2026年轻强子专题研讨会
河南 商丘

Fierz analyses on the decay properties of two- and three-gluon glueballs

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Outline

- Introduction
- Method of the Feriz rearrangement
- Excited light meson operators analyses
- Decay behavior of two- and three-gluon glueballs
- Summary

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- **Introduction**
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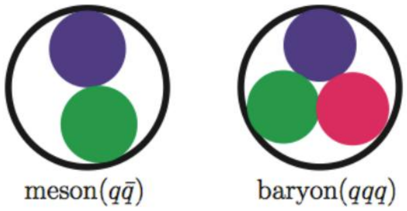
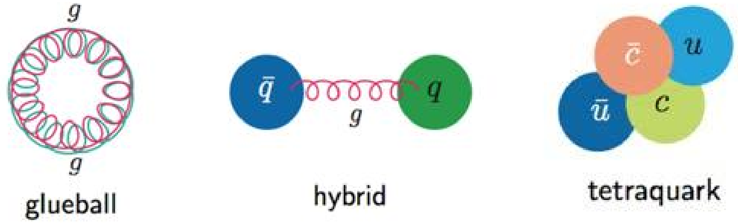
Introduction

- **Traditional quark model**



- **Exotic hadron: glueball, hybrid state, tetraquark, etc.**



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- **Exotic spin-parity quantum numbers**

$$0^{--}, 0^{+-}, 1^{-+}, 2^{+-}$$

Introduction

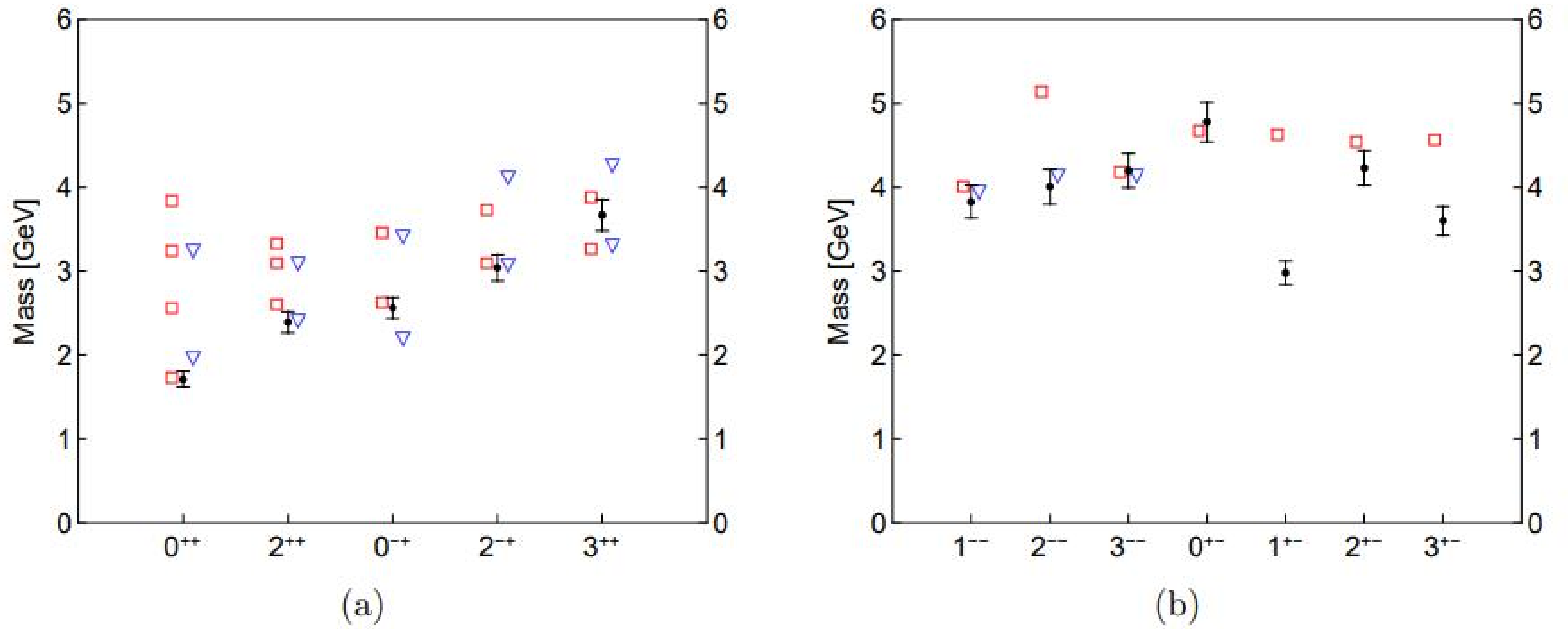


Figure 61. Mass spectra of (a) two-gluon glueballs from Ref. [800] (red squares) and Ref. [836] (blue triangles) as well as (b) three-gluon glueballs from Ref. [837] (red squares) and Ref. [799] (blue triangles). Lattice QCD calculations from Ref. [815] (black error bars) are given for comparisons.

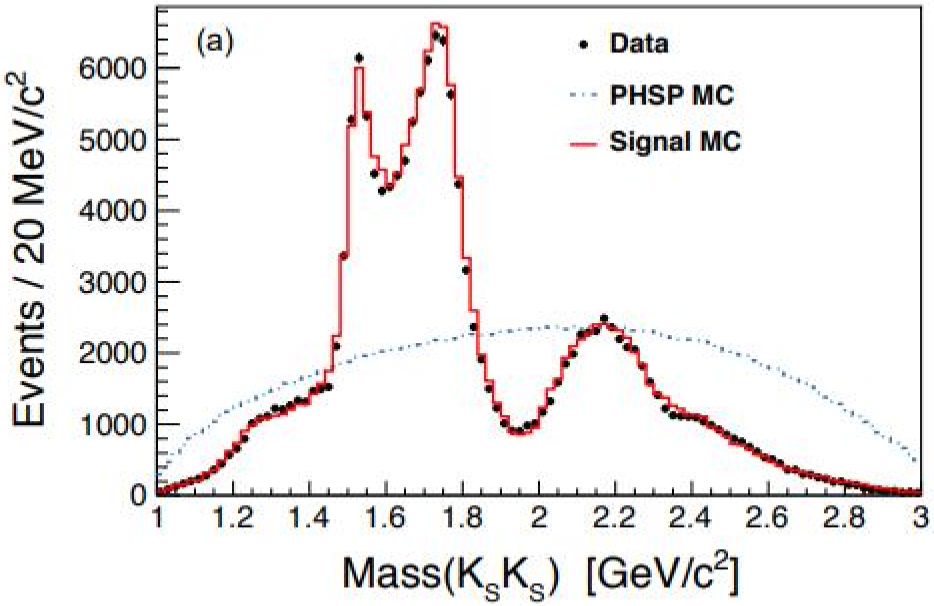
Phys. Rev. Lett. 96 (2006) 081601. Phys. Rev. D 77(2008) 114022.
Phys. Rev. D 73 (2006) 014516. Phys. Lett. B 577 (2003)61–66.
Phys. Rev. D 77 (2008) 094009.

Introduction

$f_0(1500):\ M = 1522 \pm 25$ MeV
$\Gamma = 108 \pm 33$ MeV

$f_0(1710):\ M = 1733 \pm 8$ MeV
$\Gamma = 108 \pm 11$ MeV

BESIII



Phys. Rev. D 98 (7) (2018) 072003.

Phys. Rev. D **27** (1983), 1556-1564.
Phys. Rev. D 31 (1985), 2910.
Phys. Rev. D 60 (1999), 034509.
Phys. Lett. B 124 (1983), 247-251.
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Phys. Rev. D 82 (2010), 034501.

Introduction

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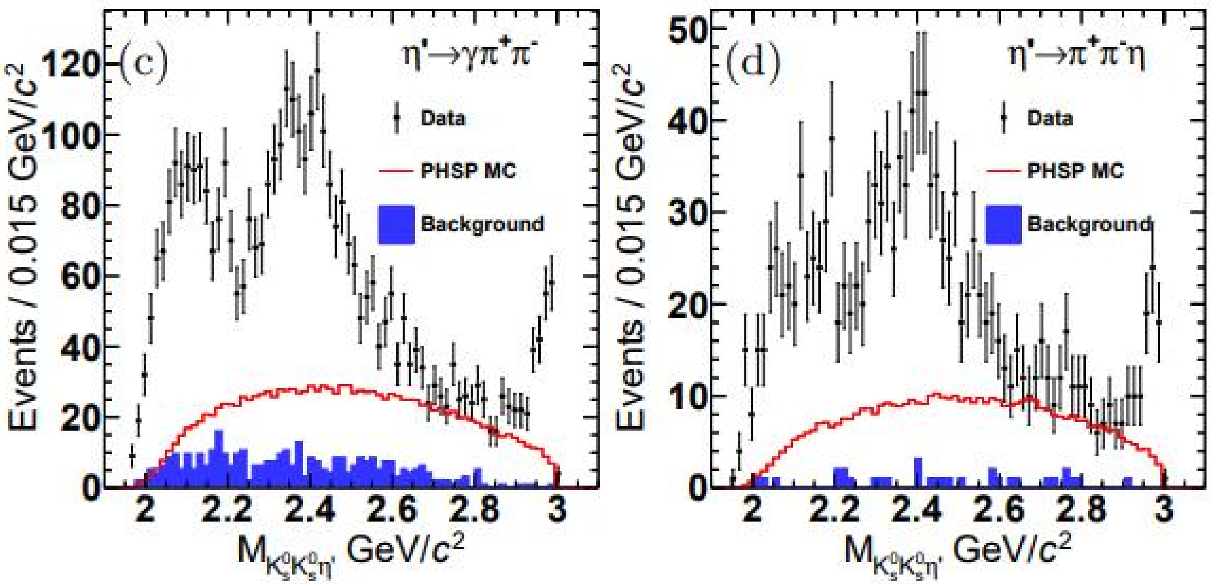

$$J/\psi \rightarrow \gamma K^0_S K^0_S \eta'$$

$$\eta(2370): \quad M = 2377 \pm 9 \text{ MeV}$$
$$\Gamma = 148^{+80}_{-28} \text{ MeV}$$

Phys.Rev.Lett. 132 (2024) 18, 181901. Phys.Rev.D 103 (2021) 1, 012009.

Introduction

Table 1: Masses of two- and three-gluon glueballs, in units of GeV. Our QCD sum rule results are listed in the 2nd column, and Lattice QCD results are listed in the 3rd-6th columns, taken from Refs. [12–14] (quenched) and Ref. [15] (unquenched).

Glueball	QCD sum rules	Ref. [12]	Ref. [13]	Ref. [14]	Ref. [15]
$\|GG; 0^{++}\rangle$	$1.79^{+0.14}_{-0.16}$	$1.71 \pm 0.05 \pm 0.08$	$1.73 \pm 0.05 \pm 0.08$	$1.48 \pm 0.03 \pm 0.07$	1.80 ± 0.06
$\|GG; 2^{++}\rangle$	$1.86^{+0.14}_{-0.17}$	$2.39 \pm 0.03 \pm 0.12$	$2.40 \pm 0.03 \pm 0.12$	$2.15 \pm 0.03 \pm 0.10$	2.62 ± 0.05
$\|GG; 0^{-+}\rangle$	$2.15^{+0.11}_{-0.11}$	$2.56 \pm 0.04 \pm 0.12$	$2.59 \pm 0.04 \pm 0.13$	$2.25 \pm 0.06 \pm 0.10$	–
$\|GG; 2^{-+}\rangle$	$2.12^{+0.11}_{-0.12}$	$3.04 \pm 0.04 \pm 0.15$	$3.10 \pm 0.03 \pm 0.15$	$2.78 \pm 0.05 \pm 0.13$	3.46 ± 0.32
$\|GGG; 0^{++}\rangle$	$4.21^{+0.18}_{-0.20}$	–	$2.67 \pm 0.18 \pm 0.13$	$2.76 \pm 0.03 \pm 0.12$	3.76 ± 0.24
$\|GGG; 2^{++}\rangle$	$3.90^{+0.18}_{-0.26}$	–	–	$2.88 \pm 0.10 \pm 0.13$	–
$\|GGG; 0^{-+}\rangle$	$4.00^{+0.17}_{-0.28}$	–	$3.64 \pm 0.06 \pm 0.18$	$3.37 \pm 0.15 \pm 0.15$	4.49 ± 0.59
$\|GGG; 2^{-+}\rangle$	$4.02^{+0.20}_{-0.23}$	–	–	$3.48 \pm 0.14 \pm 0.16$	–
$\|GGG; 1^{+-}\rangle$	$3.19^{+0.15}_{-0.17}$	$2.98 \pm 0.03 \pm 0.14$	$2.94 \pm 0.03 \pm 0.14$	$2.67 \pm 0.07 \pm 0.12$	3.27 ± 0.34
$\|GGG; 2^{+-}\rangle$	> 4.13	$4.23 \pm 0.05 \pm 0.20$	$4.14 \pm 0.05 \pm 0.20$	–	–
$\|GGG; 3^{+-}\rangle$	$3.63^{+0.21}_{-0.23}$	$3.60 \pm 0.04 \pm 0.17$	$3.55 \pm 0.04 \pm 0.17$	$3.27 \pm 0.09 \pm 0.15$	3.85 ± 0.35
$\|GGG; 1^{--}\rangle$	$5.10^{+0.21}_{-0.16}$	$3.83 \pm 0.04 \pm 0.19$	$3.85 \pm 0.05 \pm 0.19$	$3.24 \pm 0.33 \pm 0.15$	–
$\|GGG; 2^{--}\rangle$	$4.35^{+0.21}_{-0.27}$	$4.01 \pm 0.05 \pm 0.20$	$3.93 \pm 0.04 \pm 0.19$	$3.66 \pm 0.13 \pm 0.17$	4.59 ± 0.74
$\|GGG; 3^{--}\rangle$	$5.47^{+0.28}_{-0.19}$	$4.20 \pm 0.05 \pm 0.20$	$4.13 \pm 0.09 \pm 0.20$	$4.33 \pm 0.26 \pm 0.20$	–

Introduction

Some limitations

- Most theoretical analyses rely, to varying degrees, on specific model assumptions, leading to discrepancies among results obtained from different approaches and making it difficult to reach consistent conclusions.

- Due to the possible significant mixing between glueballs and conventional meson states, relying solely on mass spectrum information is often insufficient to unambiguously determine their nature.

- How to characterize, based on the fundamental symmetries of QCD, the decay mechanism of glueballs into light hadron final states—particularly the feature that glueballs should exhibit flavor-universal couplings in their decay processes—remains an open issue.

Therefore, in this work, we will apply the Fierz rearrangement method to systematically analyze the decay properties of two-gluon and three-gluon glueballs, and to determine the glueball components among the three glueball candidates $f_0(1500)$, $f_0(1710)$ and $\eta(2370)$.

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Method of the Feriz rearrangement

$$J_0 = g_s^2 G_i^{\mu\nu} G^i_{\mu\nu} \,,$$
$$\widetilde{J}_0 = g_s^2 G_i^{\mu\nu} \widetilde{G}^i_{\mu\nu} \,,$$
$$\eta_0 = f^{ijk} g_s^3 G_i^{\mu\nu} G_{j,\nu\rho} G^{\rho}_{k,\mu} \,,$$
$$\eta_1^{\alpha\beta} = d^{ijk} g_s^3 G_i^{\mu\nu} G_{j,\mu\nu} G_k^{\alpha\beta} \,,$$

$$G^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g_s f^{ijk} A^j_\mu A^k_\nu \,.$$

⬇

$$A^i_\mu \to \lambda^i_{ab} \times \bar{q}^a \gamma_\mu q^b \,,$$

● **some simplifying assumptions**：

1. the decay of glueballs is assumed to be dominated by two- or three-constituent gluons
2. each gluon is converted into a quark–antiquark pair
3. mesons are treated as quark–antiquark bound states
4. final-state interactions among the produced mesons are neglected

● **the merits**：

1. use of the universal symmetries of QCD for analyzing the algebraic structure of glueball decay channels.
2. retain a certain degree of generality across different model settings

Method of the Feriz rearrangement

color side

$$\lambda^i_{ab}\lambda^i_{cd} = 2\delta_{ad}\delta_{cb} - \boxed{\frac{2}{3}\delta_{ab}\delta_{cd}\,,}$$

$$f^{ijk}\lambda_i^{ab}\lambda_j^{cd}\lambda_k^{ef} = +2i\delta^{af}\delta^{bc}\delta^{de} - 2i\delta^{ad}\delta^{be}\delta^{cf}\,,$$

$$\begin{aligned} d^{ijk}\lambda_i^{ab}\lambda_j^{cd}\lambda_k^{ef} &= +2\delta^{af}\delta^{bc}\delta^{de} + 2\delta^{ad}\delta^{be}\delta^{cf} \\ &\quad - \frac{4}{3}\delta^{ab}\delta^{de}\delta^{cf} - \frac{4}{3}\delta^{af}\delta^{cd}\delta^{be} \\ &\quad - \frac{4}{3}\delta^{ad}\delta^{bc}\delta^{ef} + \boxed{\frac{8}{9}\delta^{ab}\delta^{cd}\delta^{ef}}\,. \end{aligned}$$

Lorentz side

$$\begin{pmatrix} \mathbf{1}\otimes\gamma_5 \\ \gamma_\mu\otimes\gamma^\mu\gamma_5 \\ \sigma_{\mu\nu}\otimes\sigma^{\mu\nu}\gamma_5 \\ \gamma_\mu\gamma_5\otimes\gamma^\mu \\ \gamma_5\otimes\mathbf{1} \end{pmatrix}_{ab,cd} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\ -1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \\ 3 & 0 & -\frac{1}{2} & 0 & 3 \\ 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \mathbf{1}\otimes\gamma_5 \\ \gamma_\mu\otimes\gamma^\mu\gamma_5 \\ \sigma_{\mu\nu}\otimes\sigma^{\mu\nu}\gamma_5 \\ \gamma_\mu\gamma_5\otimes\gamma^\mu \\ \gamma_5\otimes\mathbf{1} \end{pmatrix}_{ad,bc}.$$

Method of the Feriz rearrangement

$$
\begin{aligned}
|\mathrm{GG};0^{++}\rangle \longleftrightarrow\ & J_0 = G_i^{\mu\nu} \times G_{\mu\nu}^i \times g_s^2 \qquad (1)\\
\longrightarrow\ & C_1 \times (\partial^\mu A_i^\nu - \partial^\nu A_i^\mu + g_s f_{ijk} A_j^\mu A_k^\nu) \times (\partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g_s f^{ijk} A_\mu^j A_\nu^k)\\
\longrightarrow\ & C_2 \times \lambda_{ab}^i \lambda_{cd}^i \times (\partial^\mu \bar{q}_1^a \gamma^\nu q_2^b + \bar{q}_1^a \gamma^\nu \partial^\mu q_2^b) \times (\partial_\mu \bar{q}_3^c \gamma_\nu q_4^d + \bar{q}_3^c \gamma_\nu \partial_\mu q_4^d) + \cdots\\
\xrightarrow{\text{color}}\ & C_3 \times \delta_{ad}\delta_{cb} \times \partial^\mu \bar{q}_1^a \gamma^\nu q_2^b \times \partial_\mu \bar{q}_3^c \gamma_\nu q_4^d + \cdots\\
\xrightarrow{\text{Fierz}}\ & C_4 \times \delta_{ad}\delta_{cb} \times (-\partial^\mu \bar{q}_1^a q_4^d \times \partial_\mu \bar{q}_3^c q_2^b + \frac{1}{2}\partial^\mu \bar{q}_1^a \gamma^\nu q_4^d \times \partial_\mu \bar{q}_3^c \gamma_\nu q_2^b\\
& + \frac{1}{2}\partial^\mu \bar{q}_1^a \gamma^\nu \gamma_5 q_4^d \times \partial_\mu \bar{q}_3^c \gamma_\nu \gamma_5 q_2^b + \partial^\mu \bar{q}_1^a \gamma_5 q_4^d \times \partial_\mu \bar{q}_3^c \gamma_5 q_2^b) + \cdots\\
\longrightarrow\ & C_5 \times -(\partial^\mu [\bar{q}_1^a q_4^a]_A - [\bar{q}_1^a \overleftrightarrow{\partial}^\mu q_4^a]_A) \times (\partial_\mu [\bar{q}_3^b q_2^b]_B - [\bar{q}_3^b \overleftrightarrow{\partial}_\mu q_2^b]_B) + \cdots\\
\longrightarrow\ & + \frac{1}{4}[\bar{u}_1^a \gamma_\mu \gamma_5 d_4^a]_A \boxed{[\bar{d}_3^b \gamma_\mu \gamma_5 u_2^b]_B} \times (q_A^\nu + q_B^\nu)^2 + \frac{1}{2}[\bar{u}_1^a \gamma_\mu \gamma_5 d_4^a]_A [\bar{d}_3^b \gamma_\nu \gamma_5 u_2^b]_B \times (q_A^\mu + q_B^\mu)(q_A^\nu + q_B^\nu)\\
& + \frac{3}{4}[\bar{u}_1^a \gamma_5 d_4^a]_A [\bar{d}_3^b \gamma_5 u_2^b]_B \times (q_A^2 + q_B^2 + 2 q_A \cdot q_B) + \frac{1}{4}[\bar{u}_1^a \gamma_\mu d_4^a]_A [\bar{d}_3^b \gamma_\mu u_2^b]_B \times (q_A^\nu + q_B^\nu)^2\\
& + \frac{1}{2}[\bar{u}_1^a \gamma_\mu d_4^a]_A [\bar{d}_3^b \gamma_\nu u_2^b]_B \times (q_A^\mu + q_B^\mu)(q_A^\nu + q_B^\nu) - \frac{1}{2}[\bar{u}_1^a \sigma_{\mu\alpha} d_4^a]_A [\bar{d}_3^b \sigma_{\nu\alpha} u_2^b]_B \times (q_A^\mu + q_B^\mu)(q_a^\nu + q_b^\nu)\\
& + \frac{1}{8}[\bar{u}_1^a \sigma_{\mu\nu} d_4^a]_A [\bar{d}_3^b \sigma_{\mu\nu} u_2^b]_B \times (q_A^2 + q_B^2 + 2 q_A \cdot q_B) - \frac{3}{4}[\bar{u}_1^a d_4^a]_A [\bar{d}_3^b u_2^b]_B \times (q_A^2 + q_B^2 + 2 q_A \cdot q_B)\\
& - \frac{3}{2}[\bar{u}_1^a \overleftrightarrow{\partial}^\mu d_4^a]_A \boxed{[\bar{d}_3^b \overleftrightarrow{\partial}_\mu u_2^b]_B} + \frac{3}{2}[\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_5 d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\mu \gamma_5 u_2^b]_B + \frac{1}{2}[\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_\mu d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\mu \gamma^\mu u_2^b]_B\\
& + \frac{1}{2}[\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_\nu d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\mu \gamma^\nu u_2^b]_B + \frac{1}{2}[\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_\nu d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\nu \gamma^\mu u_2^b]_B\\
& + \frac{1}{2}[\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_\mu \gamma_5 d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\mu \gamma^\mu \gamma_5 u_2^b]_B + \frac{1}{2}[\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_\nu \gamma_5 d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\mu \gamma^\nu \gamma_5 u_2^b]_B\\
& + \frac{1}{2}[\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_\nu \gamma_5 d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\nu \gamma^\mu \gamma_5 u_2^b]_B - \frac{1}{2}[\bar{u}_1^a \overleftrightarrow{\partial}^\mu \sigma_{\mu\alpha} d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\nu \sigma^{\nu\alpha} u_2^b]_B\\
& - \frac{1}{2}[\bar{u}_1^a \overleftrightarrow{\partial}^\mu \sigma_{\nu\alpha} d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\nu \sigma^{\mu\alpha} \gamma_5 u_2^b]_B + \frac{1}{4}[\bar{u}_1^a \overleftrightarrow{\partial}^\mu \sigma_{\nu\alpha} d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\mu \sigma^{\nu\alpha} \gamma_5 u_2^b]_B .
\end{aligned}
$$

Method of the Feriz rearrangement

$$\langle 0|\bar{u}_a i\gamma_5 d_a|\pi^-(q)\rangle = \lambda_\pi\,,$$
$$\langle 0|\bar{u}_a \gamma_\mu\gamma_5 d_a|\pi^-(q)\rangle = iq_\mu f_\pi\,,$$
$$\langle 0|\bar{u}_a \gamma_\mu d_a|\rho^-(q,\epsilon)\rangle = m_\rho f_\rho \epsilon_\mu\,,$$
$$\langle 0|\bar{u}_a \sigma_{\mu\nu} d_a|\rho^-(q,\epsilon)\rangle = if_\rho^T(q_\mu\epsilon_\nu - q_\nu\epsilon_\mu)\,.$$

$$\begin{aligned}\mathcal{M}_{\pi\pi} \propto\ & -\frac{3}{4}\lambda_\pi^2(q_1^2+q_2^2+2q_1\cdot q_2)\\ & -\frac{3}{4}f_\pi^2(q_1^2+q_2^2)(q_1\cdot q_2)\\ & -\frac{1}{2}f_\pi^2(q_1^2\times q_2^2+2(q_1\cdot q_2)^2)\,,\end{aligned}$$

$$\begin{aligned}\mathcal{M}_{\rho\rho} \propto\ & +\frac{1}{2}m_\rho^2 f_\rho^2(\epsilon(q_1)\cdot q_2)(\epsilon(q_2)\cdot q_1)\\ & +\frac{1}{4}m_\rho^2 f_\rho^2(\epsilon(q_1)\cdot\epsilon(q_2))(q_1^2+q_2^2+2q_1\cdot q_2)\\ & +\frac{1}{4}(f_\rho^T)^2(\epsilon(q_1)\cdot\epsilon(q_2))(q_1^2+q_2^2)(q_1\cdot q_2)\\ & +\frac{1}{2}(f_\rho^T)^2(\epsilon(q_1)\cdot\epsilon(q_2))q_1^2\times q_2^2\\ & -\frac{1}{4}(f_\rho^T)^2(\epsilon(q_1)\cdot q_2)(\epsilon(q_2)\cdot q_1)(q_1^2+q_2^2)\,,\end{aligned}$$

$$\begin{aligned}J_0 =\ & +\frac{1}{4}[\bar{u}_1^a\gamma_\mu\gamma_5 d_4^a]_A[\bar{d}_3^b\gamma_\mu\gamma_5 u_2^b]_B\times(q_A^\nu+q_B^\nu)^2\\ & +\frac{1}{2}[\bar{u}_1^a\gamma_\mu\gamma_5 d_4^a]_A[\bar{d}_3^b\gamma_\nu\gamma_5 u_2^b]_B\times(q_A^\mu+q_B^\mu)(q_A^\nu+q_B^\nu)\\ & +\frac{3}{4}[\bar{u}_1^a\gamma_5 d_4^a]_A[\bar{d}_3^b\gamma_5 u_2^b]_B\times(q_A^2+q_B^2+2q_A\cdot q_B)\\ & +\frac{1}{4}[\bar{u}_1^a\gamma_\mu d_4^a]_A[\bar{d}_3^b\gamma_\mu u_2^b]_B\times(q_A^\nu+q_B^\nu)^2\\ & +\frac{1}{2}[\bar{u}_1^a\gamma_\mu d_4^a]_A[\bar{d}_3^b\gamma_\nu u_2^b]_B\times(q_A^\mu+q_B^\mu)(q_A^\nu+q_B^\nu)\\ & -\frac{1}{2}[\bar{u}_1^a\sigma_{\mu\alpha} d_4^a]_A[\bar{d}_3^b\sigma_{\nu\alpha} u_2^b]_B\times(q_A^\mu+q_B^\mu)(q_a^\nu+q_b^\nu)\\ & +\frac{1}{8}[\bar{u}_1^a\sigma_{\mu\nu} d_4^a]_A[\bar{d}_3^b\sigma_{\mu\nu} u_2^b]_B\times(q_A^2+q_B^2+2q_A\cdot q_B)\,.\end{aligned}$$

→

$$\frac{\mathcal{B}(|\mathrm{GG};\,0^{++}\rangle\to\rho\rho)}{\mathcal{B}(|\mathrm{GG};\,0^{++}\rangle\to\pi\pi)} = 6.49\times 10^{-2}\,.$$

The amplitudes should be understood as effective couplings dominated by algebraic structures within a constituent picture.

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Excited light meson operators

construct excited light meson operators

$$J = \bar{q}_a \overleftrightarrow{D}_\alpha \Gamma q_a,$$

$\bar{q}_a \gamma_5 q_a$, $\bar{q}_a q_a$,
$\bar{q}_a \gamma_\mu q_a$, $\bar{q}_a \gamma_\mu \gamma_5 q_a$,
$\bar{q}_a \sigma_{\mu\nu} q_a$.

$\bar{q}q$, $\bar{q}s$, and $\bar{s}s$ $(q = u/d)$

$$J_\mu^{1+-} = \bar{q}_a \overleftrightarrow{D}_\mu \gamma_5 q_a, \qquad J_{\mu\nu}^{1--} = \mathcal{A}[\bar{q}_a \overleftrightarrow{D}_\mu \gamma_\nu \gamma_5 q_a],$$

$$J_\mu^{1--} = \bar{q}_a \overleftrightarrow{D}_\mu q_a, \qquad J_{\mu\nu}^{2--} = \mathcal{S}[\bar{q}_a \overleftrightarrow{D}_\mu \gamma_\nu \gamma_5 q_a],$$

$$J^{0++} = \bar{q}_a \overleftrightarrow{D}_\mu \gamma_\nu q_a \times g^{\mu\nu}, \qquad J_\alpha^{1-+} = \bar{q}_a \overleftrightarrow{D}_\mu \sigma_{\alpha\beta} q_a \times g^{\mu\beta},$$

$$J_{\mu\nu}^{1++} = \mathcal{A}[\bar{q}_a \overleftrightarrow{D}_\mu \gamma_\nu q_a], \qquad J_{\mu,\alpha\beta}^{2-+} = \mathcal{S}'[\bar{q}_a \overleftrightarrow{D}_\mu \sigma_{\alpha\beta} q_a],$$

$$J_{\mu\nu}^{2++} = \mathcal{S}[\bar{q}_a \overleftrightarrow{D}_\mu \gamma_\nu q_a], \qquad J_\alpha^{1++} = \bar{q}_a \overleftrightarrow{D}_\mu \sigma_{\alpha\beta} \gamma_5 q_a \times g^{\mu\beta},$$

$$J^{0--} = \bar{q}_a \overleftrightarrow{D}_\mu \gamma_\nu \gamma_5 q_a \times g^{\mu\nu}, \qquad J_{\mu,\alpha\beta}^{2++} = \mathcal{S}'[\bar{q}_a \overleftrightarrow{D}_\mu \sigma_{\alpha\beta} \gamma_5 q_a].$$

$$\Gamma_{\mu'\nu';\mu\nu} = g_{\mu'\mu} g_{\nu'\nu} - g_{\nu'\mu} g_{\mu'\nu}. \qquad \Gamma_{\mu'\nu';\mu\nu} = g_{\mu'\mu} g_{\nu'\nu} + g_{\nu'\mu} g_{\mu'\nu} - \frac{1}{2} g_{\mu'\nu'} g_{\mu\nu}.$$

$$\Gamma_{\alpha'\mu'\nu';\alpha\mu\nu} = g_{\alpha'\alpha} g_{\mu'\mu} g_{\nu'\nu} + g_{\mu'\alpha} g_{\alpha'\mu} g_{\nu'\nu} - \frac{1}{2} g_{\alpha'\mu'} g_{\alpha\mu} g_{\nu'\nu}.$$

Excited light meson operators

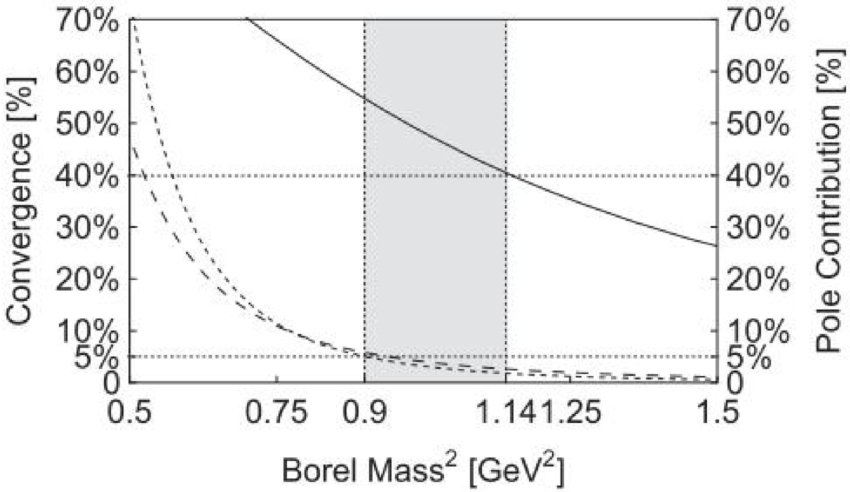
- **Convergence of OPE**

$$\mathrm{CVG}_8 \equiv \left|\frac{\Pi_{2++}^{D=8}(\infty, M_B^2)}{\Pi_{2++}(\infty, M_B^2)}\right| < 5\%,$$

$$\mathrm{CVG}_6 \equiv \left|\frac{\Pi_{2++}^{D=6}(\infty, M_B^2)}{\Pi_{2++}(\infty, M_B^2)}\right| < 10\%.$$

- **Sufficient amout of pole contribution**

$$\text{Pole contribution} \equiv \left|\frac{\Pi_{2++}(s_0, M_B^2)}{\Pi_{2++}(\infty, M_B^2)}\right| > 40\%.$$



$$0.90\mathrm{GeV}^2 \le M_B^2 \le 1.44\mathrm{GeV}^2$$

$$M_{|\bar{q}s;2^{+(+)}\rangle} = 1.44^{+0.08}_{-0.09}\ \mathrm{GeV},$$

$$f_{|\bar{q}s;2^{+(+)}\rangle} = 0.313^{+0.078}_{-0.079}\ \mathrm{GeV}^3.$$

Excited light meson operators

Operators	Meson $[J^{PC}]$	s_0^{min} $[\text{GeV}^2]$	Working Regions M_B^2 $[\text{GeV}^2]$	Working Regions s_0 $[\text{GeV}^2]$	Pole [%]	Mass [GeV]	Decay Constant	Candidate [4]
J_μ^{1--}	$\lvert\bar{q}q;1^{--}\rangle$	3.8	1.70–2.05	4.6	40–52	$1.70^{+0.14}_{-0.17}$	$0.602^{+0.173}_{-0.169}$ GeV^3	$\rho(1700)/\omega(1650)$
	$\lvert\bar{q}s;1^{-}\rangle$	3.8	1.71–2.09	4.7	40–53	$1.71^{+0.14}_{-0.17}$	$0.617^{+0.173}_{-0.170}$ GeV^3	$K^*(1680)$
	$\lvert\bar{s}s;1^{--}\rangle$	5.1	2.25–2.81	6.4	40–55	$2.00^{+0.15}_{-0.19}$	$0.968^{+0.260}_{-0.257}$ GeV^3	–
	$\lvert\bar{q}q/\bar{q}s/\bar{s}s;0^{+(-)}\rangle$	–						
J_μ^{1+-}	$\lvert\bar{q}q;1^{+-}\rangle$	2.1	1.00–1.21	2.6	40–53	$1.22^{+0.10}_{-0.13}$	$0.257^{+0.063}_{-0.062}$ GeV^3	$b_1(1235)/h_1(1170)$
	$\lvert\bar{q}s;1^{+}\rangle$	2.2	1.01–1.23	2.7	40–53	$1.27^{+0.10}_{-0.12}$	$0.271^{+0.065}_{-0.064}$ GeV^3	$K_1(1270)$
	$\lvert\bar{s}s;1^{+-}\rangle$	2.6	1.18–1.43	3.2	40–53	$1.41^{+0.10}_{-0.12}$	$0.348^{+0.086}_{-0.084}$ GeV^3	$h_1(1415)$
	$\lvert\bar{q}q/\bar{q}s/\bar{s}s;0^{-(-)}\rangle$	–						
$J_{\mu\nu}^{1++}$	$\lvert\bar{q}q;1^{++}\rangle$	2.1	0.94–1.15	2.6	40–53	$1.25^{+0.10}_{-0.13}$	$0.111^{+0.010}_{-0.010}$ GeV^2	$a_1(1260)/f_1(1285)$
	$\lvert\bar{q}s;1^{+}\rangle$	2.1	0.92–1.13	2.6	40–53	$1.28^{+0.09}_{-0.11}$	$0.106^{+0.011}_{-0.012}$ GeV^2	$K_1(1270)$
	$\lvert\bar{s}s;1^{++}\rangle$	2.5	1.08–1.33	3.1	40–53	$1.42^{+0.10}_{-0.12}$	$0.114^{+0.012}_{-0.014}$ GeV^2	$f_1(1420)$
	$\lvert\bar{q}q;1^{-+}\rangle$	3.5	2.34–2.98	4.3	40–52	$1.65^{+0.11}_{-0.13}$	$0.328^{+0.062}_{-0.066}$ GeV^2	$\pi_1(1600)$
	$\lvert\bar{q}s;1^{-}\rangle$	3.9	2.65–3.35	4.8	40–52	$1.72^{+0.12}_{-0.14}$	$0.364^{+0.070}_{-0.075}$ GeV^2	$K^*(1680)$
	$\lvert\bar{s}s;1^{-+}\rangle$	4.6	3.20–4.02	5.7	40–51	$1.85^{+0.13}_{-0.15}$	$0.431^{+0.077}_{-0.081}$ GeV^2	$\eta_1(1855)$
$J_{\mu\nu}^{2++}$	$\lvert\bar{q}q;2^{++}\rangle$	2.3	0.93–1.14	2.8	40–53	$1.36^{+0.11}_{-0.14}$	$0.286^{+0.091}_{-0.092}$ GeV^3	$a_2(1320)/f_2(1270)$
	$\lvert\bar{q}s;2^{+}\rangle$	2.4	0.90–1.14	2.9	40–54	$1.44^{+0.08}_{-0.09}$	$0.313^{+0.078}_{-0.079}$ GeV^3	$K_2^*(1430)$
	$\lvert\bar{s}s;2^{++}\rangle$	2.5	0.92–1.22	3.1	40–57	$1.52^{+0.08}_{-0.08}$	$0.358^{+0.096}_{-0.098}$ GeV^3	$f_2'(1525)$

Excited light meson operators

Operators	Meson $[J^{PC}]$	s_0^{min} $[\text{GeV}^2]$	Working Regions M_B^2 $[\text{GeV}^2]$	Working Regions s_0 $[\text{GeV}^2]$	Pole [%]	Mass [GeV]	Decay Constant	Candidate [4]
$J_{\mu\nu}^{1--}$	$\lvert\bar{q}q;1^{--}\rangle$	3.7	1.60–1.97	4.5	40–53	$1.70^{+0.12}_{-0.15}$	$0.142^{+0.015}_{-0.017}$ GeV^2	$\rho(1700)/\omega(1650)$
	$\lvert\bar{q}s;1^{-}\rangle$	3.7	1.61–2.01	4.6	40–54	$1.71^{+0.12}_{-0.15}$	$0.143^{+0.014}_{-0.016}$ GeV^2	–
	$\lvert\bar{s}s;1^{--}\rangle$	5.0	2.16–2.74	6.3	40–55	$2.00^{+0.15}_{-0.18}$	$0.166^{+0.018}_{-0.020}$ GeV^2	–
	$\lvert\bar{q}q;1^{+-}\rangle$	3.2	2.14–2.76	4.0	40–52	$1.60^{+0.10}_{-0.12}$	$0.305^{+0.057}_{-0.060}$ GeV^2	$h_1(1595)$
	$\lvert\bar{q}s;1^{+}\rangle$	3.5	2.38–2.98	4.3	40–51	$1.64^{+0.11}_{-0.13}$	$0.327^{+0.063}_{-0.067}$ GeV^2	–
	$\lvert\bar{s}s;1^{+-}\rangle$	4.5	3.10–3.94	5.6	40–52	$1.84^{+0.13}_{-0.15}$	$0.423^{+0.077}_{-0.081}$ GeV^2	–
$J_{\mu\nu}^{2--}$	$\lvert\bar{q}q;2^{--}\rangle$	3.4	1.40–1.76	4.2	40–54	$1.69^{+0.10}_{-0.12}$	$0.567^{+0.147}_{-0.147}$ GeV^3	–
	$\lvert\bar{q}s;2^{-}\rangle$	4.1	1.73–2.13	5.0	40–53	$1.81^{+0.13}_{-0.15}$	$0.728^{+0.199}_{-0.197}$ GeV^3	$K_2(1820)$
	$\lvert\bar{s}s;2^{--}\rangle$	4.5	1.90–2.39	5.6	40–55	$1.91^{+0.13}_{-0.16}$	$0.855^{+0.230}_{-0.228}$ GeV^3	–
J_{μ}^{1-+}	$\lvert\bar{q}q;0^{++}\rangle$	2.8	1.23–1.51	3.4	40–53	$1.45^{+0.12}_{-0.14}$	$0.310^{+0.030}_{-0.032}$ GeV^2	$a_0(1450)/f_0(1370)$
	$\lvert\bar{q}s;0^{+}\rangle$	2.7	1.20–1.46	3.3	40–53	$1.43^{+0.12}_{-0.15}$	$0.304^{+0.030}_{-0.032}$ GeV^2	$K_0^*(1430)$
	$\lvert\bar{s}s;0^{++}\rangle$	4.0	1.72–2.12	4.9	40–53	$1.78^{+0.13}_{-0.16}$	$0.355^{+0.039}_{-0.044}$ GeV^2	$f_0(1770)$
	$\lvert\bar{q}q/\bar{q}s/\bar{s}s;1^{-(+)}\rangle$	–						
J_{μ}^{1++}	$\lvert\bar{q}q;0^{-+}\rangle$	4.1	1.80–2.19	5.0	40–53	$1.78^{+0.13}_{-0.16}$	$0.368^{+0.037}_{-0.042}$ GeV^2	$\pi(1800)/\eta(1760)$
	$\lvert\bar{q}s;0^{-}\rangle$	4.2	1.84–2.28	5.2	40–54	$1.83^{+0.13}_{-0.15}$	$0.372^{+0.037}_{-0.042}$ GeV^2	$K(1830)$
	$\lvert\bar{s}s;0^{-+}\rangle$	5.1	2.18–2.76	6.4	40–55	$2.03^{+0.15}_{-0.18}$	$0.405^{+0.044}_{-0.050}$ GeV^2	–
	$\lvert\bar{q}q/\bar{q}s/\bar{s}s;1^{+(+)}\rangle$	–						

Outline

- Introduction
- Method of the Feriz rearrangement
- Excited light meson operators analyses
- **Decay behavior of two- and three-gluon glueballs**
- Summary

Decay behavior

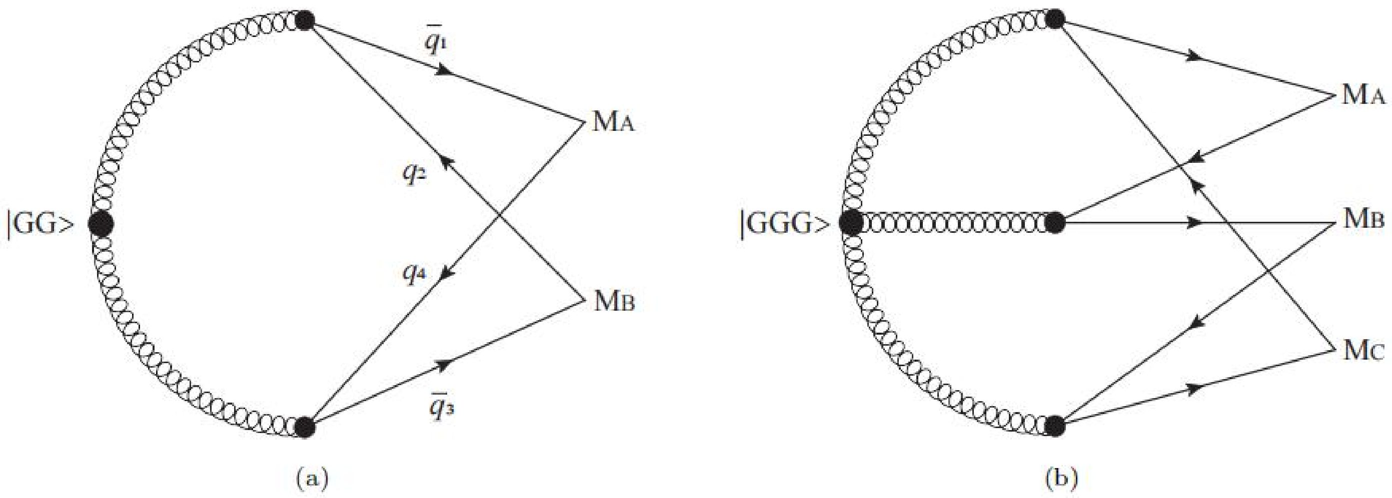


FIG. 1: Diagram (a) illustrates the schematic diagram of a two-gluon glueball decaying into two mesons via a single Fierz rearrangement, while Diagram (b) depicts the schematic diagram of a three-gluon glueball decaying into three mesons through two successive Fierz rearrangements.

$$|GG;0^{++}\rangle$$

Channel	Ratio	Channel	Ratio	Channel	Ratio	Channel	Ratio
$\pi\pi$	1.00	$K\bar{K}$	1.93	$\kappa\bar{\kappa}$	1.19	$\eta\eta$	0.43
$\rho\rho$	6.49×10^{-2}	$\sigma\sigma$	5.76×10^{-2}	$\eta'\eta'$	2.27×10^{-2}	$\omega\omega$	1.02×10^{-2}

Decay behavior

$|GG;0^{-+}\rangle$

In particular, the $\phi\phi$, $\omega\omega$, and $\phi\omega$ channel exhibit relatively large branching fractions.

Channel	Ratio	Channel	Ratio	Channel	Ratio
$f_0\eta$	1.00	$\omega\phi$	46.38	$\omega\omega$	36.72
$\rho b_1(1235)$	13.51	$\phi\phi$	11.39	$K^*\bar{K}_1(1270)$	11.28
$\omega h_1(1170)$	5.45	$\eta f_0(1770)$	4.37	$\pi a_2(1320)$	4.14
$K\bar{K}_0^*(1430)$	3.83	$K\bar{K}_2^*(1430)$	1.88	$\pi a_0(1450)$	1.85
$\kappa\bar{K}$	1.31	$\eta f_2(1270)$	1.26	$f_0\eta'$	0.31
$\eta f_2'(1525)$	0.23	$\eta' f_0(1370)$	0.10	$\eta f_0(1370)$	5.79×10^{-2}
$\eta' f_2(1270)$	4.47×10^{-2}	$\sigma\eta$	6.63×10^{-4}	$\sigma\eta'$	1.31×10^{-4}

Decay behavior

$|GGG; 0^{++}\rangle$

Channel	Ratio	Channel	Ratio	Channel	Ratio
$\pi\pi\omega$	1.00	$K\bar{K}\phi$	0.75	$K\bar{K}\rho$	0.69
$\kappa\bar{\kappa}\rho$	0.35	$K\bar{K}\omega$	0.35	$\rho\rho\omega$	0.35
$\kappa\bar{\kappa}\phi$	0.34	$K^*\bar{K}^*\rho$	0.24	$K^*\bar{K}^*\phi$	0.21
$\kappa\bar{\kappa}\omega$	0.18	$\pi\pi f_2(1270)$	0.14	$K^*\bar{K}^*\omega$	0.12
$K\bar{K}a_2(1320)$	7.59×10^{-2}	$K\bar{K}f_2(1270)$	4.72×10^{-2}	$K\bar{K}f_2'(1525)$	4.34×10^{-2}
$\kappa\bar{\kappa}a_2(1320)$	2.43×10^{-2}	$\eta\eta\phi$	2.24×10^{-2}	$\eta'\eta'\phi$	1.78×10^{-2}
$f_0f_0\phi$	1.76×10^{-2}	$\kappa\bar{\kappa}f_2(1270)$	1.52×10^{-2}	$\kappa\bar{\kappa}f_2'(1525)$	1.31×10^{-2}
$\eta\eta\omega$	8.82×10^{-3}	$\kappa\bar{\kappa}f_0$	7.41×10^{-3}	$\phi\phi\phi$	6.88×10^{-3}
$\omega\omega\omega$	5.64×10^{-3}	$\sigma\sigma\omega$	4.76×10^{-3}	$\eta\eta' f_2'(1525)$	1.68×10^{-3}
$\eta\eta f_2(1270)$	1.47×10^{-3}	$\eta'\eta'\omega$	1.43×10^{-3}	$\eta\eta f_2'(1525)$	1.35×10^{-3}
$\eta\eta' f_2(1270)$	9.02×10^{-4}	$\kappa\bar{\kappa}\sigma$	7.69×10^{-4}	$\sigma\sigma f_2(1270)$	4.85×10^{-4}
$\eta'\eta' f_2'(1525)$	3.59×10^{-4}	$f_0f_0f_2'(1525)$	2.59×10^{-4}	$f_0f_0f_0$	2.24×10^{-4}
$\eta'\eta' f_2(1270)$	1.10×10^{-4}	$\sigma\sigma\sigma$	4.67×10^{-8}		

Decay behavior

$|GGG;1^{+-}\rangle$

Channel	Ratio	Channel	Ratio	Channel	Ratio
$\pi\pi\omega$	1.00	$\pi\pi\phi$	1.49	$K\bar{K}\omega$	0.83
$\kappa\bar{\kappa}\omega$	0.29	$K\bar{K}\phi$	0.27	$\eta\eta\omega$	0.16
$K\bar{K}\rho$	0.15	$\kappa\bar{\kappa}\phi$	0.14	$\eta\eta\phi$	7.62×10⁻²
$\sigma\sigma\omega$	4.69×10⁻²	$\sigma\sigma\phi$	2.88×10⁻²	$\eta'\eta'\omega$	4.62×10⁻³
$K\bar{K}f_0$	3.65×10⁻³	$\eta'\eta'\phi$	3.51×10⁻³	$f_0f_0\omega$	5.38×10⁻⁴
$\eta\eta'\phi$	4.95×10⁻⁴	$\eta'\eta'f_0$	4.12×10⁻⁴	$\eta\eta f_0$	8.77×10⁻⁵
$\eta\eta'f_0$	4.58×10⁻⁵	$f_0f_0\phi$	3.61×10⁻⁵	$\pi\pi\sigma$	9.35×10⁻⁶
$K\bar{K}\sigma$	2.80×10⁻⁶	$\eta'\eta'f_0$	1.27×10⁻⁶	$\eta\eta\sigma$	8.05×10⁻⁸
$\eta'\eta'\sigma$	3.40×10⁻⁸	$\eta\eta'\sigma$	2.38×10⁻⁹	$K\bar{K}a_1(1260)$	1.62×10⁻¹⁵
$K\bar{K}f_1(1420)$	7.18×10⁻¹⁶	$K\bar{K}f_1(1285)$	6.25×10⁻¹⁶	$\pi\pi f_1(1285)$	3.58×10⁻¹⁶

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Summary

- We calculate the relative branching ratios of two-gluon glueballs with $J^{PC} = 0^{++}/0^{-+}$ and three-gluon glueballs with $J^{PC} = 0^{++}/1^{+-}$. In total, we derive nearly one hundred ratios for these glueballs.
- Our results suggest that the $f_0(1710)$ and $\eta(2370)$ likely contain a significant gluon component, whereas the gluon component in $f_0(1500)$ appears to be small.
- We propose observing the three-gluon glueball with $J^{PC} = 0^{++}$ in the $\pi\pi\omega$ and $K\bar{K}\phi$ channels, and the three-gluon glueball with $J^{PC} =/1^{+-}$ in the $\pi\pi\omega$, $\pi\pi\phi$, and $K\bar{K}\phi$ channels.
- Our findings provide a theoretical framework to assist the search for glueball states in high-luminosity experiments, such as BESIII, GlueX, LHCb, and PANDA.

Thanks for your attention !