



東南大學
SOUTHEAST UNIVERSITY

第11届手征有效场论研讨会 @兰州

EFT Analysis of Doubly Heavy Tetraquarks from Lattice QCD

Lu Meng (孟璐)

Southeast University

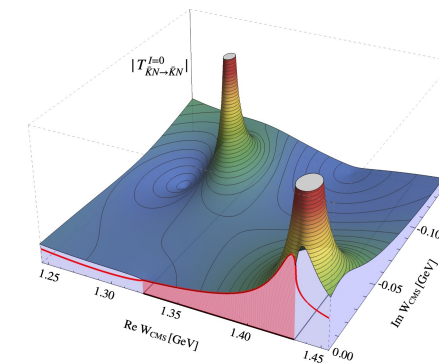
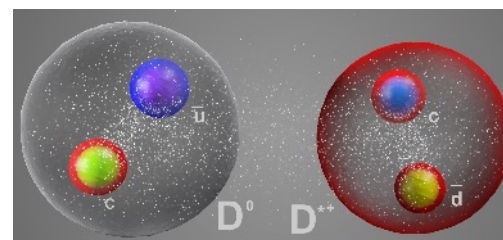
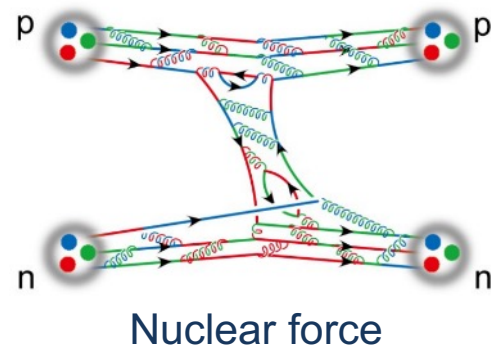
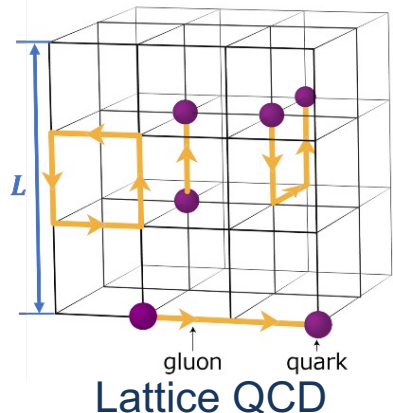
Aug. 15, 2026

Base on [PRD113\(2026\), 114041](#), [PRD111\(2025\), 034509](#), [PRD109\(2024\), L071506](#), [JHEP10\(2021\)051](#)...

With M. Abolnikov, V. Baru, E. Epelbaum, A. Filin, A.M. Gasparyan, E. Ortiz-Pacheco, M. Padmanath, and S. Prelovsek

Left-hand cut problems

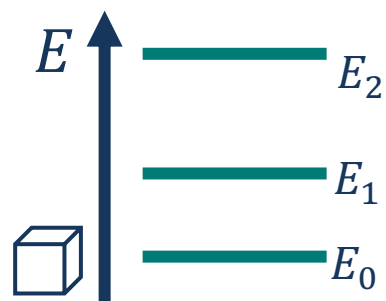
- QCD is the fundamental theory of the strong interaction
- To get the hadron-hadron interactions and resonances from the lattice QCD?



Resonance

M. Mai, '21

- From finite volume energy levels to pole positions



→
Lüscher quantization
condition (LQC)
Lüscher '90



→
Amplitude formula +
analytic continuation

Poles

Finite volume lhc problems

Infinite volume lhc problems

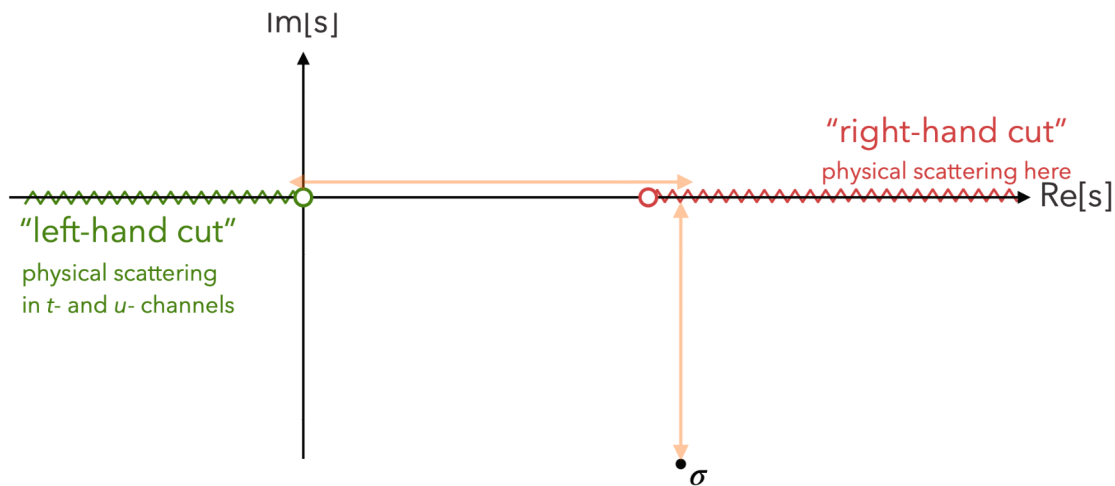
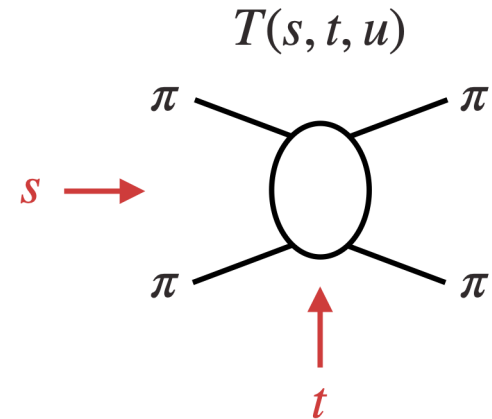
Left-hand cut (lhc) problems



Left-hand cuts in $\pi\pi$ scattering

IFV Ihc problem

- Large uncertainties of the $f_0(500)$ pole position
 - ▶ Both in Exp. and Lattice
 - ▶ One of the sources: **left-hand cuts**
 - ▶ Broad resonance: comparable distances to real axis and the Ihc
 - ▶ Solutions: dispersion relations with **crossing symmetry**



Similar analyses of Exp. data

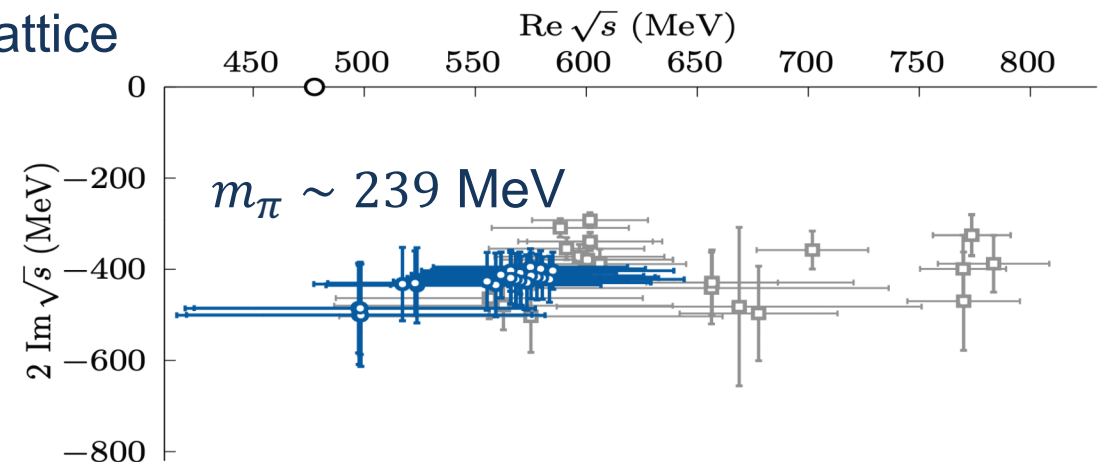
[PKU factorization] Z. Xiao and H. Q. Zheng, Nucl. Phys. A 695, 273 (2001); Z. Y. Zhou et al, JHEP 02, 043 (2005).

L. Caprini, G. Colangelo, and H. Leutwyler, PRL96, 132001 (2006)

R. Garcia-Martin, R. Kaminski, et al., PRL107, 072001 (2011)

B. Moussallam, EPJC71, 1814 (2011)

Lattice



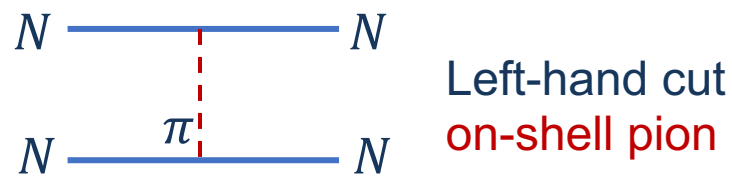
Blue: uncertainties reduced by dispersion relation

A. Rodas, J. J. Dudek, and R. G. Edwards, PRD109, 034513 (2024).

See also X.-H. Cao, Q.-Z. Li, Z.-H. Guo, and H.-Q. Zheng, PRD108, 034009 (2023)...



Left-hand cuts in NN scattering



- lhc from the **one-pion exchange**

$$V(r) = \frac{e^{-mr}}{r}, \quad V(\vec{p}, \vec{p}') = \frac{1}{(\vec{p}' - \vec{p})^2 + m^2} \quad \text{Long-range interaction \& Chiral dynamics}$$

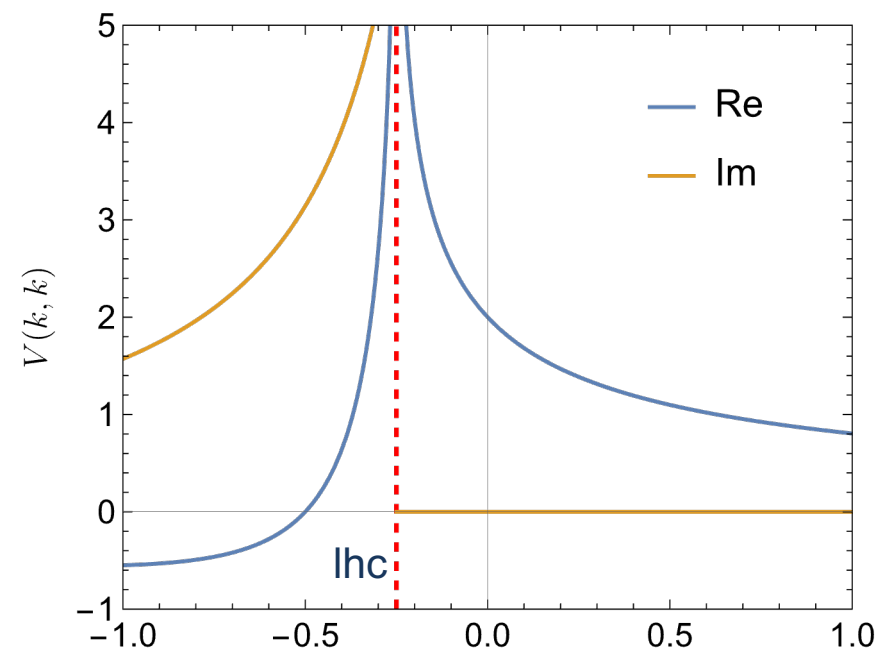
- Partial-wave decomposition, e.g. the S-wave

$$V_{l=0}(p, p') = -\frac{1}{2pp'} \log \left(\frac{(p - p')^2 + m^2}{(p + p')^2 + m^2} \right) \quad \text{Multivalued function with cuts}$$

- On-shell $p = p' = k$, with $k^2 = 2\mu E$

$$V_{l=0}(k, k) = \int_{-1}^1 dz \frac{1}{2k^2(1-z) + m^2}, \quad 2k^2(1-z) + m^2 = 0 \Rightarrow z = \frac{m^2}{2k^2} + 1, \quad \text{on-shell pion}$$

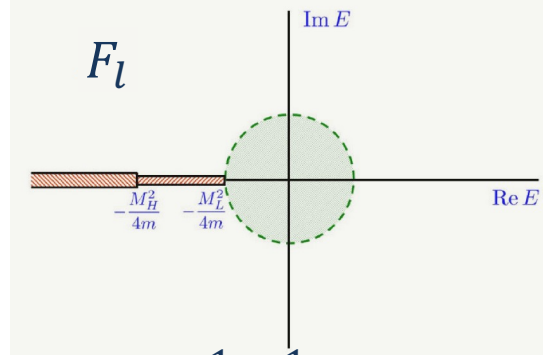
► Branch point: $k^2 < -\frac{m^2}{4}$ With imaginary part



Amplitude formalisms with lhcs

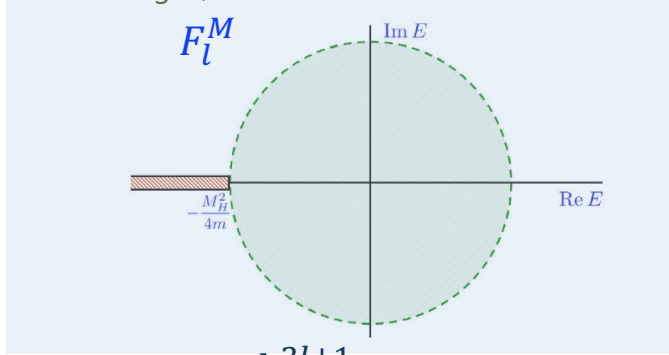
Two potential formalism $V(r) = V_L(r) + V_S(r)$

Landau, Smorodinsky '44, Bethe '49



$$F_l = -\frac{1}{a} + \frac{1}{2}rk^2 + \dots$$

van Haeringen, Kok '82



$$F_l^M \equiv M_l(q) + \frac{k^{2l+1}}{|f_l(k)|^2} [\cot(\delta_l - \sigma_l) - i]$$

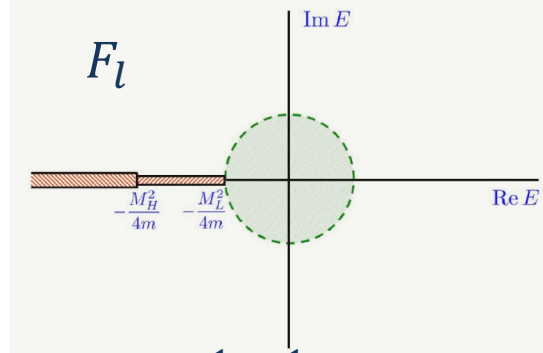
Effective range expansion (ERE) Modified Effective range expansion (MERE)



Amplitude formalisms with lhcs

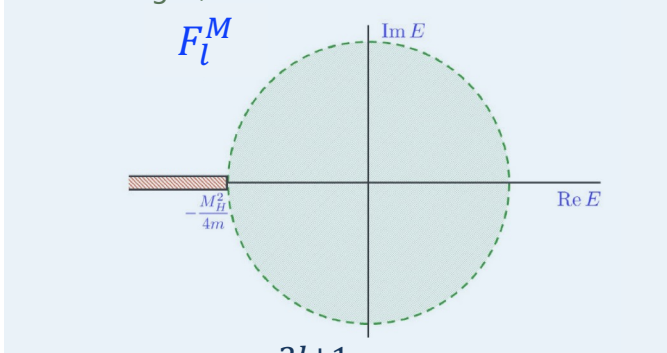
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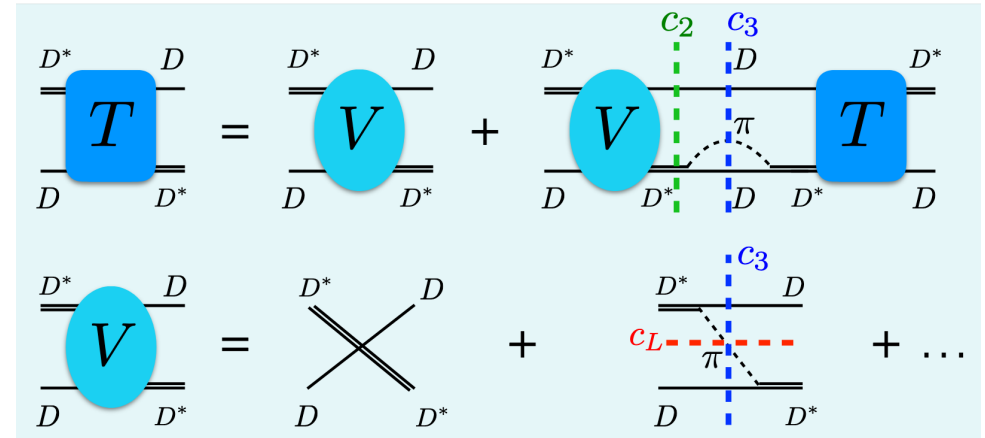
van Haeringen, Kok '82



$$F_l^M \equiv M_l(q) + \frac{k^{2l+1}}{|f_l(k)|^2} [\cot(\delta_l - \sigma_l) - i]$$

Effective range expansion (ERE) Modified Effective range expansion (MERE)

Lippmann-Schwinger equation



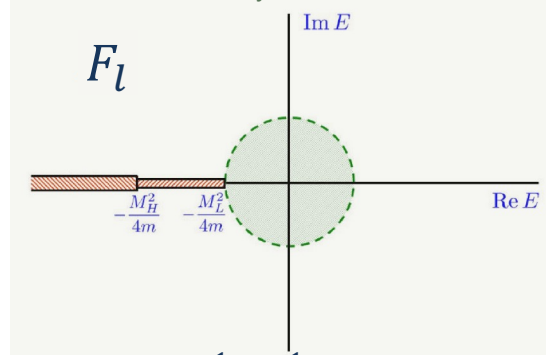
E.g. M.-L. Du, et al., PRL **131**, 131903 (2023)



Amplitude formalisms with lhcs

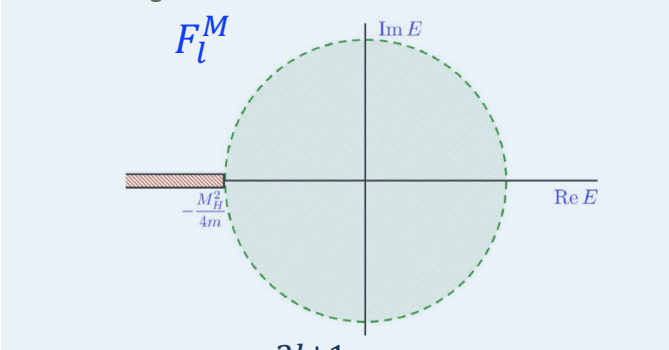
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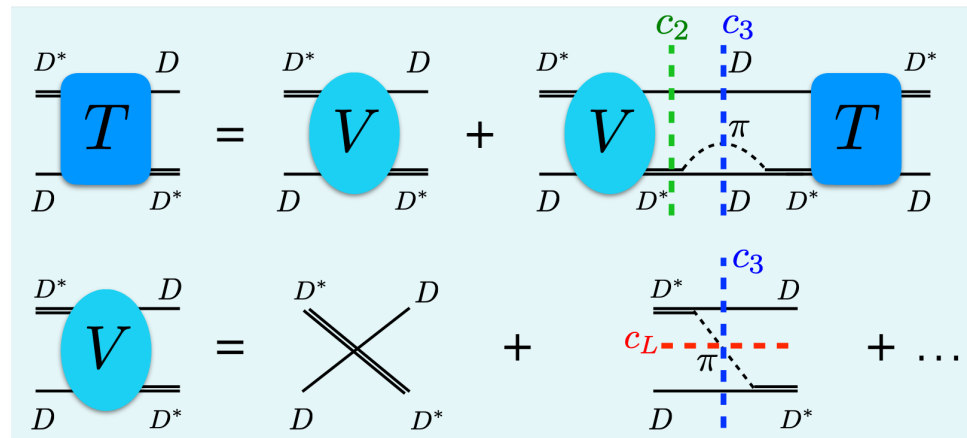
van Haeringen, Kok '82



$$F_l^M \equiv M_l(q) + \frac{k^{2l+1}}{|f_l(k)|^2} [\cot(\delta_l - \sigma_l) - i]$$

Effective range expansion (ERE) Modified Effective range expansion (MERE)

Lippmann-Schwinger equation



E.g. M.-L. Du, et al., PRL **131**, 131903 (2023)

N/D representation

$$T_l' = \frac{N_l(s)}{D_l(s)}$$

Contains left-hand cut

Contains right-hand cut

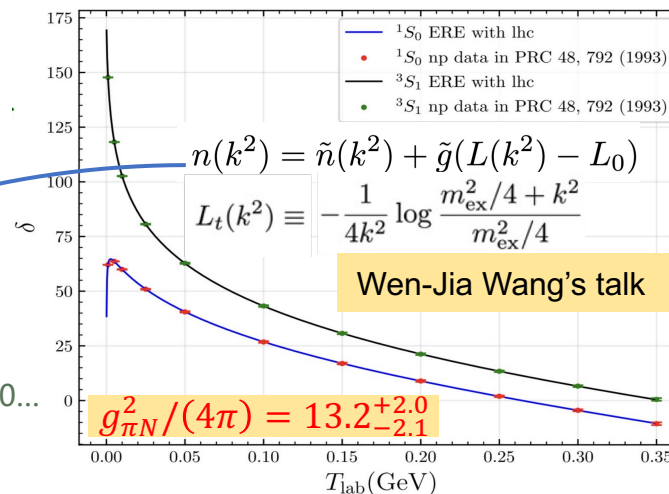
$$\text{Im } D_l = -\rho(s) N_l p^{2l}, \quad s > s_{th}$$

$$\text{Im } N_l = \text{Im } T_l' D_l, \quad s < s_{lhc}$$

$$D_l(s) = P_D(s) - \frac{(s - s_0)^n}{\pi} \int_{s_{th}}^{\infty} ds' \frac{p(s')^{2l} \rho(s') N_l(s')}{(s' - s)(s' - s_0)^n}$$

Chew & Mandelstam, '60; Bjorken, '60...

$$N_l(s) = P_N(s) + \frac{(s - s_0)^{n-l}}{\pi} \int_{-\infty}^{s_{lhc}} ds' \frac{\text{Im } T_l(s') D_l'(s')}{p(s')^{2l} (s' - s_0)^{n-l} (s' - s)}$$



M.-L. Du, F.-K. Guo, and B. Wu, PRL **135**, 011903 (2025)

B.-Y. Liu, B. Wu, J.-W. Fu et al, arXiv:2602.03115.

[Review]

D.-L. Yao, L.-Y. Dai, H.-Q. Zheng, and Z.-Y. Zhou, RPP **84**, 076201 (2021).



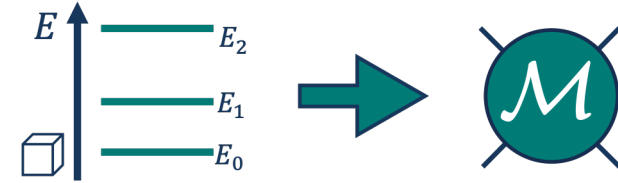
lhc problem IN Lüscher formula

FV lhc problem

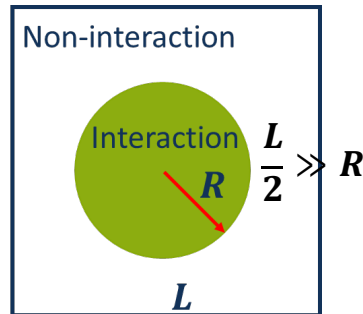
- Lüscher's formula: Lüscher '90

$$\det [G_F^{-1}(L, E^{FV}) - K(E^{FV})] = 0$$

Kinematical term K-matrix in IFV



- Valid for the short-range (compared to the box) interaction



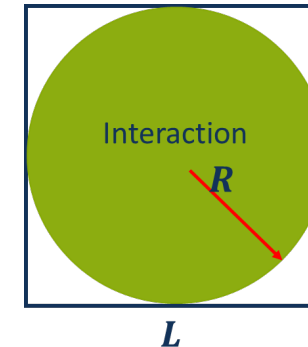
Asymptotic behavior: for $r > R$

$$(\nabla^2 + k^2)\psi_k(r) = 0,$$

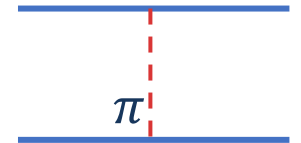
$$\psi_k(r) \sim \frac{e^{i\delta(k)} \sin[kr + \delta(k)]}{kr}.$$

Periodic Boundary condition:

$$p = \frac{2\pi}{L}n, \quad n \in \mathbb{Z}^3$$



Long-range interaction
vs small box ???



NN, D^*D systems...

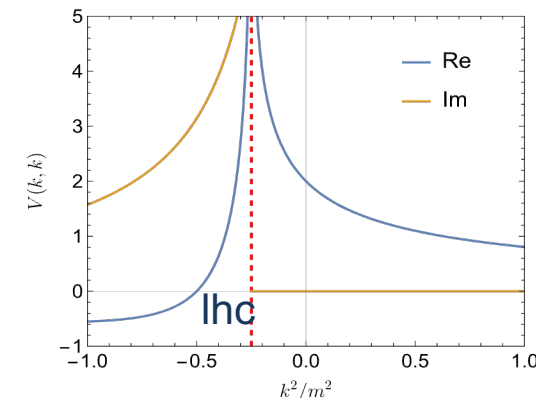
► **Exponentially suppressed effect:** $\sim e^{-m_\pi L}$

- lhc problem: $k^2 < -\frac{m^2}{4}$

$$\det [G_F^{-1}(L, E) - K(E)] \neq 0$$

Real With nonvanishing imaginary part

$$K = V + VG^{\mathcal{P}}K$$



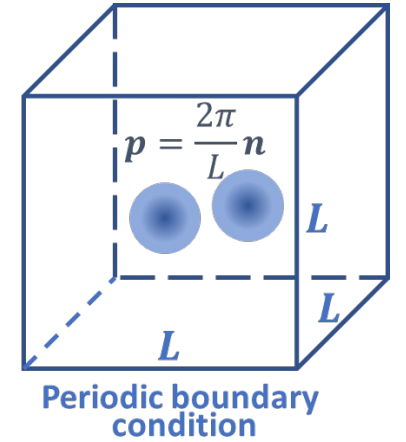
Origin of the lhc problem

- From infinite volume (IFV) to finite volume (FV)

$$\int \frac{d^3q}{(2\pi)^3} \rightarrow \frac{1}{L^3} \sum_{\vec{q}=2\pi\vec{n}/L}$$

- Poisson summation formula

$$\left(\frac{1}{L^3} \sum_{\vec{q}=2\pi\vec{n}/L} - \int \frac{d^3q}{(2\pi)^3} \right) f(\vec{q}) = \sum_{\vec{n} \neq 0, \vec{n} \in \mathbb{Z}^3} \int \frac{d^3q}{(2\pi)^3} f(\vec{q}) e^{iL\vec{q} \cdot \vec{n}}$$



► E.g.

Smooth: $f(\vec{q}) = \frac{1}{\vec{q}^2 + a^2} \rightarrow \frac{e^{-anL}}{4\pi Ln}$ exponentially suppressed

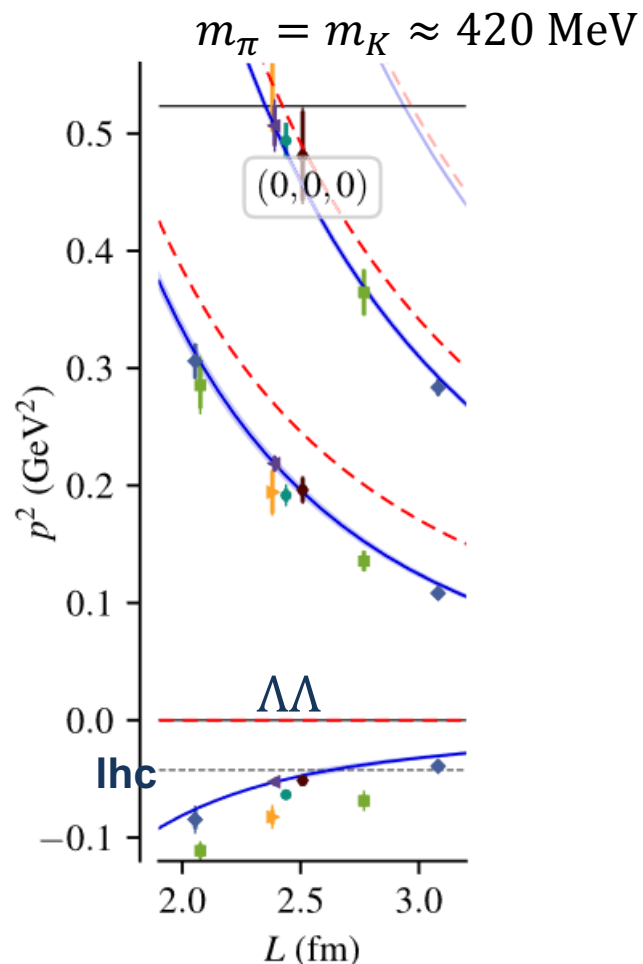
Singularity: $f(\vec{q}) = \frac{1}{\vec{q}^2 - a^2 + i\epsilon} \rightarrow \frac{e^{-ianL}}{4\pi Ln}$ power-law effect

- Lüscher QCs: only taking care of the right-hand cut (unitary cut)

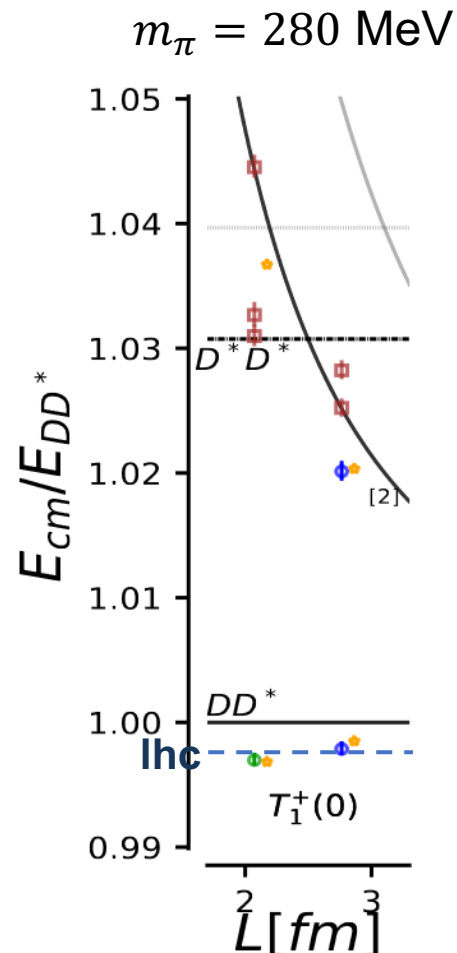


Left-hand cut problems in lattice QCD

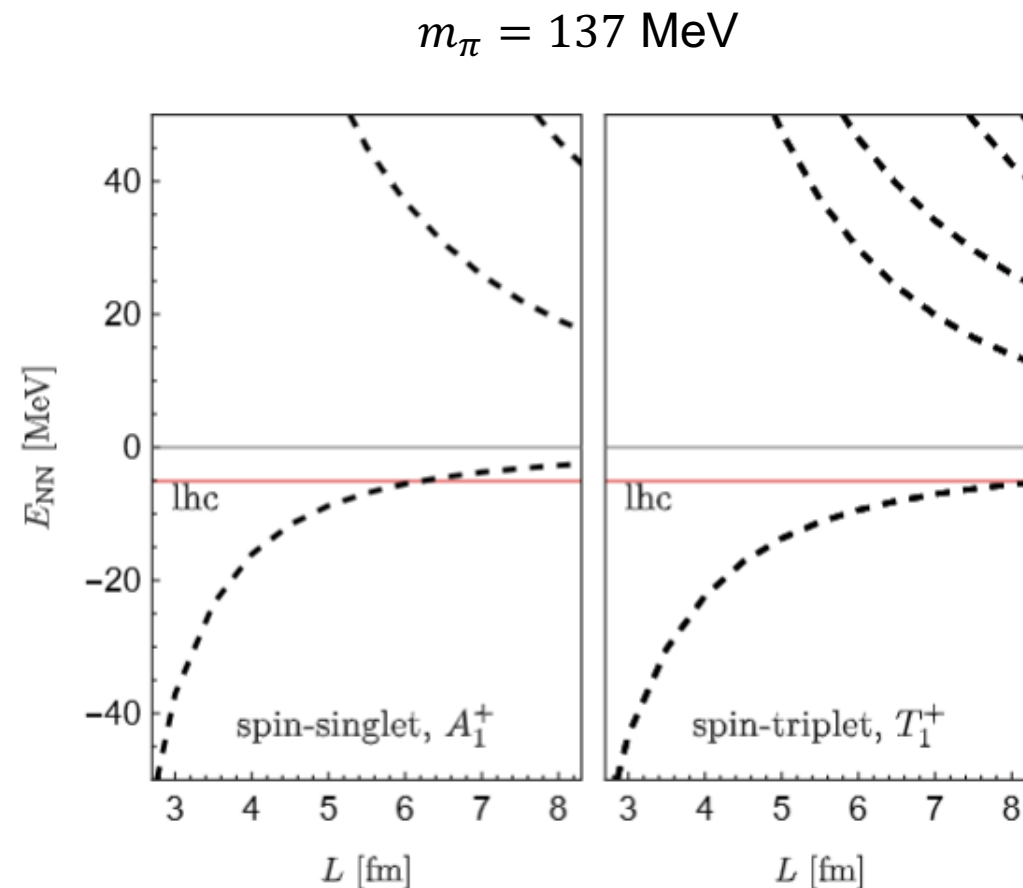
H-dibaryons ($udsuds$)



$DD^* (T_{cc})$



NN



Evaluate using Chiral nuclear force

H-dibaryons ($udsuds$)

J.R. Green, et al. PRL127 (2021) 24, 242003

M.Padmanath, et al., PRL129 (2022) 3, 032002

$DD^* (T_{cc})$

$L > 8$ fm to get rid of the lhc problem



Several methods

● Plane wave basis + Chiral effective field theory

- ▶ LM and E. Epelbaum, JHEP 10, 051 (2021); LM, V. Baru, E. Epelbaum et al., PRD109, L071506 (2024).
- ▶ S. Prelovsek, E. Ortiz-Pacheco, S. Collins et al., PRD112, 014507 (2025).
- ▶ K. Yu, G.-J. Wang, J.-J. Wu, and Z. Yang, JHEP 04, 108 (2025); K. Zhang, K. Yu, Y. Geng et al., arXiv:2605.16977

AKA Nonperturbative Hamiltonian framework

● Relativistic left-hand-cut formalism

- ▶ A. Raposo and M. Hansen, JHEP 08 (2024) 075; A. B. Raposo, R. A. Briceño, M. T. Hansen et al., JHEP 06, 186 (2025).
- ▶ N. Pitanga Lachini and R. A. Briceño, arXiv:2607.19009; A. Baião-Raposo, R. A. Briceño, N. Lang et al., arXiv:2607.24648

● Modified effective range expansion (MERE)

- ▶ R. Bubna, H-W. Hammer, F. Müller et al., JHEP 05, 168 (2024); R. Bubna, H.-W. Hammer, B.-L. Hoid, et al. JHEP 10, 197 (2025)

● Using QC3: relativistic field theory (RFT) & finite-volume unitarity (FVU)

- ▶ M. Hansen, F. Romero-López and S. Sharpe, JHEP 06 (2024) 051;
- ▶ S. M. Dawid, F. Romero-López, and S. R. Sharpe, JHEP 01, 060 (2025).
- ▶ M. Mai and M. Doring, EPJA53 (2017) 12, 240

● HAL QCD approach

- ▶ Lyu et al, PhysRevLett.131.161901

● N/D representation

- ▶ S. M. Dawid, A. W. Jackura et al., PLB 864, 139442 (2025);
- ▶ A. Rodas, L. Qiu, C. Fernández-Ramírez et al., arXiv:2605.22957

350. Dealing with left-hand cuts in multi-hadron spectroscopy

Lu Meng (Southeast University)

Lattice2026

2026/8/1 11:00



Hadronic and nuclear sp...

Plenary talk

Plenary session

<https://indico.global/event/16565/contributions/161050/>



No lhc problem in Schrödinger Eq.

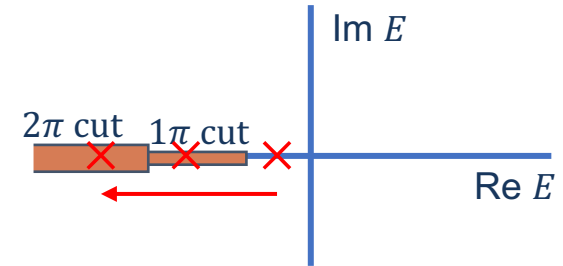
Plane wave QCs

- Schrödinger Eq. in the IFV to get the **bound state** solutions

$$\frac{\mathbf{p}^2}{2\mu}\psi(\mathbf{p}) + \int \frac{d^3\mathbf{p}'}{(2\pi)^3} V(\mathbf{p}, \mathbf{p}')\psi(\mathbf{p}') = E\psi(\mathbf{p})$$

Off-shell, for $E < 0$

- ▶ Works well even below the left-hand cut
- ▶ For the $p, p' > 0$, no lhc in potential



$$V_{l=0}(p, p') = \int_{-1}^1 dz \frac{1}{p^2 + p'^2 - 2pp'z + m^2} = -\frac{1}{2pp'} \log \left(\frac{(p - p')^2 + m^2}{(p + p')^2 + m^2} \right)$$

- FV energy levels are “bound states” trapped by the box

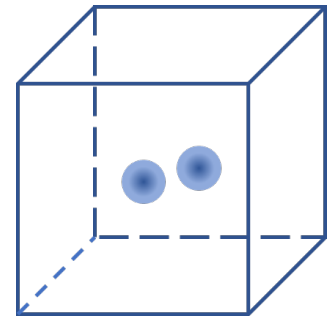
$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \rightarrow \frac{1}{L^3} \sum_{\mathbf{p}_n}$$

- Basic idea:

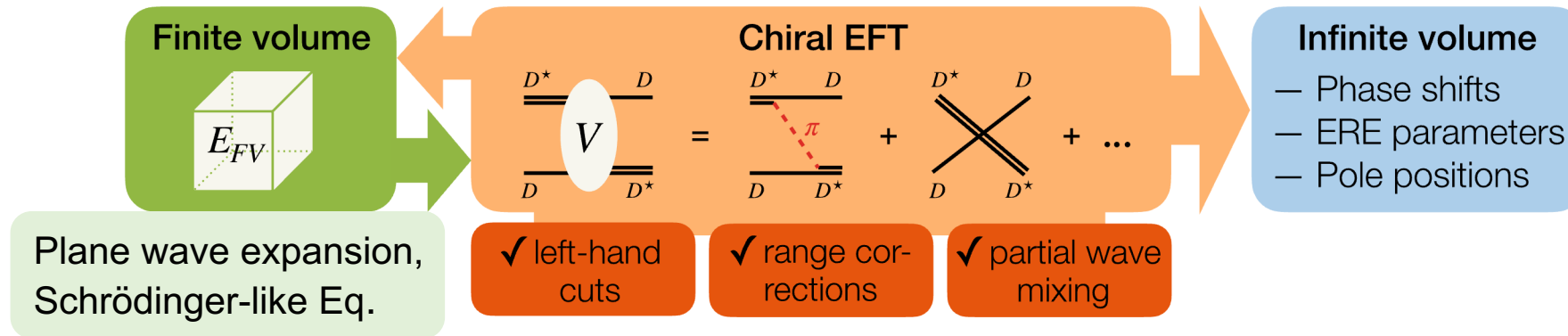
- ▶ using V to connected FV and IFV
- ▶ FV effect: Schrödinger-like Eq.

HAL QCD

- No lhc problem: S.Aoki@lattice2024
 - ▶ using Schrödinger-like Eq.
 - LO local potential:
 - ▶ easy to extract V_{long}
- Lyu et al, PhysRevLett.131.161901



Plane wave basis + Chiral EFT



- Infinite volume: Lippmann-Schwinger-like Eq.

$$T(\mathbf{p}, \mathbf{p}') = V(\mathbf{p}, \mathbf{p}') + \int \frac{d^3 \mathbf{q}}{(2\pi)^3} V(\mathbf{p}, \mathbf{q}) \frac{1}{2w_1 w_2} \frac{(w_1 + w_2)}{P_0^2 - (w_1 + w_2)^2 + i\epsilon} T(\mathbf{q}, \mathbf{p}')$$

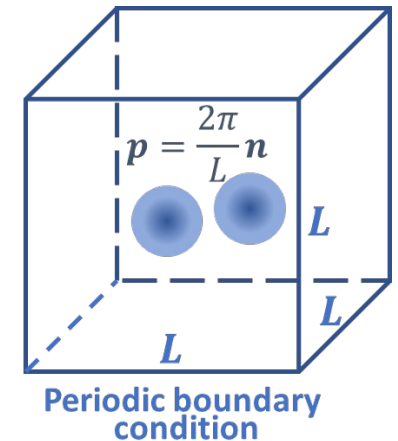
- Finite volume: plane wave (with discrete momentum) basis

$$\mathbb{T} = \mathbb{V} + \mathbb{V} \cdot \mathbb{G} \cdot \mathbb{T} \rightarrow \det(\mathbb{H} - \lambda \mathbb{I}) = 0 \rightarrow \mathbb{H} \mathbf{v} = \lambda \mathbf{v}$$

Matrix Eq. To get FV energy levels

$$\mathbb{H} \Rightarrow \text{diag}\{\mathbb{H}_{\Gamma_i}, \mathbb{H}_{\Gamma_j}, \dots\} \Rightarrow \mathbb{H}_{\Gamma} \mathbf{v} = \lambda \mathbf{v}$$

Γ : irreps.



- Additional advantage: a natural treatment of partial-wave mixing

► $\{l, m\}$ are not good quantum numbers to label states

LM and E. Epelbaum, JHEP 10, 051 (2021)
LM, V. Baru, E. Epelbaum et al., PRD109, L071506 (2024).



● Lüscher QCs

$$\chi^2/\text{dof}=3.7/5, E_{\text{pole}}^{^3S_1} = -9.9^{+3.6}_{-7.2} \text{ MeV}$$

$$a_{^3S_1} = 1.04(29)\text{fm}, \quad r_{^3S_1} = 0.96^{+0.18}_{-0.20}\text{fm}$$

$$a_{^3P_0} = 0.076^{+0.008}_{-0.009}\text{fm}^3, \quad r_{^3P_0} = 6.9(2.1)\text{fm}^{-1}$$

Four parameters:

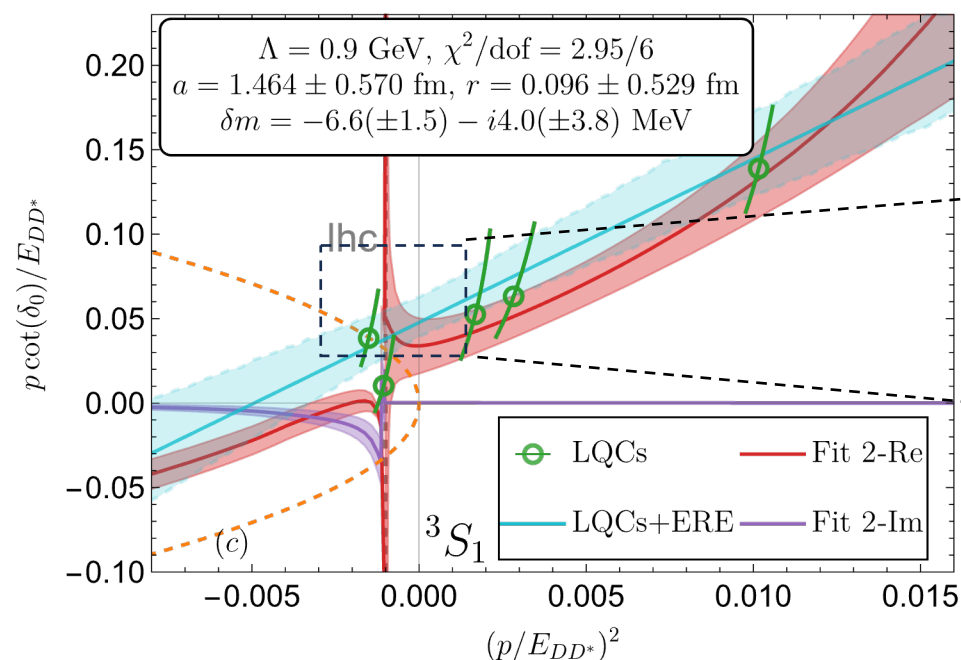
$$a_{^3S_1}, r_{^3S_1}, a_{^3P_0}, r_{^3P_0}$$

Pole:

Virtual state

● Plane wave + ChEFT

$$m_\pi \approx 280 \text{ MeV}$$

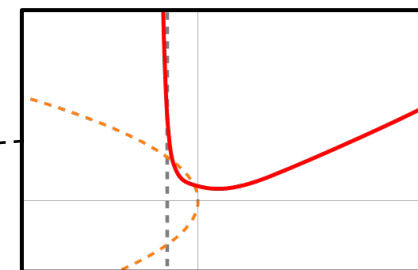
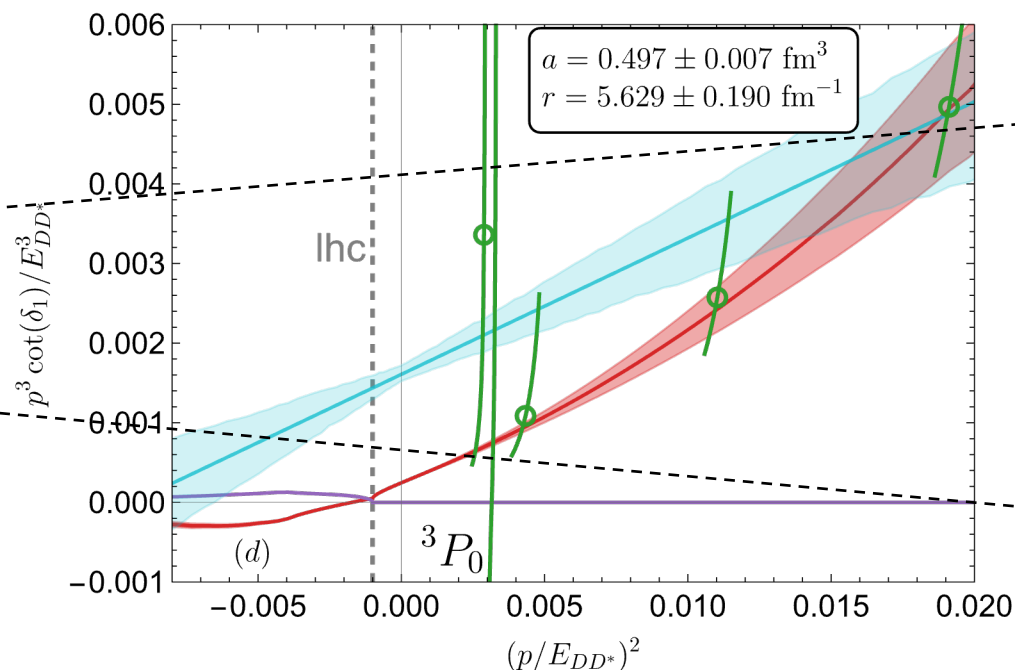


Three parameters:

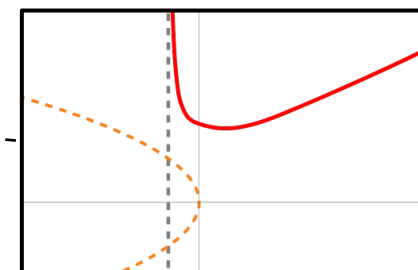
LO and NLO 3S_1 , NLO 3P_0 LECs

Pole:

More likely resonance



two virtual states



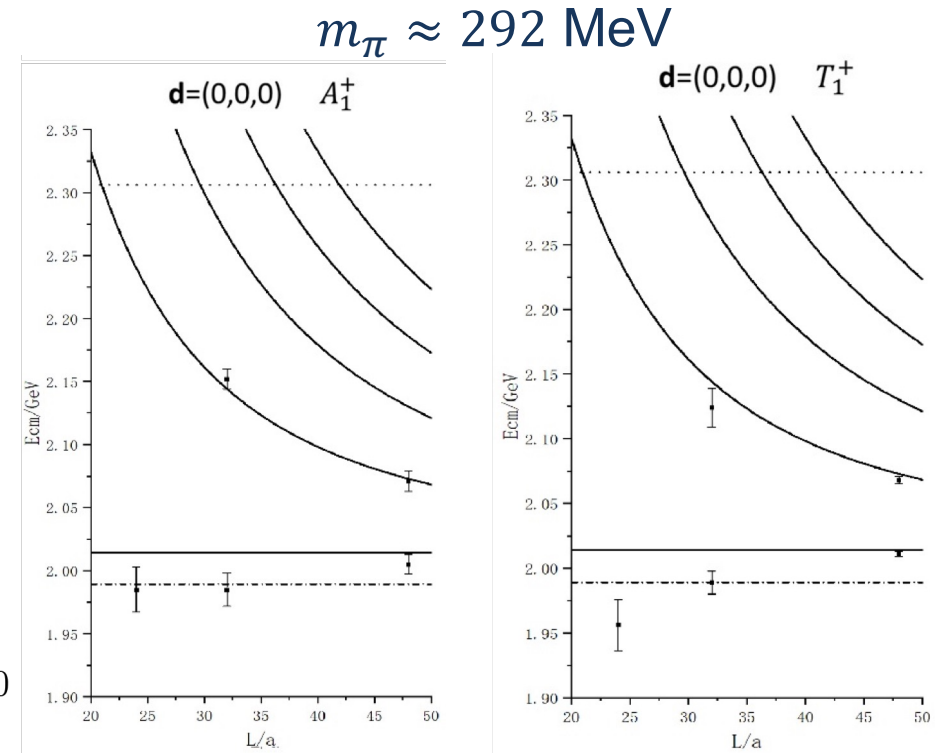
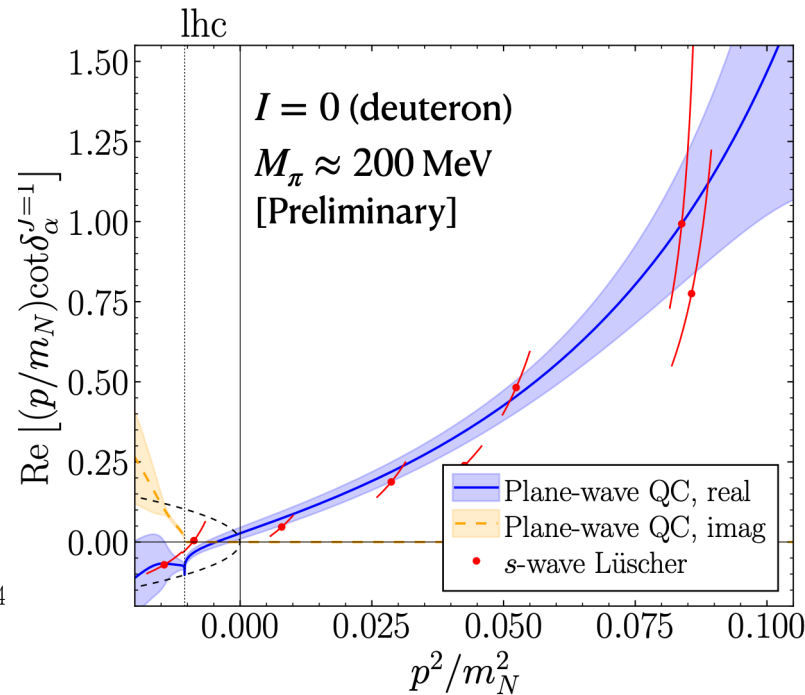
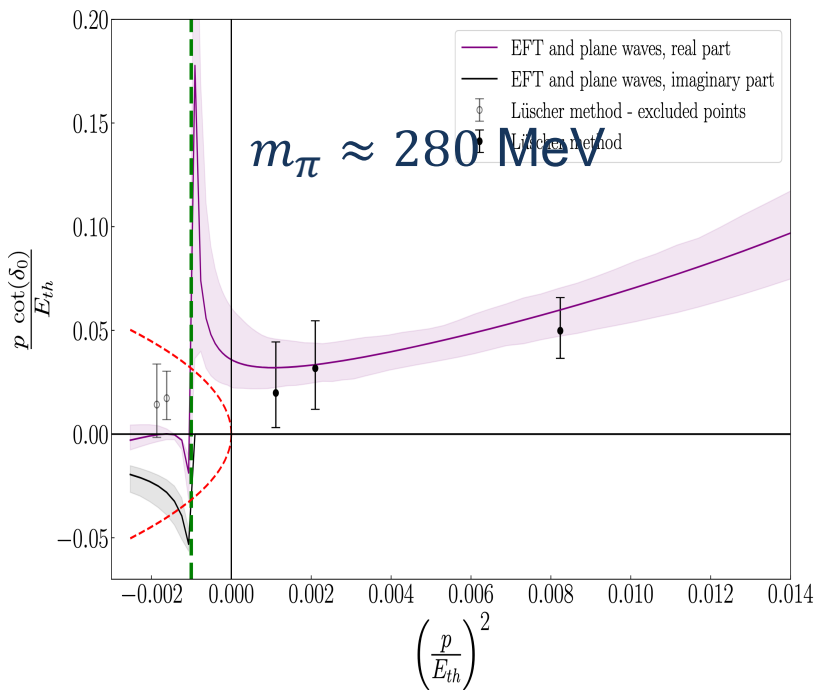
Resonance

LM, V. Baru, E. Epelbaum et al., PRD109, L071506 (2024).



Applications: NN

- T_{cc} with diquark-antidiquark operator
- [BaSc] NN with $m_\pi \approx 710, 280$ and 200 MeV
- [CLQCD] NN with $m_\pi \approx 290$ MeV



S. Prelovsek, E. Ortiz-Pacheco, S. Collins et al.
PRD112, 014507 (2025).

Significant lhc effects around threshold

CLQCD, arXiv:2605.16977 [hep-lat]

[BaSc], [Miguel Salg's talk @ lattice2026](#)

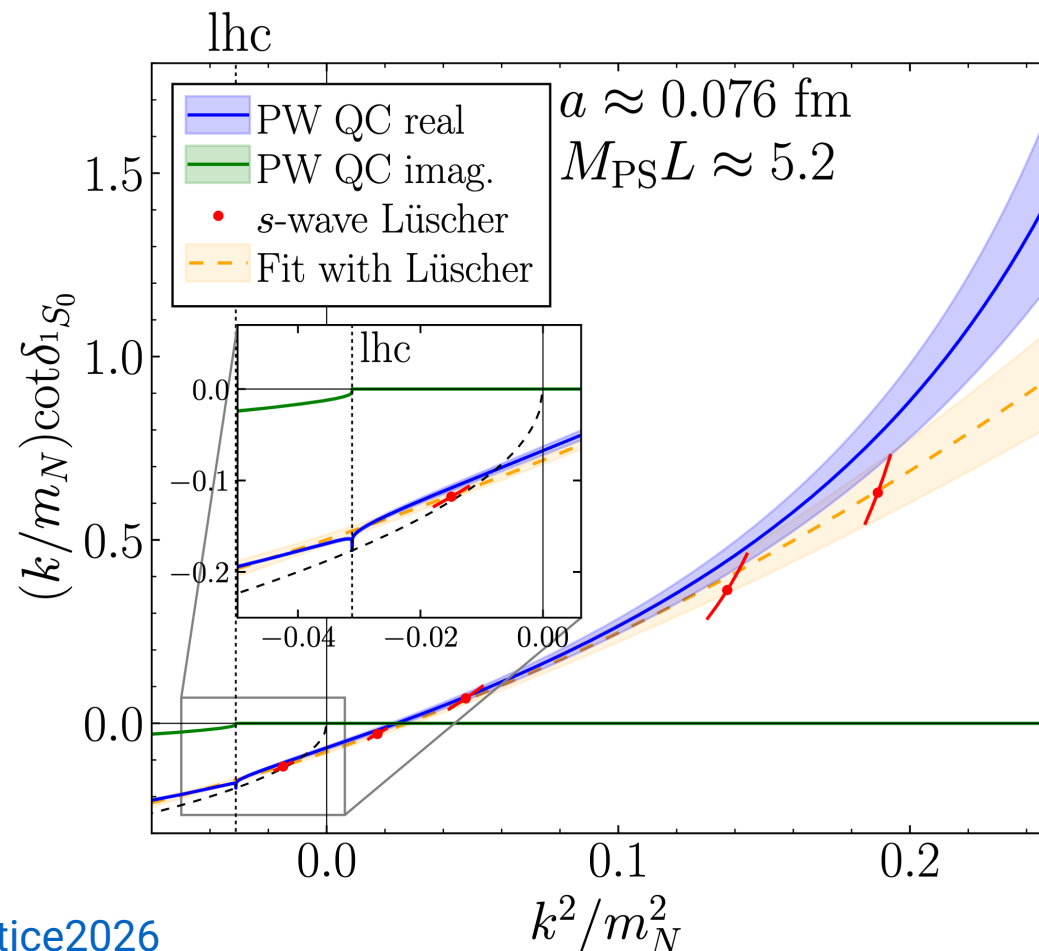
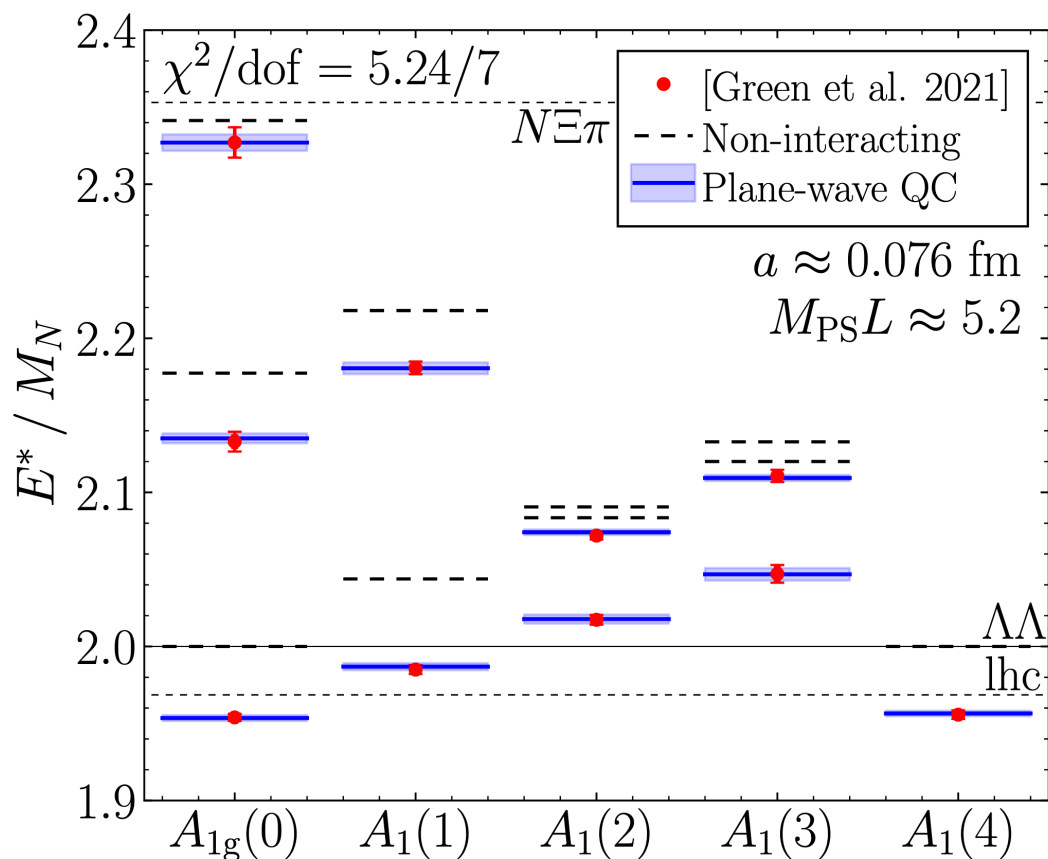
Analysis: Nonperturbative Hamiltonian framework



Applications: H-dibaryon

- Reanalyze H-dibaryon data from J.R. Green et al. PRL127 (2021) 24, 242003

- $m_\pi = m_K \approx 420$ MeV
- Original analysis: E_{FV} below lhc branch point is not used
- New analysis: smaller binding energy



[BaSc], [Jorge Baeza-Ballesteros' poster @ lattice2026](#)

Model dependence ?

- Question: (modified) Lüscher formula **vs** FV plane wave method + EFTs
Model-independent *Model-dependent?*

Our answer:

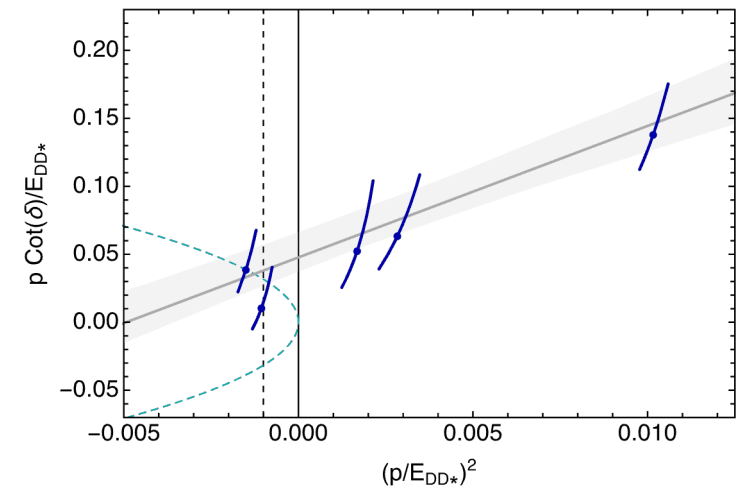
FV Plane-wave method
EFT (parametrization of kernel)

framework-dependent

vs

(modified) Lüscher QCs
Parametrization of T -matrix

also framework-dependent



EFT: Systemic calculation, controllable truncation error

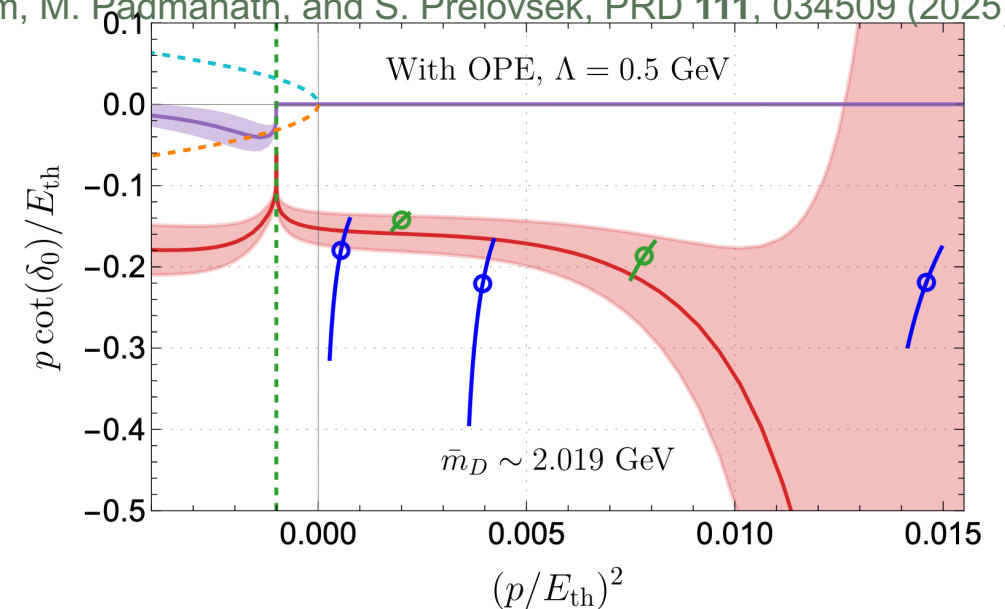
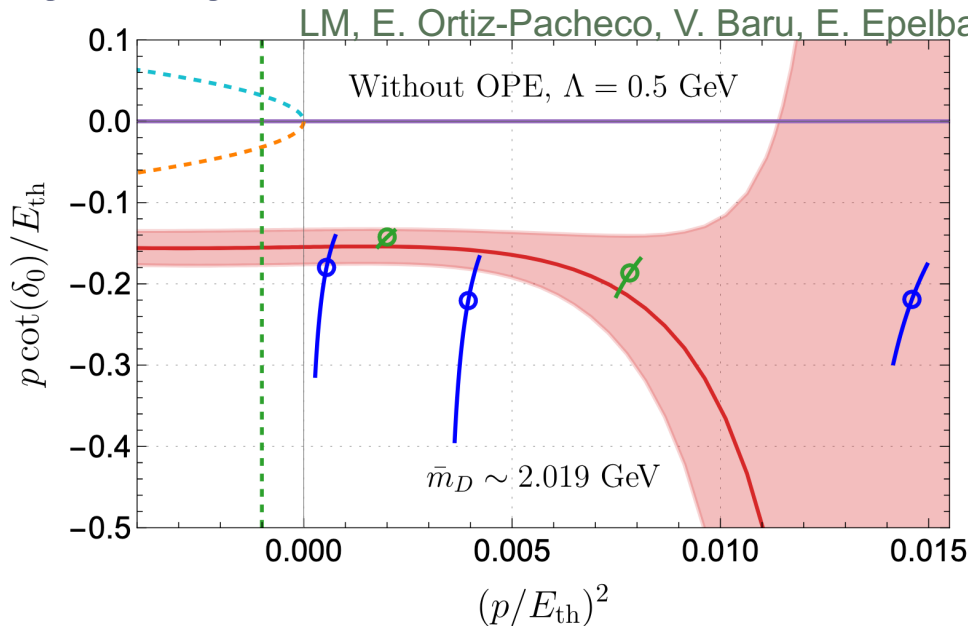
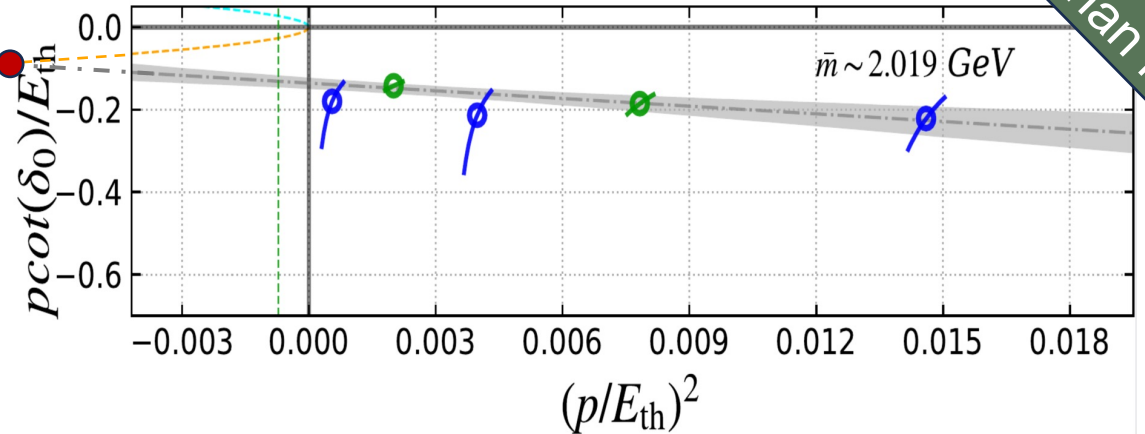


● LQCs + ERE

- ▶ Cross point: deep bound state
- ▶ Lattice data: no deep state below thresh.
- ▶ Inconsistency

● Plane wave QCs + EFT with(out) OPE

- ▶ Similar scattering length: repulsive interaction
- ▶ No deep bound state problem
- ▶ Fitting: assign the ground states

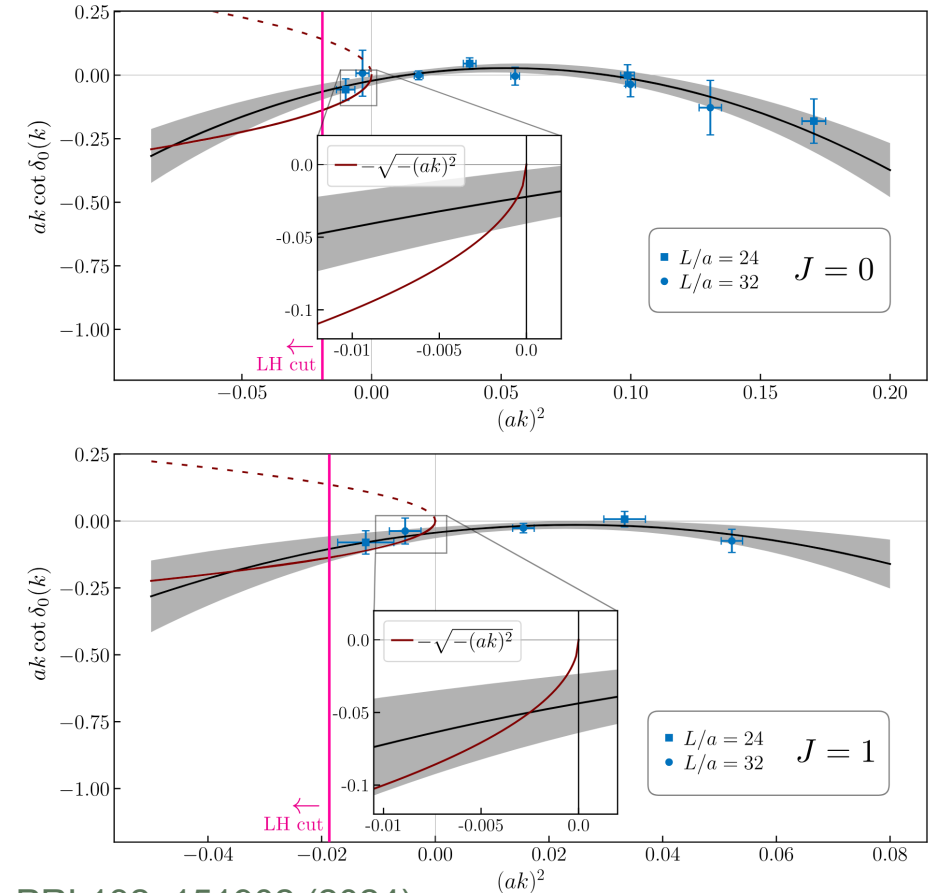
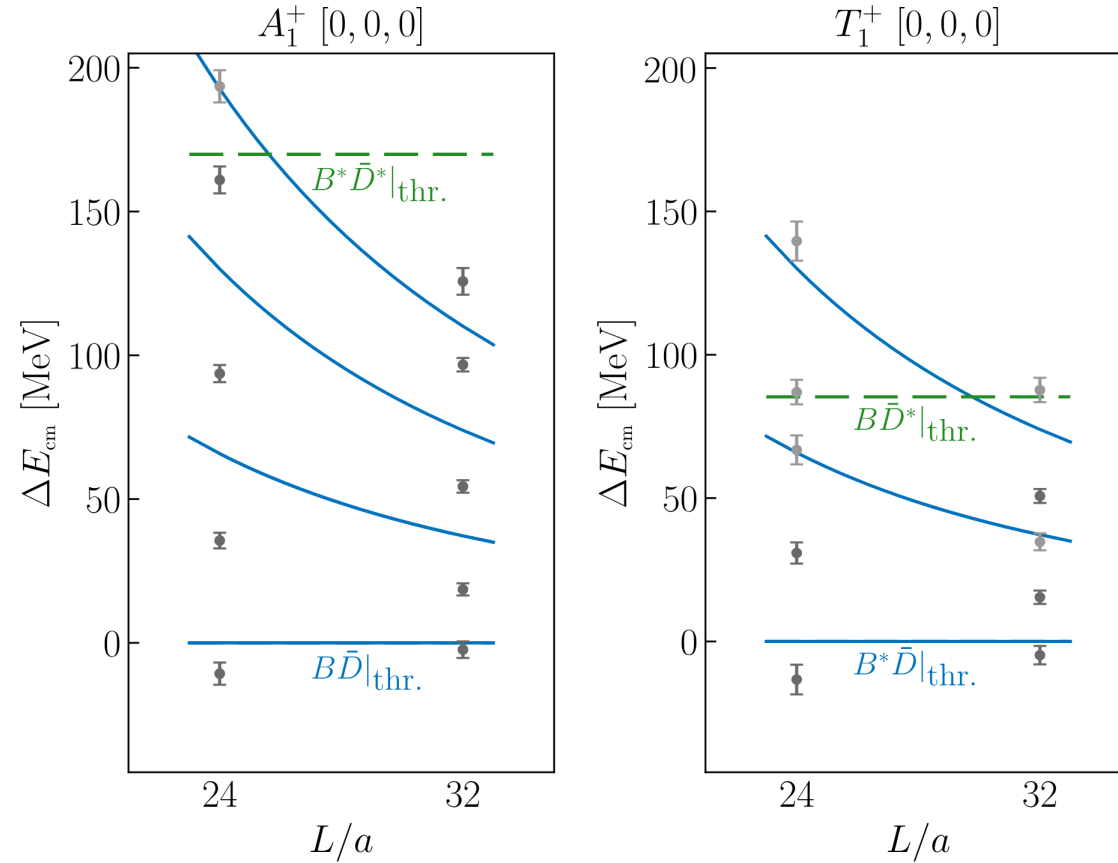


See also S. Chen, C. Shi, Y. Chen, M. Gong, Z. Liu, W. Sun, and R. Zhang, PLB 833, 137391 (2022).



$B\bar{D}$ and $B^*\bar{D}$

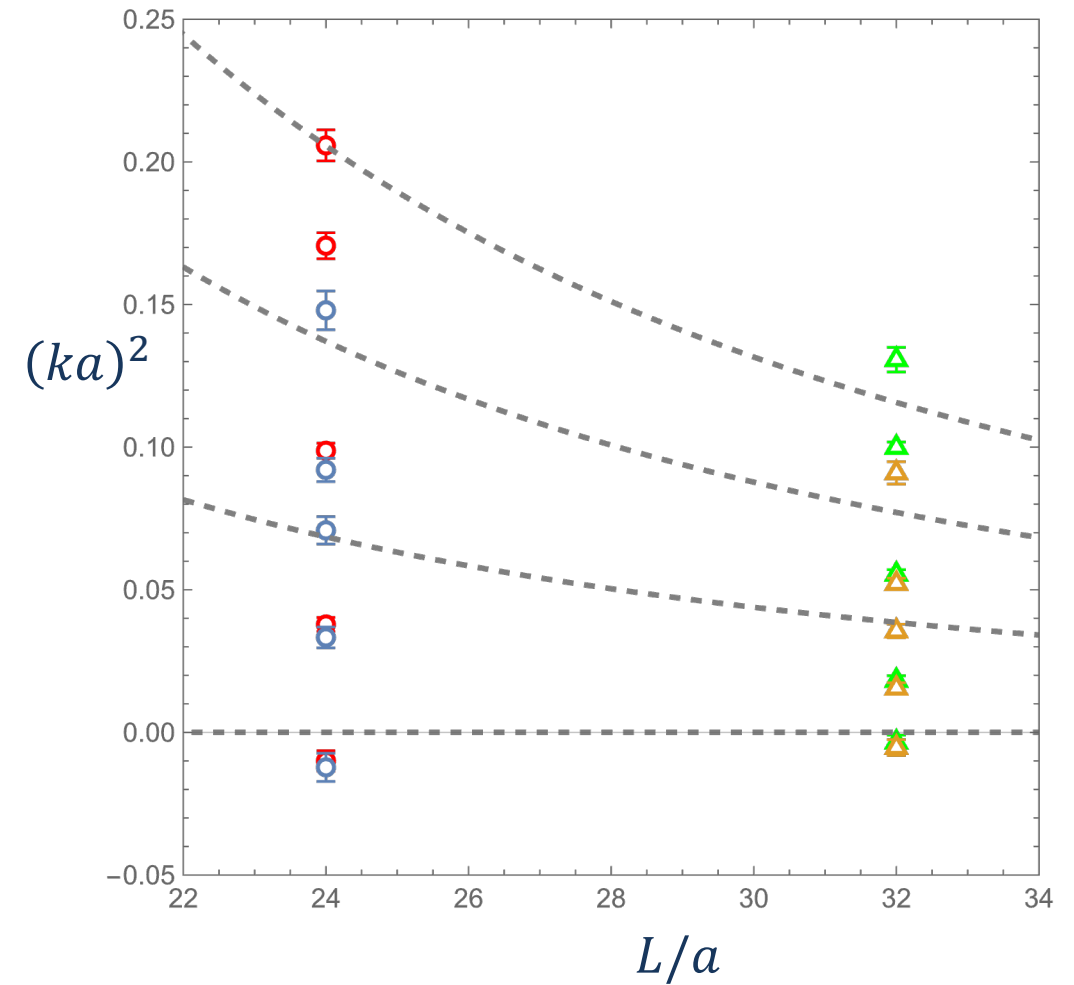
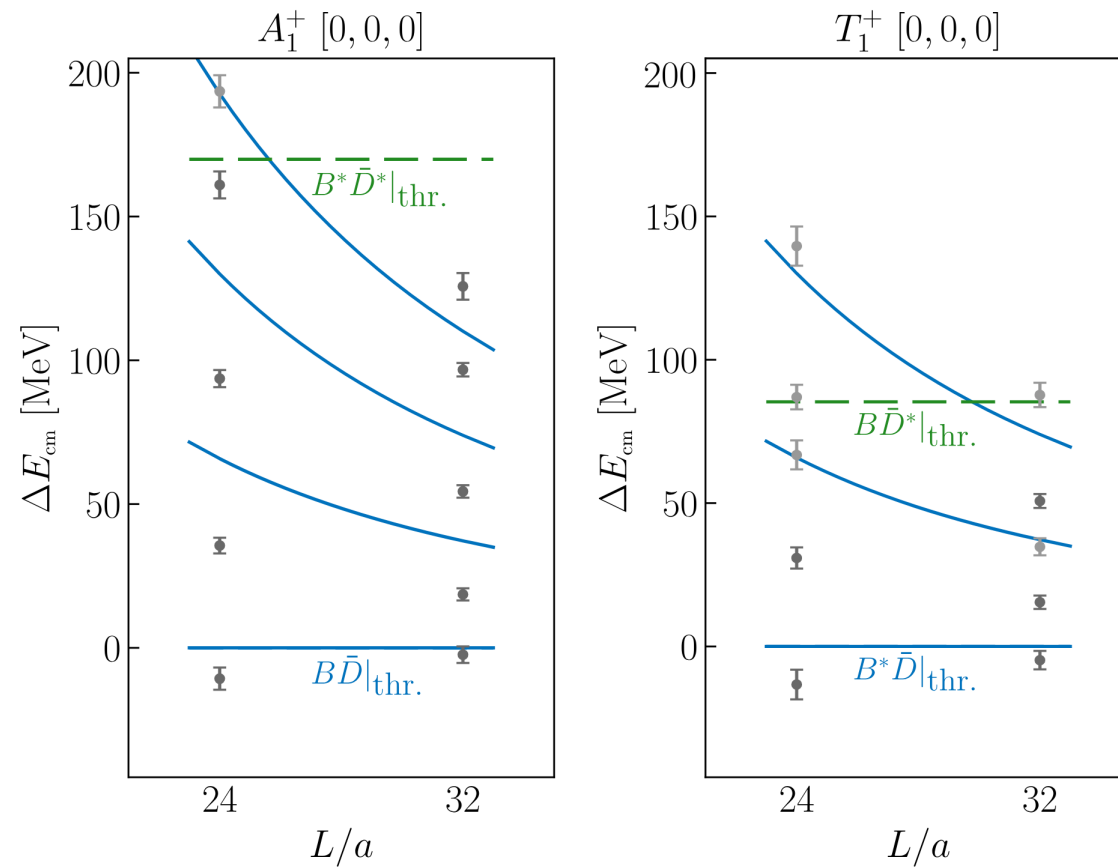
- LQCs + ERE up to $\mathcal{O}(p^4)$: shadow bound state
- No one-pion-exchange



C. Alexandrou, J. Finkenrath, T. Leontiou, S. Meinel, M. Pflaumer, and M. Wagner, PRL132, 151902 (2024).

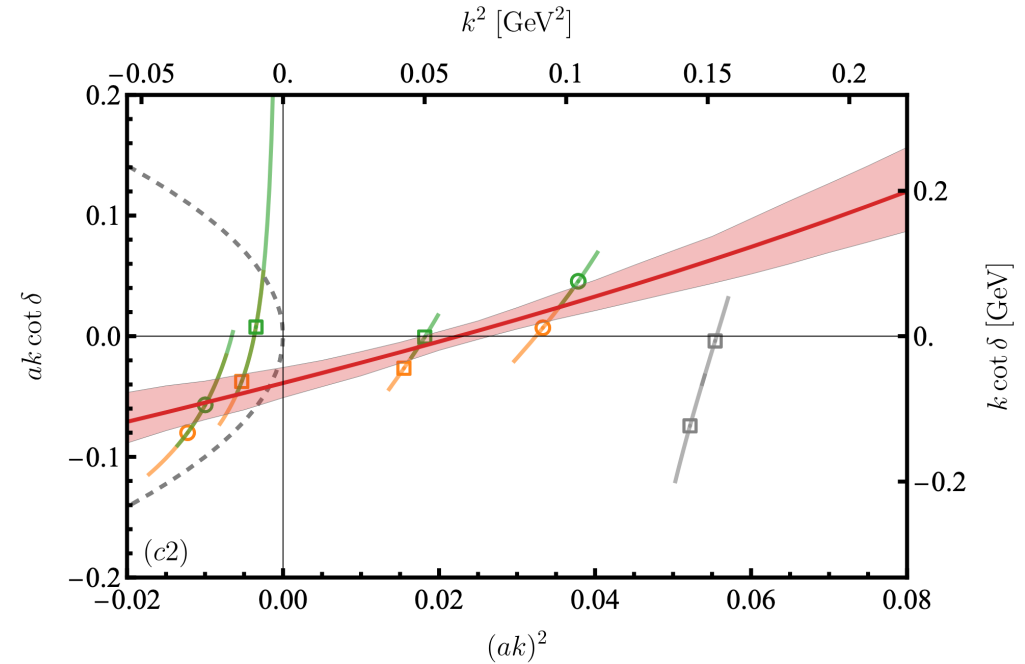
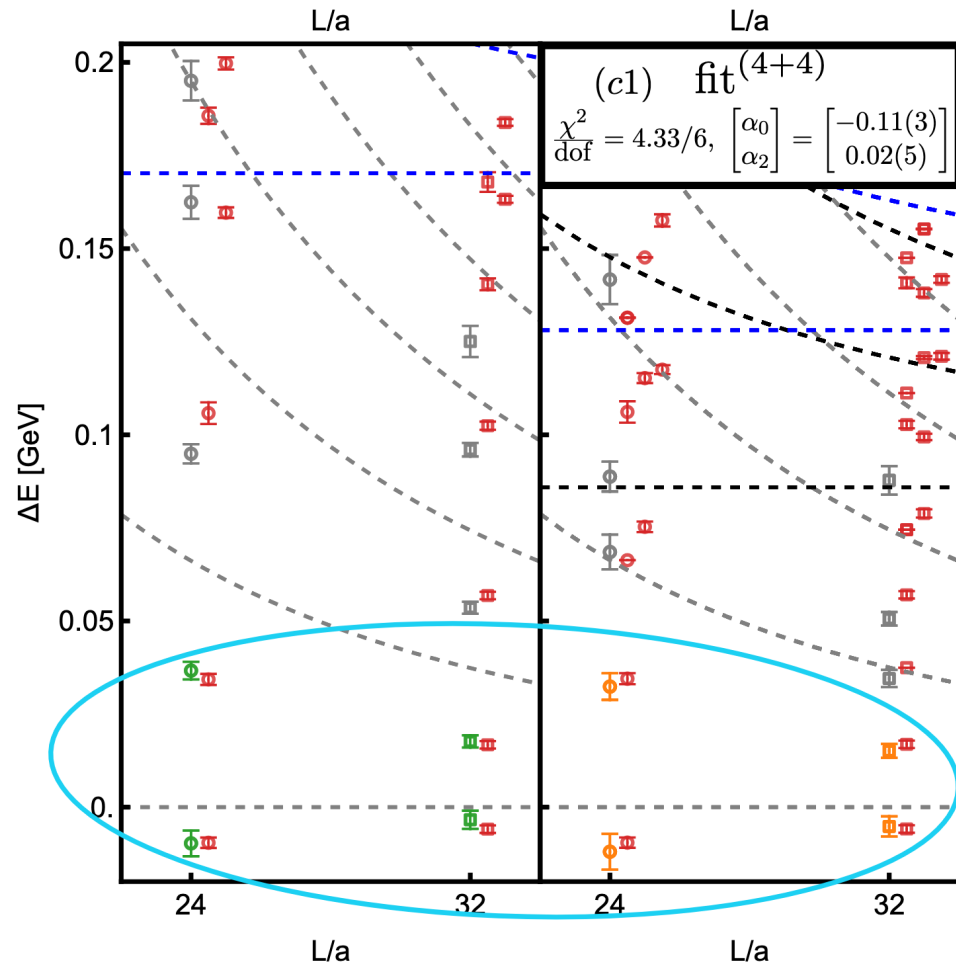


- Does **heavy quark spin symmetry** work?



● Plane wave QCs + contact EFT with **HQSS**

► Predictions of other systems



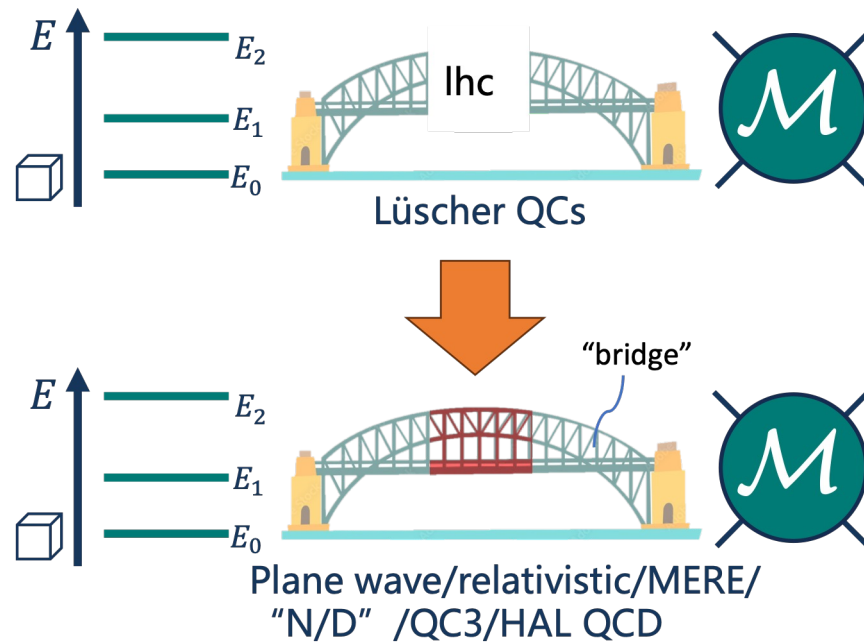
Threshold	J^P	Poles [MeV]	$1/a$ [fm ⁻¹]	r [fm]
$B\bar{D}$	0^+	$-1.8^{+1.0+1.6}_{-1.3-4.0}$	$-0.33^{+0.10+0.24}_{-0.10-0.25}$	$0.39^{+0.11+0.02}_{-0.11-0.02}$
$B^*\bar{D}$	1^+	$-1.7^{+0.9+1.6}_{-1.3-4.0}$	$-0.33^{+0.10+0.24}_{-0.10-0.25}$	$0.39^{+0.11+0.02}_{-0.11-0.02}$
$B\bar{D}^*$	1^+	$-3.6^{+1.2+2.7}_{-1.5-5.0}$	$-0.48^{+0.07+0.22}_{-0.07-0.23}$	$0.39^{+0.10+0.01}_{-0.10-0.01}$
$B^*\bar{D}^{*\dagger}$	$0^+, 1^+, 2^+$	$-3.6^{+1.2+2.7}_{-1.5-5.0}$	$-0.47^{+0.07+0.22}_{-0.07-0.23}$	$0.39^{+0.10+0.01}_{-0.10-0.01}$



M. Abolnikov, LM, V. Baru, E. Epelbaum, A. A. Filin, and A. M. Gasparyan, PRD **113**, 114041 (2026).

Summary and outlook

- Left-cut problems $\leftrightarrow$ long-range interaction, e.g. one-pion-exchange
 - ▶ Infinite volume: amplitude formulae with lhc , LSE, N/D, MERE
 - ▶ Finite volume: Lüscher QCs are invalid below the branch point of lhc
- Several approach to FV lhc problems



Developing formalisms

- Applications: H-dibaryon, NN, $T_{cc}, \dots$
 - ▶ Reanalyze existing data
 - ▶ New lattice simulations
 - ▶ Significant lhc effect for E_{FV} close and below the lhc branch point
- Other advantages:
 - ▶ Incorporating symmetries
 - ▶ No redundant deep bound state
 - ▶ Partial wave mixing effect

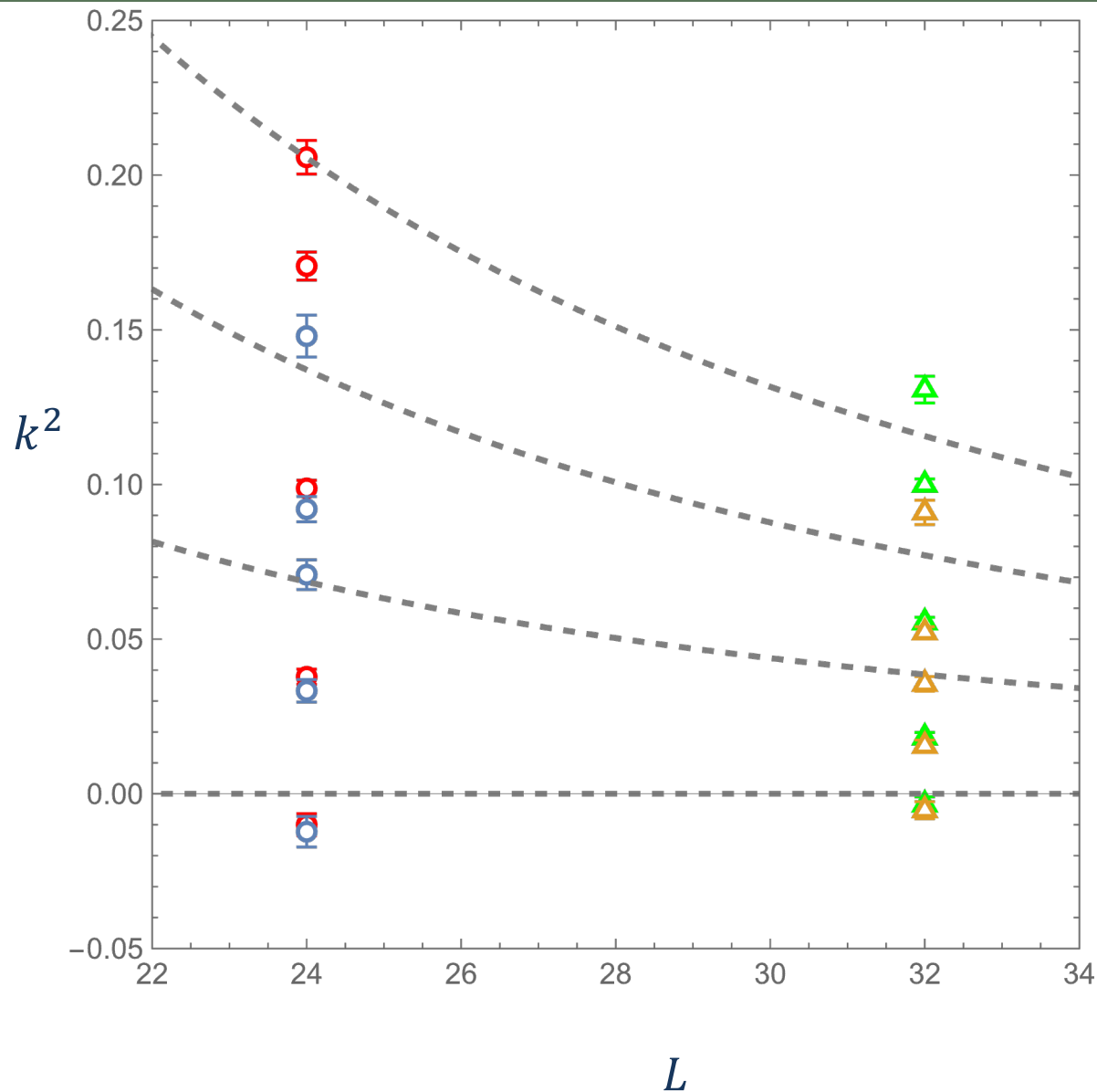
Applications, Comparisons and Benchmarks

Thanks for your attentions!



backup





$$V_C = [V_1 + V_2(p^2 + p'^2) + V_3(p^4 + p'^4) + V_4 p^2 p'^2 + \dots] \delta_{ij}$$

$$V_S = [V_4 + V_5(p^2 + p'^2) + \dots] \langle s_{1q} \cdot s_{2q} \rangle$$

$$\{s1, s2, S\} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad \langle s_{1q} \cdot s_{2q} \rangle = \begin{pmatrix} \underline{0} & -\frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{4} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{0} & \frac{1}{4} & \frac{1}{2\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & -\frac{1}{2\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \underline{\frac{1}{4}} \end{pmatrix}$$



Infinite volume

$$\mathcal{M}(s) = \text{V} + \text{V} \overset{i\epsilon}{\text{---}} \text{V} + \text{V} \overset{i\epsilon}{\text{---}} \text{V} \overset{i\epsilon}{\text{---}} \text{V} + \dots$$

$$\text{V} = \text{X} + \text{---} + \dots$$

$$\text{K} = \text{V} + \text{V} \overset{\text{p.v.}}{\text{---}} \text{V} + \text{V} \overset{\text{p.v.}}{\text{---}} \text{V} \overset{\text{p.v.}}{\text{---}} \text{V} + \dots$$

$$\text{V} \overset{i\epsilon}{\text{---}} \text{V} = \text{V} \overset{\text{p.v.}}{\text{---}} \text{V} + \text{V} \overset{i\rho(s)}{\text{---}} \text{V}$$

$i\rho(s) \propto \sqrt{s - 4M_N^2}$

$$\begin{aligned} \mathcal{M}(s) &= \text{K} + \text{K} \overset{i\rho(s)}{\text{---}} \text{K} + \text{K} \overset{i\rho(s)}{\text{---}} \text{K} \overset{i\rho(s)}{\text{---}} \text{K} + \dots \\ &= K(s) + K(s)i\rho(s)K + \dots = \frac{1}{K(s)^{-1} - i\rho(s)} \end{aligned}$$



Detailed derivation of Lüscher's formula

$$\mathcal{M}_L(s) = \text{V} + \text{V} \Sigma \text{V} + \text{V} \Sigma \text{V} \Sigma \text{V} + \dots$$

$$\text{K} = \text{V} + \text{V} \text{p.v.} \text{V} + \text{V} \text{p.v.} \text{V} \text{p.v.} \text{V} + \dots$$

$$\text{F} \equiv \Sigma - \text{p.v.}$$

Also taking on-shell

$$\text{V} = \text{X} + \text{I} + \dots$$

$$\left[\frac{1}{L^3} \sum_k - \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] f(k) = \begin{cases} \mathcal{O}(e^{-mL}) & \text{smooth } f(k) \\ \text{power of } L & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathcal{M}_L(s) &= \text{K} + \text{K} \text{F} \text{K} + \text{K} \text{F} \text{K} \text{F} \text{K} + \dots \\ &= K(s) + K(s)F(L)K + \dots = \frac{1}{K(s)^{-1} - F(L)} \end{aligned}$$

$$\text{Lüscher formula: } \det[K(s)^{-1} - F(L)] = 0$$



- Symmetry from QCD
 - Chiral symmetry and its spontaneous breaking
- Weinberg power counting
 - Systemic calculation, controllable truncation error

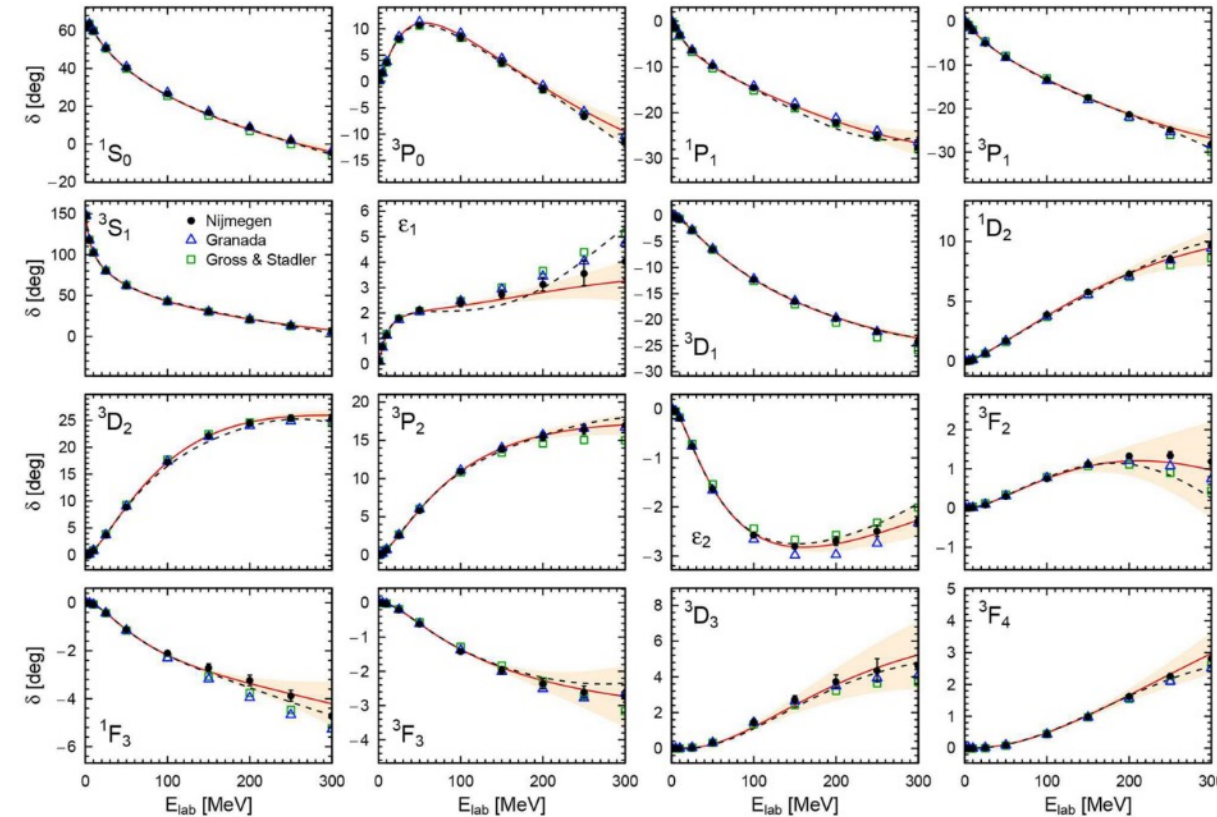
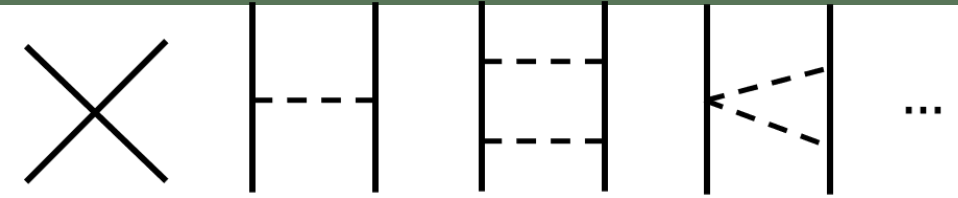
Reinert:2017usi

$$V(\vec{p}', \vec{p}) = V_{\text{contact}} + V_{1\pi} + V_{2\pi}$$

- Great success in the nuclear force
- Semilocal momentum-space regularization

$$V_{1\pi}(\vec{p}', \vec{p}) = -\frac{g_A^2}{4F_\pi^2} \left(\frac{\vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}}{q^2 + m_\pi^2} + C(m_\pi) \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) e^{-\frac{q^2 + m_\pi^2}{\Lambda^2}}$$

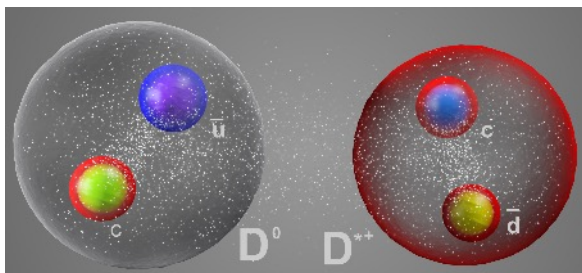
- Long-range interaction: $V_{1\pi}$ is known
- Short-range interaction: contact interaction
 - Unknown low energy constants (LECs)
 - fitting lattice QCD data



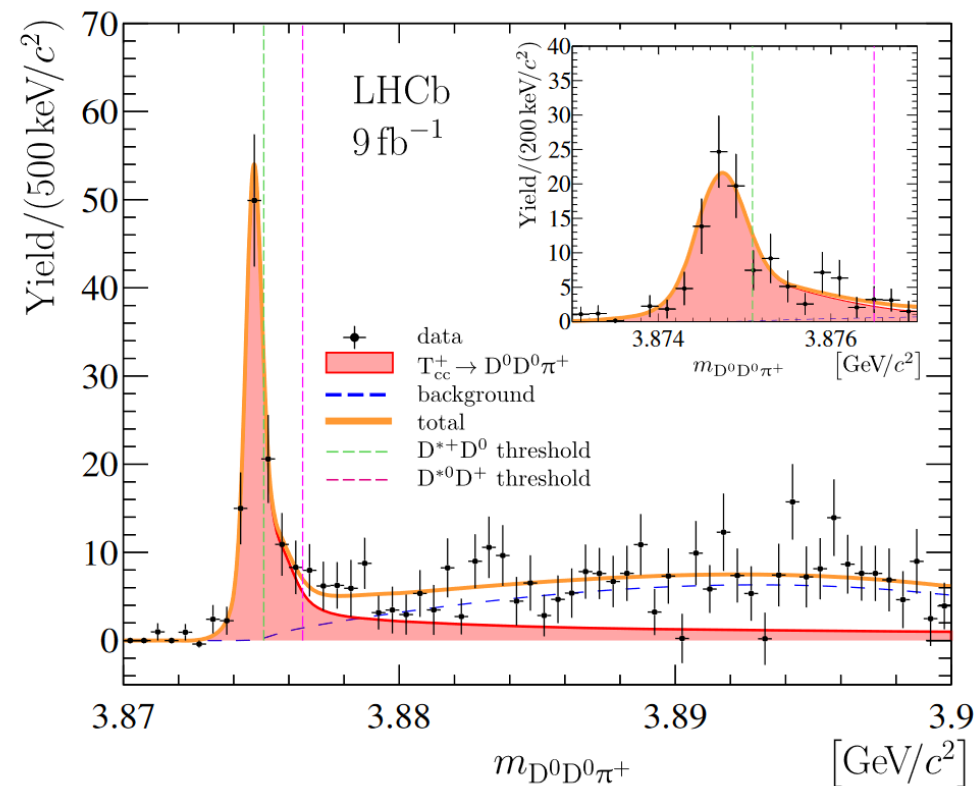
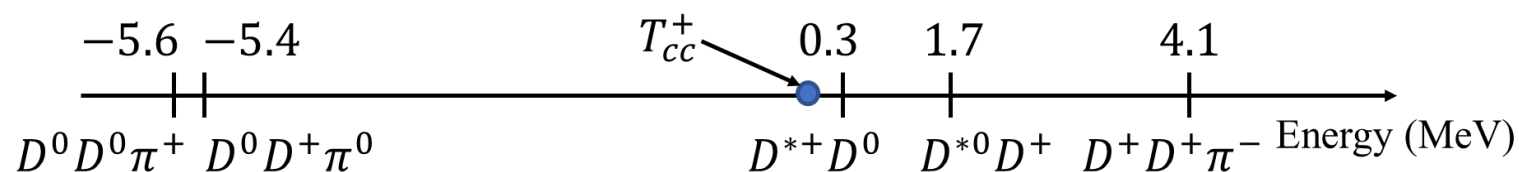
$T_{cc}(3875)^+$ state

LHCb Collaboration

- $T_{cc}(3875)^+$ was observed in 3-body final states: $D^0 D^0 \pi^+$
- Very close to $D^0 D^{*+}$ thresholds: $\delta m_U \approx -360 \text{ keV}$, $\Gamma \approx 48 \text{ keV}$
- Exotic hadrons: minimal quark content: $cc\bar{u}\bar{d}$
- Good candidates of $D^0 D^{*+}$ molecule



- 3-body dynamics could be important



lhcb:2021vvq, lhcb:2021auc, Du:2021zzh, Meng:2021jnw...

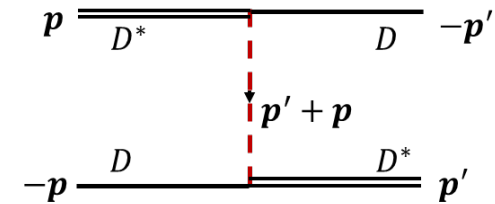


● LQCD setting: $m_\pi \approx 280$ MeV, $m_D \approx 1927$ MeV, $m_{D^*} \approx 2049$ MeV, $L \approx 2.07, 2.76$ fm, $a \approx 0.086$ fm

● Some quick estimations

Padmanath:2022cvi

- ▶ $m_{\text{eff}}^2 = m_\pi^2 - (m_{D^*} - m_D)^2 > 0$, $m_{\text{eff}} \approx 252$ MeV
- ▶ $p_{\text{lhc}}^2 \approx -\left(\frac{m_{\text{eff}}}{2}\right)^2 = -(126 \text{ MeV})^2$
- ▶ $p_{\text{rhc3}}^2 \approx 2\mu_{DD^*}(2m_D + m_\pi - m_D - m_{D^*}) \approx (560 \text{ MeV})^2$



$$\frac{-1}{k^2 + m_\pi^2 - k_0^2 - i\epsilon} = \frac{-1}{k^2 + m_{\text{eff}}^2 - i\epsilon}$$

● A conventional procedure to the IFV:

- ▶ Using Lüscher formula to get phase shift
- ▶ Use ERE to parameterize K -matrix

● Conclusion: virtual states

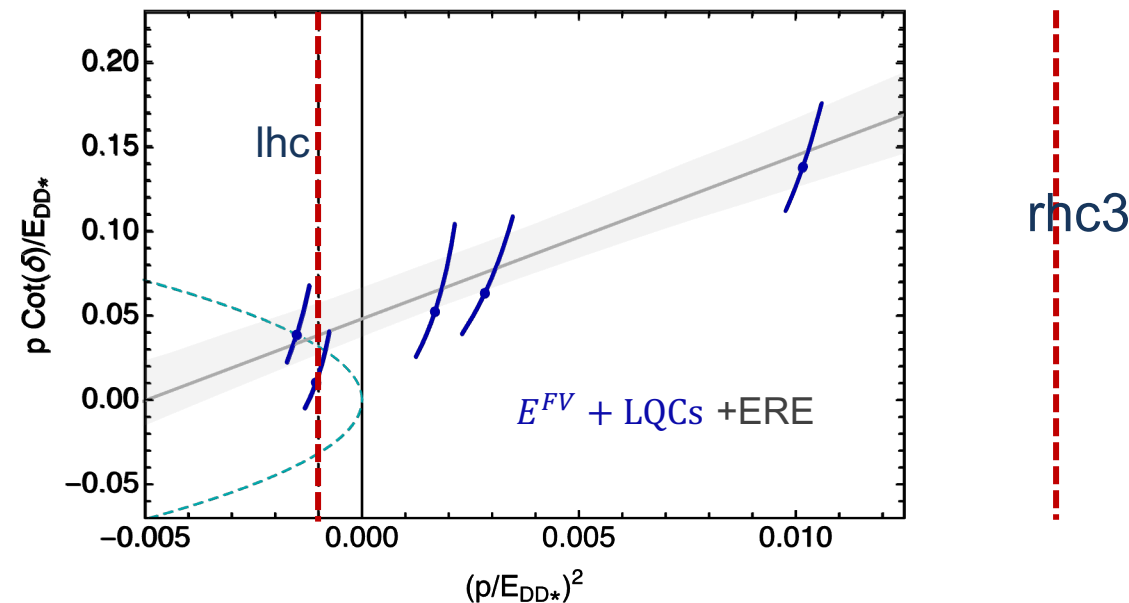
● Limitations

- ▶ $m_{\text{eff}}L = 2.6, 3.5$

exponential effect can be important

- ▶ left-hand cut

~~Lüscher formula, effective range expansion~~



Other lattice results: Cheung:2017tnt, Junnarkar:2018twb, Chen:2022vpo, Lyu:2023xro, Whyte:2024ihh...

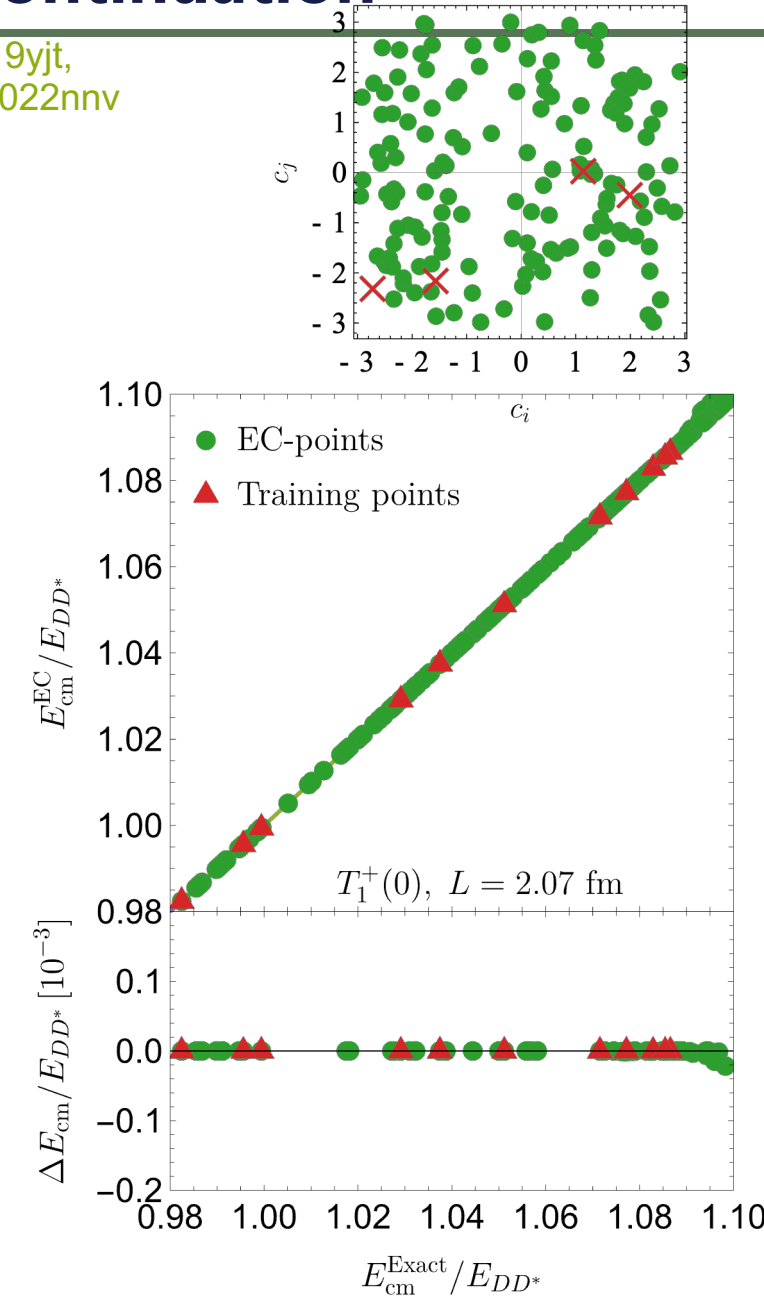
Towards a practical approach: eigenvector continuation

- Plane wave basis+Eigenvector continuation
 - Eigenvector continuation (EC) with subspace learning
- To fit or quantify uncertainty: solve eigenvalue problem with different $\{c_i\}$ repeatedly
- EC basis: eigenvectors from a selection of parameter sets $\{c_i\}_1, \{c_i\}_2, \dots$ (training point)
- Naturalness of LEC in EFT (~ 1) makes the EC more reliable
- dim is linear function

$$\dim^{EC} = \frac{p_{max}}{2\pi/L} \sim \mathcal{O}(10), \quad p_{max} \approx 0.6 \text{ GeV}$$

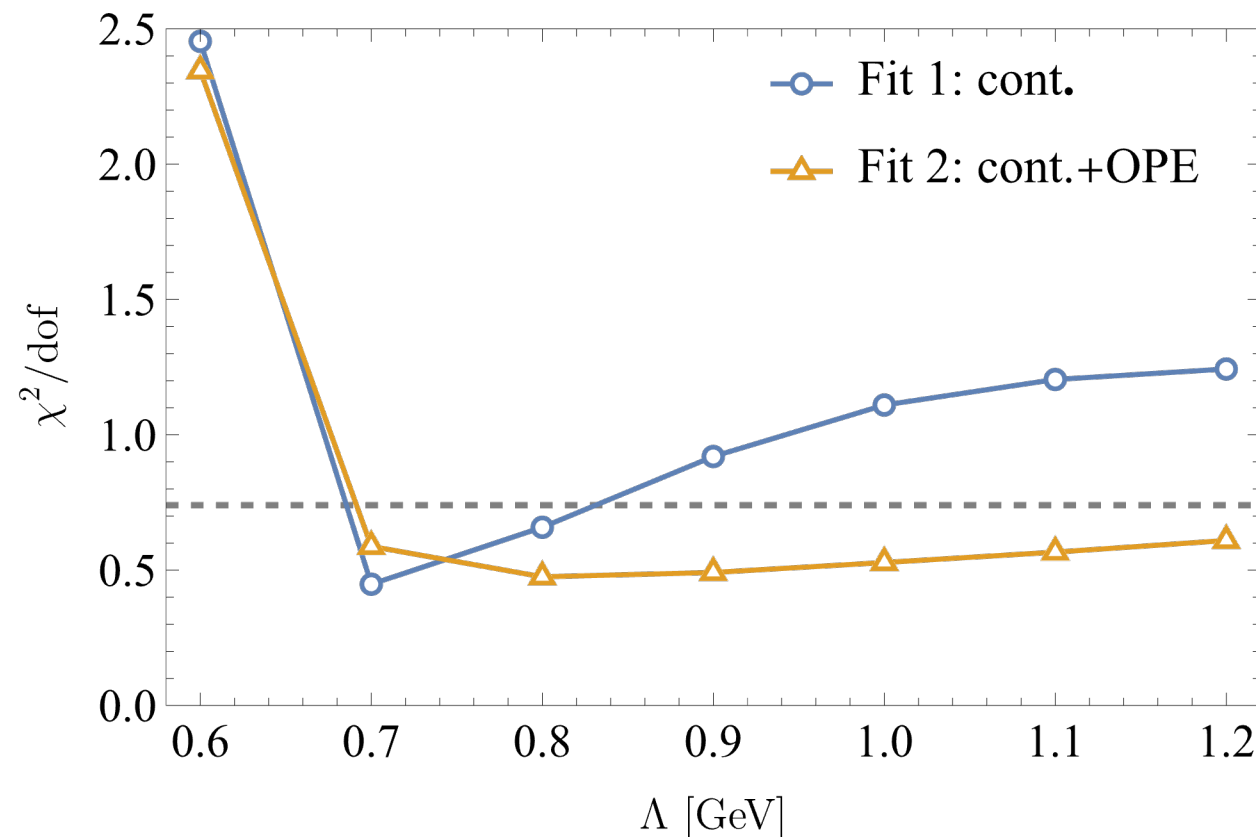
- The subspace learning is the one-time cost
- Make the calculation fast and accurate

Frame:2017fah, Demol:2019yjt,
Furnstahl:2020abp, Yapa:2022nnv



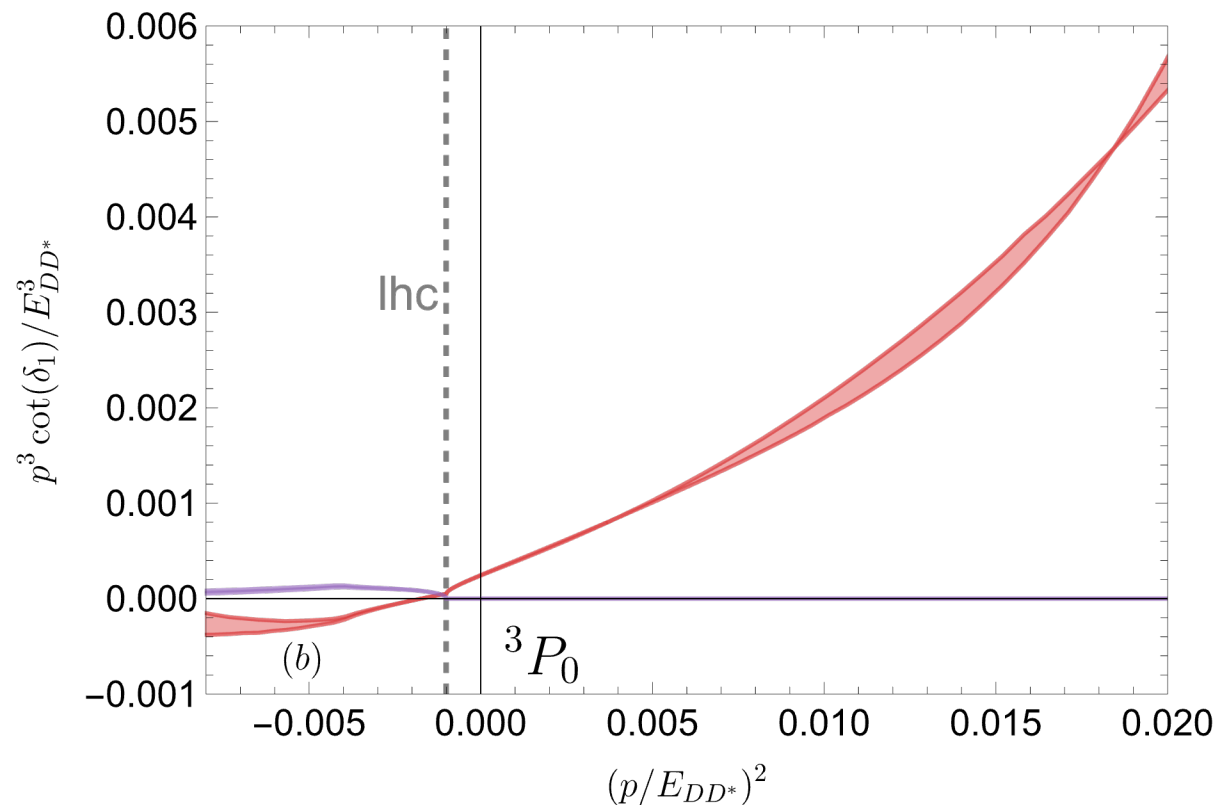
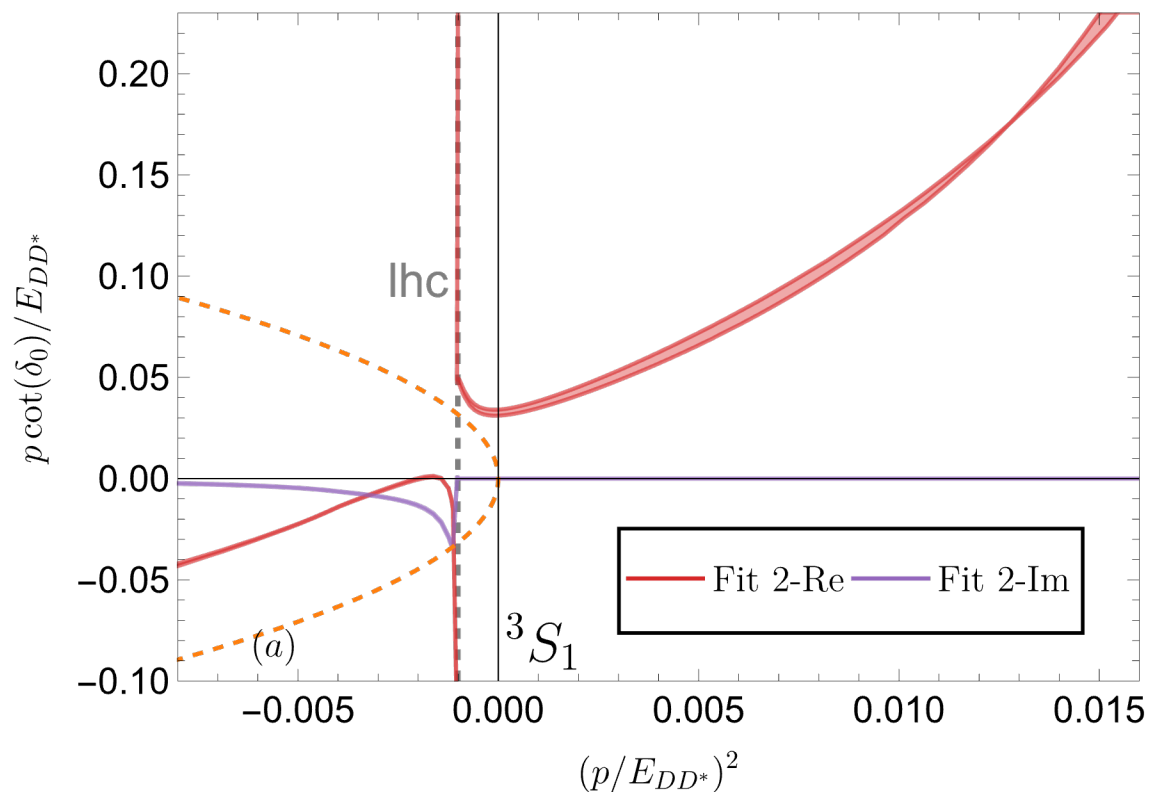
Cutoff dependence of χ^2

- 3 LECs: LO and NLO 3S_1 contact terms, NLO 3P_0
- In V_{ctc} fit, the P-wave dominate states control Λ -dependence of the χ^2
 - ▶ The shape of the of $k^3 \cot \delta_1$ is determined by regulator and cutoff
 - ▶ Sensitive to Λ
- The $V_{\text{ctc}} + V_{1\pi}$ fit is stable with Λ
- The $V_{\text{ctc}} + V_{1\pi}$ fit is even better than QCs

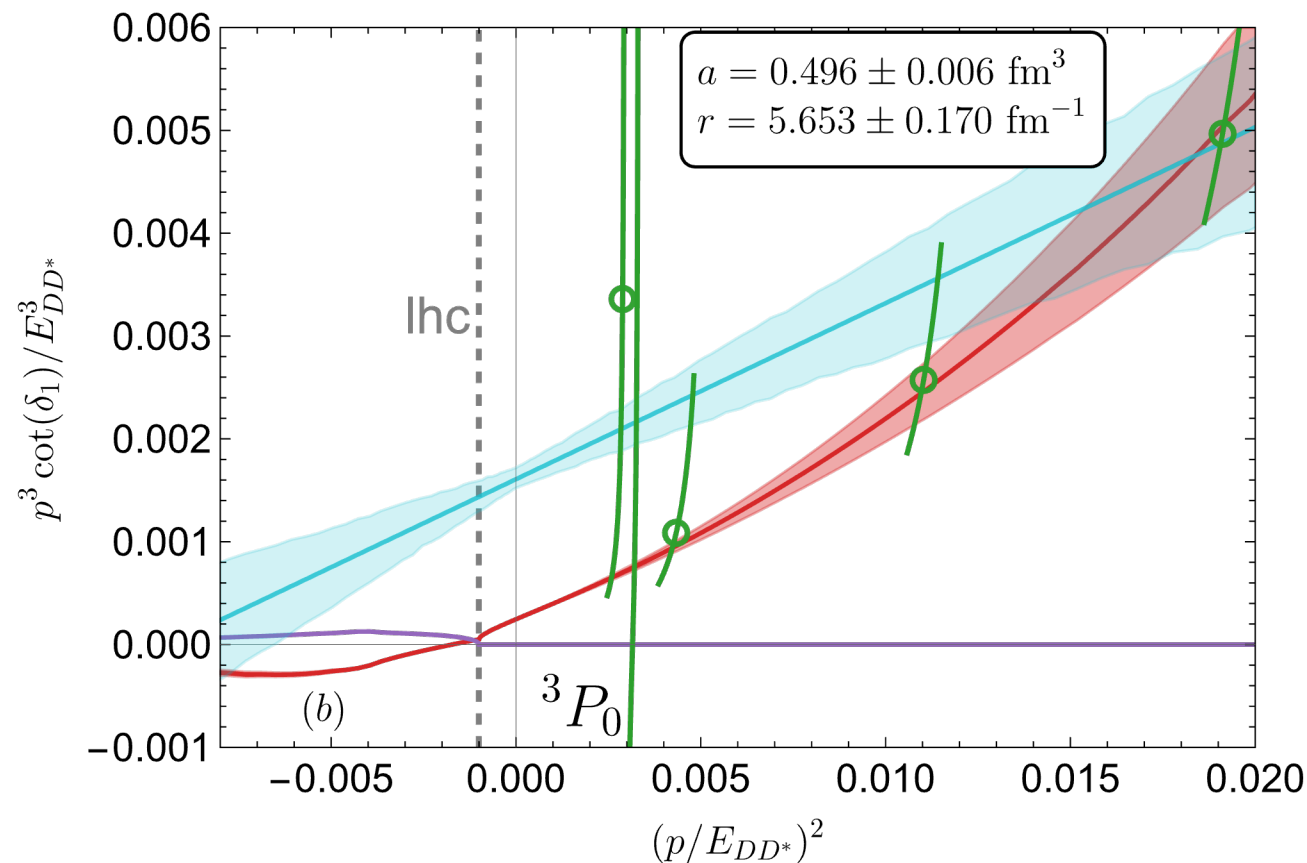
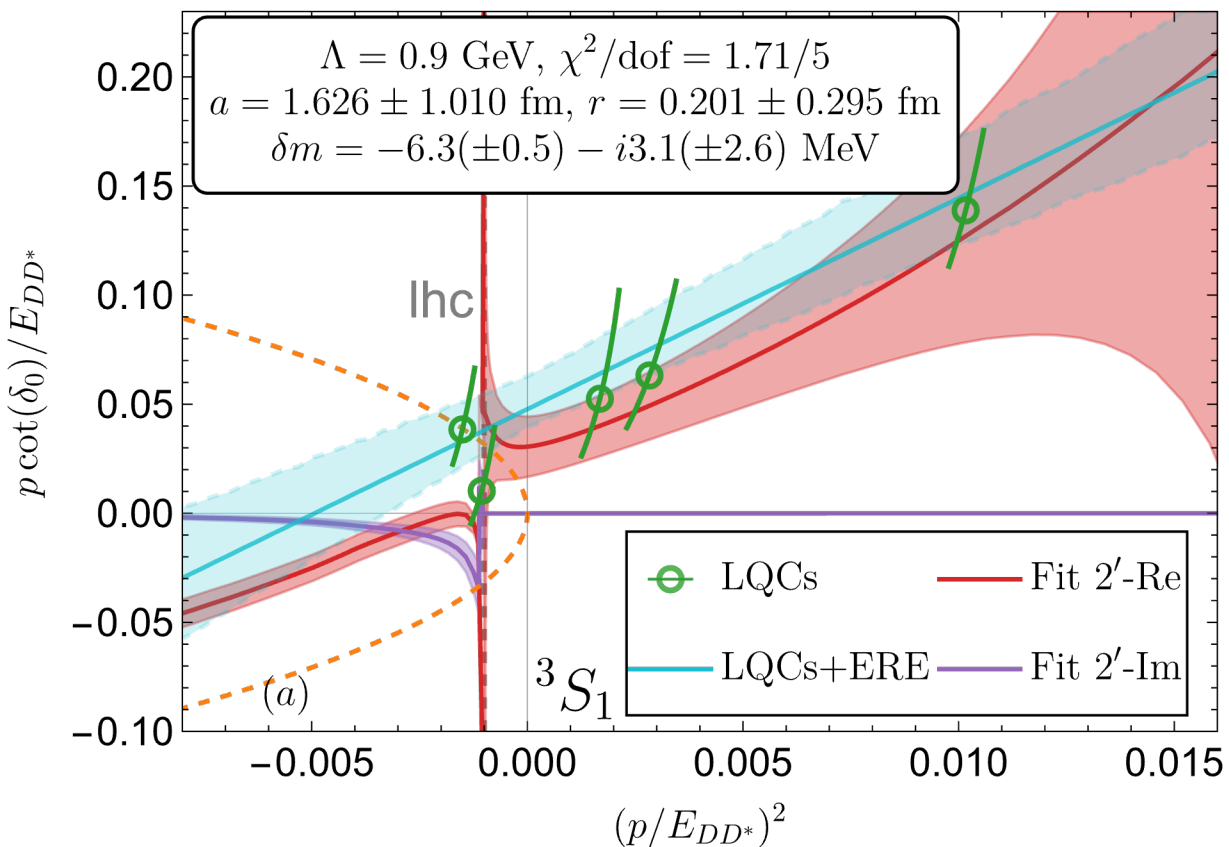


Cutoff dependence of phase shift

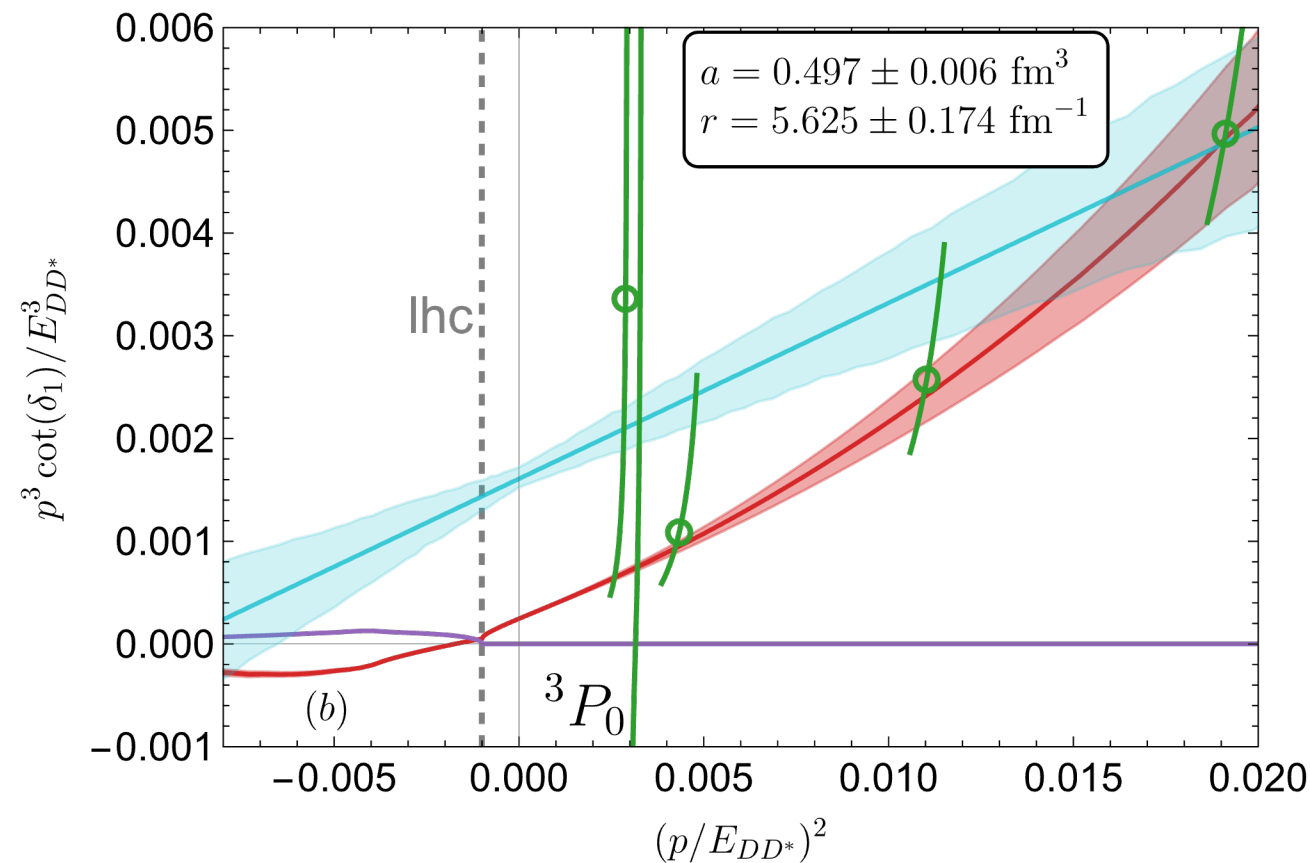
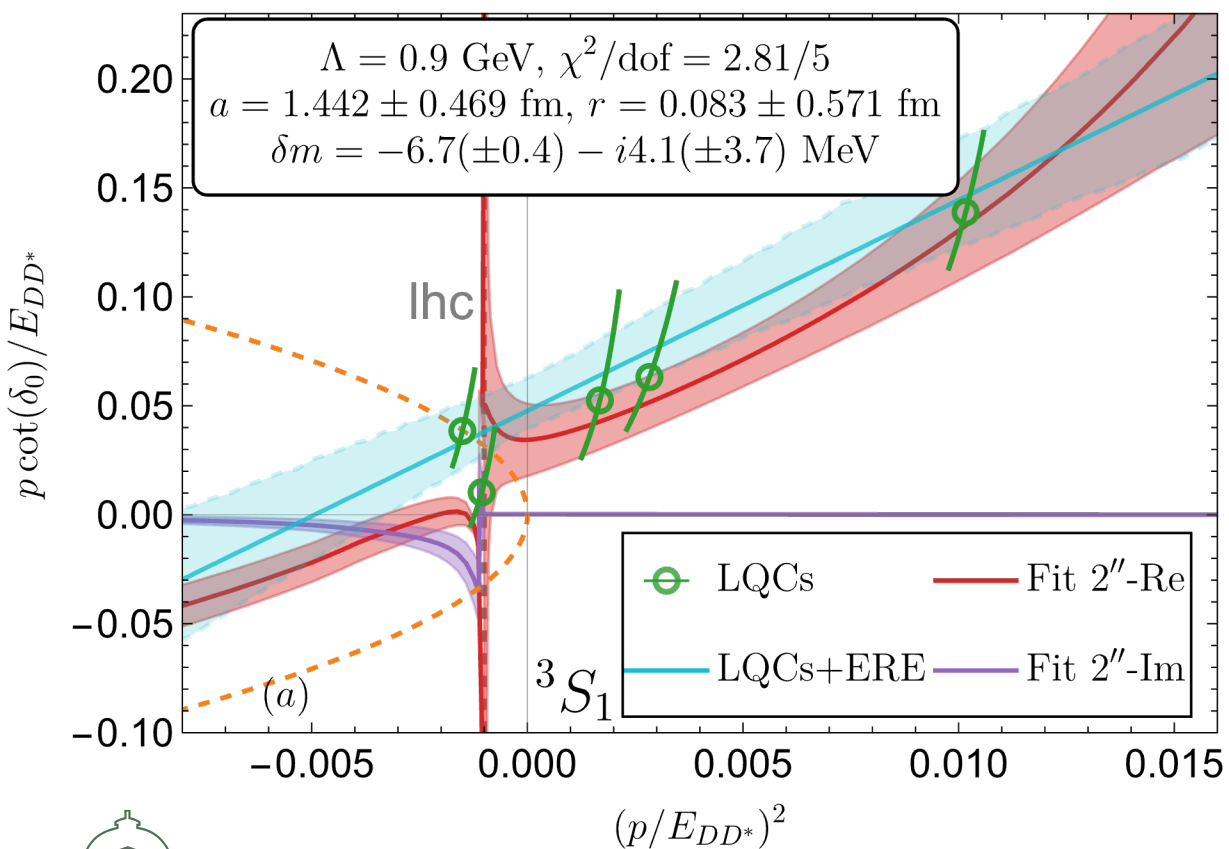
$$\Lambda = 0.7 - 1.2 \text{ GeV}$$



Including SD transition terms



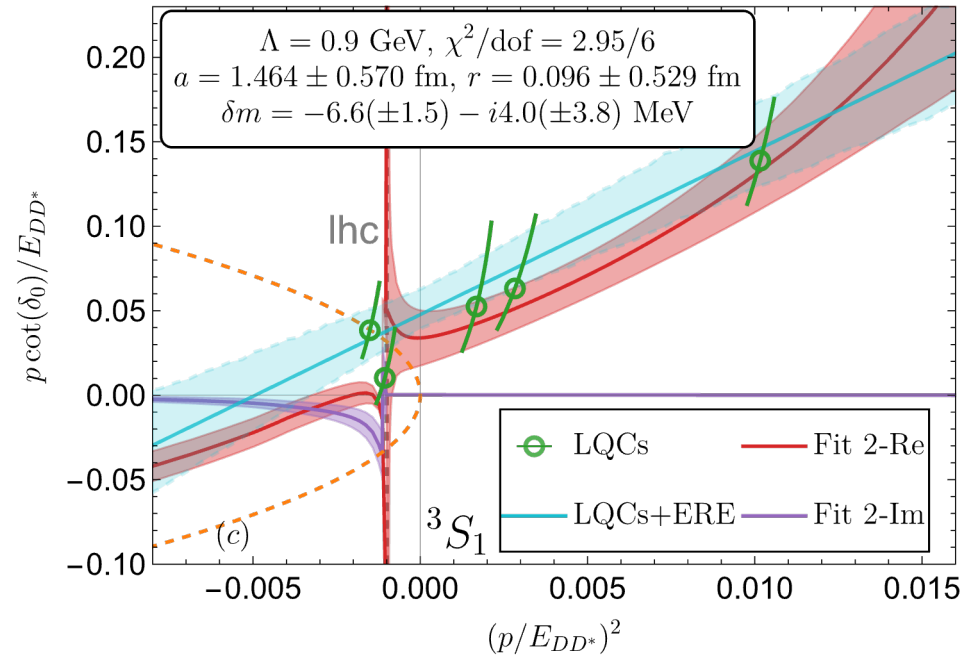
Including 3P2 term



Amplitude zero and left-hand cut

LO

$$\frac{m^2}{4k^2} \log\left(1 + \frac{4k^2}{m^2}\right) = 1 + \frac{m^2}{\tilde{g}}$$



M.-L. Du, F.-K. Guo, and B. Wu, PRL**135**, 011903 (2025)

Higher order

$$\frac{m^2}{4k^2} \log\left(1 + \frac{4k^2}{m^2}\right) = 1 + \frac{\tilde{n}(k^2)m^2}{g}$$

