Lattice QCD Study of Heavy Quark Diffusion

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[PRD 103 (2021) 1, 014511]
[PRL 130 (2023) 23, 231902]
[PRL 132 (2024) 5, 051902]
[PRD 109 (2024) 11, 114505]
[JHEP 09 (2025) 180]

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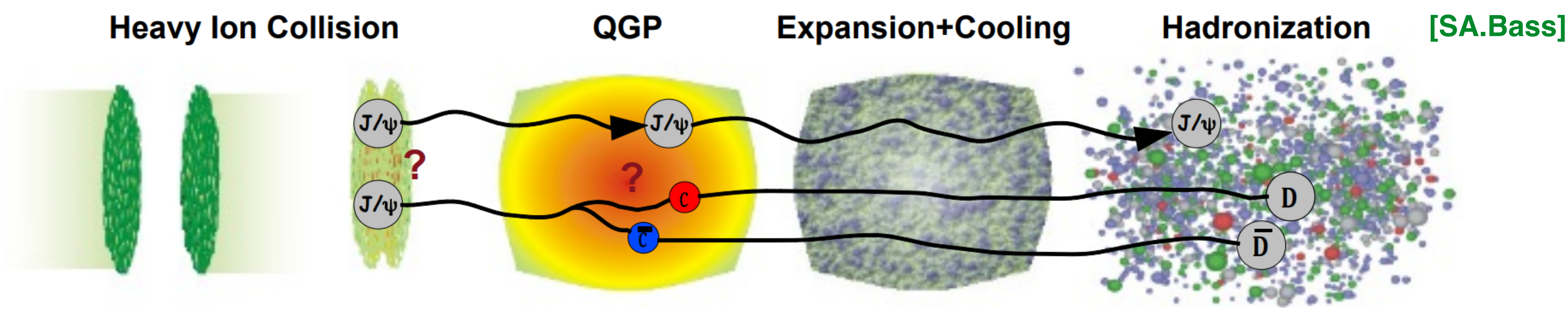
Heavy quark diffusion in HICs



Release constituents equilibrate via diffusion process

how fast do heavy quarks equilibrate?

- Perturbative estimates: [G. Moore and D. Teaney, PRC.71.064904]

  $\tau_{\mathrm{kin,charm}} \sim 6\ \mathrm{fm/c} \gg \tau_{\mathrm{kin,light}} \sim 1\ \mathrm{fm/c}$
- Experimental estimates (RHIC): [STAR Collaboration, PRL,106 (2011) 159902]

  $\tau_{\mathrm{kin,charm}} \approx \tau_{\mathrm{kin,light}}$

Need non-perturbative ab-initio determination for equilibration time!

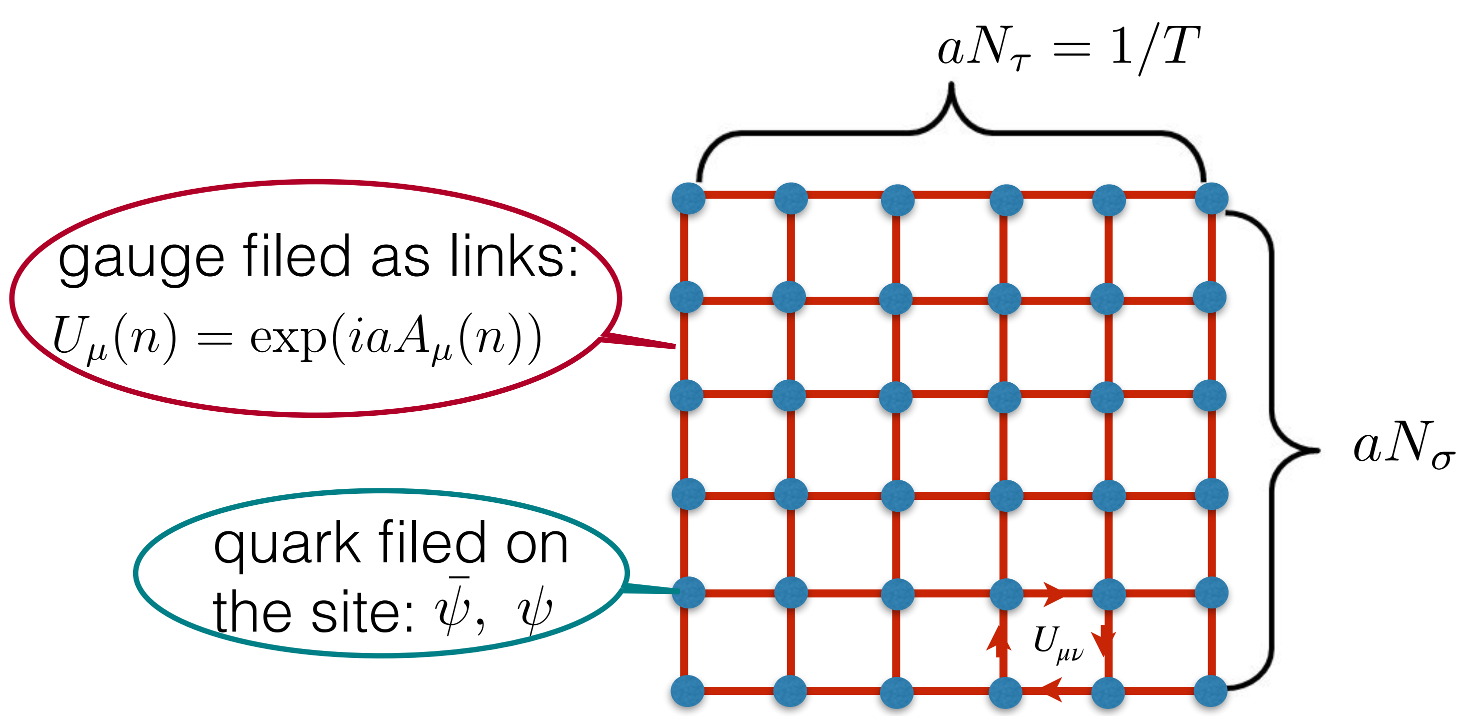
$$\tau_{\mathrm{kin}} = \frac{1}{\eta_D} = \frac{D M_{\mathrm{kin}}}{T} = \frac{2M_{\mathrm{kin}}}{T^2}\frac{1}{\kappa/T^3}$$

Introduction to Lattice QCD

LQCD is designed for non-perturbative physics: hadron structure, QCD vacuum, hadron spectrum, thermal physics, …

$$S_F = \int \mathrm{d}^4x \; \bar{\psi}\big((\partial + iA_\mu)\gamma^\mu + m\big)\psi, \qquad S_G[C] = \frac{\beta}{3}\sum_{n\in\Lambda}\sum_{\mu\le\nu} Re \; \mathrm{Tr}[\mathbb{1} - U_{\mu\nu}(n)]$$

- Basic quantities in LQCD:
  - ✳ lattice volume
  - ✳ lattice spacing
  - ✳ gauge field SU(3) matrix
  - ✳ quark field Dirac spinor (vector)
  - ✳ quark mass



Discretization of space

Discrete space $\xrightarrow[N_\sigma \to \infty]{a \to 0}$ Continuum space

Heavy quark diffusion under HQEFT

- Langevin equations of heavy quark motion

$$\partial_t p_i = -\eta_D p_i + \xi_i(t) \qquad \langle \xi_i(t)\xi_j(t')\rangle = \kappa \delta_{ij}\delta(t-t')$$

- Mass dependent **momentum** diffusion coefficient

$$\kappa^{(M)} \equiv \frac{M^2\omega^2}{3T\chi_q}\sum_i \frac{2T\rho_V^{ii}(\omega)}{\omega}\bigg|^{\eta \ll |\omega| \lesssim \omega_{\mathrm{UV}}}$$

- Large quark mass limit in HQ effective field theory

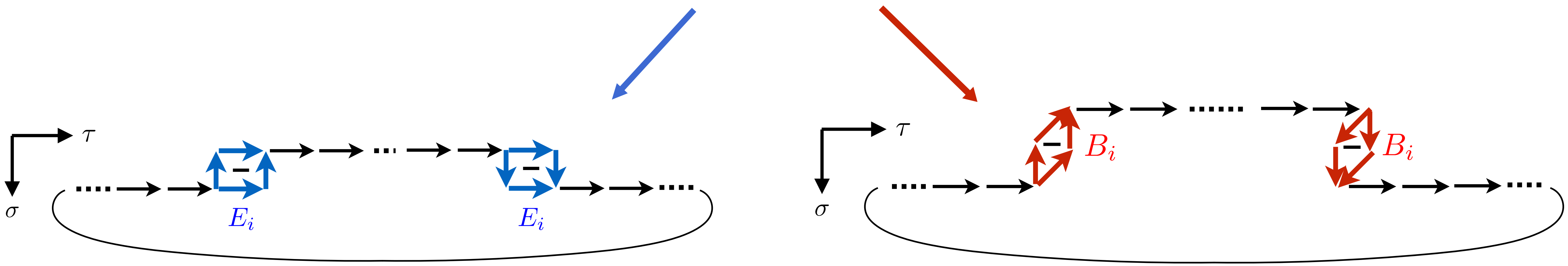
$$\kappa \equiv \frac{\beta}{3}\sum_{i=1}^{3}\lim_{\omega\to 0}\left[\lim_{M\to\infty}\frac{M^2}{\chi_q}\int_{-\infty}^{\infty}\mathrm{d}t\; e^{i\omega(t-t')}\int \mathrm{d}^3\vec{x}\,\left\langle \frac{1}{2}\{\mathcal{F}^i(t,\vec{x}),\mathcal{F}^i(0,\vec{0})\}\right\rangle\right]$$

J. Casalderrey-Solana and D. Teaney, PRD 74, 085012
S. Caron-Huot et al., JHEP 0904 (2009) 053
A. Bouttefeux, M. Laine, JHEP 12 (2020) 150

$$q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = \mathbf{F}$$

Heavy quark mass limit and finite mass correction

$$\partial_t \mathbf{p} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \mathbf{F}$$



$$G(\tau, T) = \int \frac{\mathrm{d}\omega}{\pi} \frac{\cosh(\omega(\tau T - 1/2)/T)}{\sinh(\omega/(2T))} \rho(\omega, T)$$

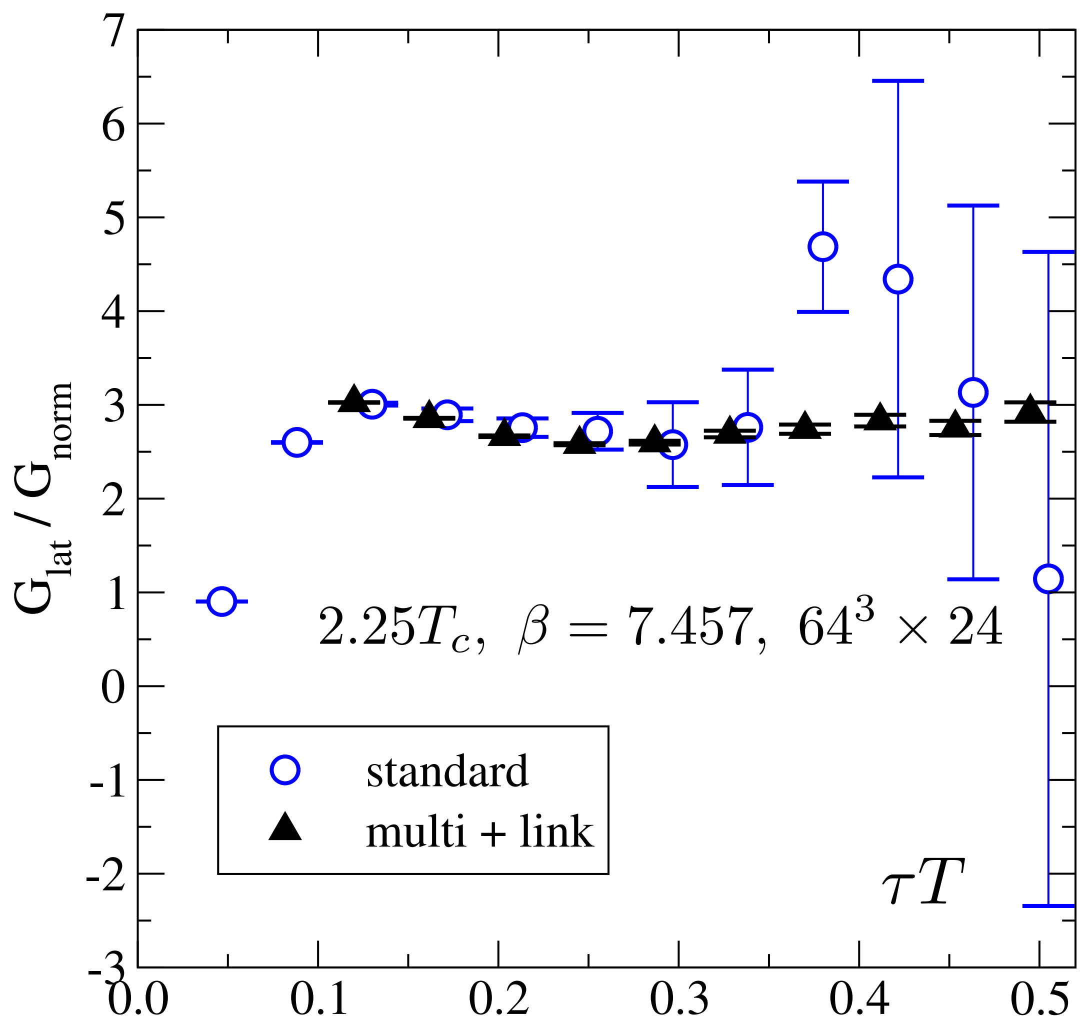
$$\frac{\kappa}{4\pi T^3} = \frac{1}{2\pi T^2} \lim_{\omega \to 0} \frac{\rho(\omega)}{\omega}$$

$$\kappa = \kappa_E + \frac{2}{3} \langle \mathbf{v^2} \rangle \kappa_{\mathbf{B}} \quad \longrightarrow \quad D_s = \frac{2T^2}{\kappa} \cdot \frac{\langle p^2 \rangle}{3MT}$$

$2/3 \cdot \langle \mathbf{v}^2 \rangle_{\mathrm{charm}}$: $18\% \sim 30\%$

$2/3 \cdot \langle \mathbf{v}^2 \rangle_{\mathrm{bottom}}$: $7\% \sim 13\%$

Previous studies using multi-level algorithm



[A. Francis, et al, PRD92 (2015)116003]

ML: independent updates in each sub-lattice

[M. Luscher and P. Weisz, JHEP 09 (2001) 010]

A long history of lattice calculations, but all using multi-level:

[S. Caron-Huot, et al., JHEP 04 (2009) 053]
[D. Banerjee, et al., P.R.D 85 (2012) 014510]
[A. Francis, et al., PRD92 (2015)116003]
[D. Banerjee, et al., JHEP 08 (2022) 128]
[D. Banerjee, et al., Nucl.Phys.A.2023.122721]
[N. Brambilla, et al., PRD107 (2023) 054508]

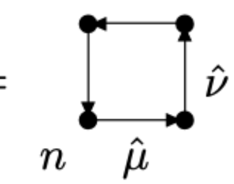
- Multi-level algorithm reduces noise in correlators
- But only applicable in quenched approximation

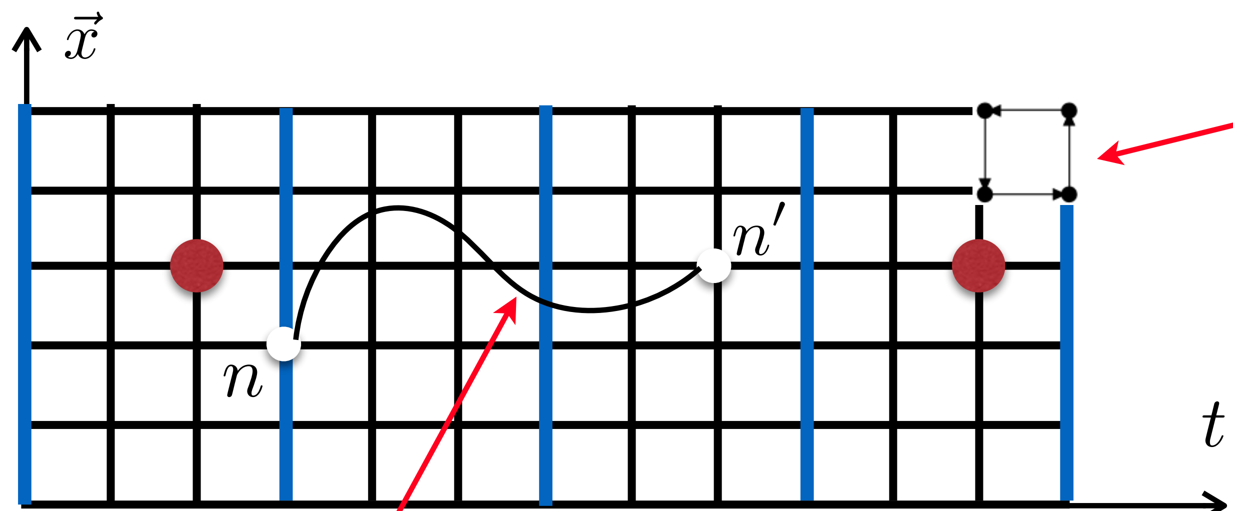
Breaking down of Multi-level algorithm in QCD

$$\mathcal{Z}(V,T) = \int [dU][d\psi][d\bar{\psi}]e^{-S_G[U]-S_F[U,\psi,\bar{\psi}]}$$

$$S_G[U] = \frac{1}{2g^2}\sum_{n,\mu,\nu} 2\mathrm{Tr}[1 - P_{\mu\nu}(n)] \qquad S_F[U,\psi,\bar{\psi}] = \bar{\psi}M_q[U]\psi$$

Action **local in quenched QCD** (sum of plaq.): $P_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U^\dagger_\mu(n+\hat{\nu})U^\dagger_\nu(n) =$





Action **non-local in full QCD** (connection between any two sites):

$$M_q(n,n';i,j)[U] = \hat{m}_q\delta_{n,n'}\delta_{ij} + \frac{1}{2}\sum_\mu \eta_\mu(n)\left((U_\mu(n))_{i,j}\delta_{n',n+\hat{\mu}} - (U^\dagger_\mu(n))_{i,j}\delta_{n,n'+\hat{\mu}}\right)$$

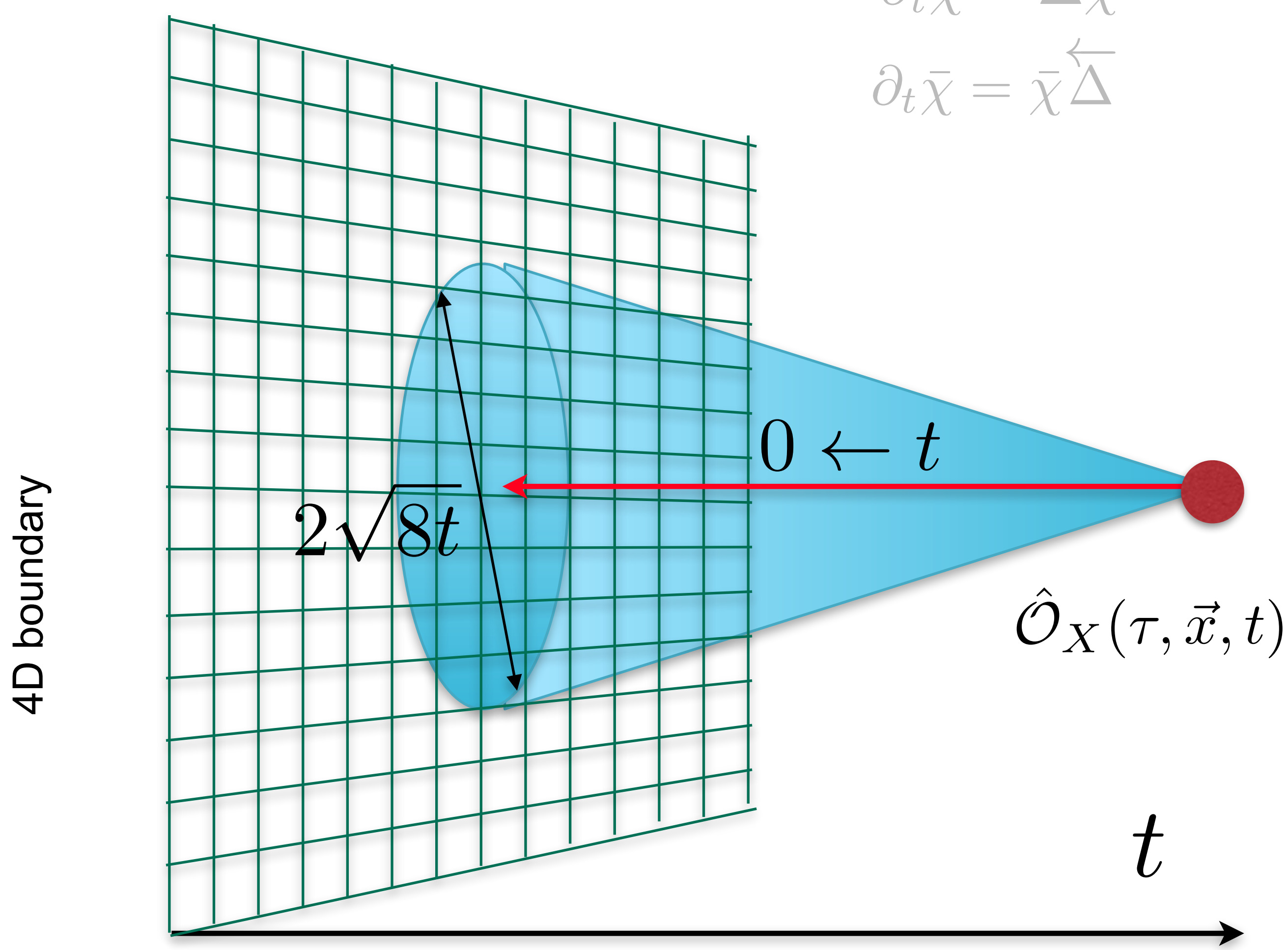
Gradient flow — the only way towards QCD

Evolve fields according to diffusion equations:

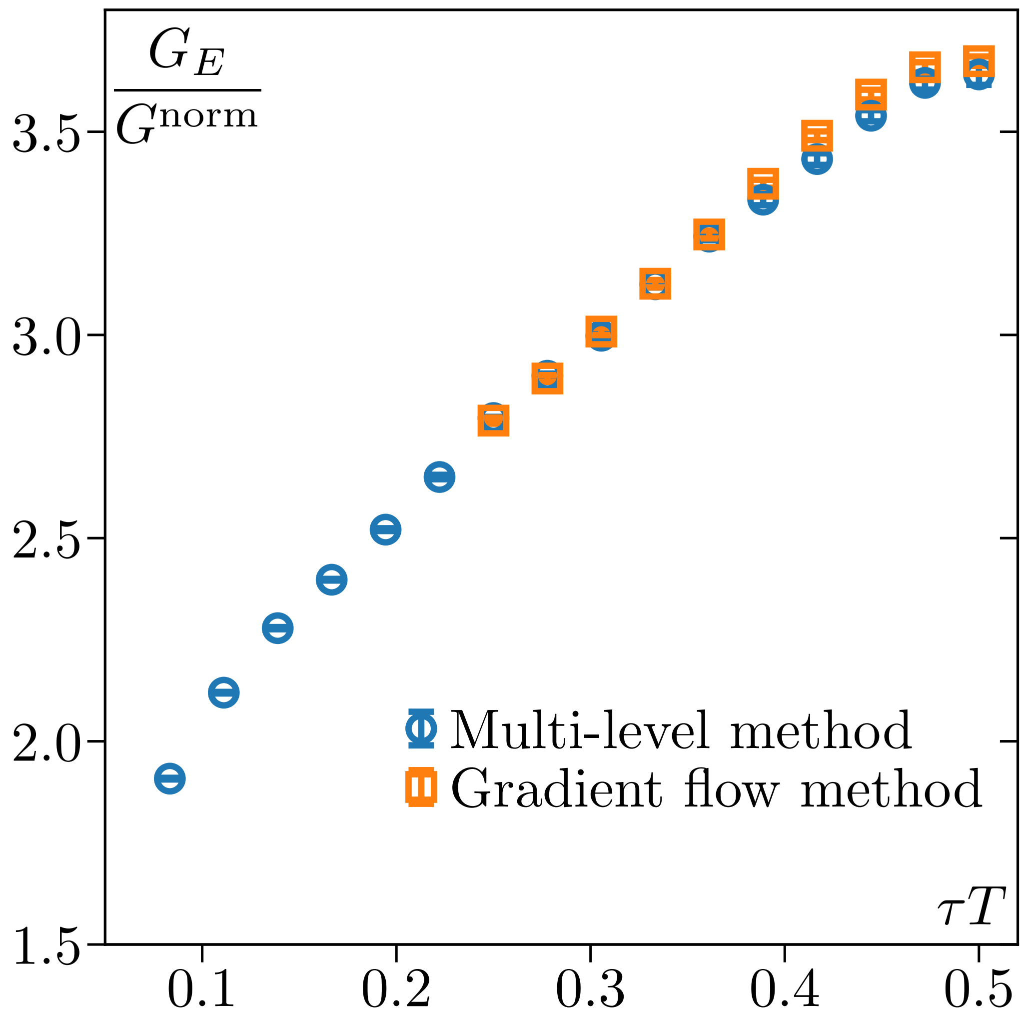
$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} \sim -\frac{\delta S_G[B_\mu(x,t)]}{\delta B_\mu(x,t)} \sim D_\nu G_{\nu\mu}(x,t)$$

$$\partial_t \chi = \Delta \chi$$
$$\partial_t \bar{\chi} = \bar{\chi} \overleftarrow{\Delta}$$

Luscher & Weisz, JHEP1102(2011)051
Narayanan & Neuberger, JHEP0603(2006)064



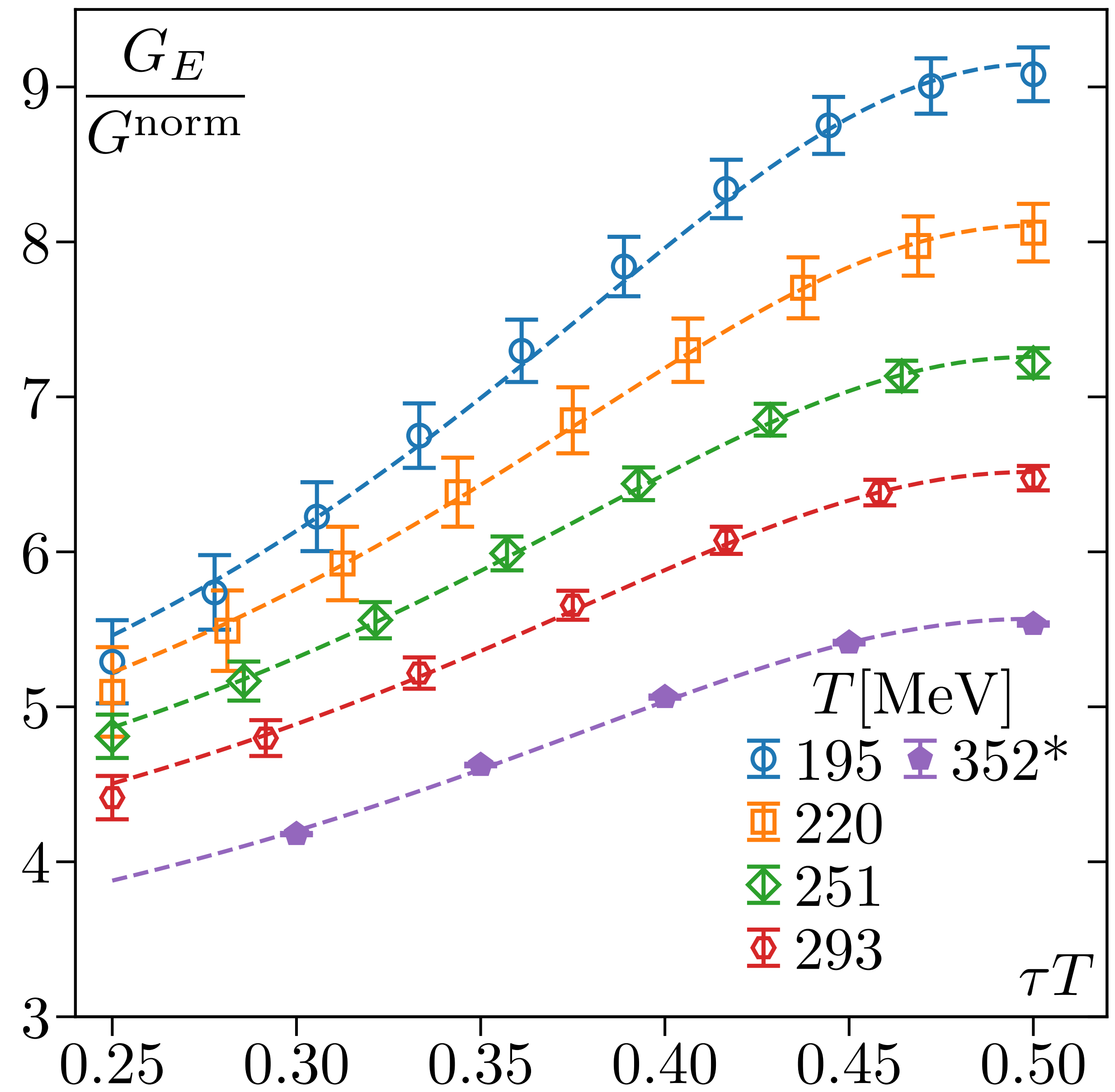
[LA, AME, OK, LM, GDM, HTS, PRD 103 (2021) 1, 014511]



- Smear gauge field + quark field simultaneously
- Well-defined renormalization
- Consistent quenched results from ML & GF

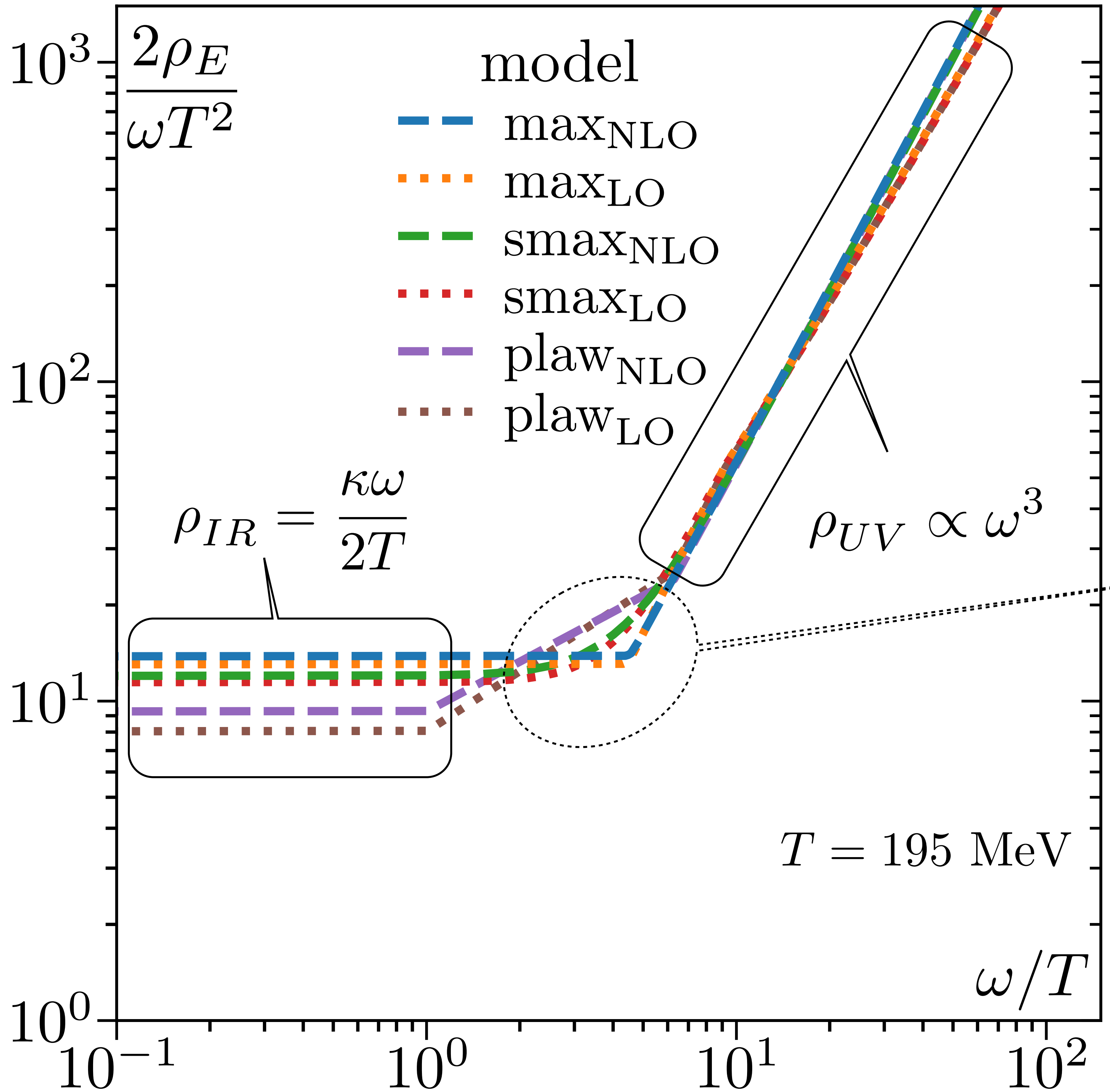
Color-electric field correlators in QCD

First QCD calculation of κ_E ! [LA, OK, RL, SM, PP, **HTS**, SS, PRL 130 (2023) 23, 231902]



- $N_f = 2 + 1$, HISQ
- 195 MeV $\leq T \leq$ 352 MeV
- Heavy pion mass: 320 MeV

Spectra analysis of color-electric field correlators



$$G(\tau, T) = \int \frac{d\omega}{\pi} K(\omega, \tau, T)\rho(\omega, T)$$

$$\begin{cases} \rho_{\max} \equiv \max(\phi_{\mathrm{IR}}, \phi_{\mathrm{UV}}) \\ \rho_{\mathrm{smax}} \equiv \sqrt{\phi_{\mathrm{IR}}^2 + \phi_{\mathrm{UV}}^2} \\ \rho_{\mathrm{plaw}} \equiv \begin{cases} \phi_{\mathrm{IR}} & \omega \le \omega_{\mathrm{IR}}, \\ a\omega^b \quad \text{for} & \omega_{\mathrm{IR}} < \omega < \omega_{\mathrm{UV}}, \\ \phi_{\mathrm{UV}} & \omega \ge \omega_{\mathrm{UV}}, \end{cases} \end{cases}$$

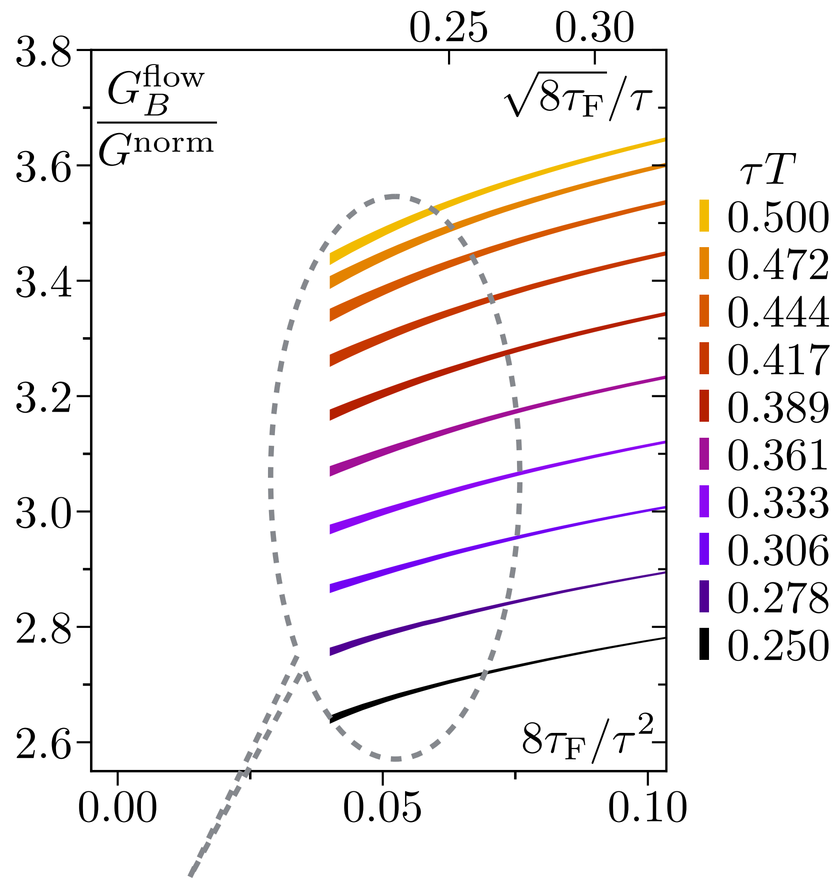
[LA, OK, RL, SM, PP, **HTS**, SS, PRL 130 (2023) 23, 231902]

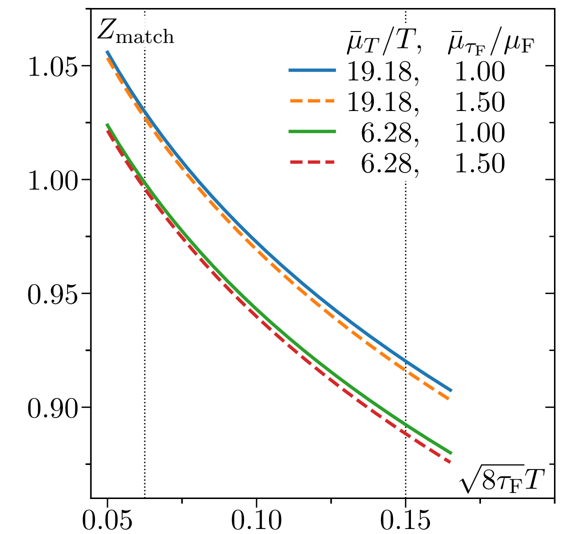
Intractabilities in color-magnetic field correlators

- Anomalous dimension in B-field
- Log divergence in flow time

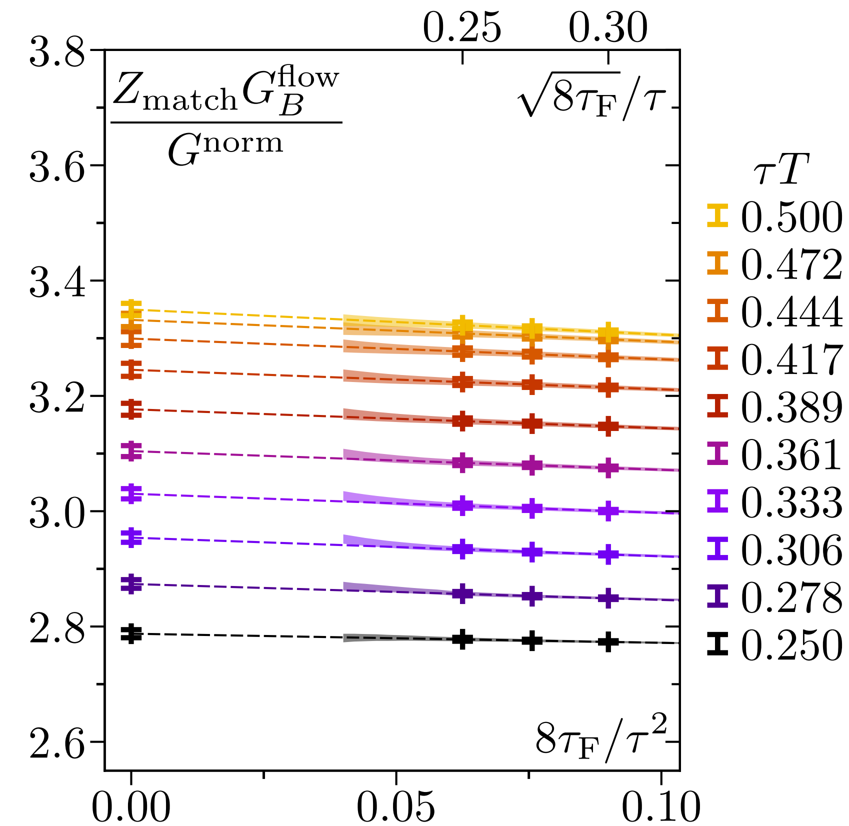
A. Bouttefeux and M. Laine, JHEP 12 (2020) 150

M. Laine, JHEP 06(2021)139





$$G_B^{\rm phys.} = Z_{\rm match} G_B^{\rm flow}(\tau_F)$$



breaks down of linear $\tau_F \to 0$ extrapolation

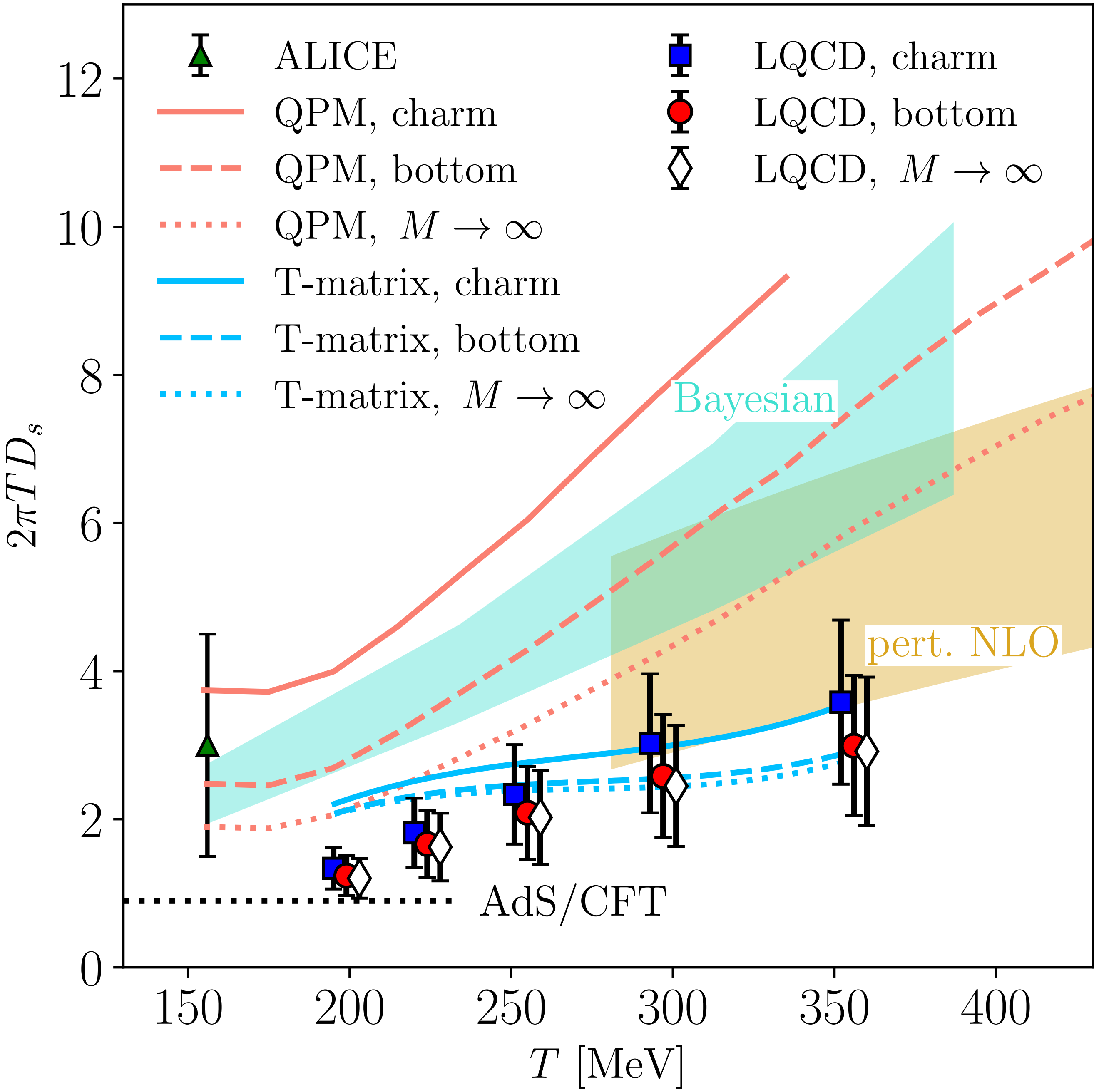
[LA, DC, OK, GDM, **HTS**, PRD 109 (2024) 11, 114505]

Renormalization issue solved by a matching factor:

$$\ln Z_{\rm match} = \int_{\bar\mu_T^2}^{\bar\mu_{\tau_F}^2} \gamma_0 g^2_{\overline{\rm MS}}(\bar\mu) \frac{d\bar\mu^2}{\bar\mu^2} + \gamma_0 g^2_{\overline{\rm MS}}(\bar\mu_T) \left[\ln \frac{\bar\mu_T^2}{(4\pi T)^2} - 2 + 2\gamma_{\rm E} \right] - \gamma_0 g^2_{\overline{\rm MS}}(\bar\mu_{\tau_F}) \left[\ln \frac{\bar\mu_{\tau_F}^2}{4\,\mu_F^2} + \gamma_{\rm E} \right]$$

Charm and bottom quark diffusion

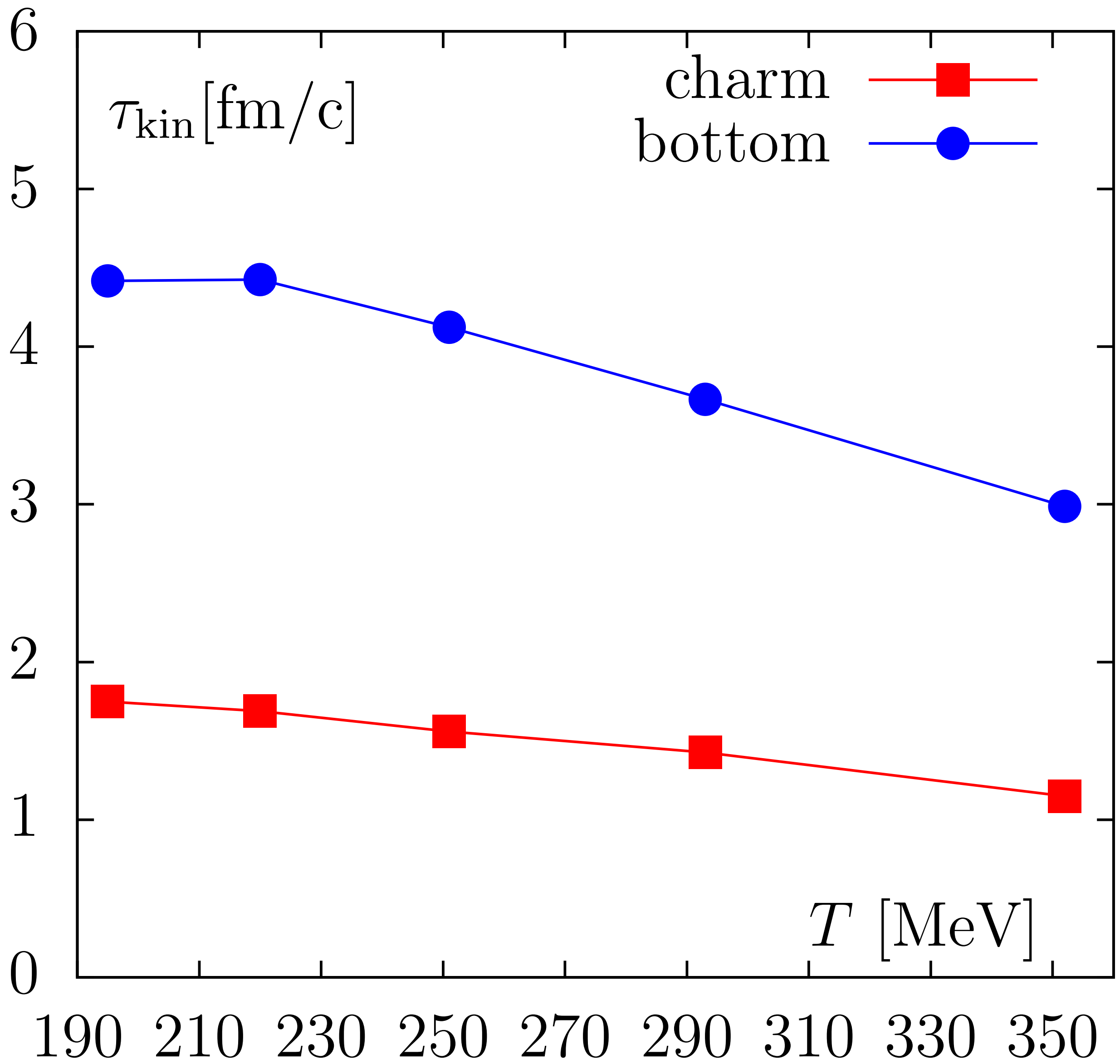
First QCD determination of charm&bottom quark diffusion!



- HQ mass dependence of HQ diffusion: mild
- Universal change pattern with quark mass
- Weak quark mass dependence in LQCD & T-matrix
- Weaker than quasi-particle model (QPM) calculations

[LA, DC, OK, RL, GDM, SM, PP, **HTS**, SS, PRL 132 (2024) 5, 051902]

Equilibration time of charm&bottom quark

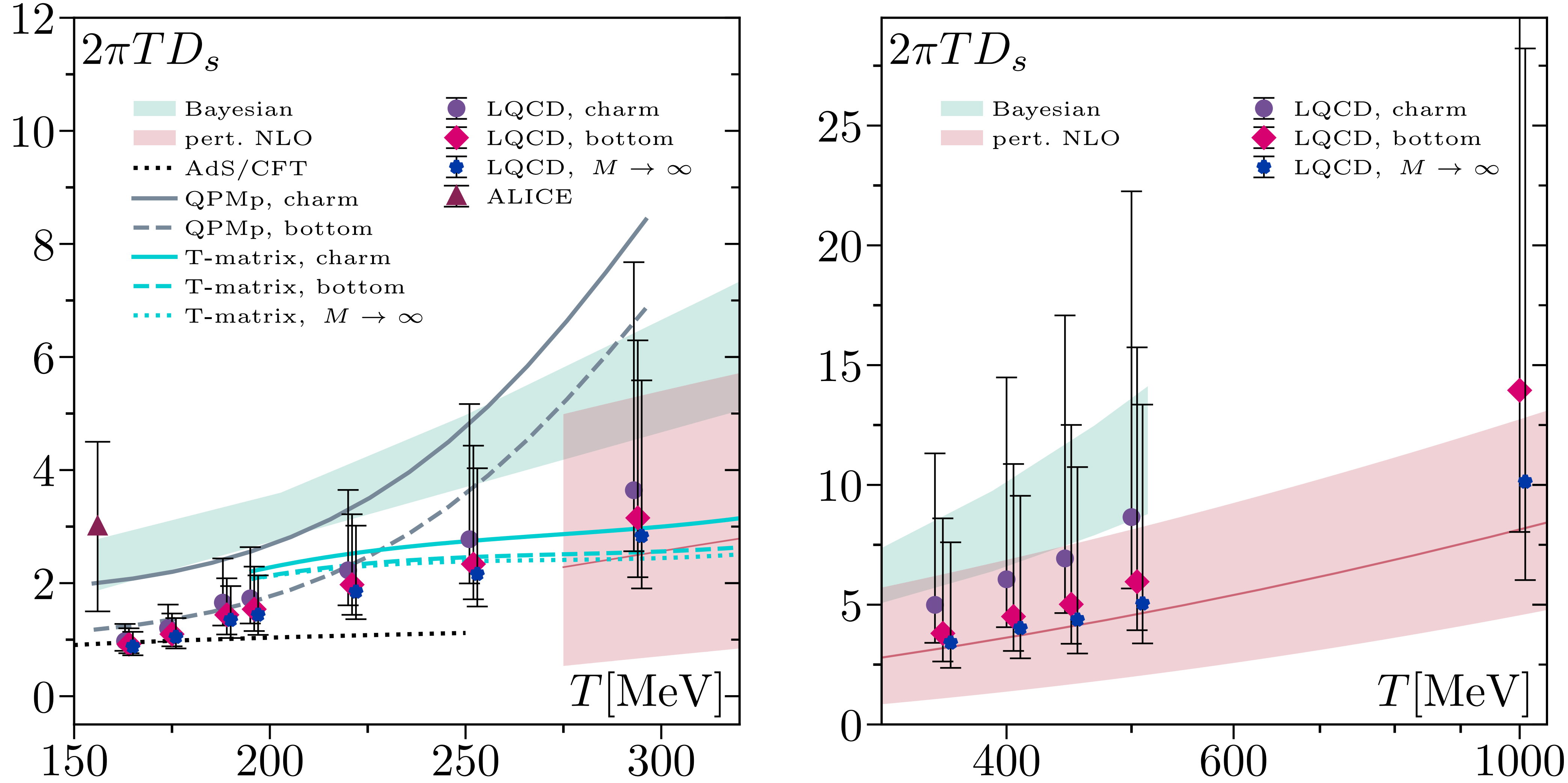


$$\tau_{\rm kin} = \frac{1}{\eta_D} = \frac{1}{\kappa/T^3}\left(\frac{T_c}{T}\right)^2\left(\frac{M}{1.5\ \mathrm{GeV}}\right)\frac{3\ \mathrm{GeV}}{T_c^2}$$

- Lattice provides short equilibration time for charm&bottom quark
- Rapid equilibrium <—> QGP is near perfect fluid
- Lattice determination of $\tau_{\rm kin}^{\rm charm}$ favors the experimental estimate (~1 fm/c for all)

[LA, DC, OK, RL, GDM, SM, PP, **HTS**, SS, PRL 132 (2024) 5, 051902]

HQ diffusion towards the physical point



[JDG, SM, PP, **HTS**, JHW, et al., JHEP 09 (2025) 180]

$m_\pi \sim 320\text{MeV}$	→	$m_\pi \sim 160\text{MeV}$
$195\text{ MeV} \leq T \leq 352\text{ MeV}$		$163\text{ MeV} \leq T \leq 10000\text{ MeV}$

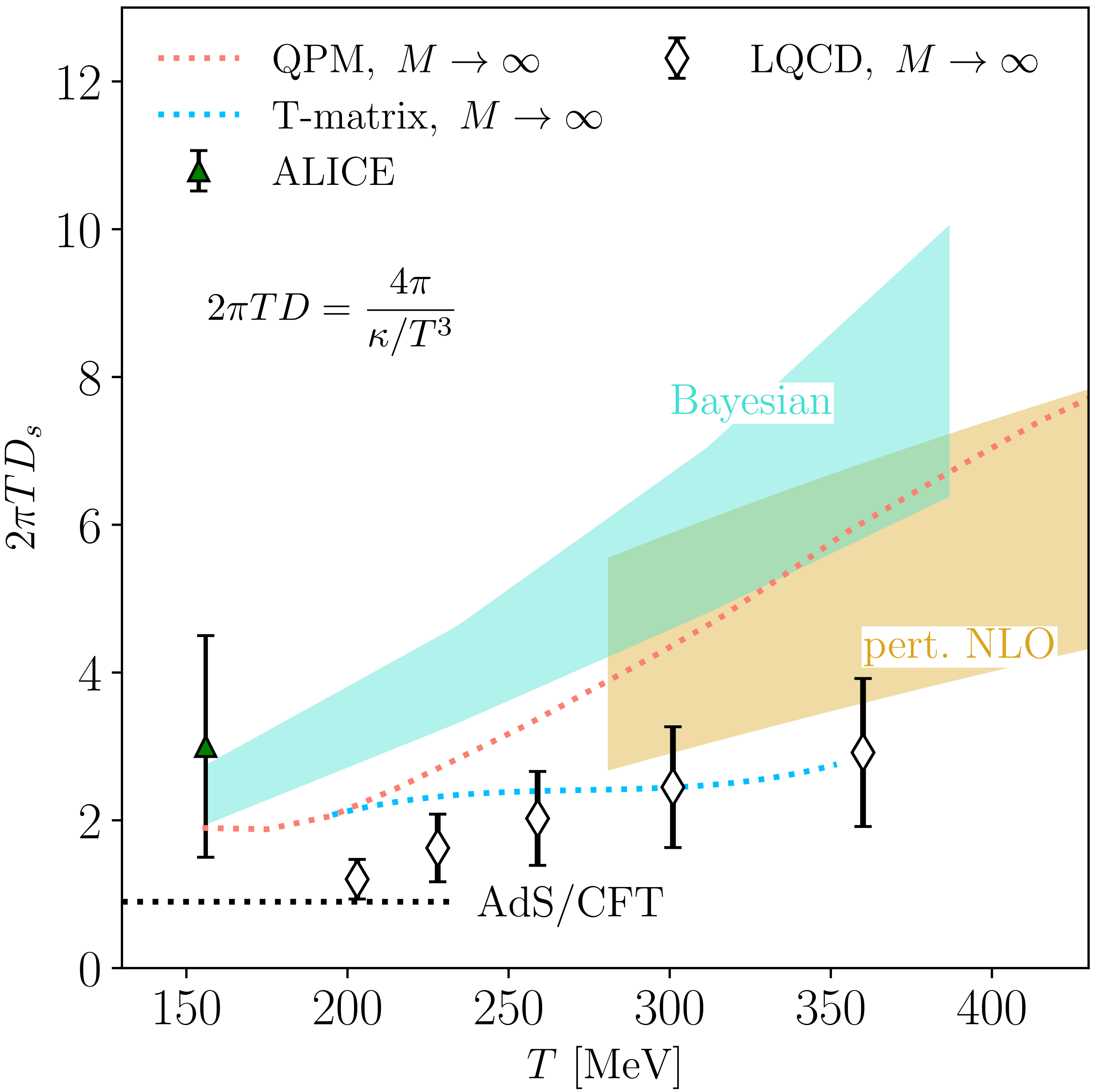
- Similar magnitudes as previous studies
- Approaching AdS/CFT limit near T_c
- Consistent with NLO at high temperature

Heavy-flavor diffusion is insensitive to changes in the light degrees of freedom across the phase transition

Summary

- Heavy quark diffusion coefficient is an important parameter in understanding the HICs
- Nonperturbative determination of HQ diffusion is a non-trivial task
- Remarkable progress has been achieved thanks to the gradient flow method
- HQ diffusion calculation at the physical point is available
- Fast equilibration of charm&bottom quark has been found
- HQ diffusion insensitive to heavy quark mass & light quark mass
- Lattice determination approaches the AdS/CFT limit near T_c
- Lattice determination is consistent with the NLO prediction at high temperature

Backup: Infinite heavy quark diffusion coefficient



[LA, OK, RL, SM, PP, **HTS**, SS, PRL 130 (2023) 23, 231902]

- Agree with AdS/CFT at $\sim T_c$
- Close to the phenomenological extraction using the ALICE data
- Agree with T-matrix estimate at moderate and high temperature
- Agree with NLO perturbative estimate at high temperature
- Lower than Bayesian&QPM estimate
- Mild temperature dependence

Backup: Finite mass correction

Physical charm & bottom quark not infinitely heavy!

$$M_c : \sim 1.3 \text{ GeV}$$

$$M_b : \sim 4.5 \text{ GeV}$$

D. Guazzini, et al., JHEP 10 (2007) 081

$$\kappa_E : \ M_Q \to \infty$$

$$\langle \mathcal{F}(t')\mathcal{F}(t)\rangle = q^2\left\{\langle E_i(t')E_j(t)\rangle + \frac{1}{3}\langle \mathbf{v}^2\rangle\langle \delta_{ij}B_k(t')B_k(t) - B_j(t')B_i(t)\rangle\right\}$$

Infinite heavy limit

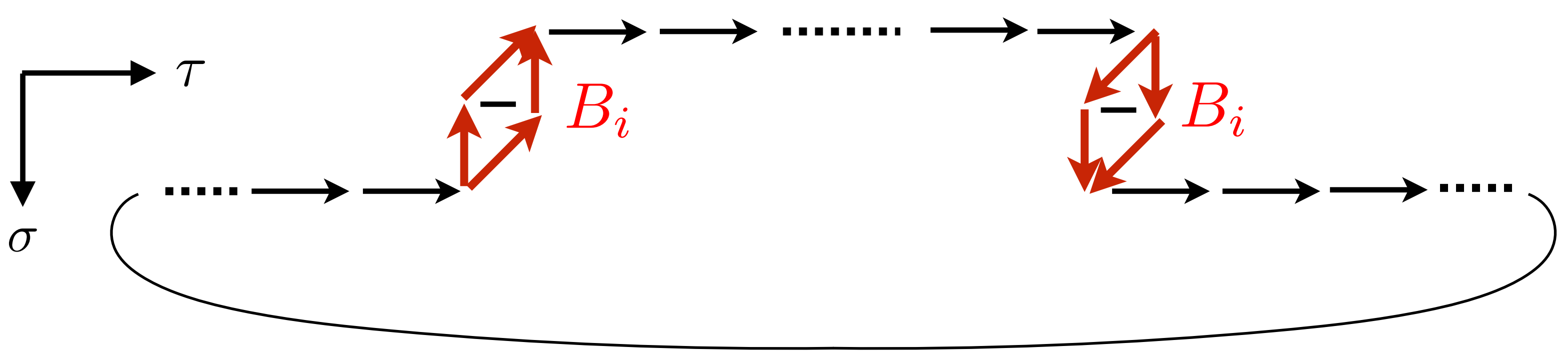
Finite mass correction

$$\kappa = \kappa_E + \frac{2}{3}\langle \mathbf{v^2}\rangle \kappa_{\mathbf{B}}$$

$$2/3 \cdot \langle \mathbf{v}^2\rangle_{\text{charm}} : \ 18\% \sim 30\%$$

$$2/3 \cdot \langle \mathbf{v}^2\rangle_{\text{bottom}} : \ 7\% \sim 13\%$$

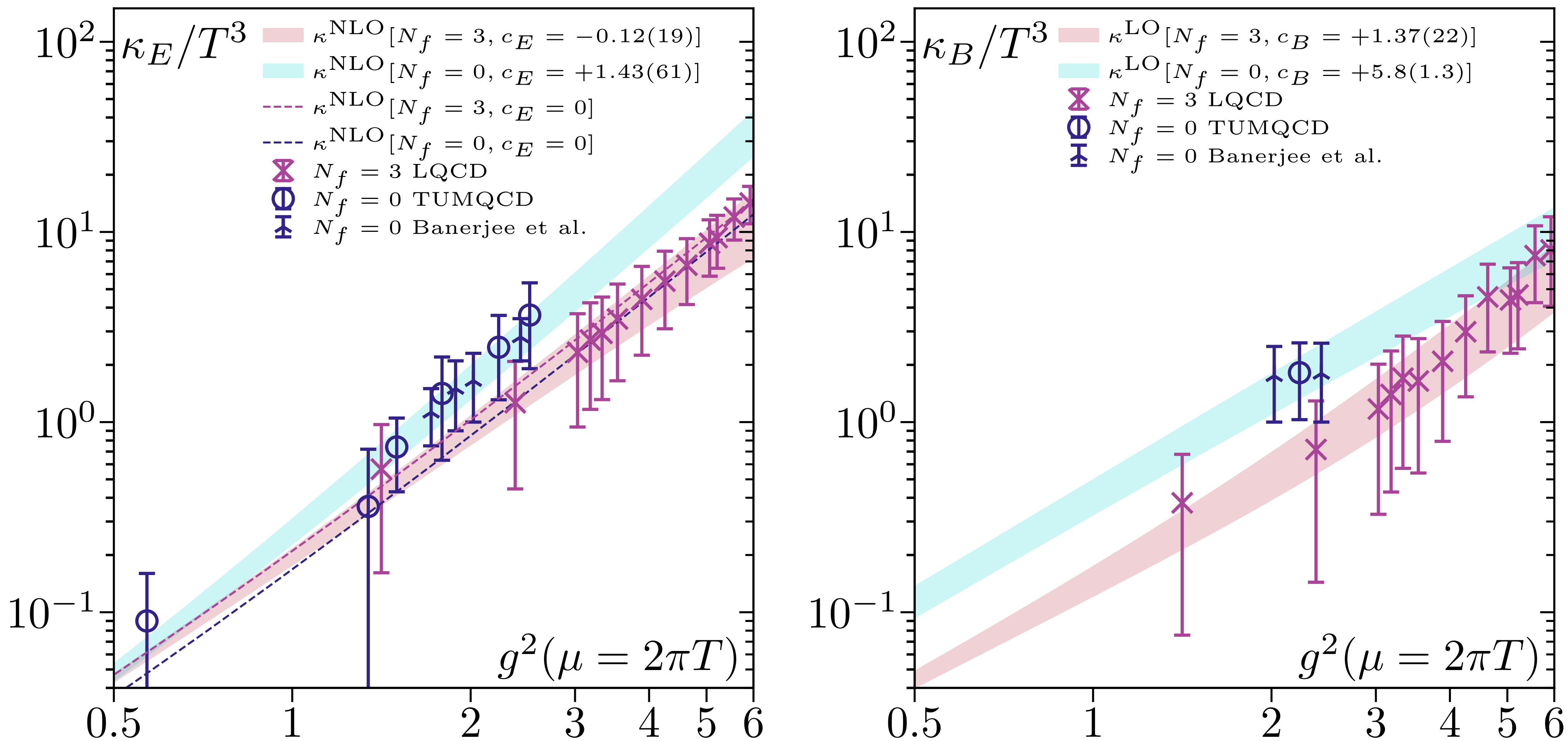
$$D_s = \frac{2T^2}{\kappa} \Longrightarrow D_s = \frac{2T^2}{\kappa} \cdot \frac{\langle p^2\rangle}{3MT}$$



Color-magnetic field correlation function

A. Bouttefeux, M. Laine, JHEP 12 (2020) 150

Backup: κ_E v.s. κ_B



[JDG, SM, PP, **HTS**, JHW, et al., JHEP 09 (2025) 180]

Perturbative prediction:

$$\kappa_E(T) = \frac{g^4 C_F T^3}{18\pi}\left(\left[N_c + \frac{N_f}{2}\right]\left[\ln\frac{2T}{m_D} + \xi\right] + \frac{N_f \ln 2}{2} + 2.3302 N_c m_D + c_E g^2\right)$$

$$\kappa_B(T) = \frac{g^4 C_F T^3}{18\pi}\left(\left[N_c + \frac{N_f}{2}\right]\ln\frac{1}{g^2} + c_B\right),$$

S. Caron-Huot and G. D. Moore, PRL. 100, 052301 (2008)

- The NLO results agree with our lattice determinations, suggesting the form $\sim g^4$

Backup: identify the heavy quark diffusion

Phenomenological diffusion picture of classical particle

Equilibrium -> Relaxation -> Equilibrium

$$\langle A(\mathbf{x})\rangle_{\text{eq}} = 0 \qquad \partial_t \langle A(\mathbf{x},t)\rangle = D\nabla^2 \langle A(\mathbf{x},t)\rangle$$

Solution:

$$\langle A(\mathbf{k},\omega)\rangle = \frac{i}{\omega + iD\mathbf{k}^2}\langle A(\mathbf{k}, t=0)\rangle$$

Linear response theory

Perturbation to Hamiltonian:

$$H(t) = H_0 - \int \mathrm{d}\mathbf{x}\ A(\mathbf{x}) h(x) e^{\epsilon t}\Theta(-t)$$

Solution:

$$\frac{\partial}{\partial t}\Big(\delta\langle A(\mathbf{k}, t=0)\rangle\Big) = -\frac{G_R(\mathbf{k},t)}{\chi_q(\mathbf{k})}\delta\langle A(\mathbf{k},0)\rangle$$

Kubo formula: $$G_R(\mathbf{k},\omega) = \frac{iD\mathbf{k}^2}{\omega + iD\mathbf{k}^2}\chi_q(\mathbf{k}) \sim \rho(\vec{k},\omega)$$

$$A \to J^\mu = \bar{\psi}\gamma^\mu\psi$$

$$D = \frac{1}{3\chi_q}\lim_{\omega\to 0}\sum_{i=1}^{3}\frac{\rho^{ii}(\omega)}{\omega}$$

Backup: full QCD setup

$N_f = 2+1, \text{ HISQ}, \; m_\pi = 320 \text{ MeV}$

T [MeV]	β	am_s	am_l	N_σ	N_τ	# conf.
195	7.570	0.01973	0.003946	64	20	5899
	7.777	0.01601	0.003202	64	24	3435
	8.249	0.01011	0.002022	96	36	2256
220	7.704	0.01723	0.003446	64	20	7923
	7.913	0.01400	0.002800	64	24	2715
	8.249	0.01011	0.002022	96	32	912
251	7.857	0.01479	0.002958	64	20	6786
	8.068	0.01204	0.002408	64	24	5325
	8.249	0.01011	0.002022	96	28	1680
293	8.036	0.01241	0.002482	64	20	6534
	8.147	0.01115	0.002230	64	22	9101
	8.249	0.01011	0.002022	96	24	688
352	8.249	0.01011	0.002022	96	20	2488

- Wide temperature range
- Different lattice spacings
- Large lattices towards thermodynamic limit

Backup: physical pion setup

T [MeV]	β	a [fm]	$a \times m_s$	$a \times m_l$	N_σ	N_τ	# conf.	streams
137	7.3730	0.0602	0.02500	0.00125	64	24	2273	4
149	7.3730	0.0602	0.02500	0.00125	64	22	4663	35
164	7.3730	0.0602	0.02500	0.00125	64	20	7424	36
182	7.3730	0.0602	0.02500	0.00125	64	18	6245	37
205	7.3730	0.0602	0.02500	0.00125	64	16	4785	4
133	7.5960	0.0493	0.02020	0.00101	64	30	1683	4
143	7.5960	0.0493	0.02020	0.00101	64	28	2036	4
154	7.5960	0.0493	0.02020	0.00101	64	26	9162	47
167	7.5960	0.0493	0.02020	0.00101	64	24	6669	37
182	7.5960	0.0493	0.02020	0.00101	64	22	7115	37
200	7.5960	0.0493	0.02020	0.00101	64	20	3017	4
222	7.5960	0.0493	0.02020	0.00101	64	18	4952	8
250	7.5960	0.0493	0.02020	0.00101	64	16	7130	9
153	7.8250	0.0404	0.01640	0.00082	64	32	2574	8
163	7.8250	0.0404	0.01640	0.00082	64	30	4757	24
174	7.8250	0.0404	0.01640	0.00082	64	28	14128	49
188	7.8250	0.0404	0.01640	0.00082	64	26	13911	48
204	7.8250	0.0404	0.01640	0.00082	64	24	4555	7
222	7.8250	0.0404	0.01640	0.00082	64	22	5109	7
244	7.8250	0.0404	0.01640	0.00082	64	20	4433	4
271	7.8250	0.0404	0.01640	0.00082	64	18	5340	4
305	7.8250	0.0404	0.01640	0.00082	64	16	6238	4

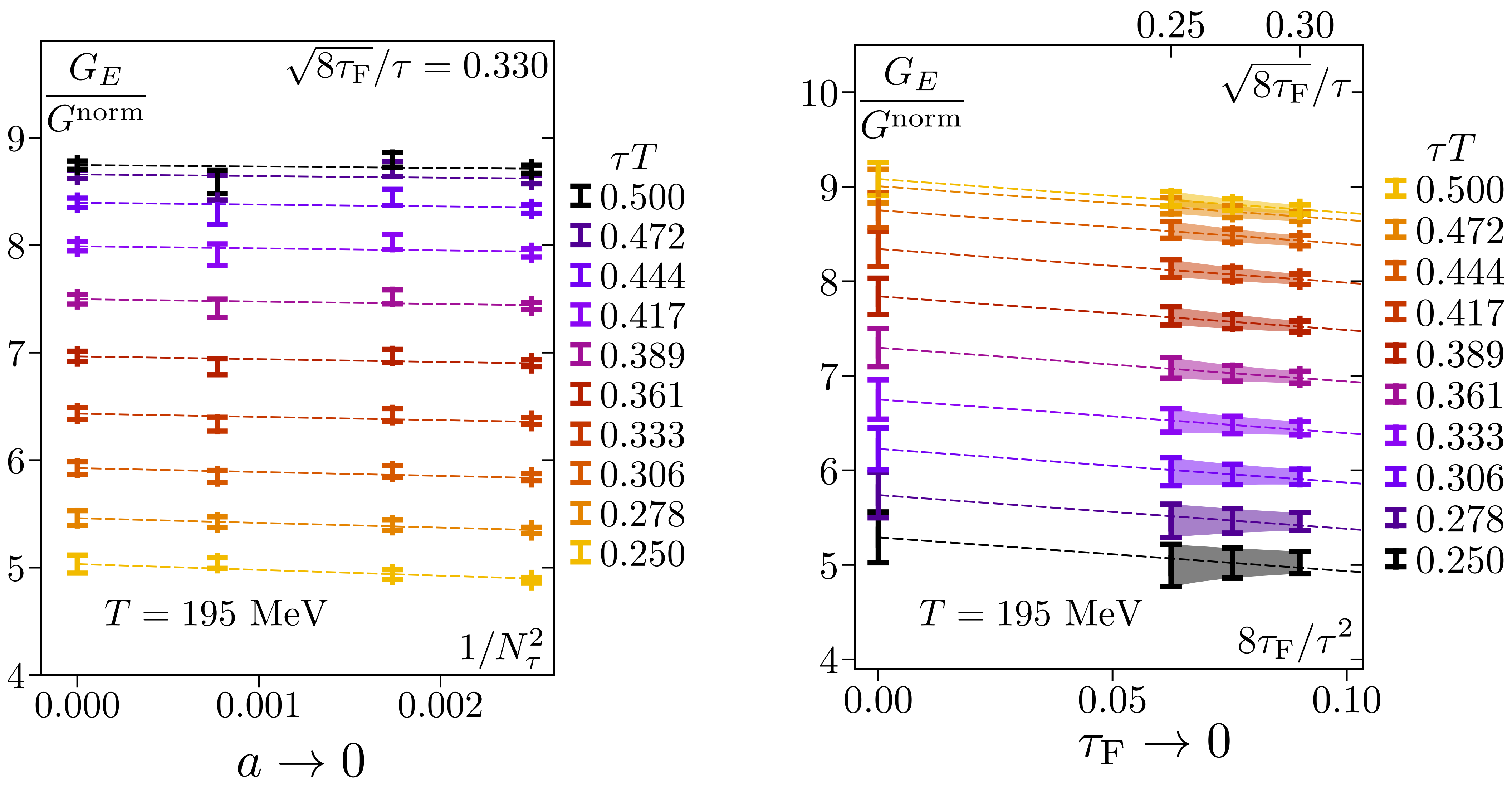
$$m_\pi = 160 \text{ MeV}$$

$$153\text{MeV} \leq T \leq 10000 \text{ MeV}$$

T [MeV]	β	a [fm]	$a \times m_s$	$a \times m_l$	N_σ	N_τ	# conf.	streams
286	8.4000	0.0247	0.008870	0.0017740	64	28	4234	16
308	8.4000	0.0247	0.008870	0.0017740	64	26	4841	16
333	8.4000	0.0247	0.008870	0.0017740	64	24	4728	4
364	8.4000	0.0247	0.008870	0.0017740	64	22	5272	4
400	8.4000	0.0247	0.008870	0.0017740	64	20	5664	4
444	8.4000	0.0247	0.008870	0.0017740	64	18	6188	4
500	8.4000	0.0247	0.008870	0.0017740	64	16	5971	4
195	7.5700	0.0505	0.019730	0.0039460	64	20	5911	12
195	7.7770	0.0421	0.016010	0.0032020	64	24	5480	4
195	8.2490	0.0280	0.010110	0.0020220	96	36	4082	4
220	7.7040	0.0449	0.017230	0.0034460	64	20	7933	12
220	7.9130	0.0374	0.014000	0.0028000	64	24	5754	4
220	8.2490	0.0280	0.010110	0.0020220	96	32	2522	2
251	7.8570	0.0393	0.014790	0.0029580	64	20	9443	4
251	8.0680	0.0327	0.012040	0.0024080	64	24	5336	12
251	8.2490	0.0280	0.010110	0.0020220	96	28	4043	4
293	8.0360	0.0336	0.012410	0.0024820	64	20	9287	4
293	8.1470	0.0306	0.011150	0.0022300	64	22	9105	12
293	8.2490	0.0280	0.010110	0.0020220	96	24	1375	3
352	8.2490	0.0280	0.010110	0.0020220	96	20	6167	4
352	8.1260	0.0311	0.011380	0.0022760	64	18	4214	12
352	8.3620	0.0255	0.009095	0.0018190	64	22	3609	16
400	8.2763	0.0274	0.009861	0.0019722	64	18	3441	16
400	8.6165	0.0205	0.007174	0.0014348	64	24	4097	24
444	8.2612	0.0278	0.010004	0.0020008	64	16	4462	16
444	8.6376	0.0202	0.007036	0.0014072	64	22	4025	16
500	8.5398	0.0219	0.007703	0.0015406	64	18	3617	16
500	8.6647	0.0197	0.006862	0.0013724	64	20	4266	24
500	8.8815	0.0164	0.005626	0.0011252	64	24	3808	24
1000	9.3653	0.0110	0.003635	0.0007270	64	18	1566	8
1000	9.4910	0.0099	0.003248	0.0006496	64	20	2047	4
1000	9.7085	0.0082	0.002675	0.0005350	64	24	1346	4
10000	12.1034	0.00110	0.0003221	0.00006442	64	18	1373	8
10000	12.2281	0.00099	0.0028855	0.00005771	64	20	1479	4
10000	12.4438	0.00082	0.0023860	0.00004772	64	24	943	4

Backup: double extrapolation

First QCD calculation of kappa (u+d+s quarks in the sea)



- Extrapolation Ansatz describes lattice data well

[LA, OK, RL, SM, PP, **HTS**, SS, PRL 130 (2023) 23, 231902]

Backup: anomalous dimension of B-field

- Anomalous dimension in MSbar-scheme $$Z_B = 1 + \frac{g^2 C_A}{(4\pi)^2}\left[\frac{1}{\epsilon} + 2\ln\left(\frac{\bar{\mu}e^{\gamma_E}}{4\pi T}\right) - 2\right] + \mathcal{O}(g^4)$$
- Gradient flow-scheme —> MSbar-scheme —> physical values
- Scale dependence must go for "WeWant" and $\langle BB \rangle_{\tau_F}$

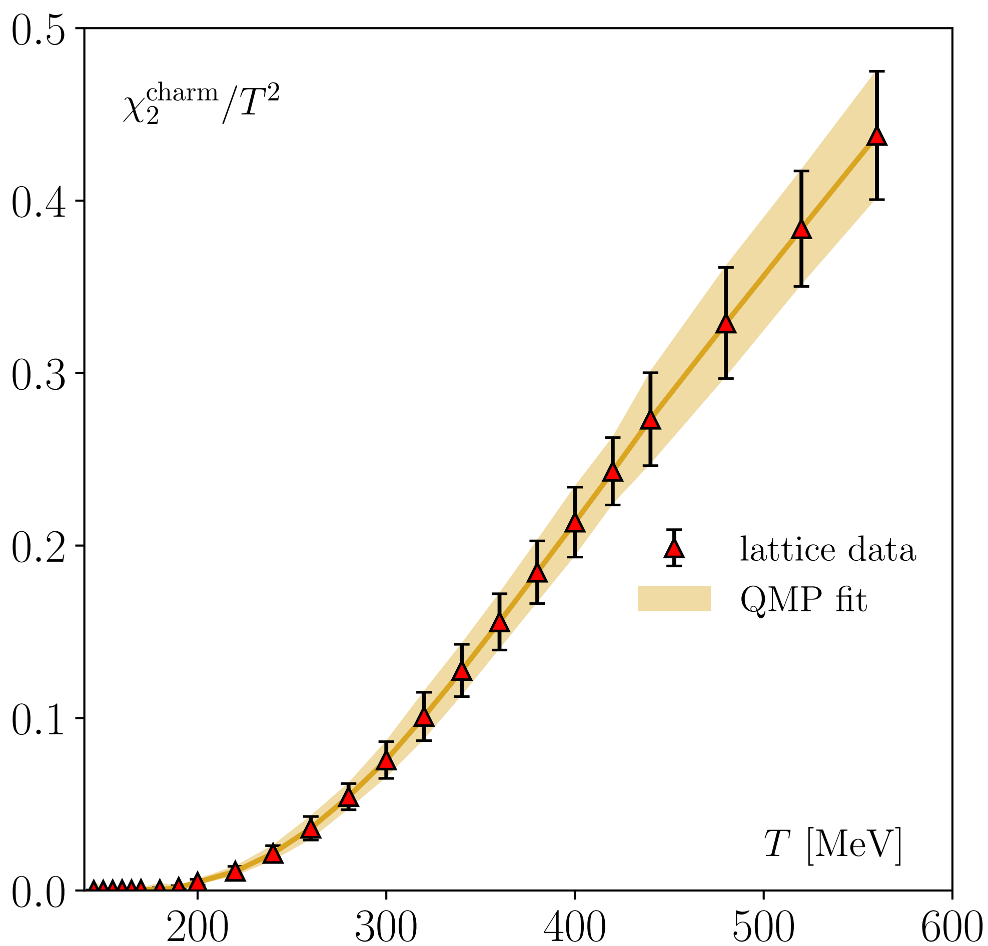
$$Z^2 = \left(1 - 2\frac{g^2 C_A}{16\pi^2}\ln(\mu^2 \tau_F)\right)\left(1 + 2K\frac{g^2 C_A}{16\pi^2}\right) \equiv Z_f^2 Z_K^2$$

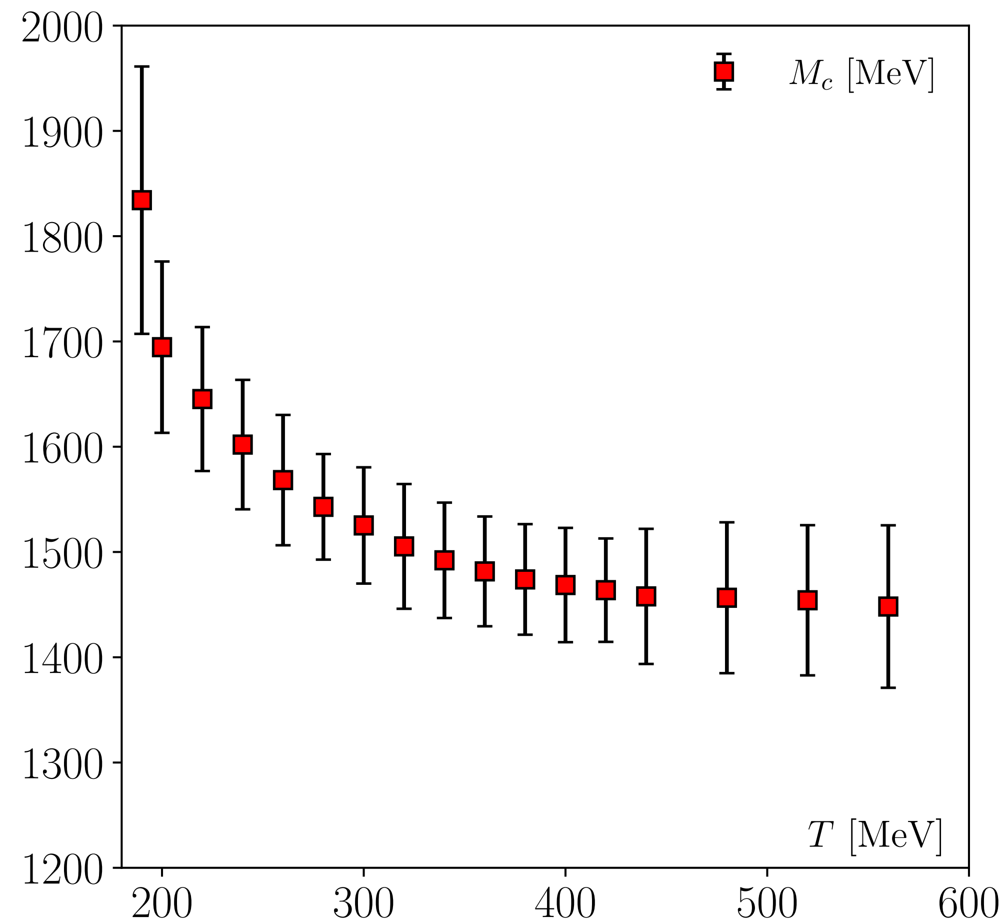
$$\mathrm{WeWant} = Z_B^2 \langle BB \rangle_{\mathrm{MS}}$$
$$\langle BB \rangle_{\tau_F} \equiv Z^{-2} \langle BB \rangle_{\mathrm{MS}}$$
$$\mathrm{WeWant} = Z_B^2 Z^2 \langle BB \rangle_{\tau_F}$$

- Determination of the matching factor

$$\ln Z_{\mathrm{match}} = \int_{\bar{\mu}_T^2}^{\bar{\mu}_{\tau_F}^2} \gamma_0 g^2_{\overline{\mathrm{MS}}}(\bar{\mu})\frac{d\bar{\mu}^2}{\bar{\mu}^2} + \gamma_0 g^2_{\overline{\mathrm{MS}}}(\bar{\mu}_T)\left[\ln\frac{\bar{\mu}_T^2}{(4\pi T)^2} - 2 + 2\gamma_{\mathrm{E}}\right] - \gamma_0 g^2_{\overline{\mathrm{MS}}}(\bar{\mu}_{\tau_F})\left[\ln\frac{\bar{\mu}_{\tau_F}^2}{4\mu_{\mathrm{F}}^2} + \gamma_{\mathrm{E}}\right]$$

Backup: T-dependent charm quark mass





[PRL 132 (2024) 5, 051902]

$$\frac{\chi_2^{\text{charm}}}{T^2} = \frac{4N_c}{(2\pi T)^3}\int d^3p\; e^{-E_p/T}$$

$$E_p^2(T) = m^2(T) + p^2$$

$\longrightarrow m(T)$

$$\langle v^2\rangle = \left(\int d^3p\;\frac{p^2}{E_p^2}e^{-E_p/T}\right) \Big/ \left(\int d^3p\; e^{-E_p/T}\right)$$

$$\langle p^2\rangle = \left(\int d^3p\; p^2 e^{-E_p/T}\right) \Big/ \left(\int d^3p\; e^{-E_p/T}\right)$$

$$\kappa = \kappa_E + \frac{2}{3}\langle \mathbf{v^2}\rangle \kappa_{\mathbf{B}}$$

$$D_s = \frac{2T^2}{\kappa}\cdot\frac{\langle p^2\rangle}{3MT}$$

Backup: scattering from various models