

Non-Abelian Discrete Symmetry Approach to Lepton Mixing and CP-Violation (Predictions and Experimental Tests)

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International Workshop on
New Opportunities for Particle Physics
IHEP, Beijing, China
August 14-16, 2026

The Problem

Understanding the origin of the peculiar pattern of neutrino mixing of two large and one small mixing angles is one of the major problems in neutrino physics.

It is a part of the more general highly challenging and still unresolved fundamental problem in particle physics of understanding the origins of the patterns of the charged lepton and neutrino masses and of neutrino mixing, of quark masses and mixing, and of CP violation in the quark and lepton sector, i.e., understanding the origins of the lepton and quark flavours.

“Asked what single mystery, if he could choose, he would like to see solved in his lifetime, Weinberg doesn’t have to think for long: he wants to be able to explain the observed pattern of quark and lepton masses.”

From “Model Physicist”, CERN Courier, 13 October 2017.

The renewed attempts to seek new better solutions of the flavour problem than those already proposed were stimulated primarily by the remarkable progress made in the studies of neutrino oscillations, which began 24 years ago with the discovery of oscillations of atmospheric ν_μ and $\bar{\nu}_\mu$ by SuperKamiokande experiment. This lead, in particular, to the determination of the pattern of the 3-neutrino mixing, which turn out to consist of two large and one small mixing angles.

The Lepton Flavour Problem

Consists of three basic elements (sub-problems), namely, understanding:

- Why $m_{\nu_j} \lll m_{e,\mu,\tau}, m_q$, $q = u, c, t, d, s, b$ ($m_{\nu_j} \lesssim 0.5$ eV, $m_l \geq 0.511$ MeV, $m_q \gtrsim 2$ MeV);
- The origins of the patterns of
 - i) neutrino mixing of 2 large and 1 small angles ($\theta_{12}^l = 33.4^\circ$, $\theta_{23}^l = 42.4^\circ$ (49.0°), $\theta_{13}^l = 8.59^\circ$), and of ii) Δm_{ij}^2 , i.e., of $\Delta m_{21}^2 \ll |\Delta m_{31}^2|$, $\Delta m_{21}^2/|\Delta m_{31}^2| \cong 1/30$.
- The origin of the hierarchical pattern of charged lepton masses: $m_e \ll m_\mu \ll m_\tau$, $m_e/m_\mu \cong 1/200$, $m_\mu/m_\tau \cong 1/17$.

The first two added new important aspects to the flavour problem.

$$m_{\nu_j} \lll m_{e,\mu,\tau}, m_q, \quad q = u, c, t, d, s, b:$$

seesaw mechanism(s), Weinberg operator, radiative ν mass generation, extra dimensions. However, additional input (symmetries) needed to explain the pattern of lepton mixing and to get specific testable predictions.

The Neutrino Mixing Problem

The most elegant, simple and testable solution of the neutrino mixing problem is arguably provided by the **non-Abelian discrete symmetry approach**.

In what concerns the lepton flavour problem, in the last 8 years a very successful and attractive approach based on **Modular Invariance** has been and continues to be developed.

The Framework: the Reference 3- ν Mixing Scheme

$$\nu_{l\text{L}} = \sum_{j=1}^3 U_{lj} \nu_{j\text{L}} \quad l = e, \mu, \tau.$$

The PMNS matrix U - 3×3 unitary.

$\nu_j, m_j \neq 0$: Dirac or Majorana particles.

Data: 3 ν s are light: $\nu_{1,2,3}, m_{1,2,3} \lesssim 0.5$ eV.

3- ν mixing: 3-flavour neutrino oscillations possible.

ν_μ, E ; at distance L : $P(\nu_\mu \rightarrow \nu_{\tau(e)}) \neq 0, P(\nu_\mu \rightarrow \nu_\mu) < 1$

$$P(\nu_l \rightarrow \nu_{l'}) = P(\nu_l \rightarrow \nu_{l'}; E, L; U; m_2^2 - m_1^2, m_3^2 - m_1^2)$$

Three Neutrino Mixing

$$\nu_{lL} = \sum_{j=1}^3 U_{lj} \nu_{jL} .$$

U is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) neutrino mixing matrix,

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

- U - $n \times n$ unitary:

	n	2	3	4
mixing angles:	$\frac{1}{2}n(n-1)$	1	3	6

CP-violating phases:

- ν_j - Dirac: $\frac{1}{2}(n-1)(n-2)$ 0 1 3
- ν_j - Majorana: $\frac{1}{2}n(n-1)$ 1 3 6

$n = 3$: 1 Dirac and
2 additional CP-violating phases, Majorana phases

S.M. Bilenky, J. Hosek, S.T.P., 1980

PMNS Matrix: Standard Parametrization

$$U = V P, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix},$$

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

- $s_{ij} \equiv \sin \theta_{ij}$, $c_{ij} \equiv \cos \theta_{ij}$, $\theta_{ij} = [0, \frac{\pi}{2}]$,
- δ - Dirac CPV phase, $\delta = [0, 2\pi]$; CP inv.: $\delta = 0, \pi, 2\pi$;
- α_{21} , α_{31} - Majorana CPV phases; CP inv.: $\alpha_{21(31)} = k(k')\pi$, $k(k') = 0, 1, 2...$
S.M. Bilenky et al., 1980
- $\Delta m_{\odot}^2 \equiv \Delta m_{21}^2 \cong 7.34 \times 10^{-5} \text{ eV}^2 > 0$, $\sin^2 \theta_{12} \cong 0.305$, $\cos 2\theta_{12} \gtrsim 0.306$ (3σ),
- $|\Delta m_{31(32)}^2| \cong 2.448$ (2.502) $\times 10^{-3} \text{ eV}^2$, $\sin^2 \theta_{23} \cong 0.545$ (0.551), NO (IO) ,
- θ_{13} - the CHOOZ angle: $\sin^2 \theta_{13} = 0.0222$ (0.0223)
F. Capozzi et al. (Bari Group), arXiv:2003.08511.

- $\text{sgn}(\Delta m_{\text{atm}}^2) = \text{sgn}(\Delta m_{31(32)}^2)$ not determined

$$\Delta m_{\text{atm}}^2 \equiv \Delta m_{31}^2 > 0, \quad \text{normal mass ordering (NO)}$$

$$\Delta m_{\text{atm}}^2 \equiv \Delta m_{32}^2 < 0, \quad \text{inverted mass ordering (IO)}$$

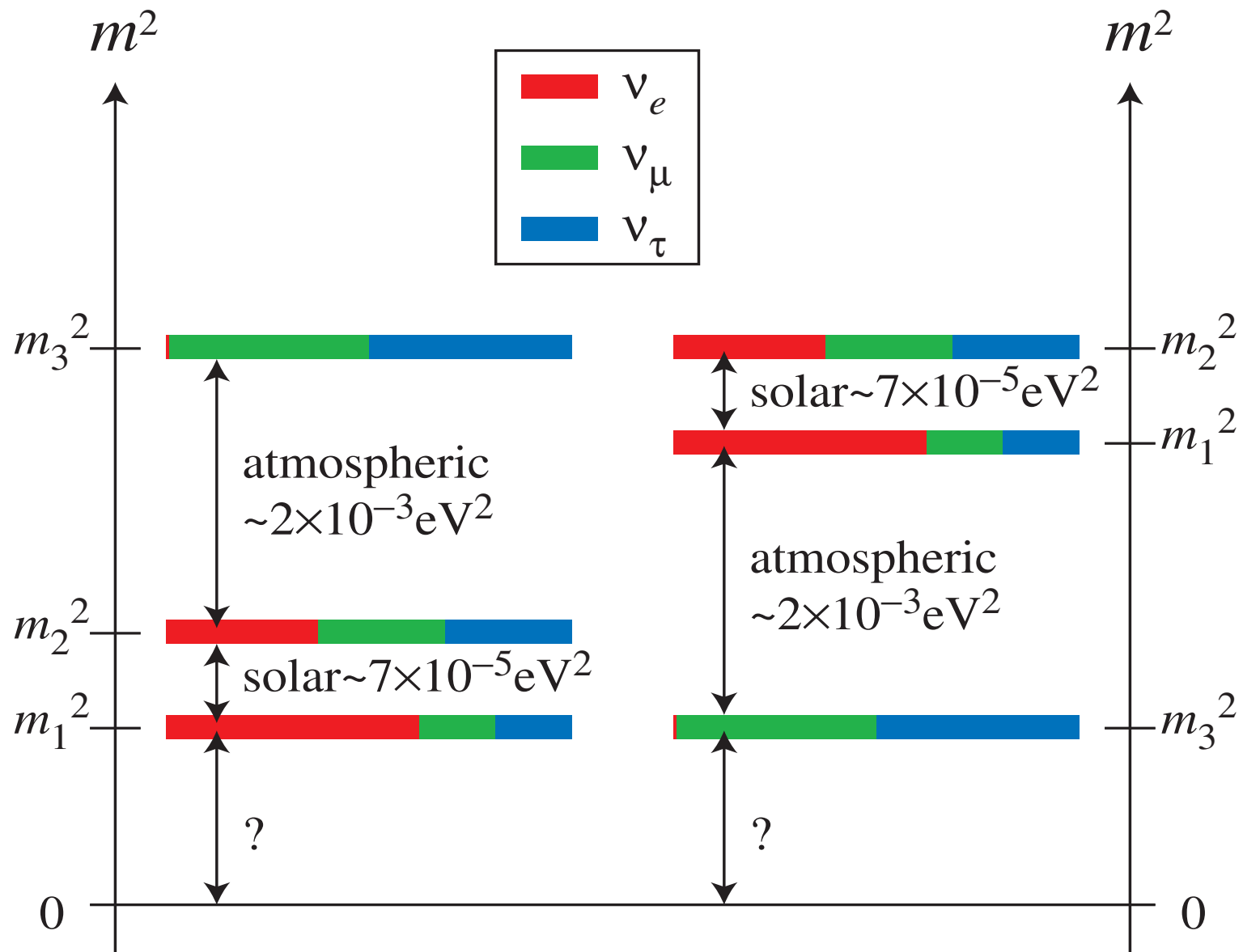
Convention: $m_1 < m_2 < m_3$ - NO, $m_3 < m_1 < m_2$ - IO

$$m_1 \ll m_2 < m_3, \quad \text{NH,}$$

$$m_3 \ll m_1 < m_2, \quad \text{IH,}$$

$$m_1 \cong m_2 \cong m_3, \quad m_{1,2,3}^2 \gg |\Delta m_{31(32)}^2|, \quad \text{QD; } m_j \gtrsim 0.10 \text{ eV.}$$

- $m_2 = \sqrt{m_1^2 + \Delta m_{21}^2}, \quad m_3 = \sqrt{m_1^2 + \Delta m_{31}^2}$ - NO;
- $m_1 = \sqrt{m_3^2 + \Delta m_{23}^2 - \Delta m_{21}^2}, \quad m_2 = \sqrt{m_3^2 + \Delta m_{23}^2}$ - IO;



• **Dirac phase δ :** $\nu_l \leftrightarrow \nu_{l'}, \bar{\nu}_l \leftrightarrow \bar{\nu}_{l'}, l \neq l'$; $A_{\text{CP}}^{(l,l')} \propto J_{\text{CP}} \propto \sin \theta_{13} \sin \delta$:

3ν —mixing:

P.I. Krastev, S.T.P., 1988

$$A_{\text{CP}}^{(l,l')} \equiv P(\nu_l \rightarrow \nu_{l'}) - P(\bar{\nu}_l \rightarrow \bar{\nu}_{l'}) , \quad l \neq l' = e, \mu, \tau$$

$$A_{\text{T}}^{(l,l')} \equiv P(\nu_l \rightarrow \nu_{l'}) - P(\nu_{l'} \rightarrow \nu_l), \quad l \neq l'$$

$$A_{\text{CP(T)}}^{(e,\mu)} = A_{\text{CP(T)}}^{(\mu,\tau)} = -A_{\text{CP(T)}}^{(e,\tau)}$$

In vacuum: $A_{\text{CP(T)}}^{(e,\mu)} = 4 J_{\text{CP}} F_{\text{osc}}^{\text{vac}}$ ($A_{\text{CP(T)}}^{(e,\mu)} = A_{\text{CP(T)}}^{(\mu,\tau)} = -A_{\text{CP(T)}}^{(e,\tau)}$)

$$J_{\text{CP}} = \text{Im} \{ U_{e1} U_{\mu 2} U_{e2}^* U_{\mu 1}^* \} = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta$$

$$F_{\text{osc}}^{\text{vac}} = \sin\left(\frac{\Delta m_{21}^2}{2E} L\right) + \sin\left(\frac{\Delta m_{32}^2}{2E} L\right) + \sin\left(\frac{\Delta m_{13}^2}{2E} L\right)$$

P.I. Krastev, S.T.P., 1988

In matter: Matter effects violate

CP : $P(\nu_l \rightarrow \nu_{l'}) \neq P(\bar{\nu}_l \rightarrow \bar{\nu}_{l'})$

CPT : $P(\nu_l \rightarrow \nu_{l'}) \neq P(\bar{\nu}_{l'} \rightarrow \bar{\nu}_l)$

P. Langacker et al., 1987

Can conserve the T-invariance (constant density or density profile symmetric relative to the middle point, e.g., **Earth**)

$$P(\nu_l \rightarrow \nu_{l'}) = P(\nu_{l'} \rightarrow \nu_l), \quad l \neq l'$$

In matter with constant density (T2K, NO ν A, T2HK, DUNE):

$$J_{\text{CP}}^{\text{mat}} = J_{\text{CP}}^{\text{vac}} R_{\text{CP}}, \quad A_{\text{T}}^{(e,\mu)} = J_{\text{CP}}^{\text{mat}} F_{\text{osc}}^{\text{mat}}$$

R_{CP} - real, does not depend on θ_{23} and δ ; $|R_{\text{CP}}| \lesssim 2.5$

P.I. Krastev, S.T.P., 1988

2018:

$R_{\text{CP}} > 0$, $R_{\text{CP}} \leq 1.2$; numerically R_{CP} (Krastev,Petcov;1988)=Naumov-HS factor (from 1991).

S.T.P., Y.-L. Zhou, 1806.09112

Current data: $|J_{CP}| \lesssim 0.040$ (can be relatively large!); b.f.v. with $\delta = 3\pi/2$:
 $J_{CP} \cong -0.035$.

- **Majorana phases** α_{21}, α_{31} :

- $\nu_l \leftrightarrow \nu_{l'}, \bar{\nu}_l \leftrightarrow \bar{\nu}_{l'}$ not sensitive;

S.M. Bilenky, J. Hosek, S.T.P., 1980;

P. Langacker, S.T.P., G. Steigman, S. Toshev, 1987

- $|\langle m \rangle|$ in $(\beta\beta)_{0\nu}$ -decay depends on α_{21}, α_{31} ;

- $\Gamma(\mu \rightarrow e + \gamma)$ etc. in SUSY theories depend on $\alpha_{21,31}$;

- BAU, leptogenesis scenario: $\delta, \alpha_{21,31}$!

Parameter	Ordering	Best fit	1 σ range	2 σ range	3 σ range	"1 σ " (%)
$\Delta m_{21}^2/10^{-5} \text{ eV}^2$	NO, IO	$7.49^{+0.19}_{-0.19}$	—	—	6.92 – 8.05	2.5
$\sin^2 \theta_{12}/10^{-1}$	NO, IO	$3.08^{+0.012}_{-0.011}$	—	—	2.75 – 3.45	4.5
$\Delta m_{31}^2/10^{-3} \text{ eV}^2$	NO	$2.513^{+0.021}_{-0.019}$	—	—	2.451 – 2.578	1.1
$\Delta m_{32}^2/10^{-3} \text{ eV}^2$	IO	-2.484 ± 0.020	—	—	- 2.547 – (- 2.421)	1.1
$\sin^2 \theta_{13}/10^{-2}$	NO	$2.215^{+0.056}_{-0.058}$	—	—	2.0300 – 2.388	3.0
	IO	2.231 ± 0.056	—	—	2.060 – 2.409	3.1
$\sin^2 \theta_{23}/10^{-1}$	NO	$4.70^{+0.017}_{-0.013}$	—	—	4.35 – 5.85	5.5
	IO	$5.50^{+0.012}_{-0.015}$	—	—	4.40 – 5.84	4.4
δ	NO	212^{+26}_{-41}	—	—	124 – 364	19
	IO	274^{+22}_{-25}	—	—	201 – 335	8

$$\Delta\chi_{\text{IO-NO}}^2 \quad \text{IO-NO} \quad +6.1 (2.45\sigma)$$

Global 3ν analysis of oscillation parameters: best-fit values and allowed ranges at $N_\sigma = 3$ for either NO or IO, including all data (also the SK atmospheric neutrino data). The latter column shows the formal “1 σ fractional accuracy” for each parameter, defined as 1/6 of the 3 σ range, divided by the best-fit value and expressed in percent. The last row reports the difference between the χ^2 minima in IO and NO. The results on the NO and CPV due to δ are inconclusive.

I. Esteban et al., arXiv:2410.05380 (NuFit-6.0 results).

θ_{12} , θ_{23} - **large**, θ_{13} - **small** (very different from the quark mixing angles).
 $\sin^2 \theta_{23}$ - **relatively large uncertainty**.

$$\Delta m_{21}^2/|\Delta m_{31}^2| \cong 1/30.$$

Reference 3- ν Mixing Scheme

$$\nu_{l\text{L}} = \sum_{j=1}^3 U_{lj} \nu_{j\text{L}} \quad l = e, \mu, \tau.$$

Data: 3 ν s are light: $\nu_{1,2,3}$, $m_{1,2,3} \lesssim 0.5$ eV;

KATRIN: $m_{\bar{\nu}_e} < 0.45$ eV;

Cosmology: $\sum_j m_j < 0.064 - 0.616$ eV (95% CL; 2503.07752).

The value of $\min(m_j)$ and “mass ordering” unknown.

Δm_{21}^2 , $|\Delta m_{31}^2|$ - known ($\text{sgn}(\Delta m_{31}^2)$ - unknown).

ν_j , $m_j \neq 0$: nature - Dirac or Majorana - unknown.

The PMNS matrix U - 3×3 unitary: θ_{12} , θ_{13} , θ_{23} - known; **CPV phases δ , α_{21} , α_{31} - unknown.**

Thus, 5 known + 4 unknown parameters + MO.

“Known” = measured; “unknown” = not measured.

m_e , m_μ , m_τ also known - used as input.

Of fundamental importance are:

- determining the status of CP symmetry in the lepton sector and high precision measurement of δ (T2K, NO ν A T2HK, DUNE); leptonic CPV might be at the origin of matter-antimatter (or baryon) asymmetry of the Universe; critical test of symmetry origin of the ν -mixing pattern.
- the determination of the status of lepton charge conservation and the nature - Dirac or Majorana - of massive neutrinos (which is one of the most challenging and pressing problems in present day elementary particle physics) (LEGEND (GERDA, MJORANA), KamLAND-Zen II, CUORE (CUPID), nEXO (EXO), SNO+, PANDA-X, NEXT, ...);
- determination of the type of spectrum neutrino masses possess, or the “neutrino mass ordering” (T2K + NO ν A; JUNO; ORCA; T2HK+HK(atm.data); DUNE; INO);
- determination of the absolute neutrino mass scale, or $\min(m_j)$ (KATRIN, new ideas; cosmology).
- High precision determination of $\sin^2 \theta_{23}$ (T2HK+HK(atm.data); DUNE) - relevant, e.g., for ν -osc. tomography of the Earth (ORCA), and $\sin^2 \theta_{12}$ (JUNO); both crucial for tests of ideas on origin of the ν -mixing pattern.

The program of research extends beyond 2040.

- Understanding at fundamental level the mechanism giving rise to the ν -masses and mixing and to the L_l -non-conservation. Includes understanding
 - the origin of the observed patterns of ν -mixing and ν -masses ;
 - the physical origin of CPV phases in U_{PMNS} ;
 - Are the observed patterns of ν -mixing and of $\Delta m_{21,31}^2$ related to the existence of a new symmetry?
 - Is there any relations between q -mixing and ν - mixing? Is $\theta_{12} + \theta_c = \pi/4$?
 - Is $\theta_{23} = \pi/4$, or $\theta_{23} > \pi/4$ or else $\theta_{23} < \pi/4$?
 - Is there any correlation between the values of CPV phases and of mixing angles in U_{PMNS} ?
- Progress in the theory of ν -mixing might lead to a better understanding of the origin of the BAU.
 - Can the Majorana and/or Dirac $CPVP$ in U_{PMNS} be the leptogenesis CPV parameters at the origin of BAU?

The Non-Abelian Discrete Symmetry Approach

With the observed pattern of neutrino mixing Nature is sending us a Message. The Message is encoded in the values of the neutrino mixing angles, leptonic CP violation phases and neutrino masses. In my opinion, Nature gave us also a hint what the content of Nature's Message is.

Neutrino Mixing: New Symmetry?

- $\theta_{12} = \theta_{\odot} \cong \frac{\pi}{5.4}, \quad \theta_{23} = \theta_{\text{atm}} \cong \frac{\pi}{4}(\text{?}), \quad \theta_{13} \cong \frac{\pi}{20}$

$$U_{\text{PMNS}} \cong \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & \epsilon \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}}(\text{?}) \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}}(\text{?}) \end{pmatrix};$$

Very different from the CKM-matrix!

- $\theta_{12} \cong \sin^{-1} \frac{1}{\sqrt{3}} (= \frac{\pi}{5}) - 0.020; \quad \theta_{12} \cong \pi/4 - 0.20,$
 $\theta_{13} \cong 0 + \pi/20, \quad \theta_{23} \cong \pi/4 \mp 0.10.$
- U_{PMNS} due to new approximate symmetry?

A Natural Possibility: $U_{\text{PMNS}} = U_l^\dagger U_\nu$

$$U_{\text{PMNS}} = U_l^\dagger(\theta_{ij}^\ell, \delta^\ell) \mathbf{Q}(\psi, \omega) U_{\text{TBM, BM, LC, ...}} \bar{P}(\xi_1, \xi_2),$$

with

$$U_{\text{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}; \quad U_{\text{BM}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \pm \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \pm \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \mp \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

- $U_{\text{lep}}^\dagger(\theta_{ij}^\ell, \delta^\ell)$ - from diagonalization of the l^- mass matrix;
- $U_{\text{TBM, BM, LC, ...}} \bar{P}(\xi_1, \xi_2)$ - from diagonalization of the ν mass matrix;
- $\mathbf{Q}(\psi, \omega)$, - from diagonalization of the l^- and/or ν mass matrices.

P. Frampton, STP, W. Rodejohann, 2003

$U_{\text{LC}}, U_{\text{GRAM}}, U_{\text{GRBM}}, U_{\text{HGM}}$:

$$U_{\text{LC}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{c_{23}^\nu}{\sqrt{2}} & \frac{c_{23}^\nu}{\sqrt{2}} & s_{23}^\nu \\ \frac{s_{23}^\nu}{\sqrt{2}} & -\frac{s_{23}^\nu}{\sqrt{2}} & c_{23}^\nu \end{pmatrix}; \quad \mu - \tau \text{ symmetry : } \theta_{23}^\nu = \mp \pi/4;$$

$$U_{\text{GR}} = \begin{pmatrix} c_{12}^\nu & s_{12}^\nu & 0 \\ -\frac{s_{12}^\nu}{\sqrt{2}} & \frac{c_{12}^\nu}{\sqrt{2}} & -\sqrt{\frac{1}{2}} \\ -\frac{s_{12}^\nu}{\sqrt{2}} & \frac{c_{12}^\nu}{\sqrt{2}} & \sqrt{\frac{1}{2}} \end{pmatrix}; \quad U_{\text{HGM}} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \theta_{12}^\nu = \pi/6.$$

U_{GRAM} : $\sin^2 \theta_{12}^\nu = (2 + r)^{-1} \cong 0.276$, $r = (1 + \sqrt{5})/2$

(GR: $r/1$; $a/b = a + b/a$, $a > b$)

U_{GRBM} : $\sin^2 \theta_{12}^\nu = (3 - r)/4 \cong 0.345$.

GRB and HG mixing: W. Rodejohann et al., 2009.

$U_{\text{TBM(BM)}} \dots$: Groups A_4 , T' , S_4 (S_4), ... (vast literature)

E. Ma, G. Rajasekaran, hep-ph/0106291; K. Babu, E. Ma, J.F.W. Valle, hep-ph/0206292;
G. Altarelli, F. Feruglio, hep-ph/0512103; C.S. Lam, 0708.3665 and 0804.2622; W. Grimus,
L. Lavoura, 0809.0226; Z.-Z. Xing, 1106.3244; S. Zhou, 1205.0761; F. Feruglio, C. Hagedorn,
R. Ziegler, 1211.5560; M. Holthausen, M. Lindner, M.A. Schmidt, 1211.6953; A. Meroni,
S.T.P., M. Spinrath, 1312.1966; S.T.P., 1405.6006; ...
(Reviews: G. Altarelli, F. Feruglio, arXiv:1002.0211; M. Tanimoto et al., arXiv:1003.3552;
S. King, Ch. Luhn, arXiv:1301.1340;...)

• U_{GRA} : Group $A_5, \dots$; $s_{13}^2 = 0$ and possibly $s_{12}^2 = 0.276$
and $s_{23}^2 = 1/2$ must be corrected.

L. Everett, A. Stuart, arXiv:0812.1057;...

• U_{LC} : alternatively $U(1)$, $L' = L_e - L_\mu - L_\tau$

S.T.P., 1982

• U_{LC} : $s_{12}^2 = 1/2$, $s_{13}^2 = 0$, s_{23}^ν - free parameter;
 $s_{13}^2 = 0$ and $s_{12}^2 = 1/2$ must be corrected.

• U_{GRB} : Group $D_{10}, \dots$; $s_{13}^2 = 0$ and possibly $s_{12}^2 = 0.345$ and $s_{23}^2 = 1/2$ must be corrected.

• U_{HG} : Group $D_{12}, \dots$; $s_{13}^2 = 0$, $s_{12}^2 = 0.25$ and possibly $s_{23}^2 = 1/2$ must be corrected.

For all symmetry forms considered we have: $\theta_{13}^\nu = 0$, $\theta_{23}^\nu = \mp\pi/4$.

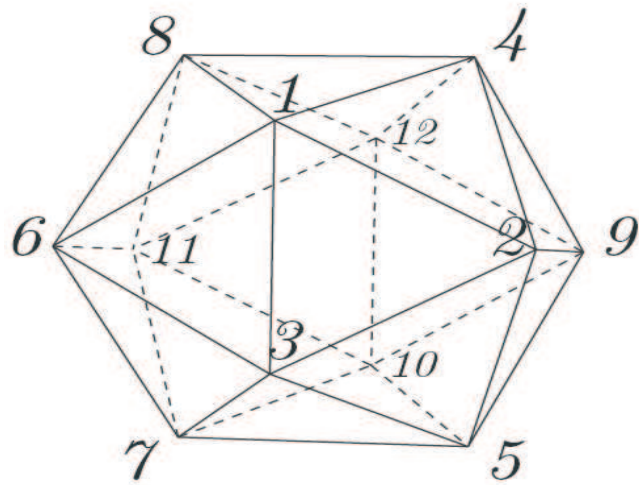
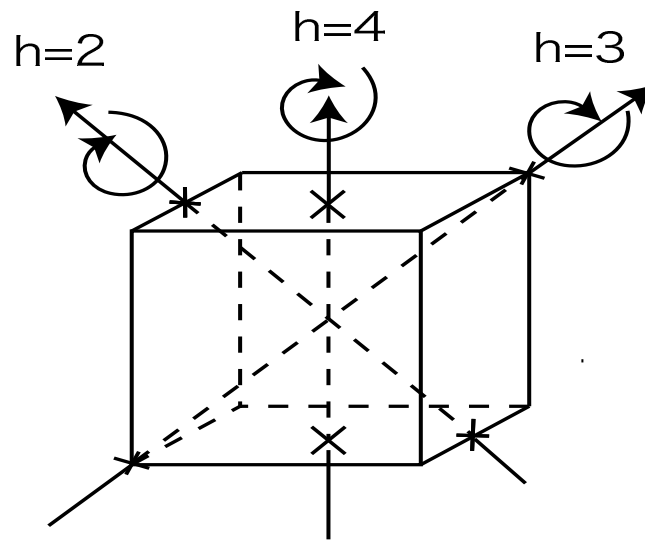
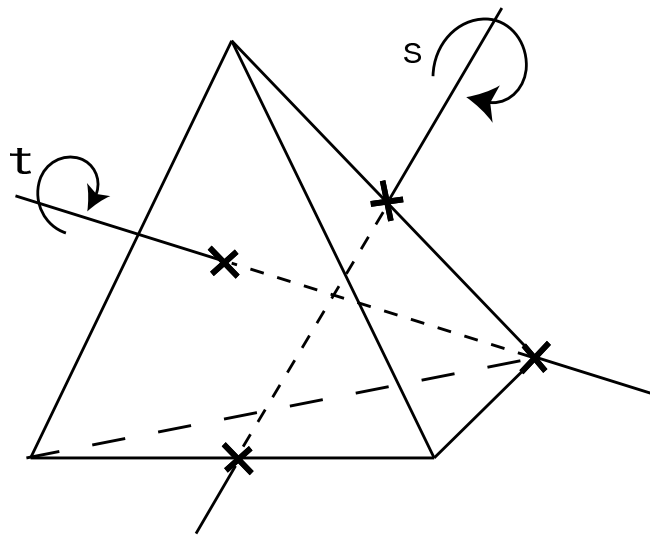
They differ by the value of θ_{12}^ν :

TBM, BM, GRA, GRB and HG forms correspond to $\sin^2 \theta_{12}^\nu = 1/3; 0.5; 0.276; 0.345; 0.25$.

The observed pattern of 3- ν mixing, two large and one small mixing angles,
 $\theta_{12} \cong 33^\circ$, $\theta_{23} \cong 45^\circ \pm 6^\circ$ and $\theta_{13} \cong 8.4^\circ$,
can most naturally be explained by extending the
Standard Model (SM) with a flavour symmetry cor-
responding to a non-Abelian discrete (finite) group
 G_f .

$$G_f = A_4, T', S_4, A_5, D_{10}, D_{12}, \dots$$

Vast literature; reviews: G. Altarelli, F. Feruglio, 1002.0211; H. Ishimori et al., 1003.3552; M. Tanimoto, AIP Conf.Proc. 1666 (2015) 120002; S. King, Ch. Luhn, 1301.1340; D. Meloni, 1709.02662; STP, 1711.10806; F. Feruglio, A. Romanino, 1912.06028



Examples of symmetries: A_4 , S_4 , A_5

From M. Tanimoto et al., arXiv:1003.3552

Group	Number of elements	Generators	Irreducible representations
S_4	24	$S, T \ (U)$	$1, 1', 2, 3, 3'$
S'_4	48	$S, T \ (R)$	$1, 1', 2, 3, 3', \hat{1}, \hat{1}', \hat{2}, \hat{3}, \hat{3}'$
A_4	12	S, T	$1, 1', 1'', 3$
T'	24	$S, T \ (R)$	$1, 1', 1'', 2, 2', 2'', 3$
A_5	60	$\tilde{S}, \tilde{T}$	$1, 3, 3', 4, 5$
A'_5	120	$\tilde{S}, \tilde{T}$	$1, 3, 3', 4, 5, \hat{2}, \hat{2}', \hat{4}, \hat{6}.$

Number of elements, generators and irreducible representations of S_4 , S'_4 , A_4 , $A'_4 \equiv T'$, A_5 and A'_5 discrete groups.

Predictions and Correlations

$$U_\nu = U_{\text{TBM,BM,GRA,GRB,HG}} \bar{P}(\xi_1, \xi_2); \quad \theta_{12}^\nu;$$

$$U_\ell^\dagger = R_{12}(\theta_{12}^\ell) Q, \quad Q = \text{diag}(e^{i\varphi}, 1, 1); \quad \theta_{12}^\ell, \varphi$$

(the “minimal” = simplest case ($SU(5) \times T', \dots$))

$$U_\ell^\dagger = R_{12}(\theta_{12}^\ell) R_{23}(\theta_{23}^\ell) Q, \quad Q = \text{diag}(1, e^{-i\psi}, e^{-i\omega}),$$

(next-to-minimal case): $\theta_{12}^\ell, \hat{\theta}_{23}^\ell, \phi$

$$\theta_{12}, \theta_{23}, \theta_{13}, \delta \text{ in terms of } \theta_{12}^\ell, \hat{\theta}_{23}^\ell, \phi + \theta_{12}^\nu$$

$$\cos \delta = \cos \delta(\theta_{12}, \theta_{23}, \theta_{13}; \theta_{12}^\nu, \dots),$$

$$J_{CP} = J_{CP}(\theta_{12}, \theta_{23}, \theta_{13}, \delta) = J_{CP}(\theta_{12}, \theta_{23}, \theta_{13}; \theta_{12}^\nu, \dots),$$

$\theta_{12}^\nu, \dots$ - known (fixed) parameters, depend on the underlying symmetry.

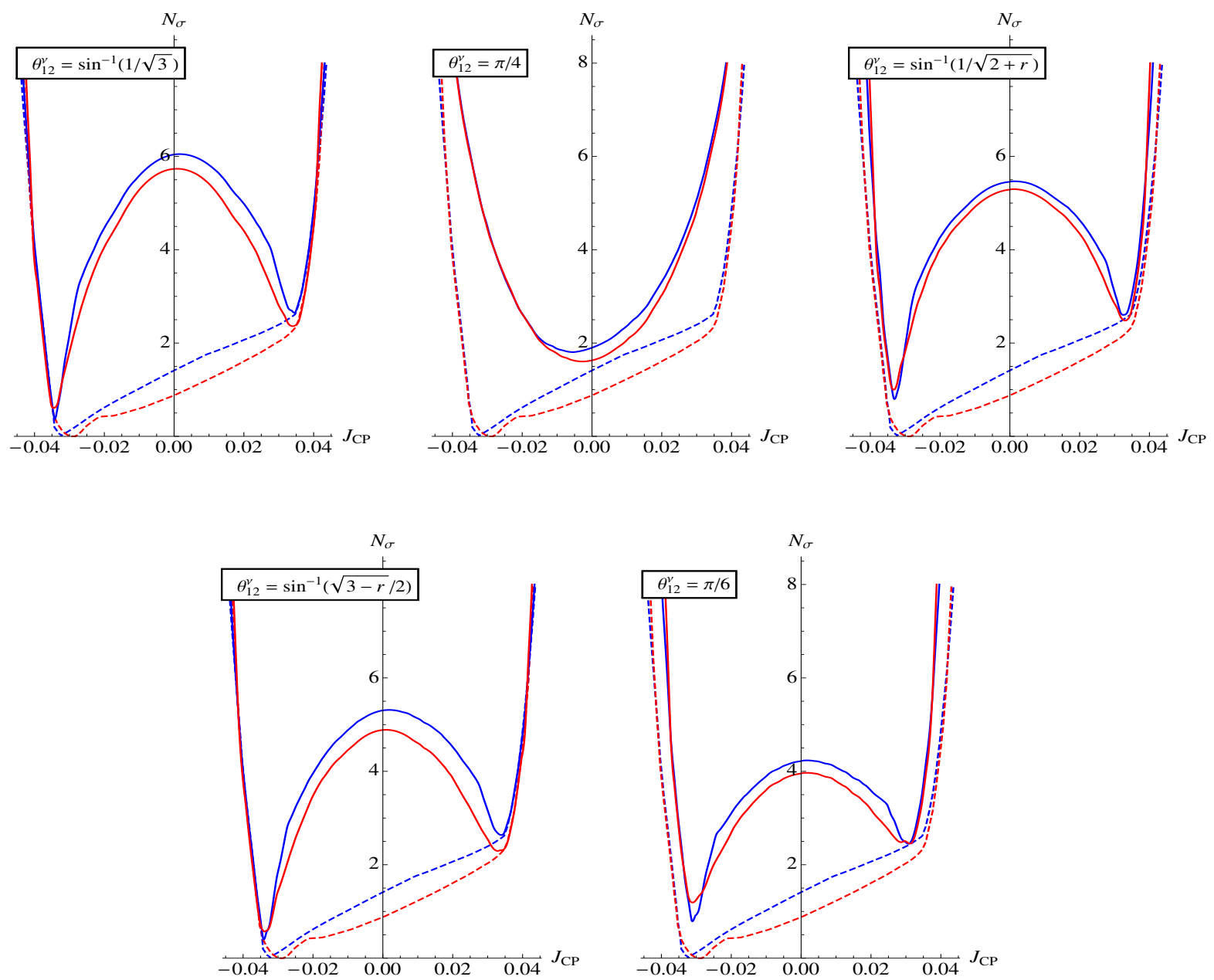
For arbitrary fixed θ_{12}^ν and any θ_{23}
(“minimal” and “next-to-minimal” cases):

$$\cos \delta = \frac{\tan \theta_{23}}{\sin 2\theta_{12} \sin \theta_{13}} \left[\cos 2\theta_{12}^\nu + (\sin^2 \theta_{12} - \cos^2 \theta_{12}^\nu) (1 - \cot^2 \theta_{23} \sin^2 \theta_{13}) \right] .$$

S.T.P., arXiv:1405.6006

This results is exact.

“Minimal” case: $\sin^2 \theta_{23} = \frac{1}{2} \frac{1 - 2 \sin^2 \theta_{13}}{1 - \sin^2 \theta_{13}} .$



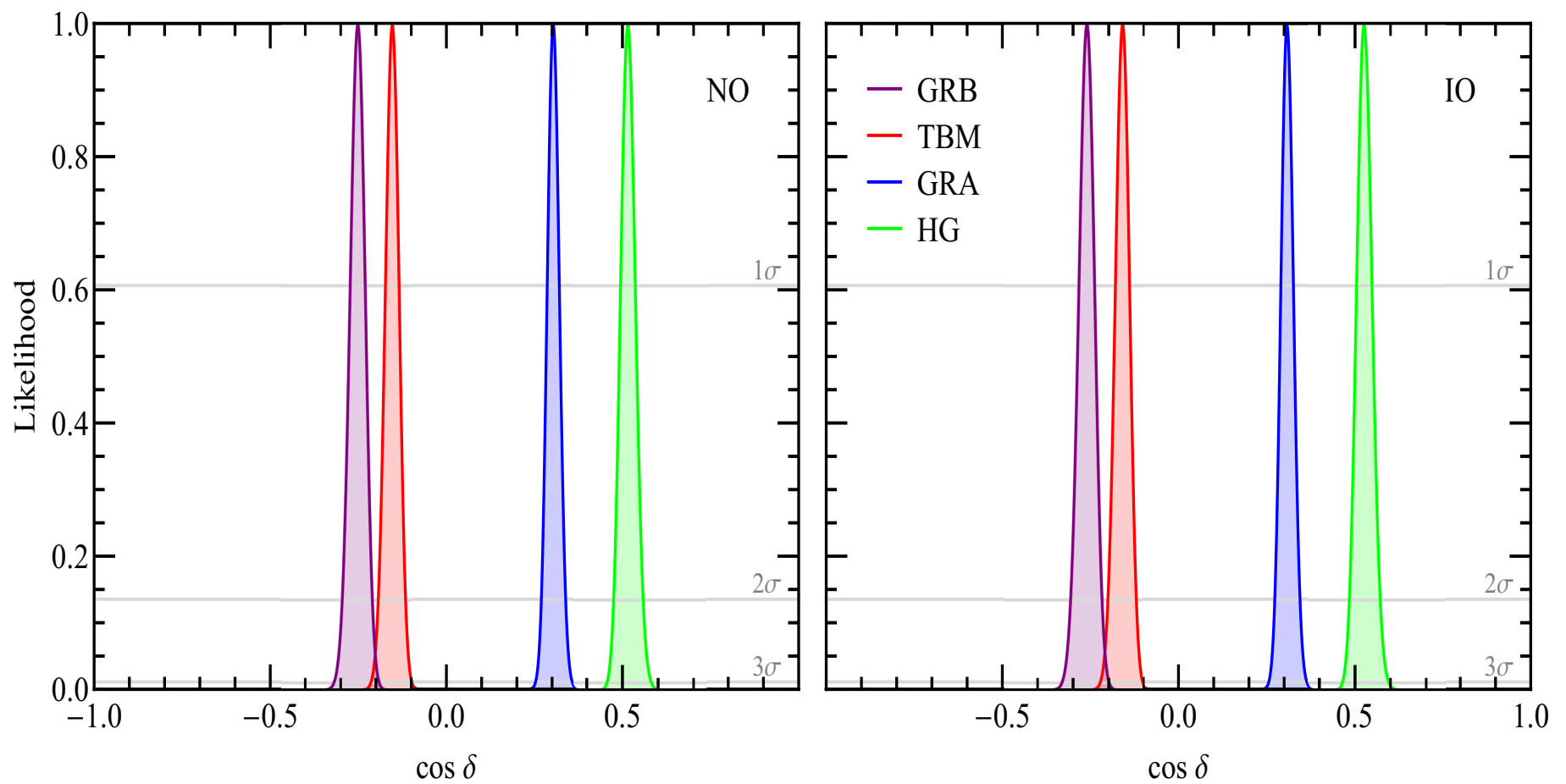
I. Girardi, S.T.P., A. Titov

Prospective precision:

$$\delta(\sin^2 \theta_{12}) = 0.7\% \text{ (JUNO)},$$

$$\delta(\sin^2 \theta_{13}) = 3\% \text{ (Daya Bay)},$$

$$\delta(\sin^2 \theta_{23}) = 3\% \text{ (T2K, NO}\nu\text{A combined)}.$$

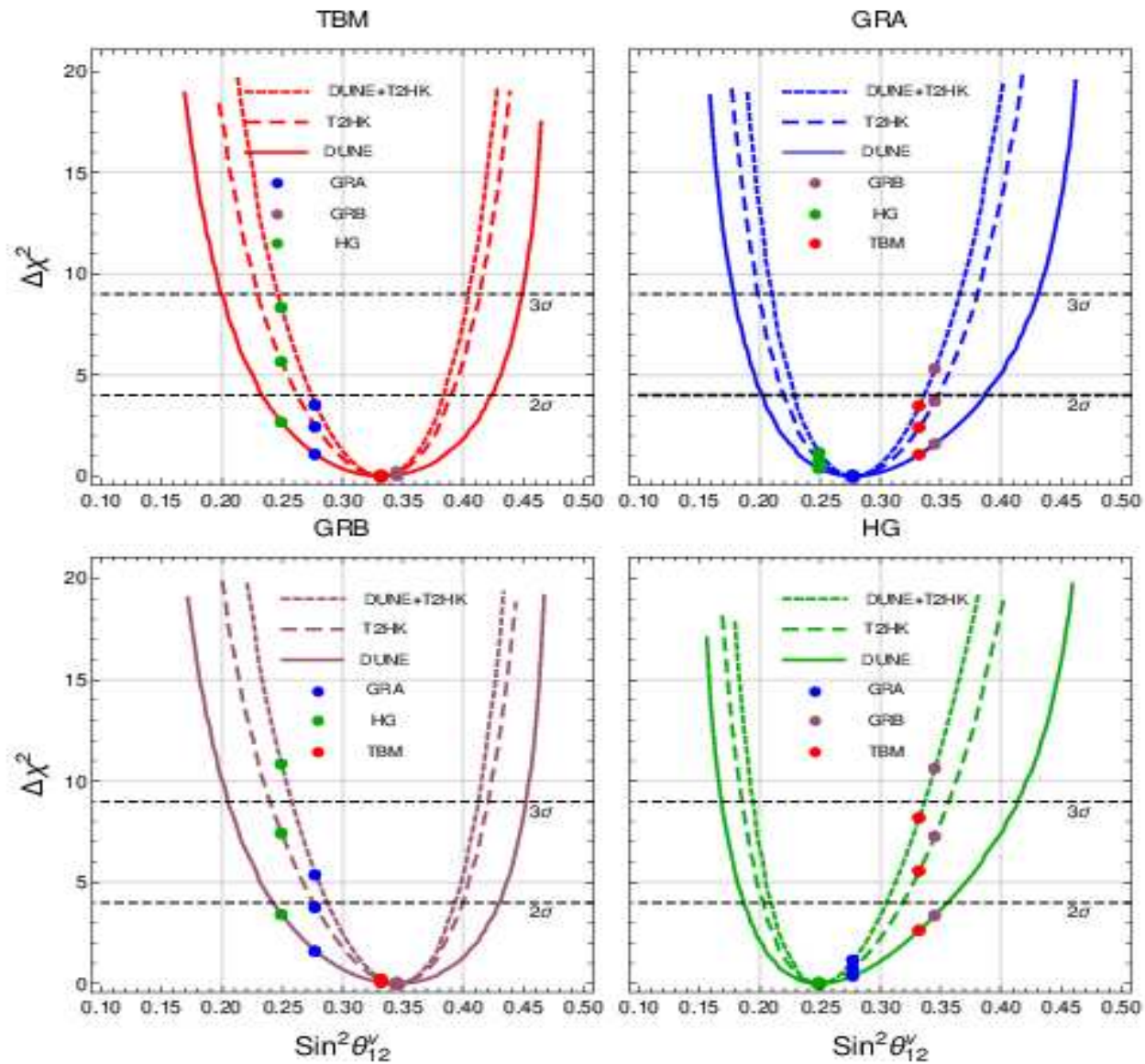


I. Girardi, S.T.P., A. Titov

b.f.v. of $\sin^2 \theta_{ij}$ (Esteban et al., Jan., 2018) + the prospective precision used.

$$\cos \delta = \frac{\tan \theta_{23}}{\sin 2\theta_{12} \sin \theta_{13}} \left[\cos 2\theta_{12}^\nu + (\sin^2 \theta_{12} - \cos^2 \theta_{12}^\nu) (1 - \cot^2 \theta_{23} \sin^2 \theta_{13}) \right].$$

$\delta(\sin^2 \theta_{23}) = 3\%$ (T2HK, DUNE).



Agarwalla, Chatterjee, STP, Titov, arXiv:1711.02107

**GRB - HG $> 3\sigma$; GRA - GRB $\geq 2\sigma$; TBM - HG $\cong 3\sigma$; TBM - GRA $\cong 2\sigma$.
DUNE + T2HK prospective data used.**

S.T. Petcov, NOPP, IHEP, Beijing, China, 14-16/08/2026

How does it Work.

Choose G_f .

$\nu_{eL}(x), \nu_{\mu L}(x), \nu_{\tau L}(x)$: assigned to $\rho^{(\nu)}(g_f)$ - irreducible representation of G_f , where g_f is an element of G_f .

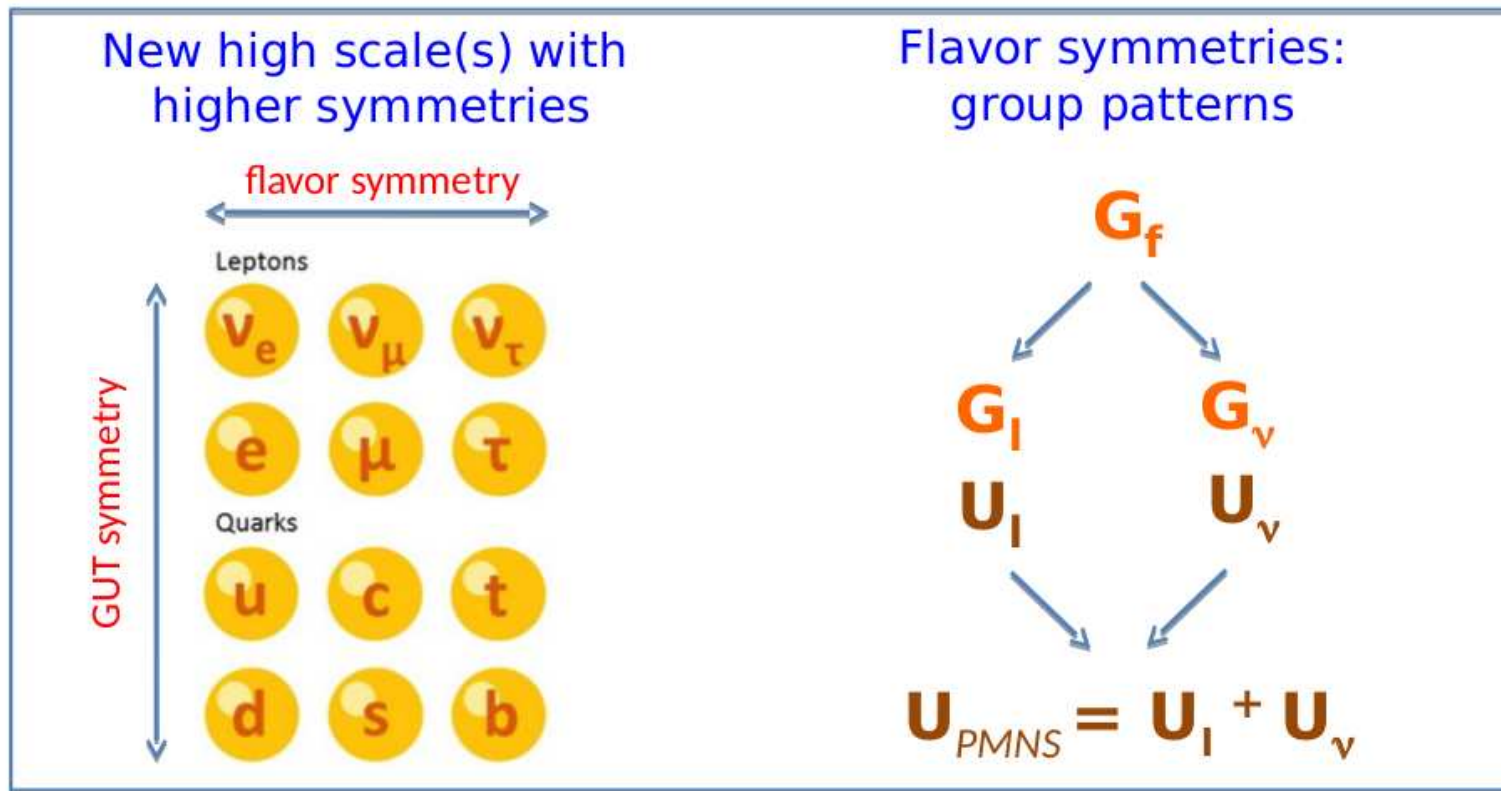
$e_L(x), \mu_L(x), \tau_L(x)$: assigned to $\rho^{(e)}(g_f)$ - IRREP of G_f .

$G_f = S_4, A_4, T', A_5$: $\rho^{(\nu)}(g_f), \rho^{(e)}(g_f)$ - **triplet IRREP**.

$e_R(x), \mu_R(x), \tau_R(x)$: singlets of G_f .

How Does it Work

Model building with symmetries



ν_j , Majorana mass term, $m_j \neq m_k$, $j \neq k = 1, 2, 3$: $G_\nu = Z_2 \times Z_2$, Z_2 E. Lisi, TAUP 2019

$G_e = Z_2$; Z_n , $n > 2$; $Z_n \times Z_m$, $n, m \geq 2$

M_e - charged lepton mass matrix (L-R convention).

$$U_e: U_e^\dagger M_e M_e^\dagger U_e = \text{diag}(m_e^2, m_\mu^2, m_\tau^2).$$

G_e - residual symmetry group of $M_e M_e^\dagger$:

$$\rho^{(e)}(g_e)^\dagger M_e M_e^\dagger \rho(g_e) = M_e M_e^\dagger,$$

$\rho^{(e)}(g_e)$ generator(s) of G_e in the triplet rep.

$\rho^{(e)}(g_e)$ and $M_e M_e^\dagger$ commute: both are diagonalised by U_e .

$\rho^{(e)}(g_e)$ - known! Thus, U_e - fixed!

M_ν - neutrino Majorana mass matrix (R-L convention).

$$U_\nu: U_\nu^T M_\nu U_\nu = \text{diag}(m_1, m_2, m_3).$$

G_ν - residual symmetry group of M_ν :

$$\rho(g_\nu)^T M_\nu \rho(g_\nu) = M_\nu,$$

g_ν : an element of G_ν ,

$\rho(g_\nu)$ generator of G_ν in the triplet repr.

$\rho(g_\nu)$ and $M_\nu^\dagger M_\nu$ commute: both are diagonalised by U_ν .

$\rho(g_\nu)$ - known! Thus, U_ν -fixed.

$$U_{\text{PMNS}} = U_e^\dagger U_\nu$$

$$A_4: G_e = Z_3^T = \{1, T, T^2\}, G_\nu = Z_2^S = \{1, S\}$$

$$(S^2 = T^3 = (ST)^3 = \mathbf{I})$$

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \omega = e^{i2\pi\tau/3} \quad (\text{A} - \text{F}).$$

$$U_e = \mathbf{I}, U_{\text{PMNS}} = U_e^\dagger U_\nu = U_{\text{TBM}} U_{13}(\theta_{13}^\nu, \alpha), \theta_{13}^\nu, \alpha - \text{free.}$$

W. Grimus, L. Lavoura, 2008

$$\sin^2 \theta_{12} = \frac{1}{3(1 - \sin^2 \theta_{13})} \cong 0.34;$$

$$\cos \delta = \frac{\cos 2\theta_{23} \cos 2\theta_{13}}{\sin 2\theta_{23} \sin \theta_{13} (2 - 3 \sin^2 \theta_{13})^{\frac{1}{2}}}; \text{ if } \theta_{23} = \frac{\pi}{4}, \delta = \pm \frac{\pi}{2}.$$

The Symmetry Breaking

The correct lepton mixing pattern in a model with non-Abelian discrete symmetry G_f is determined by the appropriate choice of residual symmetries G_e and G_ν and is not directly related to the charged lepton and neutrino mass generation.

The breaking of G_f has to ensure the correct generation of the fermion masses and keep G_e and G_ν intact.

Examples of Predictions and Correlations II.

- $\sin^2 \theta_{23} = \frac{1}{2}$.
- $\sin^2 \theta_{23} \cong \frac{1}{2} (1 \mp \sin^2 \theta_{13}) + O(\sin^4 \theta_{13}) \cong \frac{1}{2} (1 \mp 0.022)$.
- $\sin^2 \theta_{23} = 0.455; 0.463; 0.537; 0.545; 0.604$.
- $\sin^2 \theta_{12} \cong \frac{1}{3} (1 + \sin^2 \theta_{13}) + O(\sin^4 \theta_{13}) \cong 0.340$.
- $\sin^2 \theta_{12} \cong \frac{1}{3} (1 - 2 \sin^2 \theta_{13}) + O(\sin^4 \theta_{13}) \cong 0.319$.
- **and/or** $\cos \delta = \cos \delta(\theta_{12}, \theta_{23}, \theta_{13}; \theta_{12}^\nu, \dots)$,

$$J_{\text{CP}} = J_{\text{CP}}(\theta_{12}, \theta_{23}, \theta_{13}, \delta) = J_{\text{CP}}(\theta_{12}, \theta_{23}, \theta_{13}; \theta_{12}^\nu, \dots),$$

$\theta_{12}^\nu, \dots$ - known (fixed) parameters, depend on the underlying symmetry.

The Approach is testable/falsifiable experimentally!

The measurement of the Dirac phase in the PMNS mixing matrix, together with an improvement of the precision on the mixing angles θ_{12} , θ_{13} and θ_{23} , can provide unique information about the possible existence of new fundamental symmetry in the lepton sector.

Prospective (useful/requested) precision:

$$\delta(\sin^2 \theta_{12}) = 0.7\% \text{ (JUNO),}$$

$$\delta(\sin^2 \theta_{13}) = 3\% \text{ (Daya Bay),}$$

$$\delta(\sin^2 \theta_{23}) = 3\% \text{ (T2HK, DUNE; T2K+NO}\nu\text{A(?)).}$$

$$\delta(\delta) \leq 14^\circ \text{ at } \delta = 3\pi/2 \text{ } (\delta(\delta) = 10^\circ)$$

(ESS ν SB: $\sim 8\%$, A. Alekou et al., EPJ ST 231 (2022) 379; THKK?; DUNE: accounting for both the 1st and 2nd probability maxima, Jogesh Rout, Poonam Mehta et al., PRD 2021, S. Goswami et al., 2012.04958)

The Power of Data

Systematic analysis (I. Girardi *et al.*):
all possible combinations of residual symmetries G_e and G_ν of the lepton flavour symmetry groups $G_f = S_4, A_4, T'$ and A_5 , leading to correlations between some of the three neutrino mixing angles and/or between the neutrino mixing angles and the Dirac CPV phase δ , were considered.

- (A)** $G_e = Z_2$ and $G_\nu = Z_k, k > 2$ or $Z_m \times Z_n, m, n \geq 2$;
- (B)** $G_e = Z_k, k > 2$ or $Z_m \times Z_n, m, n \geq 2$ and $G_\nu = Z_2$;
- (C)** $G_e = Z_2$ and $G_\nu = Z_2$.

In these cases $U_e^\dagger$ and/or U_ν of $U = U_e^\dagger U_\nu = (\tilde{U}_e)^\dagger \psi \tilde{U}_\nu Q_0$, are partially (or fully) determined by residual discrete symmetries of $G_f = S_4, A_4, T', A_5$.

$$G_f = A_4, S_4, T', A_5.$$

A_4 : 3 Z_2 , 4 Z_3 , 1 $Z_2 \times Z_2$ subgroups (total 8).

T' : similar to A_4 .

S_4 : 9 Z_2 , 4 Z_3 , 3 Z_4 , 4 $Z_2 \times Z_2$ subgroups (total 20).

A_5 : has 15 Z_2 , 10 Z_3 , 6 Z_5 , 5 $Z_2 \times Z_2$ subgroups (36).

In the case of A_4 (T') symmetry only there are 64 models (up to permutation of rows and columns). Of these only 8 lead to distinct predictions for U_{PMNS} , while only 5 cases a priori can be phenomenologically viable, i.e., they lead to U_{PMNS} without zero entries. Similar analyses can be performed for the S_4 and A_5 symmetries.

A_4 :

$$(G_e, G_\nu) = (Z_2, Z_3), \text{ A1 - A3;}$$

$$(G_e, G_\nu) = (Z_2, Z_2), \text{ A1 - A3;}$$

$$(G_e, G_\nu) = (Z_3, Z_2), \text{ B1 - B3;}$$

$$(G_e, G_\nu) = (Z_2 \times Z_2, Z_2), \text{ B1 - B3;}$$

$$(G_e, G_\nu) = (Z_2, Z_2), \text{ C1 - C9.}$$

For A_4 , S_4 and A_5 the total number of models to be analysed is extremely large. However, a total of only 14 models survive the 3σ constraints on $\sin^2 \theta_{ij}$ from the current data and the requirement $|\cos \delta| \leq 1$.

More specifically:

A. $G_e = Z_2$, $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$;
 U_ν fixed; $U_e = U_{ij}(\theta_{ij}^e, \delta_{ij})$, $ij = 12(A1); 13(A2), 23(A3)$;
 θ_{23} , $\cos \delta$ (θ_{12} , θ_{13}) predicted.

B. $G_e = Z_n$, $n > 2$ or $G_e = Z_n \times Z_m$, $n, m \geq 2$, $G_\nu = Z_2$;
 $U_\nu = U_{\text{sym}} U_{ij}(\theta_{ij}^\nu, \delta_{ij}) Q_0$, $ij = 13, 12, 23(\mathbf{B1}, \mathbf{B2}, \mathbf{B3})$;
 U_e fixed: θ_{12} , $\cos \delta$ (θ_{23} , θ_{13}) predicted.

C. $G_e = Z_2$ and $G_\nu = Z_2$: U_e - up to $U_{ij}(\theta_{ij}^e, \delta_{ij}^\varepsilon)$,
 U_ν - up to $U_{ij}(\theta_{ij}^\nu, \delta_{ij}^\nu)$, $ij = 12; 13, 23$ (**C1 - C9**);
 θ_{12} or θ_{23} or $\cos \delta$ predicted.

For A_4 , S_4 and A_5 the total number of models to be analysed is extremely large. However, a total of only 14 models survive the 3σ constraints on $\sin^2 \theta_{ij}$ from the current data and the requirement $|\cos \delta| \leq 1$.

Pattern A: $G_e = Z_2$ and $G_\nu = Z_k$, $k > 2$ or $Z_m \times Z_n$, $m, n \geq 2$.

Cases A1-A2, θ_{23}, δ predicted.

Case A1 (U_e up to U in 1-2 plane)

$$\sin^2 \theta_{23} = 1 - \frac{\cos^2 \theta_{13}^\circ \cos^2 \theta_{23}^\circ}{1 - \sin^2 \theta_{13}},$$

$$\cos \delta = \frac{\cos^2 \theta_{13} (\sin^2 \theta_{23}^\circ - \cos^2 \theta_{12}) + \cos^2 \theta_{13}^\circ \cos^2 \theta_{23}^\circ (\cos^2 \theta_{12} - \sin^2 \theta_{12} \sin^2 \theta_{13})}{\sin 2\theta_{12} \sin \theta_{13} |\cos \theta_{13}^\circ \cos \theta_{23}^\circ| (\cos^2 \theta_{13} - \cos^2 \theta_{13}^\circ \cos^2 \theta_{23}^\circ)^{\frac{1}{2}}},$$

where θ_{13}° and θ_{23}° are fixed once the flavour symmetry group G_f and the residual symmetry subgroups G_e and G_ν are specified.

Case A2 (U_e up to U in 1-3 plane)

$$\sin^2 \theta_{23} = \frac{\sin^2 \theta_{23}^\circ}{1 - \sin^2 \theta_{13}},$$

$$\cos \delta = -\frac{\cos^2 \theta_{13} (\cos^2 \theta_{12}^\circ \cos^2 \theta_{23}^\circ - \cos^2 \theta_{12}) + \sin^2 \theta_{23}^\circ (\cos^2 \theta_{12} - \sin^2 \theta_{12} \sin^2 \theta_{13})}{\sin 2\theta_{12} \sin \theta_{13} |\sin \theta_{23}^\circ| (\cos^2 \theta_{13} - \sin^2 \theta_{23}^\circ)^{\frac{1}{2}}},$$

where θ_{12}° and θ_{23}° are fixed once G_f , G_e and G_ν are specified.

Case A3 (U_e up to U in 2-3 plane):

$\sin^2 \theta_{13} = \sin^2 \theta_{13}^\circ$ and $\sin^2 \theta_{12} = \sin^2 \theta_{12}^\circ$ are predicted, while $\cos \delta$ remains unconstrained.

Pattern B: $G_e = Z_k$, $k > 2$ or $Z_m \times Z_n$, $m, n \geq 2$ and $G_\nu = Z_2$.

Cases B1-B2, θ_{12}, δ predicted.

Case B1 (U_ν up to U in 1-3 plane)

$$\sin^2 \theta_{12} = \frac{\sin^2 \theta_{12}^\circ}{1 - \sin^2 \theta_{13}},$$

$$\cos \delta = -\frac{\cos^2 \theta_{13}(\cos^2 \theta_{12}^\circ \cos^2 \theta_{23}^\circ - \cos^2 \theta_{23}) + \sin^2 \theta_{12}^\circ(\cos^2 \theta_{23} - \sin^2 \theta_{13} \sin^2 \theta_{23})}{\sin 2\theta_{23} \sin \theta_{13} |\sin \theta_{12}^\circ| (\cos^2 \theta_{13} - \sin^2 \theta_{12}^\circ)^{\frac{1}{2}}},$$

where θ_{12}° and θ_{23}° are fixed once the symmetries are specified.

Case B2 (U_ν up to U in the 2-3 plane)

$$\sin^2 \theta_{12} = 1 - \frac{\cos^2 \theta_{12}^\circ \cos^2 \theta_{13}^\circ}{1 - \sin^2 \theta_{13}},$$

$$\cos \delta = \frac{\cos^2 \theta_{13}(\sin^2 \theta_{12}^\circ - \cos^2 \theta_{23}) + \cos^2 \theta_{12}^\circ \cos^2 \theta_{13}^\circ(\cos^2 \theta_{23} - \sin^2 \theta_{13} \sin^2 \theta_{23})}{\sin 2\theta_{23} \sin \theta_{13} |\cos \theta_{12}^\circ \cos \theta_{13}^\circ| (\cos^2 \theta_{13} - \cos^2 \theta_{12}^\circ \cos^2 \theta_{13}^\circ)^{\frac{1}{2}}}.$$

Case B3 (U_ν up to U in the 1-2 plane)

$\sin^2 \theta_{13} = \sin^2 \theta_{13}^\circ$ and $\sin^2 \theta_{23} = \sin^2 \theta_{23}^\circ$ are predicted; and no sum rule for $\cos \delta$.

Pattern C: $G_e = Z_2$ and $G_\nu = Z_2$. In this case, both U_e and U_ν are determined up to $U(2)$ transformations in the i - j and k - l planes, respectively. Depending on the planes in which the free $U(2)$ transformations are performed, we have nine possibilities. We number them as cases C1–C9. Four of them lead to sum rules for $\cos \delta$, which we summarise below.

$$\begin{aligned}
 \text{C1, } (ij, kl) = (12, 13): \quad \cos \delta &= \frac{\sin^2 \theta_{23}^\circ - \cos^2 \theta_{12} \sin^2 \theta_{23} - \cos^2 \theta_{23} \sin^2 \theta_{12} \sin^2 \theta_{13}}{\sin \theta_{13} \sin 2\theta_{23} \sin \theta_{12} \cos \theta_{12}}, \\
 \text{C3, } (ij, kl) = (12, 23): \quad \cos \delta &= \frac{\sin^2 \theta_{12} \sin^2 \theta_{23} - \sin^2 \theta_{13}^\circ + \cos^2 \theta_{12} \cos^2 \theta_{23} \sin^2 \theta_{13}}{\sin \theta_{13} \sin 2\theta_{23} \sin \theta_{12} \cos \theta_{12}}, \\
 \text{C4, } (ij, kl) = (13, 23): \quad \cos \delta &= \frac{\sin^2 \theta_{12}^\circ - \cos^2 \theta_{23} \sin^2 \theta_{12} - \cos^2 \theta_{12} \sin^2 \theta_{13} \sin^2 \theta_{23}}{\sin \theta_{13} \sin 2\theta_{23} \sin \theta_{12} \cos \theta_{12}}, \\
 \text{C8, } (ij, kl) = (13, 13): \quad \cos \delta &= \frac{\cos^2 \theta_{12} \cos^2 \theta_{23} - \cos^2 \theta_{23}^\circ + \sin^2 \theta_{12} \sin^2 \theta_{23} \sin^2 \theta_{13}}{\sin \theta_{13} \sin 2\theta_{23} \sin \theta_{12} \cos \theta_{12}}.
 \end{aligned}$$

The two cases, C5 and C9, yield correlations between $\sin^2 \theta_{12}$ and $\sin^2 \theta_{13}$:

$$\text{C5, } (ij, kl) = (23, 13): \quad \sin^2 \theta_{12} = \frac{\sin^2 \theta_{12}^\circ}{1 - \sin^2 \theta_{13}},$$

$$\text{C9, } (ij, kl) = (23, 23): \quad \sin^2 \theta_{12} = \frac{\sin^2 \theta_{12}^\circ - \sin^2 \theta_{13}}{1 - \sin^2 \theta_{13}}.$$

In cases C2 and C7, instead, there are correlations between $\sin^2 \theta_{23}$ and $\sin^2 \theta_{13}$:

$$\text{C2, } (ij, kl) = (13, 12): \quad \sin^2 \theta_{23} = \frac{\sin^2 \theta_{23}^\circ}{1 - \sin^2 \theta_{13}},$$

$$\text{C7, } (ij, kl) = (12, 12): \quad \sin^2 \theta_{23} = \frac{\sin^2 \theta_{23}^\circ - \sin^2 \theta_{13}}{1 - \sin^2 \theta_{13}}.$$

Finally, in case C6, $(ij, kl) = (23, 12)$, $\sin^2 \theta_{13}$ is predicted to be equal to $\sin^2 \theta_{13}^\circ$.

In cases C2, C5, C6, C7 and C9, $\cos \delta$ remains unconstrained.

$G_f = A_4$: **B1** is the only one phenomenologically viable case with $(G_e, G_\nu) = (Z_3, Z_2)$, which yields $(\sin^2 \theta_{12}^\circ, \sin^2 \theta_{23}^\circ) = (1/3, 1/2)$ and corresponds to the **TBM** mixing matrix corrected from the right by a $U(2)$ transformation in the 1-3 plane;
 $\cos \delta = -0.353$, $\sin^2 \theta_{12} = 0.341$.

In the case of $G_f = S_4$ there are 8 viable cases. We summarise them in following Table.

[t]

(G_e, G_ν)	Case	$\sin^2 \theta_{ij}^\circ$	$\cos \delta$	$\sin^2 \theta_{ij}$
(Z_3, Z_2)	B1	$(\sin^2 \theta_{12}^\circ, \sin^2 \theta_{23}^\circ) = (1/3, 1/2)$	-0.353	$\sin^2 \theta_{12} = 0.341$
	B2S₄	$(\sin^2 \theta_{12}^\circ, \sin^2 \theta_{13}^\circ) = (1/6, 1/5)$	0.167	$\sin^2 \theta_{12} = 0.318$
(Z_2, Z_2)	C1	$\sin^2 \theta_{23}^\circ = 1/4$	-1^*	not fixed
	C2S₄	$\sin^2 \theta_{23}^\circ = 1/2$	not fixed	$\sin^2 \theta_{23} = 0.511$
	C3	$\sin^2 \theta_{13}^\circ = 1/4$	-1^*	not fixed
	C4	$\sin^2 \theta_{12}^\circ = 1/4$	1^*	not fixed
	C7S₄	$\sin^2 \theta_{23}^\circ = 1/2$	not fixed	$\sin^2 \theta_{23} = 0.489$
	C8	$\sin^2 \theta_{23}^\circ = 3/4$	1^*	not fixed

The viable cases for $G_f = S_4$. The values of $\cos \delta$ and $\sin^2 \theta_{12}/\sin^2 \theta_{23}$ are obtained using the best fit values of the relevant (not fixed) mixing angles for NO. **In the cases marked with an asterisk**, physical values of $\cos \delta$ cannot be obtained employing the best fit values of the mixing angles, but they are achievable fixing two angles to their best fit values and varying the third one in its 3σ range.

For $G_f = A_5$, requiring the compatibility with the data in the way explained above, we find 13 viable cases. They are presented in the Table that follows.

(G_e, G_ν)	Case	$\sin^2 \theta_{ij}^\circ$	$\cos \delta$	$\sin^2 \theta_{ij}$
(Z_2, Z_3)	A1A₅	$(\sin^2 \theta_{13}^\circ, \sin^2 \theta_{23}^\circ) = (0.226, 0.436)$	0.727	$\sin^2 \theta_{23} = 0.554$
	A2A₅	$(\sin^2 \theta_{12}^\circ, \sin^2 \theta_{23}^\circ) = (0.226, 0.436)$	−0.727	$\sin^2 \theta_{23} = 0.446$
(Z_3, Z_2)	B1	$(\sin^2 \theta_{12}^\circ, \sin^2 \theta_{23}^\circ) = (1/3, 1/2)$	−0.353	$\sin^2 \theta_{12} = 0.341$
(Z_5, Z_2)	B1A₅	$(\sin^2 \theta_{12}^\circ, \sin^2 \theta_{23}^\circ) = (0.276, 1/2)$	−0.405	$\sin^2 \theta_{12} = 0.283$
$(Z_2 \times Z_2, Z_2)$	B2A₅	$(\sin^2 \theta_{12}^\circ, \sin^2 \theta_{13}^\circ) = (0.095, 0.276)$	−0.936	$\sin^2 \theta_{12} = 0.331$
	B2A₅II	$(\sin^2 \theta_{12}^\circ, \sin^2 \theta_{13}^\circ) = (1/4, 0.127)$	1*	$\sin^2 \theta_{12} = 0.331$
(Z_2, Z_2)	C1	$\sin^2 \theta_{23}^\circ = 1/4$	−1*	not fixed
	C3A₅	$\sin^2 \theta_{13}^\circ = 0.095$	1*	not fixed
	C3	$\sin^2 \theta_{13}^\circ = 1/4$	−1*	not fixed
	C4A₅	$\sin^2 \theta_{12}^\circ = 0.095$	−0.799	not fixed
	C4	$\sin^2 \theta_{12}^\circ = 1/4$	1*	not fixed
	C8	$\sin^2 \theta_{23}^\circ = 3/4$	1*	not fixed
	C9A₅	$\sin^2 \theta_{12}^\circ = 0.345$	not fixed	$\sin^2 \theta_{12} = 0.331$

The exact algebraic forms of the irrational values of $\sin^2 \theta_{ij}^\circ$ in the Table have been found in arXiv:1509.02502 They are related to the golden ratio $r = (1 + \sqrt{5})/2$ as follows: $2/(4r^2 - r) \approx 0.226$, $r/(6r - 6) \approx 0.436$, $1/(2 + r) \approx 0.276$, $1/(4r^2) \approx 0.095$, $1/(3 + 3r) \approx 0.127$, and $(3 - r)/4 \approx 0.345$.

We note that case B1 is common to all the three flavour symmetry groups A_4 , S_4 and A_5 , while cases C1, C3, C4 and C8 are shared by S_4 and A_5 . Thus, we have 16 cases in total, which lead to different predictions for $\sin^2 \theta_{12}$ or $\sin^2 \theta_{23}$ and/or $\cos \delta$. As we will see in the next subsection performing a statistical analysis of these predictions, two cases, namely, C4 and B2A₅II, are globally disfavoured at more than 3σ confidence level.

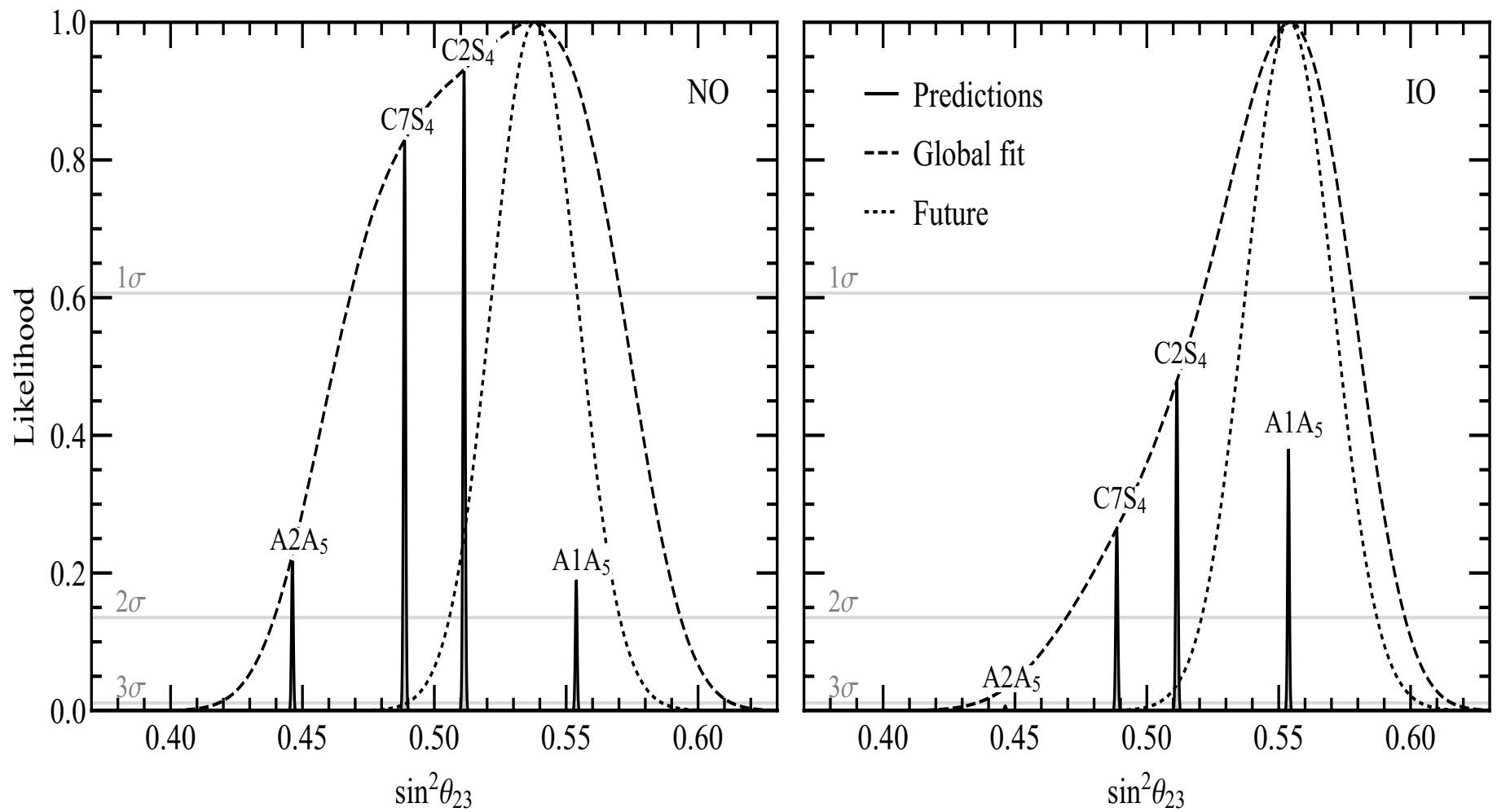
Thus, the total number of phenomenologically viable cases reduces to 14.

Phenomenologically Viable Predictions

A1 (A2), A_5 ($G_e = Z_2$, $G_\nu = Z_3$ (**Dirac** ν_j)):
 $\sin^2 \theta_{23} \cong 0.553$ (0.447); $\cos \delta \cong 0.716$ (-0.716).

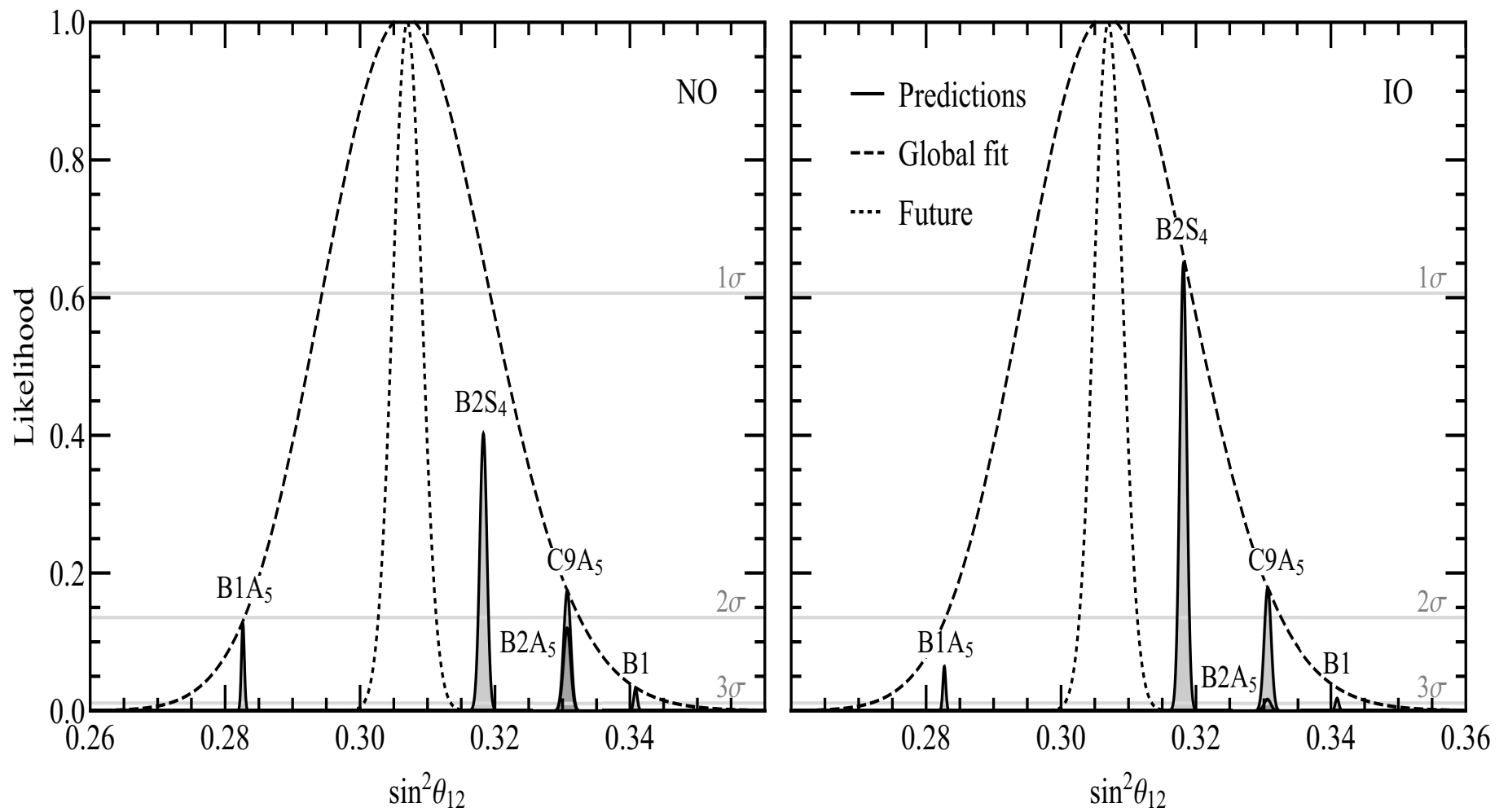
B1, A_4 (T' , S_4 , A_5) ($G_e = Z_3^T$, $G_\nu = Z_2^S$):
 $U_{\text{PMNS}} = U_{\text{TBM}} U_{13}(\theta_{13}^\nu, \delta_{13}) Q_0$;
 $\sin^2 \theta_{12} = 1/(3 \cos^2 \theta_{13}) \cong 0.340$; $\cos \delta \cong 0.570$.

B2, S_4 ($G_e = Z_3^T$, $G_\nu = Z_2^{SU}$):
 $\sin^2 \theta_{12} \cong (1 - 2 \sin^2 \theta_{13})/3 = 0.319$; $\cos \delta \cong -0.269$.



S.T.P., A. Titov, arXiv:1804.00182

Future: $\delta(\sin^2 \theta_{23}) = 3\%$ (T2HK, DUNE).



S.T.P., A. Titov, arXiv:1804.00182

Future: $\delta(\sin^2\theta_{12}) = 0.7\%$ (JUNO).

A total of 6 models would survive out of the currently viable 14 (of the extremely large number) considered if $\delta(\sin^2 \theta_{23}) = 3\%$, $\delta(\sin^2 \theta_{12}) = 0.7\%$ and the current b.f.v. would not change:

A1A₅, C3, C3A₅, C4A₅, C8, C2S₄.

A1A₅: $\cos \delta = 0.727$, $\sin^2 \theta_{23} = 0.554$;

C2S₄: $\sin^2 \theta_{23} = 0.511$.

C3, C3A₅, C8, C4A₅: predict $\cos \delta$.

C4A₅: $\cos \delta = -0.799$;

C3A₅, C8: $\cos \delta = 1^*$; **C3:** $\cos \delta = -1^*$.

In the cases marked with an asterisk, physical values of $\cos \delta$ cannot be obtained employing the best fit values of the mixing angles, but they are achievable fixing two angles to their best fit values and varying the third one in its 3σ range.

Will be constrained further by the data on δ .

JUNO

20 kt LS detector of reactor $\bar{\nu}_e$ via IBD

$\bar{\nu}_e + p \rightarrow n + e^+$; $E_{res} = 3\%/\sqrt{E}$; $L \cong 53$ km;

thermal power of the used reactors: 26.6 GW;

Sphere with a diameter of 38 m.

Cost: 300×10^6 US Dollars.

Built in China by international collaboration of more than 700 scientists from 74 Institutions in 17 countries/regions. Started data-taking on August 25, 2025.

After 6 years of operation: NMO at 3σ (using reactor ν data only). Adding ν_{atm} data can improve the sensitivity by $(0.8 - 1.4)\sigma$.

The idea put forward in S.T.P., M. Piai, PLB 553 (2002) 94 (hep-ph/0112074).

Based on: $P_{NO}(\bar{\nu}_e \rightarrow \bar{\nu}_e) \neq P_{IO}(\bar{\nu}_e \rightarrow \bar{\nu}_e)$

$$P^{NO}(\bar{\nu}_e \rightarrow \bar{\nu}_e) = 1 - \frac{1}{2} \sin^2 2\theta_{13} \left(1 - \cos \frac{\Delta m_{atm}^2 L}{2 E_\nu}\right) - \frac{1}{2} \cos^4 \theta_{13} \sin^2 2\theta_{12} \left(1 - \cos \frac{\Delta m_\odot^2 L}{2 E_\nu}\right) \\ + \frac{1}{2} \sin^2 2\theta_{13} \sin^2 \theta_{12} \left(\cos \left(\frac{\Delta m_{atm}^2 L}{2 E_\nu} - \frac{\Delta m_\odot^2 L}{2 E_\nu}\right) - \cos \frac{\Delta m_{atm}^2 L}{2 E_\nu}\right), \quad \Delta m_\odot^2 \equiv \Delta m_{21}^2,$$

$$P^{IO}(\bar{\nu}_e \rightarrow \bar{\nu}_e) = 1 - \frac{1}{2} \sin^2 2\theta_{13} \left(1 - \cos \frac{\Delta m_{atm}^2 L}{2 E_\nu}\right) - \frac{1}{2} \cos^4 \theta_{13} \sin^2 2\theta_{12} \left(1 - \cos \frac{\Delta m_\odot^2 L}{2 E_\nu}\right) \\ + \frac{1}{2} \sin^2 2\theta_{13} \cos^2 \theta_{12} \left(\cos \left(\frac{\Delta m_{atm}^2 L}{2 E_\nu} - \frac{\Delta m_\odot^2 L}{2 E_\nu}\right) - \cos \frac{\Delta m_{atm}^2 L}{2 E_\nu}\right).$$

$$\Delta m_{atm}^2 = \Delta m_{31(32)}^2(NO), \quad \Delta m_{atm}^2 = \Delta m_{32(31)}^2(IO),$$

$$\bar{\nu}_e + p \rightarrow e^+ + n$$

Spectrum of e^+ - sensitive to the difference between $P^{NO}(\bar{\nu}_e \rightarrow \bar{\nu}_e)$ and $P^{IO}(\bar{\nu}_e \rightarrow \bar{\nu}_e)$ - can be used to determine neutrino mass ordering. Optimal L exists; $E_{res} \sim 3\%/\sqrt{E}$ required.

S.T.P., M. Piai, 2001

JUNO (China, International collaboration)

S. Choubey, S.T.P., M. Piai, PRD 68 (2003) 113006 ((hep-ph/0306017):
can measure $\sin^2 \theta_{12}$, Δm_{21}^2 and Δm_{31}^2 with excep-
tionally high precision.

After 6 years of data taking:

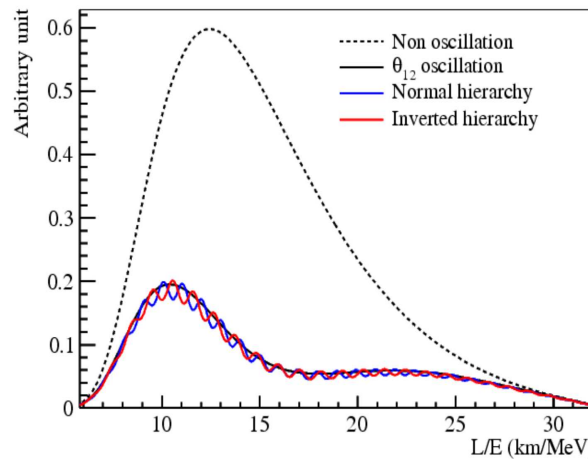
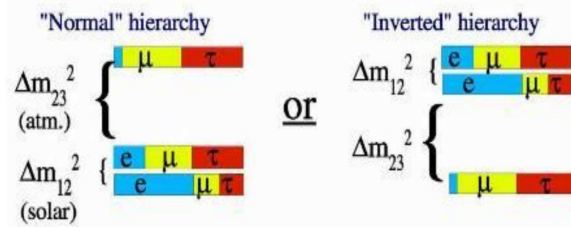
$\sin^2 \theta_{12}$: 0.5%; Δm_{21}^2 : 0.3%; Δm_{31}^2 : 0.2% (1σ)

(Y. Wang, talk given at CERN on March 20, 2024).

Wide program of research: atmospheric ν oscilla-
tions, solar neutrinos, SN neutrinos, geo-neutrinos,
nucleon decay; distant future: $(\beta\beta)_{0\nu}$ decay.



Mass Ordering by Reactor Neutrinos



$$P_{ee}(L/E) = 1 - P_{21} - P_{31} - P_{32}$$

$$P_{21} = \cos^4(\theta_{13}) \sin^2(2\theta_{12}) \sin^2(\Delta_{21})$$

$$P_{31} = \cos^2(\theta_{12}) \sin^2(2\theta_{13}) \sin^2(\Delta_{31})$$

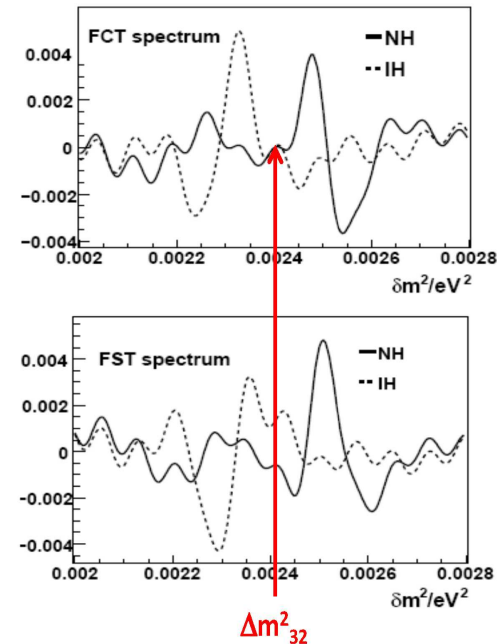
$$P_{32} = \sin^2(\theta_{12}) \sin^2(2\theta_{13}) \sin^2(\Delta_{32})$$

$$\Delta m_{31}^2 = \Delta m_{32}^2 + \Delta m_{21}^2$$

$$\text{NH: } |\Delta m_{31}^2| = |\Delta m_{32}^2| + |\Delta m_{21}^2|$$

$$\text{IH: } |\Delta m_{31}^2| = |\Delta m_{32}^2| - |\Delta m_{21}^2|$$

$$\frac{\Delta m_{21}^2}{|\Delta m_{32}^2|} \sim 3\%$$



S. Petcov and Piai, Phys. Lett. B 553, 94-106(2002)
 J. Learned et al., PRD 78(2008)071302
 L. Zhan, YFW et al., PRD 78(2008)111103

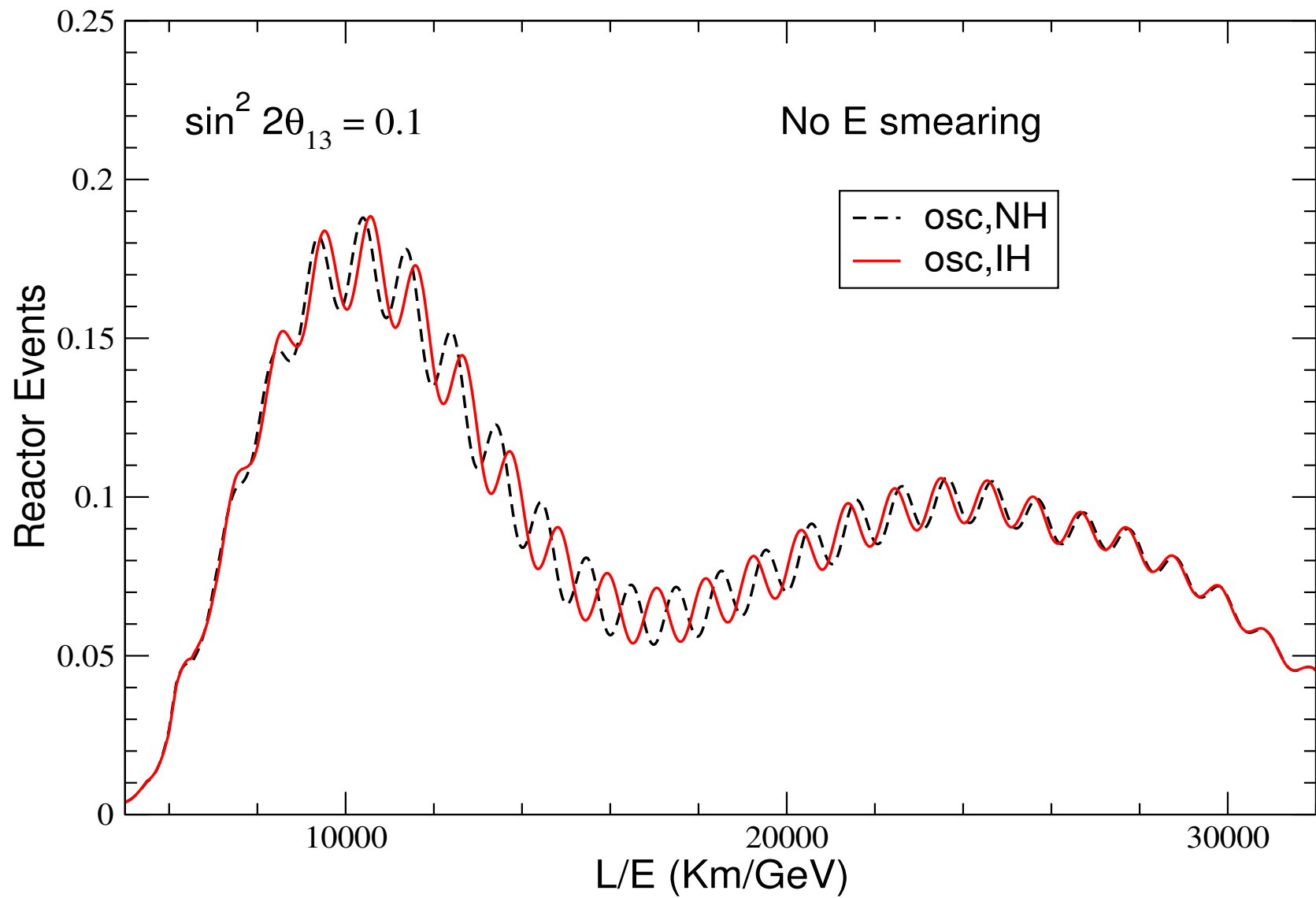
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Y. Wang, talk given at CERN on March 20, 2024

“ Very large detector, $\sin \theta_{13} \gtrsim 0.10$, $E_{res} \sim 3\%/\sqrt{E}$,... required. Probably, the experiment will never be realised... ”

“However, as it is well known, “Only those who wager can win.” (W. Pauli, Letter to Participants of a Physics Meeting in Tübingen, (H. Geiger et al.), Germany, December 4, 1930.”

S.T.P., M. Piai, PLB 553 (2002) 94 (hep-ph/0112074)

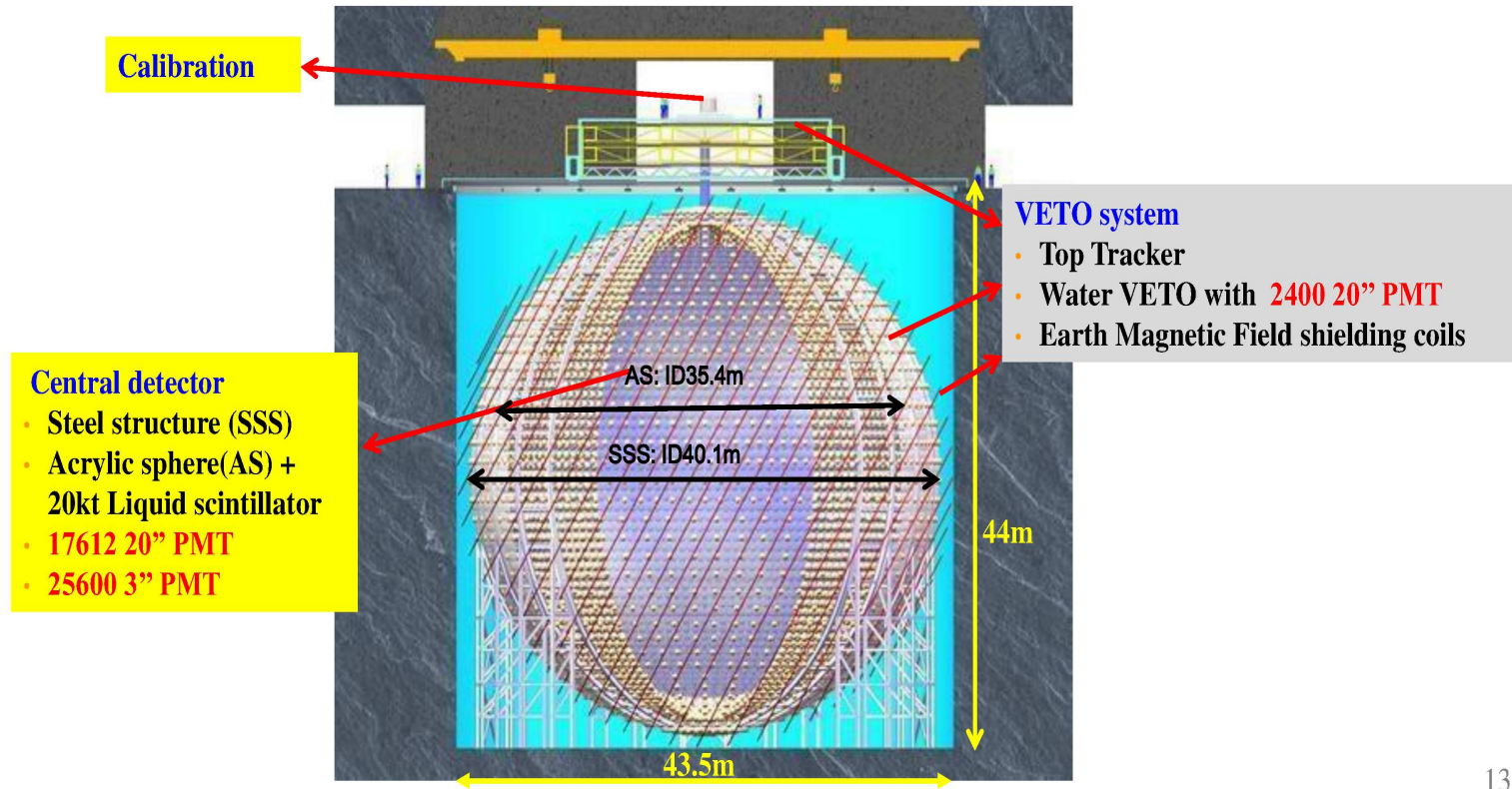


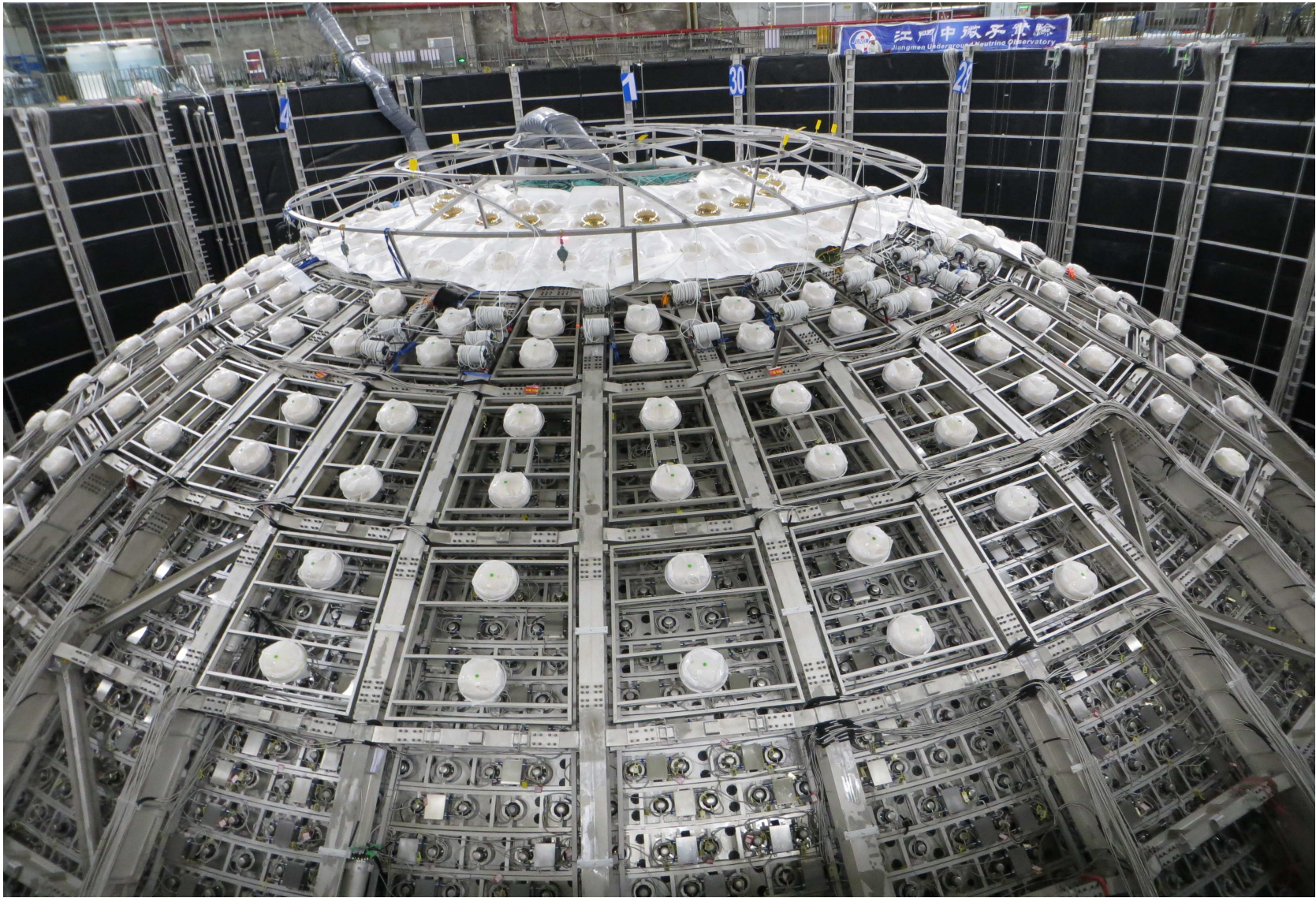
P. Ghoshal, S.T.P., JHEP 03 (2011) 058 (arXiv:1011.1646)

Concept of JUNO for Mass Ordering



- Two-layer structure for simplicity and cost: stainless steel frame + Acrylic tank
- Water as VETO and Buffer → radiopurity control of water





S.T. Petcov, NOPP, IHEP, Beijing, China, 14-16/08/2026



S.T. Petcov, NOPP, IHEP, Beijing, China, 14-16/08/2026

JUNO started taking physics data on August 26, 2025. The first results were released on November 19, 2025 (after 59.1 days):

$$\sin^2 \theta_{12} = 0.3092 \pm 0.0087, \quad \Delta m_{12}^2 = (7.50 \pm 0.12) \times 10^{-5} \text{ eV}^2$$

Improved by a factor of $\cong 1.4$ (1.6) the precision on $\sin^2 \theta_{12}$ (Δm_{12}^2).

JUNO released new data in June 2026 at $\nu 2026$:

$$\sin^2 \theta_{12} = 0.3036 \pm 0.0064, \quad \Delta m_{12}^2 = (7.388 \pm 0.078) \times 10^{-5} \text{ eV}^2$$

Further improved the precision on $\sin^2 \theta_{12}$ (Δm_{12}^2) by a factor of $\cong 1.4$ (1.5).

The results reported below are from A.V. Titov, S.T.P., 1511.19408 (PL B874 (2026) 140975) (the analytic predictions are from I. Girardi et al., 1509.02502), and from G.-J. Ding et al., 2512.03809 (JHEP 07 (2026) 256).

$\sin^2 \theta_{12}$ from Non-Abelian discrete symmetries

$$U_{PMNS} = U_e^\dagger U_\nu$$

Case 1: $A_4, S_4, A_5 \rightarrow (G_e, G_\nu) = (Z_3^{ST}, Z_2^S)$,
 $U_e = \mathbf{I}$, $U_\nu = U_{TB} U_{13}(\tilde{\theta}_{13}, \tilde{\phi}_{13})$

$$\sin^2 \theta_{12} = \frac{1}{3 \cos^2 \theta_{13}} \cong 0.341, \text{ TB2}$$

Ruled out at 5.84σ

Case 2: $A_5 \rightarrow (G_e, G_\nu) = (Z_5, Z_2^S)$,
 $U_e = \mathbf{I}$, $U_\nu = U_{GR} U_{13}(\tilde{\theta}_{13}, \tilde{\phi}_{13})$

$$\sin^2 \theta_{12} = \frac{1}{(2+r) \cos^2 \theta_{13}} \cong 0.283, \quad r = (1+\sqrt{5})/2 \cong 1.618.$$

Disfavoured at 3.2σ

Case 3: $A_5 \rightarrow (G_e, G_\nu) = (Z_2 \times Z_2, Z_2)$

$$\sin^2 \theta_{12} = 1 - \frac{1+r}{4 \cos^2 \theta_{13}} \cong 0.331.$$

Ruled out at 4.28σ

Case 4: $S_4 \rightarrow (G_e, G_\nu) = (Z_3^{ST}, Z_2^U),$
 $U_e = \mathbf{I}, U_\nu = U_{TB} U_{13}(\tilde{\theta}_{23}, \tilde{\phi}_{23})$

$$\sin^2 \theta_{12} = \frac{1 - 3 \sin^2 \theta_{13}}{3 \cos^2 \theta_{13}} \cong 0.318, \text{ TB1}.$$

Disfavored at 2.25σ

In all these cases $\cos \delta$ is also predicted in terms of $\sin^2 \theta_{13}$ and $\sin^2 \theta_{23}$.

Case 4: $\sin^2 \theta_{12} = 0.318$; for NO (IO) one has:

$$\cos \delta = - \frac{(1 - 5 \sin^2 \theta_{13}) \cot 2\theta_{23}}{2\sqrt{2} \sin \theta_{13} (1 - 3 \sin^2 \theta_{13})^{\frac{1}{2}}} = -0.131 \text{ (0.219)},$$

where the best fit values of $\sin^2 \theta_{13}$ and $\sin^2 \theta_{23}$ from I. Esteban et al., arXiv:2410.05380 (NuFit-6.0 results) were used to get the numerical values.

The results presented above are obtained employing non-Abelian discrete symmetries without imposing the gCP-symmetry. The next two slides show examples of predictions obtained with additionally imposed gCP symmetry in the charged lepton or/and neutrino sectors of models with i) one (continuous) real parameter, ii) two (continuous) real parameters: $G_f \rtimes H_{CP}$ is broken down to $\mathcal{G}_\ell = Z_n$ ($n \geq 3$) or K_4 and $\mathcal{G}_\nu = Z_2 \times H_{CP}^\nu$ in the charged lepton and neutrino sectors, respectively; $G_f = \Delta(6n^2), A_5, \Sigma(168)$ ($\Delta(24) \equiv S_4$, $\Sigma(168) \equiv PSL(2, Z_7)$). These examples are from G.-J. Ding et al., 2512.0389 (JHEP 07 (2026) 256).

A Note of Caution

Since any of the non-Abelian symmetries considered above are assumed to be realised at some high scale, the question arises how our results might be modified by renormalisation group (RG) running effects. The evaluation of the RG effects requires additional specific assumptions about the scale at which the symmetry is realised, the underlying model (Standard Model (SM), minimal supersymmetric SM, etc.), the nature — Dirac or Majorana — of massive neutrinos and the mechanism of neutrino mass generation (SM minimally extended with right-handed singlet neutrinos and conserved total lepton charge for Dirac neutrinos, or seesaw mechanism/Weinberg dimension-5 operator for Majorana neutrinos, etc.). They further depend on the type of spectrum of neutrino masses (NO or IO), on the value of the lightest neutrino mass and, in the case of massive Majorana neutrinos, on the Majorana phases.

However, the RG effects are known to be negligible for the NO mass spectrum if the lightest neutrino mass is smaller than approximately 0.01 eV; for the IO spectrum and massive Majorana neutrinos, the RG effects on θ_{12} can be large, but are known to be particularly strongly suppressed for, e.g., specific values of the Majorana phases (i.e., $\alpha_{21} = (2k + 1)\pi$, $k = 0, 1, \dots$, see S. Antusch et al., arxiv:0305273). Thus, we can conclude conservatively that the results presented above are valid at least in the indicated cases.

Order	Case	(φ_1, φ_2)	θ_{bf}/π	χ^2_{min}	$\sin^2 \theta_{12}$	$\sin^2 \theta_{13}$	$\sin^2 \theta_{23}$	δ_{CP}/π	α_{21}/π (mod 1)	α_{31}/π (mod 1)
NO	U^I	$(-\frac{\pi}{2}, \frac{\pi}{2})$	0.9170	5.308	0.318	0.02215	0.5	1.5	0	0
	U_1^{II}	$(0, \pi)$	0.1918	11.862	0.341	0.02208	0.5	1.5	0	0
	U_2^{II}	$(0, \pi)$	0.3083	11.859	0.341	0.02214	0.5	1.5	0	0
	U^{III}	—	0.0560	9.600	0.283	0.02217	0.5	1.5	0	0
	U_1^{IV}	—	0.9053	6.414	0.331	0.02214	0.523	1	0	0
	U_2^{IV}	—	0.9053	11.856	0.331	0.02214	0.477	0	0	0
	U_1^V	—	0.5715	8.921	0.341	0.02212	0.554	1.331*	0.161*	0.559*
	U_2^V	—	0.5746	13.427	0.341	0.02256	0.450	1.652	0.837	0.439
IO	U^I	$(-\frac{\pi}{2}, \frac{\pi}{2})$	0.9166	3.887	0.318	0.02235	0.5	1.5	0	0
	U_1^{II}	$(0, \pi)$	0.1915	10.471	0.341	0.02227	0.5	1.5	0	0
	U_2^{II}	$(0, \pi)$	0.3085	10.471	0.341	0.02227	0.5	1.5	0	0
	U^{III}	—	0.0563	8.188	0.283	0.02236	0.5	1.5	0	0
	U_1^V	—	0.5736	9.012	0.341	0.02241	0.551	1.343*	0.162*	0.561*
	U_2^V	—	0.5758	13.416	0.341	0.02274	0.452	1.646	0.837	0.438

The b.f.v. of the free parameter and the lepton mixing observables for all one-parameter mixing patterns for both NO and IO. The discrete parameters (φ_1, φ_2) specify the residual symmetries associated with flavour groups $\Delta(6n^2)$ ($n \leq 4$). For each pattern, the global minimum of the test statistic is obtained at $\theta = \theta_{\text{bf}}$, where the χ^2 function reaches χ^2_{min} . At the b.f. point, we present the predictions for the mixing angles and CPV phases θ_{12} , θ_{13} , θ_{23} , δ , α_{21} and α_{31} . Values of CPV phases marked with * correspond to scenarios in which the LH lepton doublets transform as the conjugate triplet $\bar{3}$.

$A_5 \rtimes H_{CP}$										
Order	Case	$\theta_\ell^{\text{bf}}/\pi$	$\theta_\nu^{\text{bf}}/\pi$	$\chi_{\min}^2$	$\sin^2 \theta_{12}$	$\sin^2 \theta_{13}$	$\sin^2 \theta_{23}$	δ_{CP}/π	α_{21}/π (mod 1)	α_{31}/π (mod 1)
NO	U_1^{VI}	0.2623	0.2424	6.076	0.331	0.02209	0.470	1.484	0	0.969
	U_2^{VI}	0.6309	0.2424	4.258	0.331	0.02210	0.471	1	0	0
	$U_{3,2}^{VI}$	0.9717	0.3973	5.998	0.335	0.02213	0.542	1.157	0.082	0.050
	$U_{4,1}^{VI}$	0.0054	0.3986	9.952	0.322	0.02227	0.461	0	0	0
	$U_{4,2}^{VI}$	0.0107	0.3915	1.458	0.314	0.02219	0.556	1	0	0
	$U_{4,3}^{VI}$	0.3984	0.7962	8.575	0.306	0.02212	0.547	0	0	0
	$U_{4,4}^{VI}$	0.8372	0.4333	1.479	0.314	0.02217	0.459	1	0	0
	U_5^{VI}	0.4688	0.1028	5.725	0.335	0.02218	0.546	1.128	0.901	0.927
	$U_{6,1}^{VI}$	0.8407	0.5677	1.287	0.301	0.02217	0.480	1	0	0
	$U_{6,2}^{VI}$	0.5062	0.0872	10.326	0.315	0.02207	0.499	0	0	0
IO	U_1^{VI}	0.2294	0.2418	3.672	0.330	0.02243	0.550	1.526	0	0.052
	$U_{3,1}^{VI}$	0.9714	0.3977	15.911	0.335	0.02245	0.459	1.842*	0.917*	0.950*
	$U_{3,2}^{VI}$	0.0358	0.3921	11.428	0.338	0.02236	0.554	1.205*	0.102*	0.062*
	U_5^{VI}	0.5370	0.1057	12.969	0.336	0.02240	0.555	1.152*	0.882*	0.914*
$\Sigma(168) \rtimes H_{CP}$										
Order	Case	$\theta_\ell^{\text{bf}}/\pi$	$\theta_\nu^{\text{bf}}/\pi$	$\chi_{\min}^2$	$\sin^2 \theta_{12}$	$\sin^2 \theta_{13}$	$\sin^2 \theta_{23}$	δ_{CP}/π	α_{21}/π (mod 1)	α_{31}/π (mod 1)
NO	$U_{1,1}^{VII}$	0.7307	0.3390	12.607	0.343	0.02235	0.563	1.700*	0.598*	0.253*
	$U_{2,1}^{VII}$	0.8112	0.2043	9.952	0.322	0.02227	0.461	0	0	0
	$U_{2,2}^{VII}$	0.0275	0.6463	1.460	0.314	0.02222	0.555	1	0	0
	$U_{2,3}^{VII}$	0.6396	0.2418	8.575	0.306	0.02212	0.547	0	0	0
	$U_{2,4}^{VII}$	0.6430	0.2390	1.479	0.314	0.02217	0.459	1	0	0
IO	$U_{1,1}^{VII}$	0.7302	0.3399	11.276	0.342	0.02251	0.565	1.697*	0.596*	0.252*
	$U_{1,2}^{VII}$	0.2682	0.6653	19.115	0.345	0.02261	0.445	1.278*	0.399*	0.744*

The b.f.v. of the free parameters and the lepton mixing observables for two-parameter mixing patterns from $A_5 \rtimes H_{CP}$ and $\Sigma(168) \rtimes H_{CP}$ symmetries for NO and IO. Here $\chi^2_{\min}$ the global minimum of the χ^2 function occurs at $(\theta_\ell, \theta_\nu) = (\theta_\ell^{\text{bf}}, \theta_\nu^{\text{bf}})$. The remaining conventions follow those used in the preceding table.

MINOS data with reactor constrained $\Delta m_{31(32)}^2$ **excludes** $\theta_{23} = \pi/4$ at 3σ ;

NO: $0.44 \leq \sin^2 \theta_{23} \leq 0.59$ **excluded at 3σ ;**

IO: $0.42 \leq \sin^2 \theta_{23} \leq 0.59$ **excluded at 3σ .**

Potential tension with the T2K and NO ν A data.

J. Thomas (for MINOS Collab.), talk given at Neutrino 2026

If the prospective 1σ precision on $\sin^2 \theta_{12}$ of 0.5% is reached by JUNO and the current best fit value $\sin^2 \theta_{12} = 0.3036$ will not change, the predicted values of $\sin^2 \theta_{12} = 0.318, 0.322, 0.314$ and 0.315 will be ruled out at $9.49\sigma, 12.12\sigma, 6.85\sigma$ and 7.51σ , respectively. From all listed predictions only $\sin^2 \theta_{12} = 0.301$ and 0.306 will “survive”. They can be distinguished (and both ruled out or one confirmed) by the respective predictions for $\sin^2 \theta_{23} = 0.480$ and 0.547 .

These are only the first of many remarkable results expected from the JUNO experiment, which has unique physics capabilities. These include determining the neutrino mass spectrum, measuring three neutrino oscillation parameters with exceptional precision, studying oscillations of reactor, solar, and atmospheric neutrinos, detecting neutrinos from a supernova explosion and the diffuse supernova neutrino background, measuring the flux of geoneutrinos, searching for proton decay, and probing physics beyond the Standard Model. The JUNO experiment has reported impressive new results after only 59.1 days of data taking, achieving a factors of 1.4 and 1.6 improvement in the precision of solar neutrino oscillation parameters compared with previous experiments. With its rich physics programme and unique detector capabilities, JUNO is set to be a leading experiment in neutrino and particle physics for the next 20 years and possibly beyond 2045.

Instead of Conclusions

We have presented one approach to the lepton flavour problem based on non-Abelian discrete symmetries. This approach leads to specific testable predictions. Only with the additional data from the current and upcoming experiments (T2K, NO ν A, JUNO, HK, DUNE, T2HK,..., it will be possible to perform thorough tests of these predictions. The planned high precision measurements of θ_{12} , θ_{23} and of the Dirac CPV phase δ , if successful, will allow us to perform such tests. Experimental confirmation of some of the discussed specific predictions will imply the existence of a new basic symmetry in particle physics. This will have profound implications.

I personally am looking very much forward to the upcoming new data on neutrino mixing, leptonic CP violation, neutrino mass ordering, absolute neutrino mass scale and the nature of massive neutrinos, and the better understanding of neutrinos and the associated development of the theory of neutrino masses, mixing and leptonic CP violation these data will bring.