, thus it avoid the need to solve extremely huge linear equations!!!

Tadpole example

To see how to translate the IBP problem to the generating function, let us use the simplest example, i.e., the one-loop massive tadpole, to demonstrate the idea. The integration family is

$$I(n,m,D) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2-m^2)^n} \tag{2}$$

- The IBP relation is

$$\begin{aligned} 0 &= \int \frac{d^D k}{(2\pi)^D} \frac{\partial}{\partial k^\mu} \left\{ k^\mu \frac{1}{(k^2-m^2)^n} \right\} \\ &= \int \frac{d^D k}{(2\pi)^D} \left\{ \frac{\partial}{\partial k^\mu} k^\mu \right\} \frac{1}{(k^2-m^2)^n} + \int \frac{d^D k}{(2\pi)^D} k^\mu \left\{ \frac{\partial}{\partial k^\mu} \frac{1}{(k^2-m^2)^n} \right\} \\ &= \int \frac{d^D k}{(2\pi)^D} D \frac{1}{(k^2-m^2)^n} + \int \frac{d^D k}{(2\pi)^D} \frac{-n2(k^2-m^2+m^2)}{(k^2-m^2)^{n+1}} \\ &= (D-2n) I(n,m,D) - 2nm^2 I(n+1,m,D) \end{aligned} \tag{3}$$

- From it, we can find the recurrence relation to finish the reduction

$$I(n+1,m,D) = \frac{(D-2n)}{2nm^2} I(n,m,D) \tag{4}$$

A key feature of the IBP relation is that it contains various integrals in the family. Thus, it is natural to think if there is a way to pack all integrals together? The answer is Yes: by the generating function

Let us consider the generating function of a one-loop tadpole

$$\begin{aligned} G(m, D; \eta) &= \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - m^2) - s_0 \eta} \\ &= \sum_{n=0}^{\infty} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - m^2)^{n+1}} (s_0 \eta)^n \\ &= \sum_{n=0}^{\infty} I(n+1, m, D)(s_0 \eta)^n \equiv \sum_{n=0}^{\infty} F_n \eta^{\mathbf{n}} \end{aligned} \tag{5}$$

Now we use the IBP relation to establish the differential equation of the generating function

$$
\begin{aligned}
0 &= \int \frac{d^D k}{(2\pi)^D} \frac{\partial}{k^\mu} \left\{ k^\mu \frac{1}{(k^2 - m^2) - s_0 \eta} \right\} \\
&= \int \frac{d^D k}{(2\pi)^D} \frac{D}{(k^2 - m^2) - s_0 \eta} + \int \frac{d^D k}{(2\pi)^D} \frac{-2k^2}{((k^2 - m^2) - s_0 \eta)^2} \\
&= DG(m, D; \eta) + \int \frac{d^D k}{(2\pi)^D} \frac{-2(k^2 - m^2 - s_0 \eta) - 2(m^2 + s_0 \eta)}{((k^2 - m^2) - s_0 \eta)^2} \\
&= (D-2)G(m, D; \eta) - 2(m^2 + s_0 \eta) \frac{1}{s_0} \frac{\partial}{\partial \eta} G(m, D; \eta)
\end{aligned}
\tag{6}
$$

Generating Function

To see how to deal with DE, let us see the effect of differential operators:

- The action of the operator

$$
\begin{aligned}
\frac{\partial}{\partial\eta}G &= \frac{\partial}{\partial\eta}\sum_{n=0}^{\infty}F_n\eta^n = \sum_{n=0}^{\infty}(n+1)F_{n+1}\eta^n \\
\eta G &= \eta\sum_{n=0}^{\infty}F_n\eta^n = \sum_{n=0}^{\infty}F_{n-1}\eta^n
\end{aligned}
\tag{7}
$$

  with the convention

$$
F_n \equiv 0, \quad when \;\; n < 0 \tag{8}
$$

- Putting above action back to (6), we can find **the recurrence relation**

$$
2\frac{m^2}{s_0}(n+1)F_{n+1} = (D-2-2n)F_n \tag{9}
$$

Generating Function

- From the point of view of DE, (9) is equivalent to solving

$$2\frac{m^2}{s_0}\frac{\partial}{\partial\eta}G(m,D;\eta) = (D-2)G(m,D;\eta) - 2\eta\frac{\partial}{\partial\eta}G(m,D;\eta) \tag{10}$$

- With the above correspondence, **especially the subscript of** F, we can define an important concept, i.e., the degree of the operator:

$$[\frac{\partial}{\partial\eta}] = +1, \qquad [\eta] = -1 \tag{11}$$

- Thus, finding nice reduction rule is translated to express a higher degree operator by other lower degree operators.

Now we show **operator degree determines the master integrals!!**

- Recalling that

$$2\frac{m^2}{s_0}\frac{\partial}{\partial\eta}G(m,D;\eta) = (D-2)G(m,D;\eta) - 2\eta\frac{\partial}{\partial\eta}G(m,D;\eta) \tag{12}$$

gives

$$2\frac{m^2}{s_0}(n+1)F_{n+1} = (D-2-2n)F_n \tag{13}$$

Repeating using it, we can reduce all $F_{n\geq 1} \to F_0$.

Generating Function

- If the solving is

$$2\frac{m^2}{s_0}\frac{\partial^2}{\partial\eta^2}G(m,D;\eta) = (D-2)G(m,D;\eta) - 2\frac{\partial}{\partial\eta}G(m,D;\eta) \tag{14}$$

  The corresponding recurrence relation will be

$$2\frac{m^2}{s_0}(n+2)(n+1)F_{n+2} = (D-2)F_n - 2(n+1)F_{n+1} \tag{15}$$

- The above rule can be applied to any point with $n \geq 2$. Repeating using it, we can reduce all $F_{n\geq 2} \to F_0, F_1$, i.e., there are two master integrals.

Generating Function

An important remark:

- Our aim is not to solve DE directly, i.e., find the analytic expression of the generating function in terms of external data and η. Finding the expression is an extremely difficult task.

- Our aim is to find recurrence relations for reduction, which are translated to **express some higher degree operators by other lower degree operators**, which is done by solving a small set of linear equations!

The first nontrivial example: Sunset

Sunset

For the massless sunset, there is only one sector, i.e., three propagators and two ISP's. The generating function is

$$G_{111} = \int \frac{\mathrm{d}^D \ell_1 \mathrm{d}^D \ell_2}{(\mathrm{i}\pi^{D/2})^2} \frac{\exp\left(s_0^{-1} \sum_{j=4}^{5} \eta_j \mathcal{D} j\right)}{\prod_{i=1}^{3} (\mathcal{D}_i - \eta_i s_0)} \equiv \sum_{n_i=0}^{\infty} F_{\vec{n}} \eta^{\vec{n}}, \tag{16}$$

where

$$\begin{aligned} &\mathcal{D}_1 = \ell_1^2, \quad \mathcal{D}_2 = \ell_2^2, \quad \mathcal{D}_3 = (\ell_1 + \ell_2 - K)^2, \\ &\mathcal{D}_4 = \ell_1 \cdot K, \quad \mathcal{D}_5 = \ell_2 \cdot K. \end{aligned} \tag{17}$$

An important point is that $F_{\vec{n}}$ **defined only for** $\vec{n} \geq 0$, i.e., each $n_i \geq 0$. The lower boundary of n_i is crucial for the termination of reduction, i.e., reducing to master integrals.

Sunset

- For this example, we see there is a nontrivial key generalizations: we will have **more than one fugitive parameter η**, and the equations are partial differential equations. Finding reduction rules will be more involved.
- The situation is similar to the division problem. For the single variable case, it is easy to do the partial fraction, for example,

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1} \tag{18}$$

However, for multiple variables, it is hard to do

$$\frac{x^2+y^2}{xy+ax+b} \tag{19}$$

For later ones, the **computational algebra geometry** is developed, such as the Grobner basis, etc.

Now we show new phenomena appearing in this example. First there are 6 fundamental IBP relations:

$$
\begin{aligned}
0 =& 2\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_3}G_{111} + \frac{K^2}{s_0}\frac{\partial}{\partial\eta_3}G_{111} + (D-3)G_{111} + \eta_4\frac{\partial}{\partial\eta_4}G_{111} - 2\eta_1\frac{\partial}{\partial\eta_1}G_{111} - (\eta_3+\eta_1-\eta_2)\frac{\partial}{\partial\eta_3}G_{111},\\
0 =& -2\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_1}G_{111} - 2\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_1}G_{111} - 2\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_3}G_{111} + \frac{K^2}{s_0}\frac{\partial}{\partial\eta_1}G_{111} + \frac{K^2}{s_0}\frac{\partial}{\partial\eta_3}G_{111}\\
&+ \eta_4\frac{\partial}{\partial\eta_5}G_{111} - (\eta_3-\eta_1-\eta_2)\frac{\partial}{\partial\eta_1}G_{111} - (\eta_3+\eta_2-\eta_1)\frac{\partial}{\partial\eta_3}G_{111},\\
0 =& -2\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_1}G_{111} - 2\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_3}G_{111} - 2\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_3}G_{111} + \frac{2K^2}{s_0}\frac{\partial}{\partial\eta_3}G_{111} + \frac{K^2}{s_0}\eta_4 G_{111},\\
0 =& -2\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_2}G_{111} - 2\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_2}G_{111} - 2\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_3}G_{111} + \frac{K^2}{s_0}\frac{\partial}{\partial\eta_2}G_{111} + \frac{K^2}{s_0}\frac{\partial}{\partial\eta_3}G_{111}\\
&+ \eta_5\frac{\partial}{\partial\eta_4}G_{111} - (\eta_3-\eta_1-\eta_2)\frac{\partial}{\partial\eta_2}G_{111} - (\eta_3+\eta_1-\eta_2)\frac{\partial}{\partial\eta_3}G_{111}\\
0 =& 2\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_3}G_{111} + \frac{K^2}{s_0}\frac{\partial}{\partial\eta_3}G_{111} + (D-3)G_{111} + \eta_5\frac{\partial}{\partial\eta_5}G_{111} - 2\eta_2\frac{\partial}{\partial\eta_2}G_{111} - (\eta_3-\eta_1+\eta_2)\frac{\partial}{\partial\eta_3}G_{111}\\
0 =& -2\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_2}G_{111} - 2\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_3}G_{111} - 2\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_3}G_{111} + \frac{2K^2}{s_0}\frac{\partial}{\partial\eta_3}G_{111} + \frac{K^2}{s_0}\eta_5 G_{111}
\end{aligned}
\tag{20}
$$

From it, we see

- Each equation is in the form

$$\sum_t c_t \widehat{O}_t \, G_{\vec{\mu}}(\vec{\eta}) = B. \tag{21}$$

where the $\widehat{O}_t$ are differential operators, which in a **standard form** can be written as

$$\vec{\eta}^{\vec{a}} \vec{\partial}^{\vec{b}} \equiv \prod_i \eta_i^{a_i} \partial_i^{b_i} \tag{22}$$

- From it, we define **the finer index** as the pair of vectors $(\vec{a}, \vec{b})$ and **index** as

$$\vec{o} \equiv \vec{b} - \vec{a} \tag{23}$$

- From index $\vec{o}$ we have the **degree** as

$$\left[\vec{\eta}^{\vec{a}} \vec{\partial}^{\vec{b}}\right] \equiv \sum_i \vec{o}_i \tag{24}$$

For example, $[\frac{\partial}{\partial \eta_2} \frac{\partial}{\partial \eta_5}] = 1 + 1 = 2$, $[\eta_2 \frac{\partial}{\partial \eta_5}] = (-1) + 1 = 0$

Now we solve 6 highest degree operators (here is degree-two) based on the coefficient matrix, which is

$$\left(\begin{array}{c|c|c|c|c|c|c}
\partial_1\partial_4 & \partial_2\partial_4 & \partial_3\partial_4 & \partial_1\partial_5 & \partial_2\partial_5 & \partial_3\partial_5 & \text{Source Eq.} \\
\hline
 & & & & & 2 & I_1 \\
\hline
-2 & & -2 & -2 & & & I_2 \\
\hline
-2 & & -2 & & & -2 & I_3 \\
\hline
 & -2 & & & -2 & -2 & I_4 \\
\hline
 & & 2 & & & & I_5 \\
\hline
 & & -2 & & -2 & -2 & I_6
\end{array}\right), \tag{25}$$

Using Gauss elimination , we can solve as

$$2\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_3}G_{111} = \frac{K^2}{s_0}\frac{\partial}{\partial\eta_3}G_{111} + (D-3)G_{111} + \eta_4\frac{\partial}{\partial\eta_4}G_{111} - 2\eta_1\frac{\partial}{\partial\eta_1}G_{111} - (\eta_3+\eta_1-\eta_2)\frac{\partial}{\partial\eta_3}G_{111}, \tag{26}$$

$$2\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_1}G_{111} = \frac{K^2}{s_0}\frac{\partial}{\partial\eta_1}G_{111} + \eta_4\frac{\partial}{\partial\eta_4}G_{111} + \eta_4\frac{\partial}{\partial\eta_5}G_{111} + (D-3)G_{111} - (\eta_3+\eta_1-\eta_2)\frac{\partial}{\partial\eta_1}G_{111} - 2\eta_3\frac{\partial}{\partial\eta_3}G_{111} - \eta_4\frac{K^2}{s_0}G_{111}, \tag{27}$$

$$2\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_1}G_{111} = -2(D-3)G_{111} - \eta_4\frac{\partial}{\partial\eta_4}G_{111} - \eta_5\frac{\partial}{\partial\eta_5}G_{111} + 2\eta_1\frac{\partial}{\partial\eta_1}G_{111} + 2\eta_2\frac{\partial}{\partial\eta_2}G_{111} + 2\eta_3\frac{\partial}{\partial\eta_3}G_{111} + \eta_4\frac{K^2}{s_0}G_{111}, \tag{28}$$

$$2\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_2}G_{111} = \frac{K^2}{s_0}\frac{\partial}{\partial\eta_2}G_{111} + \eta_5\frac{\partial}{\partial\eta_4}G_{111} + \eta_5\frac{\partial}{\partial\eta_5}G_{111} + (D-3)G_{111} - (\eta_3-\eta_1+\eta_2)\frac{\partial}{\partial\eta_2}G_{111} - 2\eta_3\frac{\partial}{\partial\eta_3}G_{111} - \eta_5\frac{K^2}{s_0}G_{111}, \tag{29}$$

$$2\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_3}G_{111} = \frac{K^2}{s_0}\frac{\partial}{\partial\eta_3}G_{111} + (D-3)G_{111}, + \eta_5\frac{\partial}{\partial\eta_5}G_{111} - 2\eta_2\frac{\partial}{\partial\eta_2}G_{111} - (\eta_3-\eta_1+\eta_2)\frac{\partial}{\partial\eta_3}G_{111}, \tag{30}$$

$$2\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_2}G_{111} = -2(D-3)G_{111} - \eta_4\frac{\partial}{\partial\eta_4}G_{111} - \eta_5\frac{\partial}{\partial\eta_5}G_{111} + 2\eta_1\frac{\partial}{\partial\eta_1}G_{111} + 2\eta_2\frac{\partial}{\partial\eta_2}G_{111} + 2\eta_3\frac{\partial}{\partial\eta_3}G_{111} + \eta_5\frac{K^2}{s_0}G_{111}. \tag{31}$$

We call these solutions **reduction rules** !!!

Sunset

Let us see the effect of these six **reduction rules**:

- For the rule $\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_3}$, it gives the recurrence relation

$$\begin{aligned} & 2(n_5+1)(n_3+1)F_{\vec{n}+\vec{e}_3+\vec{e}_5} \\ = & \frac{K^2}{s_0}(n_3+1)F_{\vec{n}+\vec{e}_3} + (D-3+n_4-2n_1-n_3)F_{\vec{n}} \\ & -(n_3+1)F_{\vec{n}+\vec{e}_3-\vec{e}_1} + (n_2+1)F_{\vec{n}+\vec{e}_3-\vec{e}_2} \end{aligned} \tag{32}$$

- Thus we see that for any lattice point $(n_1, n_2, n_3, n_4, n_5)$, as long as $n_3 \geq 1, n_5 \geq 1$, we can **repeatly use** this recurrence relation to reduce the **degree of lattice point, i.e.,** $\sum_{i=1}^{5} n_i$, by at least one.
- Noticing that the index of the operator is $\vec{o} = (0, 0, 1, 0, 1)$, thus the above condition to use the reduction rule can be expressed as

$$\vec{n} - \vec{o} \geq 0 \tag{33}$$

- Thus points can **NOT be reduced** using this rule are

$$\mathcal{U}_{\vec{o}} = \{\vec{m} \in \mathbb{Z}^N \mid \vec{m} \geq 0 \quad \& \quad \vec{m} - \vec{o} \ngeq 0\}. \tag{34}$$

- Now the six reduction rules have the form

$$\left\{\frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta_2}, \frac{\partial}{\partial \eta_3}\right\} \times \left\{\frac{\partial}{\partial \eta_4}, \frac{\partial}{\partial \eta_5}\right\}, \tag{35}$$

Thus, points can not be reduced using them are the intersection of 6 $\mathcal{U}$'s, which is

$$\mathcal{U}_1 = (0, 0, 0, n_4, n_5), \quad \mathcal{U}_2 = (n_1, n_2, n_3, 0, 0). \tag{36}$$

Generating new equations

Now coming to a very important problem: The 6 fundamental IBP relations contain all information, but the 6 reduction rules obtained by solving them can not be used to reduce all integrals. **What did we overlook?**

- The key observation is that there are some **hidden relations** insides these 6 fundamental relations, for example, the consistent condition (or integrability condition) shows as $[\frac{\partial}{\partial\eta_i}, \frac{\partial}{\partial\eta_j}] = 0$
- Using it, we can obtain a new relation, i.e.,

$$\frac{\partial}{\partial\eta_4}\left(\text{Rule of } \frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_1}\right) = \frac{\partial}{\partial\eta_5}\left(\text{Rule of } \frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_1}\right) \tag{37}$$

It turns out that the action $\frac{\partial}{\partial\eta_i}$ can be used to extract all hidden information!

Simplification

There is another important step, i.e., simplification using the known reduction rule:

- The new equation given by (37) is

$$
\begin{aligned}
&\frac{K^2}{s_0}\boxed{\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_1}}G_{111}+\frac{\partial}{\partial\eta_4}\eta_4\frac{\partial}{\partial\eta_4}G_{111}+\frac{\partial}{\partial\eta_4}\eta_4\frac{\partial}{\partial\eta_5}G_{111}\\
&+(D-3)\frac{\partial}{\partial\eta_4}G_{111}-(\eta_3+\eta_1-\eta_2)\boxed{\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_1}}G_{111}-2\eta_3\boxed{\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_3}}G_{111}-\frac{\partial}{\partial\eta_4}\eta_4\frac{K^2}{s_0}G_{111}\\
=\quad&-2(D-3)\frac{\partial}{\partial\eta_5}G_{111}-\eta_4\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_4}G_{111}-\frac{\partial}{\partial\eta_5}\eta_5\frac{\partial}{\partial\eta_5}G_{111}+2\eta_1\boxed{\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_1}}G_{111}\\
&+2\eta_2\boxed{\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_2}}G_{111}+2\eta_3\boxed{\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_3}}G_{111}+\eta_4\frac{K^2}{s_0}\frac{\partial}{\partial\eta_5}G_{111}
\end{aligned}
\tag{38}
$$

- Part in the box can be reduced using the 6 reduction rules, thus the highest degree of operators in this equation is reduced from 2 to 1. This is extremely important: it removes solved operators from the set of highest degree and leaves new operators, which can be solved to get new reduction rules !!!

It this way, we can produce 9 relations, where 8 of them are independent. Solving them, we find

- 3 reduction rules for degree-two operators $\frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\eta_2}$, $\frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\eta_3}$ and $\frac{\partial}{\partial\eta_2}\frac{\partial}{\partial\eta_3}$.
- 4 reduction rules for degree-one operators:

$$
\begin{aligned}
&-\left(6-3D-2\eta_4\frac{\partial}{\partial\eta_4}-2\eta_5\frac{\partial}{\partial\eta_5}\right)\frac{\partial}{\partial\eta_4}G_{111}\\
=&\frac{K^2(D-2)}{s_0}G_{111}+\frac{K^2}{s_0}\eta_5\frac{\partial}{\partial\eta_5}G_{111}+\frac{K^2}{s_0}(\eta_4+\eta_5)\frac{\partial}{\partial\eta_4}G_{111}+\frac{K^2}{s_0}\eta_3\frac{\partial}{\partial\eta_3}G_{111}\\
&+\frac{K^2}{s_0}\eta_2\frac{\partial}{\partial\eta_2}G_{111}-\frac{2K^2}{s_0}\eta_1\frac{\partial}{\partial\eta_1}G_{111}+(\eta_2+\eta_3-\eta_1)\eta_5\frac{\partial}{\partial\eta_5}G_{111}+(\eta_2\eta_5-\eta_1\eta_4)\frac{\partial}{\partial\eta_4}G_{111}\\
&+(3\eta_1-3\eta_2-\eta_3)\eta_3\frac{\partial}{\partial\eta_3}G_{111}-(\eta_2-3\eta_1+3\eta_3)\eta_2\frac{\partial}{\partial\eta_2}G_{111}+2\eta_1^2\frac{\partial}{\partial\eta_1}G_{111}\\
&+((D-3)(\eta_2+\eta_3-2\eta_1)+\frac{(K^2)^2\eta_5}{2s_0^2})G_{111}+\frac{K^2(4\eta_1\eta_4-(\eta_1+3\eta_2-\eta_3)\eta_5)}{2s_0}G_{111},
\end{aligned}
\tag{39}
$$

•

$$
\begin{aligned}
&- \left(6 - 3D - 2\eta_4 \frac{\partial}{\partial \eta_4} - 2\eta_5 \frac{\partial}{\partial \eta_5}\right) \frac{\partial}{\partial \eta_5} G_{111} \\
=& \frac{K^2\eta_4}{s_0} \frac{\partial}{\partial \eta_4} G_{111} + \frac{K^2(\eta_5+\eta_4)}{s_0} \frac{\partial}{\partial \eta_5} G_{111} + \frac{K^2(D-2)}{s_0} G_{111} + \frac{K^2\eta_3}{s_0} \frac{\partial}{\partial \eta_3} G_{111} \\
&- \frac{2K^2\eta_2}{s_0} \frac{\partial}{\partial \eta_2} G_{111} + \frac{K^2\eta_1}{s_0} \frac{\partial}{\partial \eta_1} G_{111} + (\eta_1 - \eta_2 + \eta_3)\eta_4 \frac{\partial}{\partial \eta_4} G_{111} + (\eta_1\eta_4 - \eta_2\eta_5) \frac{\partial}{\partial \eta_5} G_{111} \\
&- (3\eta_1 - 3\eta_2 + \eta_3)\eta_3 \frac{\partial}{\partial \eta_3} G_{111} + 2\eta_2^2 \frac{\partial}{\partial \eta_2} G_{111} - (\eta_1 - 3\eta_2 + 3\eta_3)\eta_1 \frac{\partial}{\partial \eta_1} G_{111} \\
&+ ((D-3)(\eta_1 - 2\eta_2 + \eta_3) - \frac{(K^2)^2\eta_4}{2s_0^2})G_{111} + \frac{K^2(4\eta_2\eta_5 - (3\eta_1 + \eta_2 - \eta_3)\eta_4)}{2s_0} G_{111},
\end{aligned} \tag{40}
$$

•

$$
\begin{aligned}
&\left(D-4-2\eta_1\frac{\partial}{\partial\eta_1}\right)\frac{\partial}{\partial\eta_1}G_{111}=\left(D-4-2\eta_3\frac{\partial}{\partial\eta_3}\right)\frac{\partial}{\partial\eta_3}G_{111}\\
&-2\eta_4\eta_1\frac{\partial}{\partial\eta_1}G_{111}-2\eta_4\eta_2\frac{\partial}{\partial\eta_2}G_{111}-2\eta_4\eta_3\frac{\partial}{\partial\eta_3}G_{111}+\eta_4^2\frac{\partial}{\partial\eta_4}G_{111}+\eta_4\eta_5\frac{\partial}{\partial\eta_5}G_{111}\\
&+2(D-3)\eta_4G_{111}-\frac{K^2}{2s_0}\eta_4^2G_{111},
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
&\left(D-4-2\eta_2\frac{\partial}{\partial\eta_2}\right)\frac{\partial}{\partial\eta_2}G_{111}=\left(D-4-2\eta_3\frac{\partial}{\partial\eta_3}\right)\frac{\partial}{\partial\eta_3}G_{111}\\
&-2\eta_5\eta_1\frac{\partial}{\partial\eta_1}G_{111}-2\eta_5\eta_2\frac{\partial}{\partial\eta_2}G_{111}-2\eta_5\eta_3\frac{\partial}{\partial\eta_3}G_{111}+\eta_4\eta_5\frac{\partial}{\partial\eta_4}G_{111}+\eta_5^2\frac{\partial}{\partial\eta_5}G_{111}\\
&+2(D-3)\eta_5G_{111}-\frac{K^2}{2s_0}\eta_5^2G_{111},
\end{aligned}
\tag{42}
$$

It is useful to see that $\frac{\partial}{\partial\eta_i}$ are same degree. Here we choose using $\frac{\partial}{\partial\eta_3}$ to solve $\frac{\partial}{\partial\eta_1}$ and $\frac{\partial}{\partial\eta_2}$

Up to this moment, we have $6+3=9$ degree two reduction rules and 4 degree-one reduction rules. We need to check if they can be used to finish the reduction:

- Using degree-one operators $\frac{\partial}{\partial \eta_i}, i=4,5$, we can reduce the set $\mathcal{U}_1 = (0,0,0,n_4,n_5)$ to point $(0,0,0,0,0)$.
- Using degree-one operators $\frac{\partial}{\partial \eta_a}, a=1,2$, we can reduce the set $\mathcal{U}_2 = (n_1,n_2,n_3,0,0)$ to points $(0,0,n_3,0,0)$.

There are still infinity number of points can not be reduced by the current reduction rules!

Now we need to generate new equations:

- Acting $\frac{\partial}{\partial \eta_3}$ on the reduction rule, for example, $\frac{\partial}{\partial \eta_4}$, after **simplification**, we can solve

$$\widehat{O}_{0;II} \frac{\partial}{\partial \eta_3} G_{111} = -\frac{(D-3)(3D-8)}{2} G_{111} + \frac{(5D-16)}{2} \eta_3 \frac{\partial}{\partial \eta_3} G_{111} - \eta_3^2 \frac{\partial^2}{\partial \eta_3^2} G_{111}$$
$$+ (\eta_1, \eta_2, \eta_4, \eta_5) \text{ components}, \tag{43}$$

with

$$\widehat{O}_{0;II} = \frac{(D-4)K^2}{2s_0} - \frac{K^2}{s_0} \eta_3 \frac{\partial}{\partial \eta_3}. \tag{44}$$

- Using the new rule, finally we can reduce all points $(0, 0, n_3, 0, 0)$ to point $(0, 0, 0, 0, 0)$, which is the only master integral for this family!!!

Now we have found the complete reduction rules!!!

The general setup

Generating functions

Having the above examples, now we can present the general setup:

- For L-loop integrals with E independent external momenta, a family of scalar FIs can be represented as

$$I(\vec{\nu}) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_1^{\nu_1} \cdots \mathcal{D}_K^{\nu_K}}, \tag{45}$$

- We can group different $I(\vec{\nu})$ using **generating function**

$$G_{\vec{\mu}}(\eta) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} e^{\sum_{j=1}^{N}(1-\mu_j)\eta_j s_0^{-1} \mathcal{D}_j} \frac{1}{\prod_{i=1}^{N}(\mathcal{D}_i - s_0 \eta_i)^{\mu_i}}, \tag{46}$$

with $\vec{\mu}_i = 0, 1$ for different sectors. Expanding it, we see that

$$G_{\vec{\mu}}(\eta) \quad = \quad \sum_{\vec{n} \geq 0} F_{\vec{n}} \eta^{\vec{n}} \tag{47}$$

where $F_{\vec{n}} \sim I(n_1 + 1, ..., n_K + 1, -n_{K+1}, ..., -n_N)$

Generating functions

- The general differential equations for generating functions have the form

$$\sum_t c_t \vec{\eta}^{\vec{a}_t} \frac{\partial^{\vec{b}_t}}{\partial \eta^{\vec{b}_t}} G_{\mu_1}(\vec{\eta}) = B \, , \tag{48}$$

where B is the function of generating functions of the sub-sectors of the sector $\vec{\mu}_1$ and

$$\vec{\eta}^{\vec{a}_t} = \prod_{i=1}^{N} \eta_i^{(\vec{a}_t)_i}, \qquad \frac{\partial^{\vec{b}_t}}{\partial \eta^{\vec{b}_t}} = \prod_{i=1}^{N} \frac{\partial^{(\vec{b}_t)_i}}{\partial \eta_i (\vec{b}_t)_i} \tag{49}$$

Thus the equation (48) gives the recurrence relation

$$\sum_t c_t \left(\prod_{j=1}^{N} \prod_{p=1}^{(\vec{b}_t)_j} (n_j - (\vec{a}_t)_j + p) \right) F_{\vec{n}+\vec{b}_t-\vec{a}_t} = \widetilde{B} \tag{50}$$

Operators

- For the operator $\widehat{O} = \eta^{\vec{a}} \frac{\partial^{\vec{b}}}{\partial \eta^{\vec{b}}}$, we define the **finer index** by the pair of vectors $(\vec{a}, \vec{b})$ and the corresponding **index** as the vector $\vec{o} \equiv \vec{b} - \vec{a}$, thus the **degree** of a operator is given by $|\vec{o}| \equiv \sum_{i=1}^{N} (\vec{o})_i$

Reduction rules we are looking for:

- **T1-type (Type 1):** The equation contains a single highest-degree operator index. This is the simplest case, allowing for a direct solution for that operator index.
  - **T1A-type:** The equation involves only one operator at the highest degree. It yields a rule of the form:
$$\widehat{O}_H = \sum_i c_i \widehat{O}_i + B, \quad |\vec{o}_H| > |\vec{o}_i| \ \forall i. \tag{51}$$
  - **T1B-type:** The equation involves multiple operators, all sharing the same highest-degree index $\vec{o}$. They can be bundled to yield a rule for the representative operator $\widehat{O}^F_{\vec{o}}$:
$$\widehat{O}_0 \widehat{O}^F_{\vec{o}} = \sum_i c_i \widehat{O}_i + B, \quad |\vec{o}| > |\vec{o}_i| \ \forall i. \tag{52}$$

    where $\widehat{O}_0$ is a zero-index operator.

Operators

- **T2-type (Type 2):** The equation contains multiple operator indices of the same maximal degree:
  - **T2A-type:** The chosen target index $\vec{o}_1$ corresponds to a single operator in the equation:

$$\widehat{O}_{\vec{o}_1} = \sum_j b_j \widehat{O}_{\vec{o}_j} + \sum_i c_i \widehat{O}_i + B, \quad |\vec{o}_1| = |\vec{o}_j| > |\vec{o}_i| \; \forall j, i. \tag{53}$$

  - **T2B-type:** The chosen target index $\vec{o}_1$ corresponds to multiple operators in the equation:

$$\widehat{O}_0 \widehat{O}^F_{\vec{o}_1} = \sum_j b_j \widehat{O}_{\vec{o}_j} + \sum_i c_i \widehat{O}_i + B, \quad |\vec{o}_1| = |\vec{o}_j| > |\vec{o}_i| \; \forall j, i. \tag{54}$$

The T2-type reduction rules are less direct because they do not completely lower the degree in a single step. One must therefore choose the solving strategy with care: the algorithm needs a global ordering of operator indices so that higher-priority operators are expressed in terms of lower-priority ones of the same degree. Once this ordering is fixed, the remaining operators can later be converted into T1-type reductions, and the degree is eventually lowered in a controlled way.

Operators

The use of reduction rules:

- Descendant reduction rules: From the rule (51), we get the reduction rule for many operators:

$$\widehat{Q} \quad = \quad \widehat{O}_\delta \widehat{O}_H = \sum_i c_i \widehat{O}_\delta \widehat{O}_i + \widehat{O}_\delta B \tag{55}$$

  It is extremely important for the **simplification** !!!

- **Two sets:** Reducible set by the rule

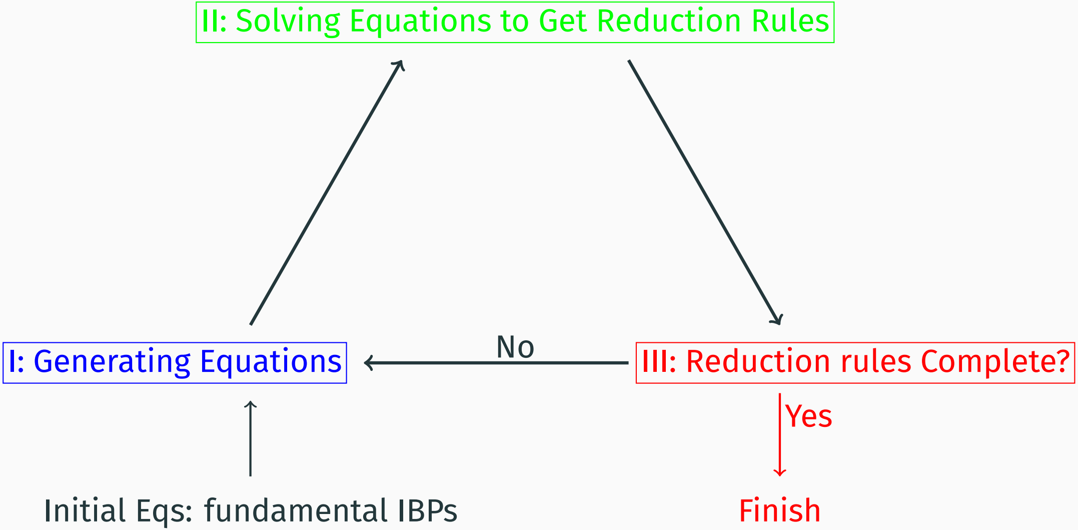
$$\mathcal{S}_{\vec{o}} \quad = \quad \{\vec{m} \in R^N | \vec{m} \geq 0 \ \ \& \ \ \vec{m} \geq \vec{o}\} \tag{56}$$

  and the irreducible set by the rule

$$\mathcal{U}_{\vec{o}} = \{\vec{m} \in R^N | \vec{m} \geq 0 \ \ \& \ \ \exists i, (\vec{m} - \vec{o}_+)_i < 0\} \tag{57}$$

Algorithm

Algorithm


II: Solving Equations to Get Reduction Rules
I: Generating Equations
No
III: Reduction rules Complete?
Yes
Initial Eqs: fundamental IBPs
Finish

Algorithm

The algorithm operates in a loop consisting of three main modules:

- **Module I:** Generating a system of differential equations:
  - The initial set of equations is given by the fundamental IBP relations.
  - Others are obtained by the acting $\frac{\partial}{\partial \eta_i}$ on reduction rules found in previous round of computation.
- **Module II:** Solving these equations to extract valid reduction rules.
- **Module III:** Checking the completeness of the derived rules:
  - If we find the complete reduction rules, the computation will stop.
  - Otherwise, we start another round of computation.

The First Block

- The initial seed of differential equations comes from fundamental IBP relations.
- New equations:
  - First **acting** $\frac{\partial}{\partial \eta_i}$ on reduction rules obtained in the previous solving block.
  - Secondly simplifying these equations using **descendant reduction rule**. It is one of the **most crucial parts** of the whole algorithm. Only after this simplification, **new unsolved operators will appear manifestly**, which will produce new reduction rules later.

The Second Block

The general principle of solving a set of equations:

- We divide them into subsets according to the degree
- Write down the coefficient matrix of the highest degree operators. Using Gauss elimination to solve **these linear equations**
- Using the new solve reduction rules to **update** (i.e, using the descendant reduction rules) unsolved equations and previous solved reduction rules

The Third Block

- The lattice points that can not be reduced by this rule are the set

$$\mathcal{U}_{\vec{o}} = \{\vec{m} \in R^N | \vec{m} \geq 0 \;\; \& \;\; \exists i, (\vec{m} - \vec{o}_+)_i < 0\} \tag{58}$$

- With a set of reduction rules, the unreduced set is given by

$$\mathcal{U}_{total} \equiv \cap_{i=1}^{n} \mathcal{U}_{\vec{o}_i} \tag{59}$$

Criteria of completeness of a set of reduction rules: When and only when the number of irreducible lattice points in the set $\mathcal{U}_{total}$ is equal to the number of master integrals in the given sector!

Several Examples

Top sector of planar double box

First round

- For the top sector, there are 7 **propagators** and 2 **ISPs**.
- There are 10 initial IBP equations.
- Using them, we get reduction rules for

$$\text{Degree one:} \quad \left\{ \frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta_3}, \frac{\partial}{\partial \eta_4}, \frac{\partial}{\partial \eta_6} \right\}, \tag{60}$$

$$\text{Degree two:} \quad \left\{ \frac{\partial}{\partial \eta_8}, \frac{\partial}{\partial \eta_9} \right\} \times \left\{ \frac{\partial}{\partial \eta_2}, \frac{\partial}{\partial \eta_5}, \frac{\partial}{\partial \eta_7} \right\}. \tag{61}$$

- The irreducible lattice points are

$$\mathcal{U}_1 = (0, n_2, 0, 0, n_5, 0, n_7, 0, 0), \qquad \mathcal{U}_2 = (0, 0, 0, 0, 0, 0, 0, n_8, n_9) \tag{62}$$

Second round

- With $\frac{\partial}{\partial \eta_i}, i = 2, 5, 7, 8, 9$ on four degree one reduction rules. We get 9 **non-trivial generated new equations**, but only 8 are independent
- Using them, we can solve:

$$\begin{aligned} &\text{Degree two:} && \frac{\partial}{\partial \eta_2}\frac{\partial}{\partial \eta_5}, \ \frac{\partial}{\partial \eta_2}\frac{\partial}{\partial \eta_7}, \ \frac{\partial}{\partial \eta_5}\frac{\partial}{\partial \eta_7}, \ \frac{\partial}{\partial \eta_8}\frac{\partial}{\partial \eta_9}, \\ &\text{Degree one:} && \frac{\partial}{\partial \eta_2}, \ \frac{\partial}{\partial \eta_7}, \ \frac{\partial}{\partial \eta_9} \end{aligned} \tag{63}$$

- The irreducible lattice points are

$$\mathcal{U}_1 = (0,0,0,0,n_5,0,0,0,0), \qquad \mathcal{U}_2 = (0,0,0,0,0,0,0,n_8,0) \tag{64}$$

Third round

- Acting $\frac{\partial}{\partial\eta_5}, \frac{\partial}{\partial\eta_8}$ on four new three degree one reduction rules and simplifying it, we have
$$0 = \left[\widehat{O}^{up}_{0;5} - 1\right] \frac{\partial^2}{\partial\eta_5^2} G + \cdots \,, \tag{65}$$
and
$$0 = \frac{s_0}{s}\left[\widehat{O}^{up}_{0;8}(\widehat{O}^{up}_{0;8}) - 1\right] \frac{\partial}{\partial\eta_8} G + \frac{t}{2s_0}\widehat{O}^{up}_{0;5} \frac{\partial}{\partial\eta_5} G + \cdots \,, \tag{66}$$
- We can solve $\frac{\partial}{\partial\eta_8}$ and $\frac{\partial^2}{\partial\eta_5^2}$, thus obtain two master integrals:
$$\{(0,0,0,0,0,0,0,0,0), \quad (0,0,0,0,1,0,0,0,0)\} \,, \tag{67}$$

Top sector of two-loop non-planar massless double-box

First round

- For the top sector, there are 7 **propagators** and 2 **ISPs**.
- There are 10 initial IBP equations.
- Using them, we get reduction rules for

$$\begin{aligned}
&\text{Degree one:} && \frac{\partial}{\partial \eta_3}, \quad \frac{\partial}{\partial \eta_1} \\
&\text{Degree two:} && \frac{\partial}{\partial \eta_7}\frac{\partial}{\partial \eta_8}, \quad \frac{\partial}{\partial \eta_6}\frac{\partial}{\partial \eta_8}, \quad \frac{\partial}{\partial \eta_5}\frac{\partial}{\partial \eta_8}, \quad \frac{\partial}{\partial \eta_4}\frac{\partial}{\partial \eta_8}, \quad \frac{\partial}{\partial \eta_2}\frac{\partial}{\partial \eta_8}, \quad \frac{\partial}{\partial \eta_2}\frac{\partial}{\partial \eta_9}, \\
& && \frac{\partial}{\partial \eta_6}\frac{\partial}{\partial \eta_9} \to \frac{\partial}{\partial \eta_7}\frac{\partial}{\partial \eta_9}, \quad \frac{\partial}{\partial \eta_4}\frac{\partial}{\partial \eta_9} \to \frac{\partial}{\partial \eta_5}\frac{\partial}{\partial \eta_9}
\end{aligned} \tag{68}$$

- The irreducible lattice points are

$$\mathcal{U}_1^{\text{1st}} = (0, n_2, 0, n_4, n_5, n_6, n_7, 0, 0), \tag{69}$$

$$\mathcal{U}_2^{\text{1st}} = (0, 0, 0, 0, n_5, 0, n_7, 0, n_9), \tag{70}$$

$$\mathcal{U}_3^{\text{1st}} = (0, 0, 0, 0, 0, 0, 0, n_8, n_9). \tag{71}$$

Second round

- With the action $\frac{\partial}{\partial \eta_i}, i \neq 1, 3$ we get 16 new equations.
- Using them, we get reduction rules for

$$\begin{aligned}
&\text{Degree two:} && \partial_2\partial_4, \partial_2\partial_5, \partial_2\partial_6, \partial_2\partial_7, \partial_4\partial_5, \partial_4\partial_7, \partial_5\partial_9, \partial_5\partial_6, \partial_6\partial_7, \partial_7\partial_9 \\
& && \partial_5\partial_7 \to \partial_4\partial_6, \ \partial_8\partial_9 \to \partial_8\partial_8 \\
&\text{Degree one:} && \widehat{O}^{up}_{0;2}\frac{\partial}{\partial \eta_2} \to \left(\widehat{O}^{up}_{0;4}\frac{\partial}{\partial \eta_4}, \widehat{O}^{up}_{0;5}\frac{\partial}{\partial \eta_5}\right), \quad \widehat{O}^{up}_{0;6}\frac{\partial}{\partial \eta_6} \to \widehat{O}^{up}_{0;4}\frac{\partial}{\partial \eta_4}, \\
& && \widehat{O}^{up}_{0;7}\frac{\partial}{\partial \eta_7} \to \widehat{O}^{up}_{0;5}\frac{\partial}{\partial \eta_5},
\end{aligned} \tag{72}$$

- The irreducible lattice points are

$$\begin{aligned}
\mathcal{U}^{2nd}_1 &= (0,0,0,n_4,0,0,0,0,0), \quad \mathcal{U}^{2nd}_2 = (0,0,0,0,n_5,0,0,0,0), \\
\mathcal{U}^{2nd}_3 &= (0,0,0,0,0,0,0,n_8,0), \quad \mathcal{U}^{2nd}_4 = (0,0,0,0,0,0,0,0,n_9).
\end{aligned} \tag{73}$$

Third round

- Acting only with $\frac{\partial}{\partial \eta_i}$ for $i = 4, 5, 8, 9$ on the 3 degree-one rules, we get 8 new equations
- Using them, we get reduction rules for

$$
\begin{aligned}
\text{Degree one:} \quad & \widehat{O}_{0;89,2} \frac{\partial}{\partial \eta_9} \to \left(\widehat{O}_{0;89,3} \frac{\partial}{\partial \eta_8}, \widehat{O}^{up}_{0;4} \frac{\partial}{\partial \eta_4} \right), \quad \widehat{O}^{up}_{0;5} \frac{\partial}{\partial \eta_5} \to \widehat{O}^{up}_{0;4} \frac{\partial}{\partial \eta_4}, \\
& \widehat{O}_{0;89,1} \frac{\partial}{\partial \eta_8} \to \widehat{O}^{up}_{0;4} \frac{\partial}{\partial \eta_4}, \\
\text{Degree two:} \quad & (1 - \widehat{O}^{up}_{0;4}) \frac{\partial^2}{\partial \eta_4^2} G
\end{aligned}
\tag{74}
$$

- Two master integrals:

$$
\{(0,0,0,0,0,0,0,0,0), \quad (0,0,0,1,0,0,0,0,0)\}\,,
\tag{75}
$$

Degenerated top sector

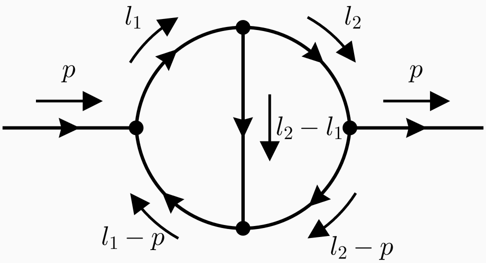
- It is

$$I(a_i) = \int \prod_{j=1}^{2} \frac{\mathrm{d}^D \ell_j}{\mathrm{i}\pi^{D/2}} \frac{1}{\prod_{i=1}^{5} \mathcal{D}_i^{a_i}}\,, \tag{76}$$

with

$$\mathcal{D}_1 = \ell_1^2\,, \quad \mathcal{D}_2 = (\ell_1 - K)^2\,, \quad \mathcal{D}_3 = \ell_2^2\,, \quad \mathcal{D}_4 = (\ell_2 - K)^2\,, \quad \mathcal{D}_5 = (\ell_1 - \ell_2)^2\,. \tag{77}$$

There is no ISP.



- The key is we have two equations with **identity operator** as the highest degree operator

$$
\begin{aligned}
0 =& \left[(\mathrm{D}-4) - 2\eta_5 \frac{\partial}{\partial \eta_5}\right] \mathrm{G} + \frac{2s_0}{s}\eta_5 \left[(D-4) - \eta_5 \frac{\partial}{\partial \eta_5}\right] G \\
&+ \left[\frac{\partial}{\partial \eta_2} B_4 - \frac{\partial}{\partial \eta_2} B_5 + \frac{\partial}{\partial \eta_1} B_3 - \frac{\partial}{\partial \eta_1} B_5\right] \\
&+ \frac{s_0}{s}\eta_5 \left[\left[\frac{\partial}{\partial \eta_2} + \frac{\partial}{\partial \eta_5}\right] B_1 - \left[\frac{\partial}{\partial \eta_5} + \frac{\partial}{\partial \eta_1}\right] B_2 - \frac{\partial}{\partial \eta_5} B_3 + \frac{\partial}{\partial \eta_5} B_4\right], \qquad (78)
\end{aligned}
$$

and

$$
\begin{aligned}
0 =& \left[(\mathrm{D}-4) - 2\eta_5 \frac{\partial}{\partial \eta_5}\right] \mathrm{G} + \frac{2s_0}{s}\eta_5 \left[(D-4) + \eta_5 \frac{\partial}{\partial \eta_5}\right] G \\
&+ \left[\frac{\partial}{\partial \eta_4} B_2 - \frac{\partial}{\partial \eta_4} B_5 + \frac{\partial}{\partial \eta_3} B_1 - \frac{\partial}{\partial \eta_3} B_5\right] \\
&+ \frac{s_0}{s}\eta_5 \left[\frac{\partial}{\partial \eta_5} B_1 + \frac{\partial}{\partial \eta_5} B_2 - \left[\frac{\partial}{\partial \eta_5} + \frac{\partial}{\partial \eta_4}\right] B_3 - \left[\frac{\partial}{\partial \eta_5} + \frac{\partial}{\partial \eta_3}\right] B_4\right]. \qquad (79)
\end{aligned}
$$

From above two equations, we can see the following conclusion:

- The rule of identity operator means the top sector can be expressed as the sum of sub-sectors, so it is **degenerate**, and there are no master integrals in this sector!
- Two different expressions for the identity operator mean **a relation** between different sub-sectors.

Conclusion

In this talk, I have presented an algorithm to find **complete symbolic reduction rules** using generating functions.

Thank you very much for your attention !