Gradient Flow in Heavy Flavor Physics

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Contents

Gradient flow smooths fields over a physical radius

QCD fields are extended to $t > 0$ with boundary values $B_\mu(0,x) = A_\mu(x)$ and $\chi(0,x) = \psi(x)$.

Gauge field

$$\partial_t B_\mu = D_\nu G_{\nu\mu} + \kappa D_\mu \partial_\nu B_\nu,$$
$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu].$$

Quark field

$$\partial_t \chi = \Delta\chi - \kappa(\partial_\mu B_\mu)\chi, \qquad \Delta = D_\mu D_\mu,$$
$$\partial_t \bar{\chi} = \bar{\chi}\overleftarrow{\Delta} + \kappa\bar{\chi}(\partial_\mu B_\mu).$$

$r_F = \sqrt{8t}$ acts as a gauge-invariant smoothing radius and UV scale.

M. Luescher, JHEP 08 (2010) 071; M. Luescher and P. Weisz, JHEP 02 (2011) 051.

Small-flow-time expansion connects lattice and continuum

Small-flow-time expansion

$$O_i(t,x) = \sum_j \zeta_{ij}(t,\mu) O_j^{\overline{\mathrm{MS}}}(x;\mu) + \mathcal{O}(t).$$

$$\mu_t \equiv \frac{1}{\sqrt{8t}} \gg M_{\mathrm{phys}}.$$

The expansion is a short-distance OPE.

- The coefficients ζ_{ij} contain the UV conversion from the flow scheme to $\overline{\mathrm{MS}}$.
- $\zeta_{ij}(t,\mu)$ is perturbatively calculable.
- Heavy quark mass effects require a mass-dependent extension.

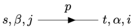
M. Luescher, JHEP 04 (2013) 123; H. Makino and H. Suzuki, PTEP 2014 (2014) 063B02.

Representative Feynman rules in gradient flow

Besides ordinary QCD, gradient flow introduces additional Feynman rules.

J. Artz et al., JHEP 06 (2019) 121.

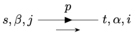
Flowed quark propagator: $\langle \chi(t,p)\bar{\chi}(s,p)\rangle$



$$= \delta_{ij}\frac{(-i\not{p}+m)_{\alpha\beta}}{p^2+m^2}\, e^{-(t+s)p^2}$$

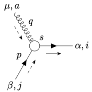
UV Suppr.

Quark flow line: $\langle \chi(t,p)\bar{\lambda}(s,p)\rangle$

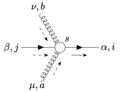


$$= \delta^{ij}\delta_{\alpha\beta}\,\theta(t-s)\, e^{-(t-s)p^2}$$

Quark-flow vertices from $\mathcal{L}_\chi$

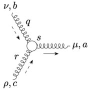


$$= ig\,\delta_{\alpha\beta}(T^a)_{ij}\int_0^\infty \mathrm{d}s\,\left(2p_\mu+(1-\kappa)q_\mu\right)$$

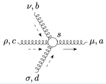


$$= g^2\delta_{\alpha\beta}\delta_{\mu\nu}\left\{T^a,T^b\right\}_{ij}\int_0^\infty \mathrm{d}s$$

Gluon-flow vertices from $\mathcal{L}_B$



$$= -igf^{abc}\int_0^\infty \mathrm{d}s\,\big(\delta_{\nu\rho}(r-q)_\mu+2\delta_{\mu\nu}q_\rho-2\delta_{\mu\rho}r_\nu + (\kappa-1)(\delta_{\mu\rho}q_\nu-\delta_{\mu\nu}r_\rho)\big)$$



$$= -g^2\int_0^\infty \mathrm{d}s\,\big(f^{abe}f^{cde}(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}) + f^{ace}f^{bde}(\delta_{\mu\nu}\delta_{\rho\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}) + f^{ade}f^{bce}(\delta_{\mu\nu}\delta_{\rho\sigma}-\delta_{\mu\rho}\delta_{\nu\sigma})\big)$$

Why gradient flow?

Lattice QCD at $t > 0$

- Flow suppresses ultraviolet noise and reduces lattice cutoff effects.
- Finite flowed operators enable controlled continuum and zero-flow-time limits.

L. Altenkort et al. [HotQCD], PRL 130 (2023) 231902.
L. Altenkort et al. [HotQCD], PRL 132 (2024) 051902.
A. Francis et al., PRL 136 (2026) 171903.
D. Bollweg et al., arXiv:2601.13070 [hep-lat], PRL accepted.

Small-flow-time expansion

$\Longrightarrow$

GF scheme $\to \overline{\mathrm{MS}}$

Continuum perturbation theory

- Infrared physics is unchanged; dimensional regularization can be adopted in matching calculations.
- Higher order matching coefficients are accessible and the perturbative expansion convergence is good.

J. Artz et al., JHEP 06 (2019) 121.
J. Borgulat et al., JHEP 05 (2024) 179.
N. Brambilla, H.-S. Chung, A. Vairo and **X.-P. Wang**, JHEP 01 (2022) 184.
N. Brambilla and **X.-P. Wang**, JHEP 06 (2024) 210.

Off-lightcone Wilson-line operators in gradient flow

N. Brambilla and **X.-P. Wang**, JHEP 06 (2024) 210.

$$W(zv,0) = \mathrm{P}\exp\left(ig\int_0^z ds\, v\cdot A(sv)\right), \qquad v^2 \neq 0.$$

Quark operator

$$\mathcal{O}_\Gamma(zv) = \bar{\psi}(zv)\Gamma W(zv,0)\psi(0).$$

- Quark quasi-PDFs/LCDAs.
- The Dirac structure Γ selects the channel.

Gluon operator

$$\mathcal{O}^{\mu\nu\alpha\beta}(zv) = g^2 F^{\mu\nu}(zv) W(zv,0) F^{\alpha\beta}(0).$$

- Gluon quasi-PDFs/LCDAs.
- Heavy quark/Quarkonium transport coefficient, P-wave quarkonium decay.

Generalized HQET turns Wilson lines into local currents

Introduce an auxiliary field h_v whose propagator is the Wilson line. Then

Quark operator

$$\mathcal{O}_\Gamma(zv) = \int d^d x\, \delta\Big(\frac{v\cdot x}{v^2} - z\Big)\, \big\langle [\bar\psi(x) h_v(x)]\Gamma[\bar h_v(0)\psi(0)]\big\rangle_{h_v}\,.$$

Gluon operator

$$\mathcal{O}_i(zv) = \int d^d x\, \delta\Big(\frac{v\cdot x}{v^2} - z\Big)\, \big\langle J_i(x)\bar J_i(0)\big\rangle_{h_v}\,, \qquad \begin{aligned} J_{\parallel\perp} &= gF^{\mu\nu}_{\parallel\perp} h_v,\\ J_{\perp\perp} &= gF^{\mu\nu}_{\perp\perp} h_v,\\ J_F &= g\bar h_v F^{\mu\nu}_{\perp\perp} h_v.\end{aligned}$$

J_F represents B field insertion into Wilson loop that defines spin-dependent potentials.

Non-local operator renormalization reduces to local-current renormalization.

Matching relations and results

Local renormalization implies local matching

$$J_a^{\mathrm{R}}(t) = c_a(t,\mu) J_a^{\overline{\mathrm{MS}}}(\mu) + \mathcal{O}(t),$$
$$\mathcal{O}_a^{\mathrm{R}}(zv,t) = \mathcal{C}_a(t,\mu) e^{\delta m\, z} \mathcal{O}_a^{\overline{\mathrm{MS}}}(zv) + \mathcal{O}(t), \qquad \mathcal{C}_a = c_a^2.$$

$$\delta m = -\frac{\alpha_s}{\sqrt{\pi}} C_R \frac{1}{\sqrt{8t}}, \quad C_R = C_A \text{ or } C_R = C_F$$

Quark current

$$c_{\psi h_v} = 1 - \frac{1}{2}\frac{\alpha_s}{4\pi} C_F \left[3L_t + \ln 432 + 2\right] + \mathcal{O}(\alpha_s^2),$$

$L_t \equiv \ln(2\mu^2 t e^{\gamma_E})$.

Gluon currents and c_F

$$c_{\parallel\perp} = 1 + \mathcal{O}(\alpha_s^2),$$
$$c_{\perp\perp} = 1 + \frac{\alpha_s}{4\pi} C_A L_t + \mathcal{O}(\alpha_s^2),$$
$$c_F = 1 + \frac{\alpha_s}{4\pi} C_A L_t + \mathcal{O}(\alpha_s^2).$$

c_F enters spin-dependent potentials.

Applications: Quasi-PDF/LCDAs, spin-dependent potentials, quarkonium decay, transport coefficient and so on...

Inclusive P-wave decay from pNRQCD + lattice QCD

N. Brambilla et al., arXiv:2607.13024.

Strongly coupled pNRQCD at leading order in v

$$\Gamma(\chi_{QJ} \to \mathrm{LH}) = \frac{|R'_{\chi_Q}(0)|^2}{m_Q^4} \left[\frac{3N_c}{2\pi} \, 2\,\mathrm{Im} f_1({}^3P_J) + \frac{2T_F}{9N_c} \frac{3N_c}{2\pi} \, 2\,\mathrm{Im} f_8({}^3S_1)\, \mathcal{E}_3 \right].$$

$$\mathcal{E}_3(\Lambda) = \frac{T_F}{N_c} \int_0^\infty d\tau\, \tau^3 \langle 0 | g E^{i,a}(\tau) \Phi^{ab}(\tau, 0) g E^{i,b}(0) | 0 \rangle.$$

- Universal across J, radial excitation, and heavy flavor.
- Its UV logarithm cancels the singlet SDC infrared logarithm.

Task: determine $\mathcal{E}_3$ in the gradient-flow scheme and match it to $\overline{\mathrm{MS}}$.

N. Brambilla et al., PRL 88 (2002) 012003; JHEP 04 (2020) 095.

Gradient flow makes $\mathcal{E}_3$ lattice accessible

$$\mathcal{E}_3(\Lambda) = \mathcal{E}_3^{\text{pt}}(\Lambda, \tau_*) + \mathcal{E}_3^{\text{np}}(\tau_*).$$

Perturbative/nonperturbative split

$$\mathcal{E}_3^{\text{pt}}(\Lambda, \tau_*) = \frac{T_F}{N_c} \int_0^{\tau_*} d\tau\, \tau^3 \mathcal{E}^{\overline{\text{MS}}}(\tau),$$
$$\mathcal{E}_3^{\text{np}}(\tau_*) = \frac{T_F}{N_c} \int_{\tau_*}^{\infty} d\tau\, \tau^3 G_{EE}^{\overline{\text{MS}}}(\tau).$$

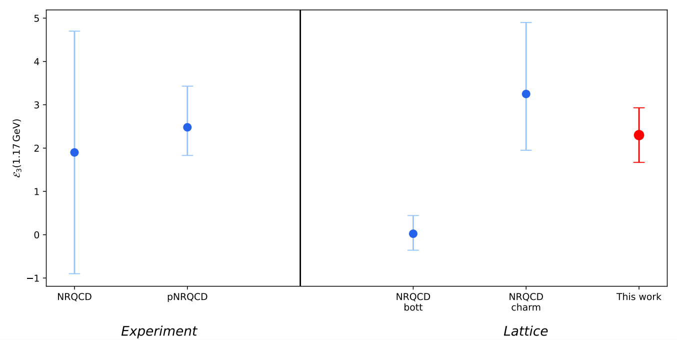
Flowed lattice correlator to $\overline{\text{MS}}$

$$G_{EE}^{\overline{\text{MS}}}(\tau) = \lim_{t_F \to 0} e^{\left(\frac{d}{\sqrt{8t_F}} + m_0^{\overline{\text{MS}}}\right)\tau} Z_{EE}^{\overline{\text{MS}}} G_{EE}(t_F, \tau).$$

$Z_{EE}^{\overline{\text{MS}}} = c_{\parallel\perp}^2 = 1 + \mathcal{O}(\alpha_s^2)$;
$d = 0.315(6)$ fitted from lattice, agrees with pQCD;
$m_0^{\overline{\text{MS}}}$ fixed by comparing lattice with NLO $\overline{\text{MS}}$ result.

$\tau_* \simeq 0.19\,\text{fm}$, $\Lambda = 1.17\,\text{GeV}$: $\mathcal{E}_3 = 2.3 \pm 0.60_{\text{stat}} \pm 0.15_{\text{syst}} \pm 0.11_{\text{h.o.}}$.

Compare with other determinations of $\mathcal{E}_3$



The values obtained from experiments are run to a common scale $\Lambda = 1.17\,\mathrm{GeV}$ using the appropriate number n_f of active massless flavors. When comparing quenched lattice results, we run to the common scale with $n_f = 0$.

- Data-driven pNRQCD and the charmonium NRQCD determination overlap with this work.
- The bottomonium NRQCD lattice result is the main outlier from universality.
- This work has the smallest uncertainty among the compatible determinations.

$\mathcal{E}_3$ from lattice agrees with χ_c data and predicts $\chi_b(1P\text{-}3P)$

Charmonium widths [MeV]

State	Prediction	Experiment
$\chi_{c0}(1P)$	13.19 ± 4.35	12.2 ± 0.5
$\chi_{c1}(1P)$	0.69 ± 0.28	0.552 ± 0.041
$\chi_{c2}(1P)$	1.66 ± 0.59	1.60 ± 0.06

The dominant charmonium error is the leading-order truncation in the velocity expansion.

Bottomonium predictions [MeV]

State	Prediction	State	Prediction
$\chi_{b0}(1P)$	1.11 ± 0.14	$\chi_{b0}(2P)$	1.42 ± 0.17
$\chi_{b1}(1P)$	0.13 ± 0.02	$\chi_{b1}(2P)$	0.17 ± 0.03
$\chi_{b2}(1P)$	0.25 ± 0.04	$\chi_{b2}(2P)$	0.32 ± 0.04
$\chi_{b0}(3P)$	1.64 ± 0.20		
$\chi_{b1}(3P)$	0.19 ± 0.04		
$\chi_{b2}(3P)$	0.36 ± 0.05		

The same strategy extends to other gluonic correlators related to quarkonium decay and production and to quasi-PDFs/LCDAs.

Heavy-quark mass effects: two-step vs. one-step matching

Y.-H. Gui, X.-Q. Li, **X.-P. Wang** and Y.-D. Yang, draft in preparation.

Two-step: $\delta \ll 1$

$$O_i(t) \xrightarrow[\mu_t \gg m_Q]{\text{SFTX}} \sum_j \zeta_{ij} O_j^{\overline{\text{MS}}}$$

$$\xrightarrow[m_Q \gg \Lambda_{\text{QCD}}]{\text{QCD}\to\text{HQET}} \sum_{j,\alpha} \zeta_{ij} C_{j\alpha} O_\alpha^{\text{HQET}},$$

$$c_{i\alpha}^{\text{2step}} = \sum_j \zeta_{ij} C_{j\alpha} + \mathcal{O}\left(\delta, \Lambda_{\text{QCD}}/m_Q\right).$$

One-step: finite δ

$$O_i(t; m_Q) = \sum_\alpha c_{i\alpha}(t, m_Q, \mu) O_\alpha^{\text{HQET}}(\mu) + \mathcal{O}\Big(\Lambda_{\text{QCD}}/m_Q, t\Lambda_{\text{QCD}}^2\Big).$$

Retains the $\delta \equiv 2m_Q^2 t$ dependences.

Flowed-QCD to QCD conversion

$$O_i(t; m_Q) = \sum_j \xi_{ij}(t, m_Q, \mu) O_j^{\overline{\text{MS}}}(\mu) + \mathcal{O}(t),$$

$$c_{i\alpha} = \sum_j \xi_{ij} C_{j\alpha}, \qquad \xi_\rho = \frac{c_\rho(t, m_Q, \mu)}{C_\rho(m_Q, \mu)} \quad \text{(no mixing case).}$$

Heavy-light current as an example (scalar)

Flowed QCD scalar current to HQET matching: $\mathring{j}(t) = \mathring{\bar{\chi}}_q(t)\mathring{\chi}_Q(t) = c_S(t, m_Q, \mu)\,\widetilde{J}_S(\mu) + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{m_Q}, \Lambda_{\rm QCD}^2 t\right)$,
with $\delta \equiv 2m_Q^2 t$, $L_m \equiv \ln\delta + \gamma_E$ and $L_t \equiv \ln(2t\mu^2) + \gamma_E$.

Full one-loop result

$$c_S = e^{\delta/2}\Bigg\{1 + \frac{\alpha_s}{4\pi}C_F\Bigg[-3 + 2\ln 2 - 3\ln 3 + \frac{3}{2}(L_m - L_t) - 8e^{-\delta/4} + \frac{4+2e^{-\delta/4}}{\delta}$$
$$- 4\sqrt{\pi\delta}\,\mathrm{Erf}\left(\frac{\sqrt{\delta}}{2}\right) + \frac{7}{2}\Gamma\left(0, \frac{\delta}{4}\right) + \int_0^1 dx\left(-\frac{6e^{-\delta/(4-6x+2x^2)}}{\delta} - \frac{(-15+2x+x^2)\,\mathrm{Ei}\left(\frac{(1+x)^2\delta}{4(-1+x)}\right)}{2(1+x)^2}\right)\Bigg]\Bigg\}.$$

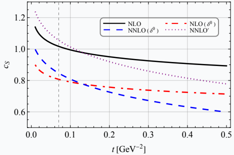
Small-δ expansion

$$c_S = 1 + \frac{\alpha_s}{4\pi}C_F\left[-4\ln 2 - 3\ln 3 - \frac{3}{2}(L_m + L_t)\right]$$
$$+ \delta\left\{\frac{1}{2} + \frac{\alpha_s}{4\pi}C_F\left[-\frac{21}{8} - 2\ln 2 - \frac{3}{2}\ln 3 - \frac{3}{4}(L_m + L_t)\right]\right\} + \mathcal{O}(\delta^2).$$

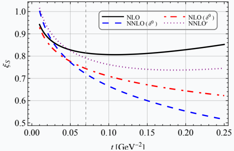
The δ^0 term reproduces SFTX $\times$ QCD-to-HQET two-step matching; higher powers are new finite-mass information.

Scalar-current mass effects in matching and conversion

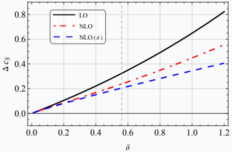
Matching coefficient $c_S(t)$



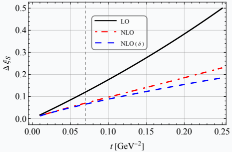
Conversion coefficient $\xi_S(t)$



Finite-mass part $\Delta c_S(\delta)$



Finite-mass part $\Delta \xi_S(t)$



Finite-mass effects are significant and more important than NNLO correction in both c_S and ξ_S.

Divide by ξ_S before the zero-flow-time extrapolation.

Summary and Outlook

Summary

- **Off-lightcone Wilson-line operators:** Matching of non-local operators is reduced to matching of local current operators, greatly simplifying the calculation.
- **P-wave quarkonium inclusive decays:** Combining pNRQCD with gradient-flow lattice QCD gives first-principle predictions that agree well with data.
- **Mass-dependent matching:** Heavy-quark mass effects are important in gradient-flow-to-$\overline{\mathrm{MS}}$ matching; we have developed a systematic framework that retains them.

Outlook

- Two-loop matching
- Mass effects for quarkonium operators
- Heavy-flavor nonperturbative parameters
- Renormalon subtractions using gradient flow

Gradient Flow Workshop 2027 in Wuhan

Gradient Flow in QCD and Strongly Coupled Field Theories

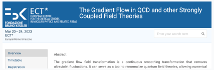
2–5 March 2027 | Wuhan, China

Hosted by C3NT, Institute of Particle Physics, Central China Normal University.

Contact: xpwang@ccnu.edu.cn

Workshop website

The international gradient-flow workshop series



Trento, Italy

20–24 March 2023

Zürich, Switzerland

12–14 February 2025

Edinburgh Gradient Flow Workshop

Edinburgh, UK

12–15 May 2026