The improved moment QCD sum rules

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July 13–19, 2026

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arxiv:

Abstract

QCD sum rules are widely used non-perturbative tools in hadron physics. Among various formulations, the Laplace and moment sum rules are the most common. However, the Laplace approach suffers from subjectivity in parameter selection, while the moment method cannot extract the coupling of the interpolating current. Moreover, the masses obtained from these two methods are often inconsistent. In this work, we propose an improved version of the moment sum rules that addresses these issues. By explicitly incorporating quark-hadron duality, which introduces an approximation condition $\boldsymbol{\delta'_n(s_0,Q_0^2)\approx\delta'_{n+1}(s_0,Q_0^2)}$ and naturally constrains the parameters as an a posteriori check --- and by further imposing rigorous dependence conditions on parameters $\boldsymbol{(Q_0^2,n)}$, our framework uniquely determines the physical scale $\boldsymbol{s_0}$ and mass $\boldsymbol{m_X}$ without any subjective input.

Laplace sum rules

We always start from two-point correlation function *Nucl.Phys.B 147 (1979) 385-447*, *Nucl.Phys.B 147 (1979) 448-518*

$$\Pi(q^2)=\mathrm{i}\int \mathrm{d}^d x\, \mathrm{e}^{\mathrm{i}q\cdot x}\langle 0|T[J(x)J^+(0)]|0\rangle$$

Consider the "narrow resonance" assumption and quark-hadron duality

$$\rho^{\mathrm{PH}}(s)=f_X^2\delta(s-m_X^2)+\rho^h(s)\theta(s-s_h)$$

$$\rho^h(s)=\frac{1}{\pi}\mathrm{Im}\Pi^{\mathrm{OPE}}(s)\theta(s-s_0)\ \text{ or }\ \int_{s_h}^{\infty}\frac{\rho^h(s)\mathrm{d}s}{s-q^2}=\int_{s_0}^{\infty}\frac{\rho^{\mathrm{OPE}}(s)\mathrm{d}s}{s-q^2}$$

The hadron mass in Laplace sum rules is extracted to be

$$m_X^2=\frac{\int_{s_N}^{s_0}\rho^{\mathrm{OPE}}(s)\,s\,\mathrm{e}^{-s/M_B^2}\mathrm{d}s}{\int_{s_N}^{s_0}\rho^{\mathrm{OPE}}(s)\,\mathrm{e}^{-s/M_B^2}\mathrm{d}s}$$

$\boldsymbol{m_X}$ depends on $\boldsymbol{s_0}$ and $\boldsymbol{M_B^2}$. They are constrained by the convergence and pole contribution (PC), both of which are inherently subjective.

$$\left|\frac{\Pi_{\mathrm{dim}=6}(M_B^2)}{\Pi_{\mathrm{pert}}(M_B^2)}\right|\le 25\%\quad \left|\frac{\Pi_{\mathrm{dim}=8}(M_B^2)}{\Pi_{\mathrm{pert}}(M_B^2)}\right|\le 10\%,\quad \left|\frac{\Pi_{\langle G^2\rangle}(M_B^2)}{\Pi_{\mathrm{pert}}(M_B^2)}\right|\le 30\%$$

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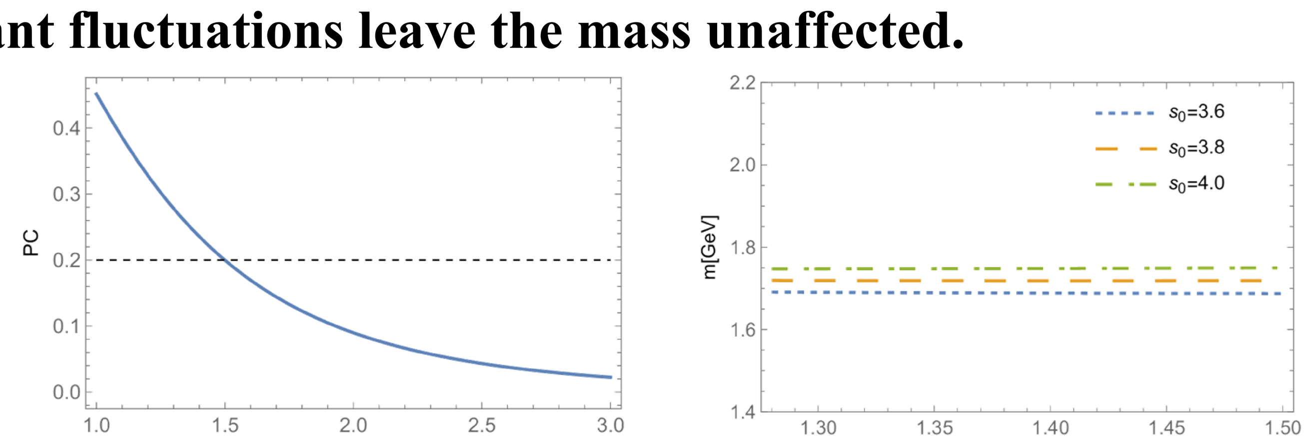
A contradiction exists in the definition of PC: within the Borel window, the mass is stable, whereas the PC varies considerably.

If PC were physical, it should not be sensitive to $\boldsymbol{M_B^2}$; nor should its significant fluctuations leave the mass unaffected.



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Moment sum rules

Instead of Borel transform, the moment is defined by *Phys.Rept. 127 (1985) 1*

$$M_n(Q_0^2)=\frac{1}{\Gamma(n+1)}\left(-\frac{\mathrm{d}}{\mathrm{d}Q^2}\right)^n\Pi(Q^2)|_{Q^2=Q_0^2}$$

We also adopt "narrow resonance" and consider the ratio of moment

$$M_n^{\mathrm{PH}}(Q_0^2)=\frac{f_X^2}{(m_X^2+Q_0^2)^{n+1}}[1+\delta_n(s_h,Q_0^2)]$$

$$r^{\mathrm{PH}}(n,Q_0^2)=\frac{M_n^{\mathrm{PH}}(Q_0^2)}{M_{n+1}^{\mathrm{PH}}(Q_0^2)}=(m_X^2+Q_0^2)\frac{1+\delta_n(s_h,Q_0^2)}{1+\delta_{n+1}(s_h,Q_0^2)}$$

For sufficiently large $\boldsymbol{n}$: $\boldsymbol{\delta_n(s_h,Q_0^2)\approx\delta_{n+1}(s_h,Q_0^2)}$. The mass is

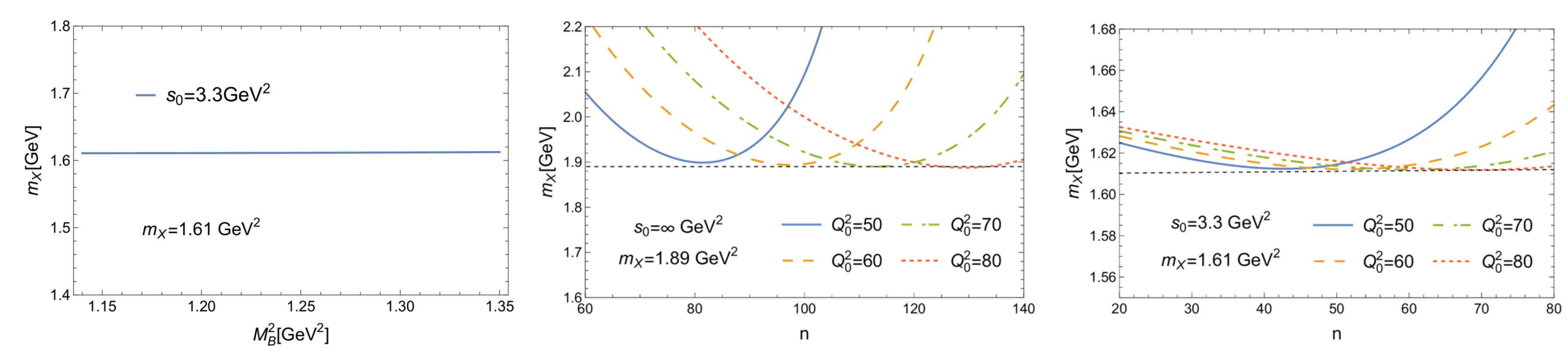
$$r^{\mathrm{OPE}}(n,Q_0^2)=m_X^2+Q_0^2\ \Rightarrow\ m_X=\sqrt{r^{\mathrm{OPE}}(n,Q_0^2)-Q_0^2}$$

Ignorance of $\boldsymbol{\rho^h(s)}$ prevents the determination of coupling $\boldsymbol{f_X}$.

A more striking issue is the systematic discrepancy between two methods: the moment method yields a larger mass than the Laplace sum rules.

The improved moment sum rules

The reason why moment sum rules yield a larger mass is the absence of quark-hadron duality. Using the same upper limit of integration $\boldsymbol{s_0}$ eliminates the discrepancy between two methods.



Then, quark-hadron duality divides moment into two parts

$$\widetilde{M}_n^{\mathrm{OPE}}(Q_0^2)=\int_{s_N}^{s_0}\frac{\rho^{\mathrm{OPE}}(s)}{(s+Q_0^2)^{n+1}}\mathrm{d}s+M_n^{\mathrm{NS}}(Q_0^2)\qquad M'^{\mathrm{OPE}}_n(Q_0^2)=\int_{s_0}^{\infty}\frac{\rho^{\mathrm{OPE}}(s)}{(s+Q_0^2)^{n+1}}\mathrm{d}s$$

$$\delta'_n(s_0,Q_0^2)=\frac{M'^{\mathrm{OPE}}_n(Q_0^2)}{\widetilde{M}_n^{\mathrm{OPE}}(Q_0^2)}\qquad \text{Duality}\Rightarrow\delta'_n(s_0,Q_0^2)=\delta_n(s_h,Q_0^2)$$

Approximation: $\boldsymbol{\delta_n(s_h,Q_0^2)\approx\delta_{n+1}(s_h,Q_0^2)\rightarrow\delta'_n(s_0,Q_0^2)\approx\delta'_{n+1}(s_0,Q_0^2)}$.

$$m_X^2+Q_0^2=\frac{\widetilde{M}_n^{\mathrm{OPE}}(s_0,Q_0^2)}{\widetilde{M}_{n+1}^{\mathrm{OPE}}(s_0,Q_0^2)}\qquad \eta(s_0,Q_0^2,n)=\frac{\delta'_n(s_0,Q_0^2)}{\delta'_{n+1}(s_0,Q_0^2)}$$

Dependence conditions: $\boldsymbol{s_0}$ is physical and $\boldsymbol{(Q_0^2,n)}$ are unphysical.

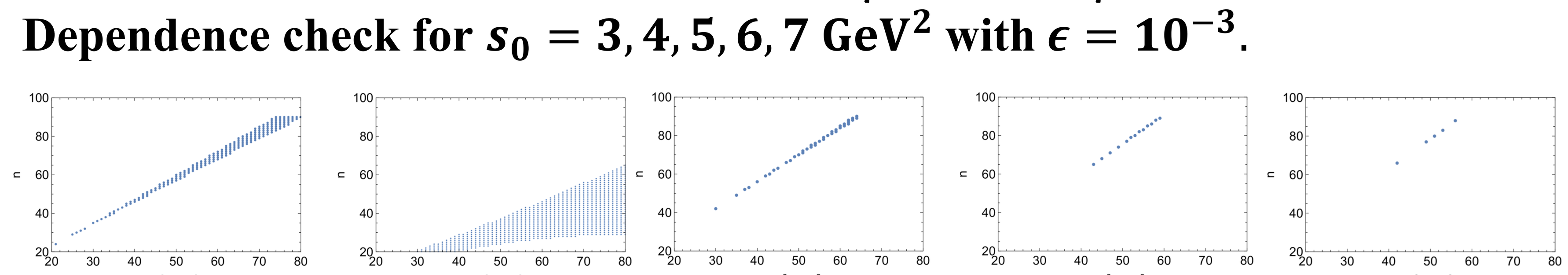
$$\frac{\partial m_X^2}{\partial Q_0^2}<\epsilon\qquad \frac{\partial m_X^2}{\partial n}<\epsilon\ \mathrm{GeV}^2$$

We can uniquely determine $\boldsymbol{s_0}$ without any artificial criteria.

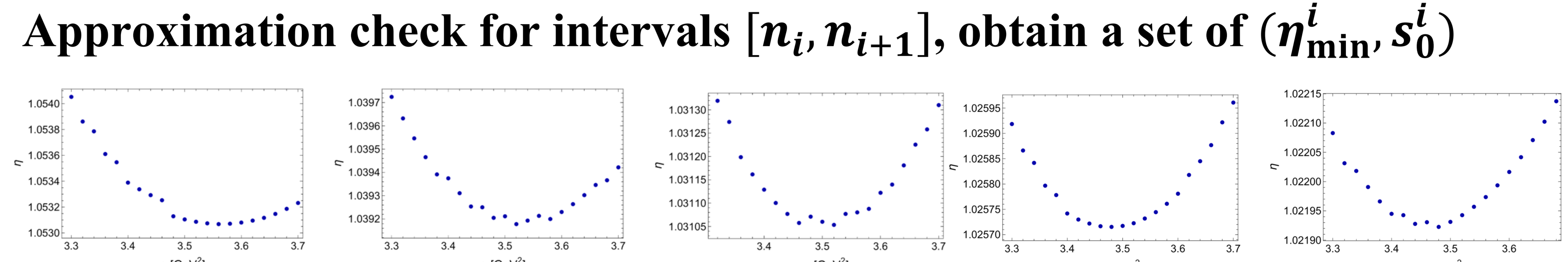
Applications

We present a specific example: $\boldsymbol{J=u_a^TC\gamma_\mu\gamma_5 d_b[\bar{d}_a\gamma_\mu C\bar{s}_b^T+(a\leftrightarrow b)]}$.

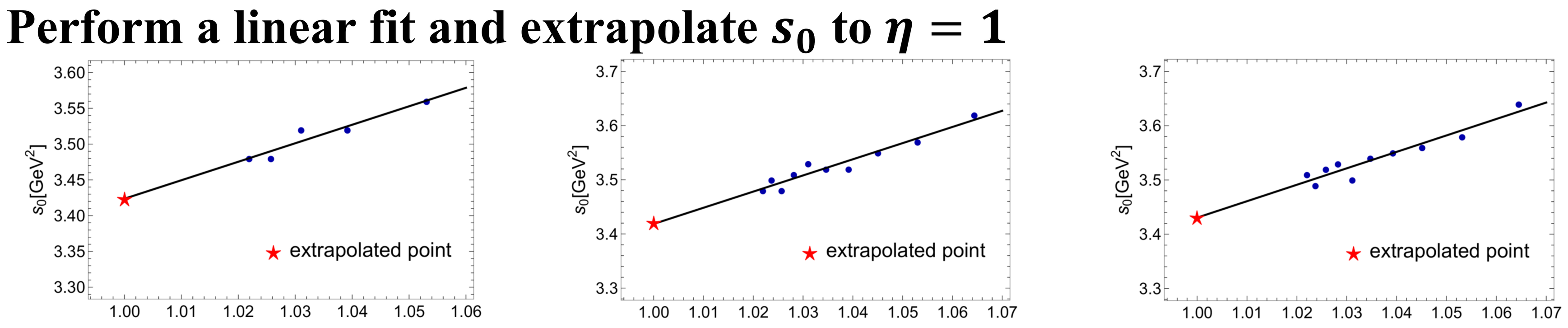
Dependence check for $\boldsymbol{s_0=3,4,5,6,7\ \mathrm{GeV}^2}$ with $\boldsymbol{\epsilon=10^{-3}}$.



Approximation check for intervals $\boldsymbol{[n_i,n_{i+1}]}$, obtain a set of $\boldsymbol{(\eta^i_{\mathrm{min}},s_0^i)}$



Perform a linear fit and extrapolate $\boldsymbol{s_0}$ to $\boldsymbol{\eta=1}$



$\Delta n=20,\Delta s_0=\frac{1}{50}\ \mathrm{GeV}^2,\epsilon=\frac{10^{-3}}{4}$ — $s_0(\eta=1)=3.42\pm0.03\ \mathrm{GeV}^2$

$\Delta n=10,\Delta s_0=\frac{1}{100}\ \mathrm{GeV}^2,\epsilon=\frac{10^{-3}}{4}$ — $s_0(\eta=1)=3.42\pm0.02\ \mathrm{GeV}^2$

$\Delta n=10,\Delta s_0=\frac{1}{100}\ \mathrm{GeV}^2,\epsilon=10^{-4}$ — $s_0(\eta=1)=3.43\pm0.03\ \mathrm{GeV}^2$

Summary

- The Laplace sum rules have long been constrained by subjective criteria in parameter selection. The physical meaning of PC is questionable.
- The conventional moment sum rules yield a larger mass than the Laplace sum rules do, and fail to extract the coupling.
- Our improved moment sum rules uniquely determine the scale $\boldsymbol{s_0}$ and mass $\boldsymbol{m_X}$ free from any artificial criteria.