Non-topological and topological strings in the grand unified theories

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Vortices/Strings in field theories

- The vortices/strings arise in field theories with the SSB of global/local $\mathrm{U}(1)$ symmetries, e.g., $\mathrm{U}(1) \to \emptyset$ and $\mathrm{U}(1)_1 \otimes \mathrm{U}(1)_2 \to \mathrm{U}(1)$. Non-trivial first homotopy group of $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$, characterized by the winding number (vorticity) $n \in \mathbb{Z}$.
- Global strings in the global Landau-Ginzburg (LG) theory

$$\mathcal{L} = |\partial_\mu \Phi|^2 - \lambda \left(|\Phi|^2 - \frac{1}{2} v^2 \right)^2 , \quad \Phi \to e^{i\alpha} \Phi . \tag{1}$$

- Local Abrikosov-Nielsen-Olesen (ANO) strings in the Abelian Higgs model (AHM)

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \Phi|^2 - \lambda \left(|\Phi|^2 - \frac{1}{2} v^2 \right)^2 , \\ & \Phi \to e^{i\alpha(x)} \Phi , \quad A_\mu \to A_\mu + \frac{1}{e} \partial_\mu \alpha(x) . \end{aligned} \tag{2}$$

Vortices/Strings in field theories

- The global string profile function in the LG theory

$$\phi_{\rm ANO} = \frac{v}{\sqrt{2}} \bar{f}(r) e^{in\varphi}\,, \quad n \in \mathbb{Z}\,. \tag{3}$$

- The global string tension

$$\mu = 2\pi \cdot \frac{v^2}{2} \int \xi d\xi \left\{ \left(\frac{d\bar{f}(\xi)}{d\xi} \right)^2 + n^2 \frac{\bar{f}^2(\xi)}{\xi^2} + \frac{1}{4} \left(\bar{f}^2(\xi) - 1 \right)^2 \right\} \sim \log \Lambda\,, \tag{4}$$

with $\xi \equiv \sqrt{2\lambda v^2} \cdot r$, and the Euler-Lagrange equation of

$$\begin{aligned} &\frac{d^2 \bar{f}(\xi)}{d\xi^2} + \frac{1}{\xi} \frac{d\bar{f}(\xi)}{d\xi} - \frac{n^2}{\xi^2} \bar{f}(\xi) - \frac{1}{2} \left(\bar{f}^2(\xi) - 1 \right) \bar{f}(\xi) = 0\,, \\ &\bar{f}(\xi = 0) = 0\,, \quad \bar{f}(\xi = \infty) = 1\,. \end{aligned} \tag{5}$$

A symmetry restored string core.

- The string profiles in the AHM

$$\Phi_{\rm ANO} = \frac{v}{\sqrt{2}} \bar f(r) e^{in\varphi} \,, \quad \vec A_{\rm ANO} = -n \frac{\bar\zeta(r)}{r} \vec e_\varphi \,, \quad n \in \mathbb{Z} \,, \tag{6}$$

and the finite tension of

$$\begin{aligned} \mu_{\rm ANO} &= 2\pi v^2 \int_0^\infty \xi d\xi \Big\{ \frac{e^2}{4} \frac{1}{\xi^2} \Big(-n \frac{d\bar\zeta(\xi)}{d\xi} \Big)^2 + \frac{1}{2} \Big[(\frac{d\bar f(\xi)}{d\xi})^2 \\ &\quad + n^2 \left(1 - e\bar\zeta(\xi)\right)^2 \frac{\bar f^2(\xi)}{\xi^2} \Big] + \frac{\beta}{4} \left(\bar f^2(\xi) - 1 \right)^2 \Big\} \,, \end{aligned} \tag{7}$$

with $\xi \equiv \frac{ev}{\sqrt{2}} r$ and $\beta \equiv \frac{m_H^2}{m_V^2} = \frac{2\lambda}{e^2}$.

- BCs: $\bar f(\xi = 0) = \bar\zeta(\xi = 0) = 0$, $\bar f(\xi = \infty) = 1$, $\bar\zeta(\xi = \infty) = \frac{1}{e}$.

Vortices/Strings in field theories

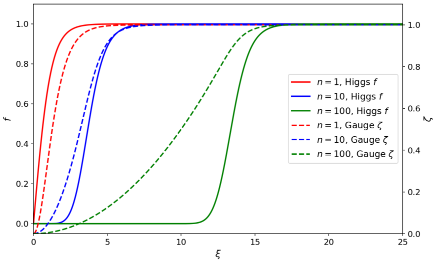


Figure: The profiles of the Higgs and gauge fields in the AHM (with $e = 1.0$). Credit: BIAN Zhengyang.

String core size scales as $\xi_n \propto \sqrt{n}$ as $n \to \infty$ [Penin, Weller, 2009.06640, 2105.12137].

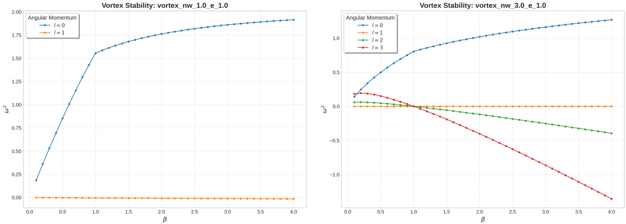


Figure: The eigenvalues for the perturbed modes in the AHM. Credit: BIAN Zhengyang.

Stable string for $n = 1$: any β, stable string for $n \geq 2$: $\beta < 1$ [1976, Bogomol'nyi]. The eigenvalues ω^2 for the coupled ODEs of [1995, Goodband, Hindmarsh]

$$\mathcal{D}^{nm}\delta\Theta_m = 0\,, \quad \delta\Theta_m = (\delta\phi e^{-i\omega t}\,, \delta\phi^\dagger e^{i\omega t}\,, \delta A_m e^{-i\omega t})^T\,. \tag{8}$$

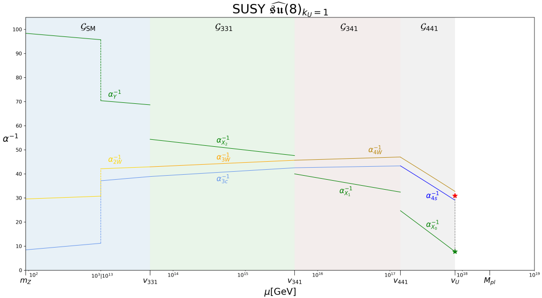
The $\mathrm{SU}(8)$ theory: motivation

- Q: Do we have strings in the non-Abelian theories, such at the GUT?
- Usually, the GUT contains ‘t Hooft-Polyakov monopole with $\mathrm{SU}(5) \to \mathrm{SU}(3)_c \otimes \mathrm{SU}(2)_W \otimes \mathrm{U}(1)_Y$, and the domain walls with discrete $\mathbb{Z}_2$ in the Higgs sector.
- A recently proposed $\mathrm{SU}(8)$ theory, by myself and Teng, Mao, Hou and etc. This is motivated to solve the SM quark/lepton flavor puzzle, by abandoning the “generations” in the UV [1979, H. Georgi]

$$\left[4 \times \overline{\mathbf{8_F}} \oplus \mathbf{28_F} \right] \bigoplus \left[5 \times \overline{\mathbf{8_F}} \oplus \mathbf{56_F} \right] \, .$$

https://arxiv.org/abs/2307.07921
https://arxiv.org/abs/2402.10471
https://arxiv.org/abs/2406.09970
https://arxiv.org/abs/2411.12979
Talk: https://www.koushare.com/live/details/40893

The RGE of the $\mathcal{N}=1$ $\widehat{\mathfrak{su}}(8)_{k_U=1}$ theory



- In the $\widehat{\mathfrak{su}}(8)_{k_U=1}$ theory [2411.12979], we find the conformal embedding of $\widehat{\mathfrak{su}}(4)_{k_s=1} \oplus \widehat{\mathfrak{su}}(4)_{k_W=1} \oplus \widehat{\mathfrak{u}}(1)_{k_1=1/4}$. The gauge coupling unification is achieved through the relation of

$$g_{4s}^2 = g_{4W}^2 = \frac{1}{4} g_{X_0}^2 = \frac{8\pi G_N}{\alpha'}\,. \tag{9}$$

The $\mathrm{SU}(8)$ theory: global symmetries

- The global symmetries of the $\mathrm{SU}(8)$ theory:

$$\widetilde{\mathcal{G}}_{\rm global}\left[\mathrm{SU}(8)\right] = \left[\widetilde{\mathrm{SU}}(4)_\omega \otimes \widetilde{\mathrm{U}}(1)_{T_2} \otimes \widetilde{\mathrm{U}}(1)_{\mathrm{PQ}_2}\right] \bigotimes \left[\widetilde{\mathrm{SU}}(5)_{\dot\omega} \otimes \widetilde{\mathrm{U}}(1)_{T_3} \otimes \widetilde{\mathrm{U}}(1)_{\mathrm{PQ}_3}\right] ,$$
$$\left[\mathrm{SU}(8)\right]^2 \cdot \widetilde{\mathrm{U}}(1)_{T_{2,3}} = 0 \,, \quad \left[\mathrm{SU}(8)\right]^2 \cdot \widetilde{\mathrm{U}}(1)_{\mathrm{PQ}_{2,3}} \neq 0 \,. \tag{10}$$

- The Higgs fields and the Yukawa couplings:

$$-\mathcal{L}_Y = Y_{\mathcal{B}} \overline{\mathbf{8_F}}^{\omega} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega} + Y_{\mathcal{T}} \mathbf{28_F} \mathbf{28_F} \mathbf{70_H} + Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\dot\omega} \mathbf{56_F} \overline{\mathbf{28_H}}_{,\dot\omega} + \frac{c_4}{M_{\rm pl}} \mathbf{56_F} \mathbf{56_F} \overline{\mathbf{28_H}}^{\dagger}_{,\dot\omega} \mathbf{63_H} + H.c. \,. \tag{11}$$

NB: $\mathbf{56_F}\mathbf{56_F}\mathbf{28_H} = 0$ ['08, S. Barr], $d=5$ operator suppressed by $1/M_{\rm pl}$ is possible, with $M_{\rm pl} = (8\pi G_N)^{1/2} = 2.4\times 10^{18}\,\mathrm{GeV}$.

The SU(8) theory: global symmetries

Higgs	$\mathcal{G}_{441} \to \mathcal{G}_{341}$	$\mathcal{G}_{341} \to \mathcal{G}_{331}$	$\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$	$\mathcal{G}_{\rm SM} \to \mathrm{SU}(3)_c \otimes \mathrm{U}(1)_{\rm EM}$
$\overline{\mathbf{8_H}}_{,\omega}$	✓$(\omega = \mathrm{IV})$	✓$(\omega = \mathrm{V})$	✓$(\omega = 3\,, \mathrm{VI})$	✓
$\overline{\mathbf{28_H}}_{,\dot\omega}$	✗	✓$(\dot\omega = \dot 1\,, \dot{\mathrm{V}}\mathrm{II})$	✓$(\dot\omega = \dot 2\,, \mathrm{V}\dot{\mathrm{I}}\mathrm{II}\,, \dot{\mathrm{I}}\mathrm{X})$	✓
$\mathbf{70_H}$	✗	✗	✗	✓

- This talk: based on $\mathrm{SU}(6) \to \mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$ toy model

$$\{\mathcal{F}_L\} \equiv \overline{\mathbf{6_F}}^\omega \oplus \mathbf{15_F}\,, \quad \{\mathcal{H}\} \equiv \overline{\mathbf{6_H}}_{,\omega} \oplus \mathbf{15_H}\,. \tag{12}$$

For the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$ breaking, we consider the non-topological and topological strings from [2604.06530]

$$\Phi_{\overline{\mathbf{3}},\omega} \equiv (\mathbf{1}\,, \overline{\mathbf{3}}\,, -\frac{1}{3})_{\mathbf{H},\omega} \subset \overline{\mathbf{6_H}}_{,\omega}\,, \quad \omega = (1\,,2)\,. \tag{13}$$

The embedded Z string in the SM

- The semilocal string: to consider a $\mathrm{U}(1) \otimes \widetilde{\mathrm{SU}}(2)$ model with $\Phi \equiv (\phi_1\,, \phi_2)^T \in \mathbb{C}$, [1991-1992, Hindmarsh, Vachaspati].
- Promote to the local $\mathrm{SU}(2)_W \otimes \mathrm{U}(1)_Y$, with the string profiles of

$$\Phi_{\mathrm{ANO}} = \frac{v}{\sqrt{2}} \bar{f}(r) e^{in\varphi} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,, \quad \vec{Z} = -n \frac{\bar{z}(r)}{r} \vec{e}_\varphi\,, \quad n \in \mathbb{Z}\,, \tag{14}$$

and the integer quantized magnetic flux of

$$\Phi_Z^{\mathrm{non-topo}} = \int d^2x\, Z_{12} = -\frac{4\pi n}{g_Z}\,, \quad g_Z^2 \equiv g_{2W}^2 + g_Y^2\,. \tag{15}$$

The embedded Z string in the SM

- In the Z string background, we have

$$d_m \Phi_{\rm ANO} = \Big[\partial_m \mathbb{I}_2 + \frac{i}{2} g_Z \begin{pmatrix} -\cos 2\vartheta_W & \\ & +1 \end{pmatrix} Z_{{\rm ANO}\,,m} \Big] \Phi_{\rm ANO}\,. \qquad (16)$$

- The classical Euler-Lagrange equations

$$\frac{d^2 \bar f(\xi)}{d\xi^2} + \frac{1}{\xi} \frac{d\bar f(\xi)}{d\xi} - \Big(1 - \frac{g_Z}{2} \bar z(\xi) \Big)^2 \frac{\bar f(\xi)}{\xi^2}$$
$$-\beta \left(\bar f^2(\xi) - 1 \right) \bar f(\xi) = 0\,, \qquad (17a)$$

$$\frac{d^2 \bar z}{d\xi^2} - \frac{1}{\xi} \frac{d\bar z}{d\xi} + \frac{4}{g_Z} \bar f^2(\xi) \Big(1 - \frac{g_Z}{2} \bar z(\xi) \Big) = 0\,, \qquad (17b)$$

with BCs of: $\bar f(\xi = 0) = \bar z(\xi = 0) = 0$, $\bar f(\xi = \infty) = 1$, and $\bar z(\xi = \infty) = \frac{2}{g_Z}$.

- $\beta \equiv \frac{m_H^2}{m_Z^2} = \frac{4\lambda}{g_Z^2}$ for $V(\Phi) = \frac{1}{2}\lambda \left(|\Phi|^2 - \frac{1}{2} v^2 \right)^2$.

The classical stability of the Z string in the SM

The procedures of analyzing the classical stability of the Z string background [hep-ph/9212301]:

- To impose the perturbations to the Z string background

$$\Phi = \begin{pmatrix} \delta\pi^+ \\ \phi_{\rm ANO} \end{pmatrix} , \quad \vec{W}^\pm = \delta \vec{W}^\pm . \tag{18}$$

- The perturbed string tension

$$\begin{aligned} \mu &= \mu_{\rm ANO} + \delta\mu_{(2)} , \\ \delta\mu_{(2)} &= \delta\mu_W[\delta\vec{W}^\pm] + \delta\mu_\pi[\delta\pi^\pm] + \delta\mu_c[\delta\pi^\pm , \delta\vec{W}^\pm] . \end{aligned} \tag{19}$$

- The gauge fixing terms

$$\begin{aligned} \mathcal{L}_{\rm fix} &= \frac{1}{2} \left(|F(W_\mu^+ , \delta\pi^+)|^2 + |F(W_\mu^- , \delta\pi^-)|^2 \right) , \quad F(W_\mu^\pm , \delta\pi^\pm) = \partial^\mu \delta W_\mu^\pm \\ &\mp igc_{\vartheta_W} Z^\mu_{\rm ANO} \delta W_\mu^\pm \pm \frac{ig}{\sqrt{2}} \phi^*_{\rm ANO}/\phi_{\rm ANO} \, \delta\pi^\pm . \end{aligned} \tag{20}$$

The classical stability of the Z string in the SM [cont.]

- An eigenvalue problem for the coupled Helmholtz equations in the basis of $\delta\Theta \equiv (\delta\pi^+ e^{i\omega t}, \delta W_\uparrow e^{i\omega t}, \delta W_\downarrow e^{i\omega t})^T$

$$\hat{\mathcal{O}}\delta\Theta = \omega^2 \delta\Theta\,, \quad \hat{\mathcal{O}} = \begin{pmatrix} \mathcal{D}_{11} & \mathcal{B}_\uparrow & \mathcal{B}_\downarrow \\ \mathcal{B}_\uparrow & \mathcal{D}_\uparrow & 0 \\ \mathcal{B}_\downarrow & 0 & \mathcal{D}_\downarrow \end{pmatrix}\,, \tag{21}$$

  with

$$\begin{aligned} \mathcal{D}_{11} &= -\vec{\nabla}_\xi^2 + 2c_W^2 \bar{f}^2 + \beta(\bar{f}^2 - 1)\,, \\ \mathcal{D}_{\uparrow/\downarrow} &= -\vec{\nabla}_\xi^2 + 2c_W^2 \bar{f}^2 \pm 4c_W^2 \frac{1}{\xi}\frac{d\bar{z}}{d\xi}\,. \end{aligned} \tag{22}$$

- $\omega^2 > 0$: stable, $\omega^2 < 0$: unstable.

The classical stability of the Z string in the SM [cont.]

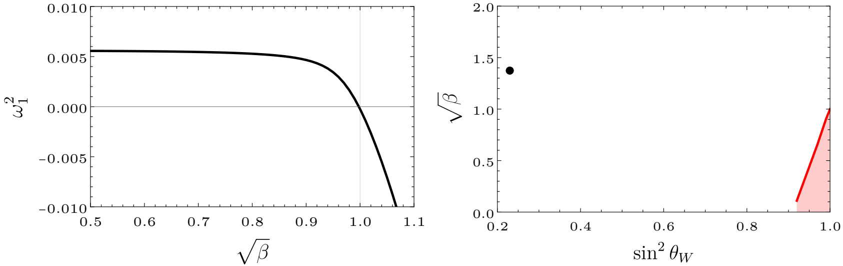


Figure: The stability region of the EW Z string: the semilocal limit of $\vartheta_W \to \frac{\pi}{2}$ and $\beta \equiv \frac{m_H^2}{m_Z^2} < 1$. Plot credit: Eto, Hamada, Jinno, Nitta, and Yamada, 2402.19471.

The classical stability of the Z string in the SM [cont.]

- The Z string stability is dynamical rather than topological.
- Two sources of the Z string instability:
  1. The instability due to the Higgs profile

     $$\mathcal{D}_{11} = -\vec{\nabla}_\xi^2 + 2c_W^2 \bar{f}^2 + \beta(\bar{f}^2 - 1)\,,$$

     with β controlling the negative contributions from the string core.
  2. The instability due to the spin-magnetic couplings

     $$\mathcal{D}_{\uparrow/\downarrow} = -\vec{\nabla}_\xi^2 + 2c_W^2 \bar{f}^2 \pm 4c_W^2 \frac{1}{\xi}\frac{d\bar{z}}{d\xi}\,.$$

     when $\pm\frac{1}{\xi}\frac{d\bar{z}}{d\xi}$ is negative.
- The semilocal limit of $\vartheta_W \to \frac{\pi}{2}$ eliminate the spin-magnetic instability.

The non-topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

- The massive off-diagonal $\mathcal{G}_{331}$ gauge bosons of $(W'^{\pm}_\mu\,, N_\mu\,, \bar N_\mu)$

$$W'^{\pm}_\mu \equiv \frac{1}{\sqrt{2}}(W^4_\mu \mp i W^5_\mu)\,, \quad N_\mu/\bar N_\mu \equiv \frac{1}{\sqrt{2}}(W^6_\mu \mp i W^7_\mu)\,, \tag{23}$$

  and the diagonal $\mathcal{G}_{331}$ gauge boson

$$\begin{aligned} Z'_\mu &\equiv c_{\vartheta_S} W^8_\mu - s_{\vartheta_S} X_\mu\,, \\ B_\mu &\equiv s_{\vartheta_S} W^8_\mu + c_{\vartheta_S} X_\mu\,. \end{aligned} \tag{24}$$

- The masses of

$$\begin{aligned} & m^2_{W'^{\pm}} = m^2_{N\,,\bar N} = \frac{1}{12} g^2_{Z'} c^2_{\vartheta_S} v^2_{331}\,, \quad m^2_{Z'} = \frac{1}{9} g^2_{Z'} v^2_{331} \\ & g^2_{Z'} \equiv 3 g^2_{3W} + g^2_X\,. \end{aligned} \tag{25}$$

The non-topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

- The $(1\,,1)$ string profile functions with unit winding

$$\vec{Z}'_{\rm ANO} = -\frac{\bar\zeta(r)}{r}\vec{e}_\varphi\,, \quad \Phi_{\overline{\mathbf{3}}\,,\omega} = \frac{V_\omega}{\sqrt{2}}\bar{f}_\omega(r)e^{i\varphi}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\,, \quad \omega = (1\,,2)\,, \tag{26}$$

  and the integer quantized magnetic flux

$$\Phi_{Z'}^{\rm non-topo} = \int d^2x\,\partial_{[1}Z'_{\rm ANO\,2]} = \frac{6\pi}{g_{Z'}}\,, \tag{27}$$

  with the BCs: $\bar\zeta(\xi=0)=0$ and $\bar\zeta(\xi=\infty) = -\frac{3}{g_{Z'}}$.

- The generic Higgs potential with two Higgs triplets

$$\begin{aligned} V(\Phi_\omega) =& \sum_\omega \frac{1}{2}\lambda_\omega\left(|\Phi_{\overline{\mathbf{3}}\,,\omega}|^2 - \frac{1}{2}V_\omega^2\right)^2 + \frac{1}{2}\lambda_3\left(\sum_\omega|\Phi_{\overline{\mathbf{3}}\,,\omega}|^2 - \frac{1}{2}\bar{V}^2\right)^2 \\ &+ \lambda_4\Big(|\Phi_{\overline{\mathbf{3}}\,,1}|^2|\Phi_{\overline{\mathbf{3}}\,,2}|^2 - |\Phi_{\overline{\mathbf{3}}\,,1}^\dagger\Phi_{\overline{\mathbf{3}}\,,2}|^2\Big) + \lambda_5\Big|\Phi_{\overline{\mathbf{3}}\,,1}^\dagger\Phi_{\overline{\mathbf{3}}\,,2} - \frac{1}{2}V_1V_2\Big|^2\,. \end{aligned} \tag{28}$$

- The string tension with the dimensionless coordinates of $\xi \equiv \frac{m_{Z'}}{\sqrt{2}} r$

$$
\begin{aligned}
\frac{\mu_{\rm ANO}}{2\pi v_{331}^2} &= \int \xi d\xi \, \rho(\xi) \, , \quad \rho(\xi) = \rho_{\bar\zeta}(\xi) + \rho_{\bar f_\omega}(\xi) + \rho_V(\xi) \, , \\
\rho_{\bar\zeta}(\xi) &= \frac{g_{Z'}^2}{36} \left(\frac{1}{\xi} \frac{d\bar\zeta(\xi)}{d\xi} \right)^2 \, , \\
\rho_{\bar f_\omega}(\xi) &= \frac{1}{2} \sum_{\omega=1\,,2} \frac{V_\omega^2}{v_{331}^2} \Big[\left(\frac{d\bar f_\omega(\xi)}{d\xi} \right)^2 + \left(1 + \frac{g_{Z'}}{3} \bar\zeta(\xi) \right)^2 \left(\frac{\bar f_\omega(\xi)}{\xi} \right)^2 \Big] \, , \\
\rho_V(\xi) &= \frac{1}{4} \beta_1 c_{\tilde\beta}^4 \left(\bar f_1^2(\xi) - 1 \right)^2 + \frac{1}{4} \beta_2 s_{\tilde\beta}^4 \left(\bar f_2^2(\xi) - 1 \right)^2 \\
&\quad + \frac{1}{4} \beta_3 \left(c_{\tilde\beta}^2 \bar f_1^2(\xi) + s_{\tilde\beta}^2 \bar f_2^2(\xi) - 1 \right)^2 + \frac{1}{2} \beta_5 s_{\tilde\beta}^2 c_{\tilde\beta}^2 \left(\bar f_1(\xi) \bar f_2(\xi) - 1 \right)^2 \, , \\
&\quad \text{where} \quad \beta_i \equiv \frac{9\lambda_i}{g_{Z'}^2} \, .
\end{aligned} \tag{29}
$$

The non-topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

- The classical field equations

$$\frac{d^2 \bar f_1(\xi)}{d\xi^2} + \frac{1}{\xi}\frac{d\bar f_1(\xi)}{d\xi} - \left(1 + \frac{g_{Z'}}{3}\bar\zeta(\xi)\right)^2 \frac{\bar f_1(\xi)}{\xi^2} = \beta_1 c_{\tilde\beta}^2 \left(\bar f_1^2(\xi) - 1\right) \bar f_1(\xi)$$
$$+\beta_3 \left(c_{\tilde\beta}^2 \bar f_1^2(\xi) + s_{\tilde\beta}^2 \bar f_2^2(\xi) - 1\right) \bar f_1(\xi)\,, \tag{30a}$$

$$\frac{d^2 \bar f_2(\xi)}{d\xi^2} + \frac{1}{\xi}\frac{d\bar f_2(\xi)}{d\xi} - \left(1 + \frac{g_{Z'}}{3}\bar\zeta(\xi)\right)^2 \frac{\bar f_2(\xi)}{\xi^2} = \beta_2 s_{\tilde\beta}^2 \left(\bar f_2^2(\xi) - 1\right) \bar f_2(\xi)$$
$$+\beta_3 \left(c_{\tilde\beta}^2 \bar f_1^2(\xi) + s_{\tilde\beta}^2 \bar f_2^2(\xi) - 1\right) \bar f_2(\xi)\,, \tag{30b}$$

$$\frac{d^2 \bar\zeta(\xi)}{d\xi^2} - \frac{1}{\xi}\frac{d\bar\zeta(\xi)}{d\xi} = \frac{6}{g_{Z'}} \left(1 + \frac{g_{Z'}}{3}\bar\zeta(\xi)\right) \left(c_{\tilde\beta}^2 \bar f_1^2(\xi) + s_{\tilde\beta}^2 \bar f_2^2(\xi)\right)\,, \tag{30c}$$

with BCs of: $\bar f_\omega(\xi = 0) = \bar\zeta(\xi = 0) = 0$, $\bar f_\omega(\xi = \infty) = -\frac{g_{Z'}}{3}\bar\zeta(\xi = \infty) = 1$.

The non-topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

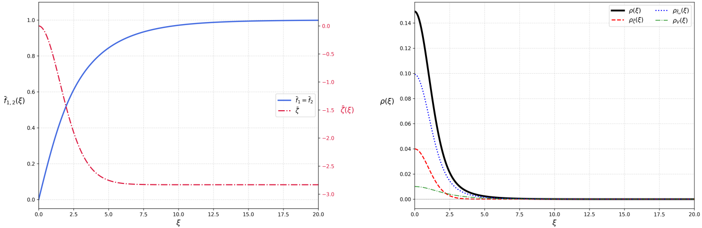


Figure: The parameters are: $\lambda_1 = \lambda_2 = \lambda_5 = 0.15$, $\lambda_3 = -0.145$, $t_{\tilde\beta} = 1.0$, and $g_{Z'} = 1.06$.

Remarks on the non-topological Z' string profile with two Higgs triplets:

- At $\xi \to \infty$, the asymptotic behaviors for the Higgs profiles read

$$\bar{f}_1(\xi) \sim 1 - \frac{1}{\sqrt{\xi}} \left(C_1 \exp\Big(-\sqrt{\beta_1 + 2\beta_3 + \beta_5} \xi \Big) + C_2 \exp\Big(-\sqrt{\beta_1} \xi \Big) \right) \,,$$
$$\bar{f}_2(\xi) \sim 1 - \frac{1}{\sqrt{\xi}} \left(C_1 \exp\Big(-\sqrt{\beta_1 + 2\beta_3 + \beta_5} \xi \Big) - C_2 \exp\Big(-\sqrt{\beta_1} \xi \Big) \right) \,, \quad (31)$$

  with $t_{\tilde{\beta}} = 1.0$, $\beta_1 = \beta_2$.

- The Higgs potential parameter constraints: $\beta_3 > -\frac{1}{2}(\beta_1 + \beta_5)$ or $\beta_3 > -\frac{1}{2}(\beta_2 + \beta_5)$.

The non-topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

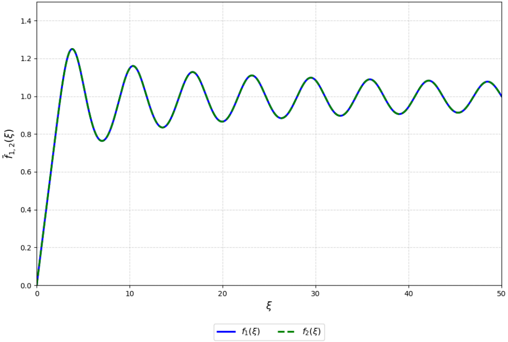


Figure: Unwanted parameters of: $\lambda_1 = \lambda_2 = 0.15$, $\lambda_3 = -0.14$, $\lambda_4 = \lambda_5 = 0$, $t_{\tilde{\beta}} = 1.0$, and $g_{Z'} = 1.06$. Credit to: GUO Mian.

The classical stability of the non-topological Z' string

The classical stability analyses:

- To impose the perturbations to the Z' string background

$$\Phi_\omega = \begin{pmatrix} \delta\pi^-_{W'\,,\omega} \\ \delta\pi^0_{\bar{N}\,,\omega} \\ \phi_{\mathrm{ANO}\,,\omega} \end{pmatrix}\,, \quad (\delta W'^{\pm}_m\,, \delta N_m\,, \delta\bar{N}_m)\,. \tag{32}$$

- The perturbed Z' string tension

$$\mu = \mu_{\mathrm{ANO}} + \delta\mu_{(2)}\,, \quad \delta\mu_{(2)} = \delta\mu_{W'} + \delta\mu_\pi + \delta\mu_c\,. \tag{33}$$

- The gauge fixing terms

$$\begin{aligned} \mathcal{L}_{\mathrm{fix}} &= F(\delta W'^+_m)\cdot F(\delta W'^-_m)\,, \\ F(\delta W'^{\pm}_m) &= \frac{m_{Z'}}{\sqrt{2}}\Big(\vec{\nabla}_\xi\cdot\delta\vec{W}'^{\pm} \mp \frac{i g_{Z'} c^2_{\vartheta_S}}{2}\frac{\bar{\zeta}(\xi)}{\xi}(s_\varphi \delta W'^+_{\xi_1} - c_\varphi \delta W'^+_{\xi_2}) \\ &\quad \pm i c_{\vartheta_S}\sqrt{\frac{3}{2}}\sum_\omega \frac{V_\omega}{v_{331}}\bar{f}_\omega(\xi) e^{\pm i\varphi}\delta\pi^{\pm}_{W'\,,\omega}\Big)\,. \end{aligned} \tag{34}$$

The classical stability of the non-topological Z' string [cont.]

- Both the $(\delta\pi^{\pm}_{W'\,,\omega}\,, \delta W'^{\pm}_m)$ and the $(\delta\pi^0_{N\,,\omega}\,, \delta\pi^0_{\bar{N}\,,\omega}\,, \delta N_m\,, \delta\bar{N}_m)$ lead to the identical results for the (in)stabilities.
- An eigenvalue problem for the coupled Helmholtz equations in the basis of $\delta\Theta \equiv (s_{\ell\,,1}\,, s_{\ell\,,2}\,, w_{\uparrow\,,\ell}, w_{\downarrow\,,\ell})^T$

$$\hat{\mathcal{O}} = \begin{pmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} & \mathcal{B}_\uparrow & \mathcal{B}_\downarrow \\ \mathcal{D}_{12} & \mathcal{D}_{22} & \mathcal{B}_\uparrow & \mathcal{B}_\downarrow \\ \mathcal{B}_\uparrow & \mathcal{B}_\uparrow & \mathcal{D}_\uparrow & 0 \\ \mathcal{B}_\downarrow & \mathcal{B}_\downarrow & 0 & \mathcal{D}_\downarrow \end{pmatrix}\,, \tag{35}$$

with $\delta\pi^+_{W'\,,\omega} = \sum_\ell s_{\ell\,,\omega} e^{-i\ell\varphi}$, $\delta W'^{+}_\uparrow = \sum_\ell -i w_{\uparrow\,,\ell} e^{-i\ell\varphi}$, and etc.

The classical stability of the non-topological Z' string [cont.]

Sources of the non-topological Z' string instability:

1. The instabilities due to the Higgs profile

$$\mathcal{D}_{11} = -\vec{\nabla}_\xi^2 + \frac{3c_{\vartheta_S}^2}{2}\sum_\omega \bar{f}_\omega^2(\xi)\frac{V_\omega^2}{v_{331}^2} \underbrace{+\beta_1 c_{\tilde{\beta}}^2(\bar{f}_1^2(\xi)-1)}_{<0(\xi\to 0)\ \text{with}\ \beta_1>0}$$
$$\underbrace{+\beta_3\left(c_{\tilde{\beta}}^2(\bar{f}_1^2(\xi)-1)+s_{\tilde{\beta}}^2(\bar{f}_2^2(\xi)-1)\right)}_{>0(\xi\to 0)\ \text{with}\ \beta_3<0} + \underbrace{\beta_4 s_{\tilde{\beta}}^2\bar{f}_2^2(\xi)}_{>0\ ,\xi\to 0}, \tag{36}$$

2. The spin-magnetic couplings of $\mathcal{D}_{\uparrow/\downarrow} = -\vec{\nabla}_\xi^2 + \frac{3c_{\vartheta_S}^2}{2}\sum_\omega \bar{f}_\omega^2\frac{V_\omega^2}{v_{331}^2} \pm g_{Z'}c_{\vartheta_S}^2\frac{1}{\xi}\frac{\partial\bar{\zeta}}{\partial\xi}$.
3. In the semilocal limit of $\vartheta_S\to\frac{\pi}{2}$, there can still be off-diagonal elements of $\mathcal{D}_{12}$ due to the mixings between $\delta\pi_{W',1}^{\pm}$ and $\delta\pi_{W',2}^{\mp}$ from the perturbed Higgs potential as well as the gauge fixing terms.

The classical stability of the non-topological Z' string [cont.]

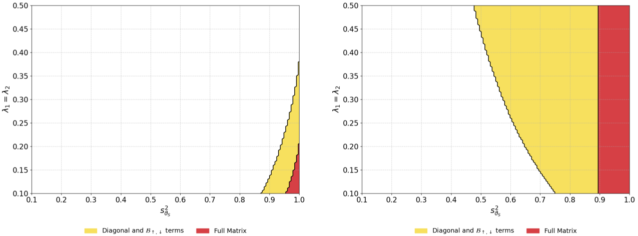


Figure: The stability regions with a fixed $\lambda_3 = -0.05$ (left panel) and a varying $\lambda_3 = -\frac{1}{2}(\lambda_1 + \lambda_5) + 0.005$ (right panel), with $g_{Z'} = 1.06$. Left panel: $\lambda_4 = \lambda_5 = 0.1$, $t_{\tilde{\beta}} = 2$ in the left panel, Right panel: $\lambda_1 = \lambda_2 = \lambda_5$, $\lambda_4 = 0$, $t_{\tilde{\beta}} = 1$. Subject to the constraint of $\lambda_3 > -\frac{1}{2}(\lambda_1 + \lambda_5)$.

The topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

- The global $\widetilde{\rm U}(1)$ symmetry between two Higgs fields in the $\mathcal{G}_{331}$: $(\Phi_{\mathbf{\overline{3}}\,,1}\,,\Phi_{\mathbf{\overline{3}}\,,2}) \to (e^{-i\alpha}\Phi_1\,, e^{i\alpha}\Phi_2)$ in Eq. (28) w.o. the λ_5 term. Similar to the situation in the 2HDM [Dvali, Senjanovic, 9305278].
- The topological $(n_1\,, n_2)$ Z' string profiles (with $n_{1\,,2} \in \mathbb{Z}$ and $n_1 \neq n_2$)

$$\Phi_{\mathbf{\overline{3}}\,,1} = \frac{V_1}{\sqrt{2}} \bar{f}_1(r) e^{in_1\varphi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\,, \quad \Phi_{\mathbf{\overline{3}}\,,2} = \frac{V_2}{\sqrt{2}} \bar{f}_2(r) e^{in_2\varphi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\,, \tag{37}$$

and the minimal structures being $(1\,,0)$ or $(0\,,1)$ strings.

- The non-integer magnetic flux of the Z' string

$$\Phi_{Z'}^{(n_1\,,n_2)} = \frac{6\pi}{g_{Z'}} \left(c_{\tilde\beta}^2 n_1 + s_{\tilde\beta}^2 n_2 \right)\,, \tag{38}$$

$\Phi_{Z'}^{(1\,,0)} = \frac{1}{2} \Phi_{Z'}^{\rm non-topo}$ in Eq. (27) when $\tilde\beta = \frac{\pi}{4}$.

The topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

- The classical field equations

$$\frac{d^2 \bar f_1(\xi)}{d\xi^2} + \frac{1}{\xi}\frac{d\bar f_1(\xi)}{d\xi} - \left(1 + \frac{g_{Z'}}{3}\bar\zeta(\xi)\right)^2 \frac{\bar f_1(\xi)}{\xi^2} = \beta_1 c_{\tilde\beta}^2 \left(\bar f_1^2(\xi) - 1\right) \bar f_1(\xi)$$
$$+\beta_3 \left(c_{\tilde\beta}^2 \bar f_1^2(\xi) + s_{\tilde\beta}^2 \bar f_2^2(\xi) - 1\right) \bar f_1(\xi)\,, \qquad (39a)$$

$$\frac{d^2 \bar f_2(\xi)}{d\xi^2} + \frac{1}{\xi}\frac{d\bar f_2(\xi)}{d\xi} - \left(\frac{g_{Z'}}{3}\bar\zeta(\xi)\right)^2 \frac{\bar f_2(\xi)}{\xi^2} = \beta_2 s_{\tilde\beta}^2 \left(\bar f_2^2(\xi) - 1\right) \bar f_2(\xi)$$
$$+\beta_3 \left(c_{\tilde\beta}^2 \bar f_1^2(\xi) + s_{\tilde\beta}^2 \bar f_2^2(\xi) - 1\right) \bar f_2(\xi)\,, \qquad (39b)$$

$$\frac{d^2 \bar\zeta(\xi)}{d\xi^2} - \frac{1}{\xi}\frac{d\bar\zeta(\xi)}{d\xi} = \frac{6}{g_{Z'}}\left[c_{\tilde\beta}^2 \left(1 + \frac{g_{Z'}}{3}\bar\zeta(\xi)\right) \bar f_1^2(\xi)\right.$$
$$\left.+s_{\tilde\beta}^2 \left(\frac{g_{Z'}}{3}\bar\zeta(\xi)\right) \bar f_2^2(\xi)\right]\,, \qquad (39c)$$

with BCs of $\bar f_1(\xi=0) = \bar\zeta(\xi=0) = 0$, $\frac{d\bar f_2}{d\xi}(\xi=0) = 0$, and $\bar f_\omega(\xi=\infty) = 1$, $\bar\zeta(\xi=\infty) = -\frac{3}{g_{Z'}}c_{\tilde\beta}^2$.

The topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

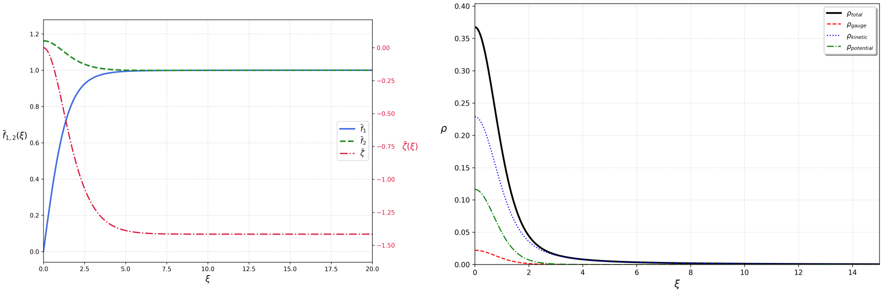


Figure: The profiles (left panel) and energy densities (right panel) for the topological $(1\,,0)$ Z' string with $\lambda_1 = \lambda_2 = \lambda_3 = 0.15$, $\lambda_5 = 0$, $g_{Z'} = 1.06$, and $t_{\tilde\beta} = 1.0$.

The $\mathcal{G}_{331}$ symmetry not full restored at the string core, and $\mu \sim \log \Lambda$.

Stable topological Z' string

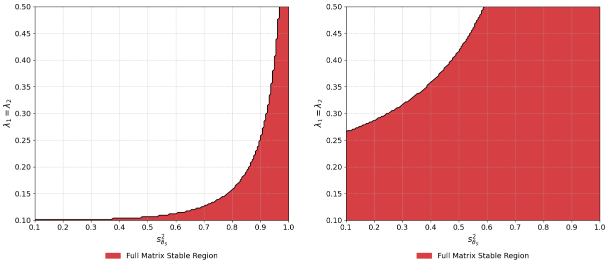


Figure: The stability regions for the topological $(1\,,0)$ Z' string with $\lambda_1 = \lambda_2$, $\lambda_3 = -0.5\lambda_1 + 0.005$, $g_{Z'} = 1.06$, $\lambda_4 = 0.025$ (left panel) and $\lambda_4 = 0.1$ (right panel).

Summary

- The non-topological/topological strings in the flavor unified $\mathrm{SU}(8)$ theory originate from the multiple Higgs fields at the intermediate symmetry breaking stages.
- Non-topological strings cannot be stable unless with the semilocal limit of $\vartheta_S \to \frac{\pi}{2}$, even if one tunes the Higgs potential parameters.
- Topological strings due to the global $\widetilde{\mathrm{U}}(1)$'s in the Higgs potential, very likely to be stable. Explicitly $\widetilde{\mathrm{U}}(1)$-breaking terms may lead to the domain-wall string composite [2311.15805].

Thank you!