Quantum information with elementary particles: from neutrino oscillations to electroweak processes

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International Workshop on New Opportunities for Particle Physics 2026 - Beijing, August 14 2026

Summary

1. Entanglement in neutrino mixing and oscillations
2. Quantum Field Theory of neutrino mixing and oscillations
3. Quantum correlations & nonlocality in neutrino oscillations
4. Chiral oscillations & lepton-antineutrino entanglement
5. Entanglement dynamics in QED processes
6. Entanglement saturation
7. Entanglement in electroweak processes (preliminary results)

Motivations

- Entanglement in relativistic systems as a possible resource in Quantum Information;
- Investigation of fundamental properties of (elementary) particles via quantum correlations
- Entanglement in high energy processes as a probe for new physics beyond Standard Model.
- Understanding the mechanism of entanglement generation in fundamental processes;

Entanglement in particle physics[¶]

- Entanglement, decoherence, Bell inequalities for the $K^0\bar{K}^0$ (or $B^0\bar{B}^0$) system[*];

- Quantumness in neutrino oscillations (entanglement[†], Leggett-Garg inequalities[‡], quantum correlations);

- Entanglement in scattering processes[§].

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[*] R.A.Bertlmann and B.Hiesmayr, Phys.Rev. A (2001); A.Di Domenico (Ed.) Frascati Physics Series (2007).

[†] M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2009).

[‡] J.A. Formaggio et al., Phys. Rev. Lett. (2016).

[§] G. Aad et al. [ATLAS], Nature (2024)

[¶] R.A.Bertlmann, *Entanglement, Bell inequalities and decoherence in particle physics*, Lect. Not. Phys. (2006);
Y.Shi, *Historical origins of quantum entanglement in particle physics*, Int. J. Mod. Phys. A (2025)

Entanglement in neutrino mixing and oscillations

Neutrino oscillations in QM *

Pontecorvo mixing relations

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

– Time evolution:

$$|\nu_e(t)\rangle = \cos\theta\, e^{-iE_1 t}\,|\nu_1\rangle + \sin\theta\, e^{-iE_2 t}\,|\nu_2\rangle$$

– Flavor oscillations:

$$P_{\nu_e\to\nu_e}(t) \;=\; |\langle\nu_e|\nu_e(t)\rangle|^2 = 1 - \sin^2 2\theta\ \sin^2\left(\frac{\Delta E}{2}t\right) = 1 - P_{\nu_e\to\nu_\mu}(t)$$

– Flavor conservation:

$$|\langle\nu_e|\nu_e(t)\rangle|^2 + |\langle\nu_\mu|\nu_e(t)\rangle|^2 = 1$$

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*S.M.Bilenky and B.Pontecorvo, Phys. Rep. (1978)

Entanglement in neutrino mixing[†]

– Flavor mixing (neutrinos)

$$|\nu_e\rangle = \quad \cos\theta\,|\nu_1\rangle \,+\, \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle \,+\, \cos\theta\,|\nu_2\rangle$$

• Correspondence with two-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2 \equiv |10\rangle, \qquad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2 \equiv |01\rangle,$$

$|\rangle_i$ denotes states in the Hilbert space for neutrinos with mass m_i.

$\Rightarrow$ flavor states are entangled superpositions of the mass eigenstates:

$$|\nu_e\rangle = \quad \cos\theta\,|10\rangle \,+\, \sin\theta\,|01\rangle.$$

[†]M.B., F.Dell'Anno, S.De Siena, M.Di Mauro and F.Illuminati, PRD (2008).

Composite structure of Hilbert space for neutrinos

• Necessity of tensor-product structure of Hilbert space for two generations:

Orthogonality of Hilbert spaces for fields with different masses[‡]

Example: two scalar fields with different masses

$$(\Box + \mu_1^2)\phi_1(x) = 0 \quad , \qquad (\Box + \mu_2^2)\phi_2(x) = 0$$

with boundary conditions $\phi_1(0, \mathbf{x}) = \phi_2(0, \mathbf{x})$ and $\dot{\phi}_1(0, \mathbf{x}) = \dot{\phi}_2(0, \mathbf{x})$

One obtains

$$ {}_1\langle 0|0\rangle_2 \, \simeq \, \exp\left\{ -\frac{V}{64\pi^2}\int_0^\infty dk \, \frac{(\mu_1^2 - \mu_2^2)^2}{k^2} \right\}$$

which vanishes in the infinite volume limit.

[‡]G.Barton, Introduction to Advanced Field Theory, Intersc. Publ. (1963)

Entanglement - mathematical definition

- Given a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, a system is entangled, iff

$$\rho_{AB} \neq \sum_k p_k \, \rho_k^{(A)} \otimes \rho_k^{(B)}$$

with $0 \leq p_i \leq 1, \quad \sum_k p_k = 1.$

- For a generic pure state of the form:

$$|\psi\rangle_{AB} = \sum_{ij} c_{ij} |i\rangle_A \otimes |j\rangle_B$$

the condition for entanglement reads

$$|\psi\rangle_{AB} \neq |\phi\rangle_A \otimes |\chi\rangle_B$$

Single-particle entanglement*

- A state like $|\psi\rangle_{A,B} = |0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B$ is entangled;

– entanglement among field modes, rather than particles;

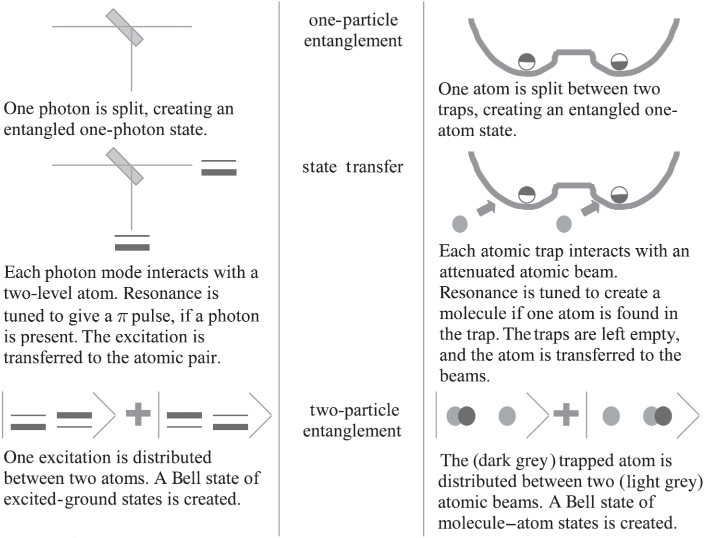
– entanglement is a property of composite systems, rather than of many-particle systems;

– entanglement and non-locality are not synonyms;

– single-particle entanglement is as good as two-particle entanglement for applications (quantum cryptography, teleportation, violation of Bell inequalities, etc..).

* J.van Enk, Phys. Rev. A (2005), (2006);
J.A.Dunningham and V.Vedral, Phys. Rev. Lett. (2007).

Protocols for extraction of single-particle entanglement [†]



[†] M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007)

Multipartite entanglement in neutrino mixing[‡]

– Neutrino mixing (three flavors):

$$|\underline{\nu}_f\rangle \quad = \quad U(\tilde{\theta}, \delta)\, |\underline{\nu}_m\rangle$$

with $|\underline{\nu}_f\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$ and $|\underline{\nu}_m\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T$.

– Mixing matrix (PMNS)

$$U(\tilde{\theta}, \delta) \quad = \quad \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

where $(\tilde{\theta}, \delta) \equiv (\theta_{12}, \theta_{13}, \theta_{23}; \delta)$, $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$.

- Correspondence with three-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2|0\rangle_3 \equiv |100\rangle, \qquad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2|0\rangle_3 \equiv |010\rangle,$$

$$|\nu_3\rangle \equiv |0\rangle_1|0\rangle_2|1\rangle_3 \equiv |001\rangle$$

[‡] M.B., F.Dell'Anno, S.De Siena, M.Di Mauro and F.Illuminati, PRD (2008).

(Flavor) Entanglement in neutrino oscillations§

• Quantification of (static) neutrino entanglement via linear entropy: take $\rho^{(e)} = |\nu_e\rangle\langle\nu_e|$ and trace over mode 2 $\Rightarrow \rho_1^{(e)} = Tr_2[\rho^{(e)}]$. Then one obtains

$$S_L^{(1;2)}(\rho^{(e)}) = 2(1 - Tr_1[(\rho_1^{(e)})^2]) = \sin^2 2\theta.$$

– Flavor neutrino state at time t can be written in terms of flavor states at time $t = 0$ as

$$|\nu_\alpha(t)\rangle = A_{\alpha e}(t)|\nu_e\rangle + A_{\alpha\mu}(t)|\nu_\mu\rangle, \quad \alpha = e, \mu.$$

– Transition probability for $\nu_\alpha \to \nu_\beta$

$$P_{\nu_\alpha \to \nu_\beta}(t) = |\langle\nu_\beta|\nu_\alpha(t)\rangle|^2 = |A_{\alpha\beta}(t)|^2 .$$

§ M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2009).

Entanglement in neutrino oscillations: two-flavors

- We now take the flavor states at initial time as our qubits:

$$|\nu_e\rangle \equiv |1\rangle_e |0\rangle_\mu \equiv |10\rangle_f, \quad |\nu_\mu\rangle \equiv |0\rangle_e |1\rangle_\mu \equiv |01\rangle_f,$$

- Starting from $|10\rangle_f$ or $|01\rangle_f$, time evolution generates the (entangled) Bell-like states:

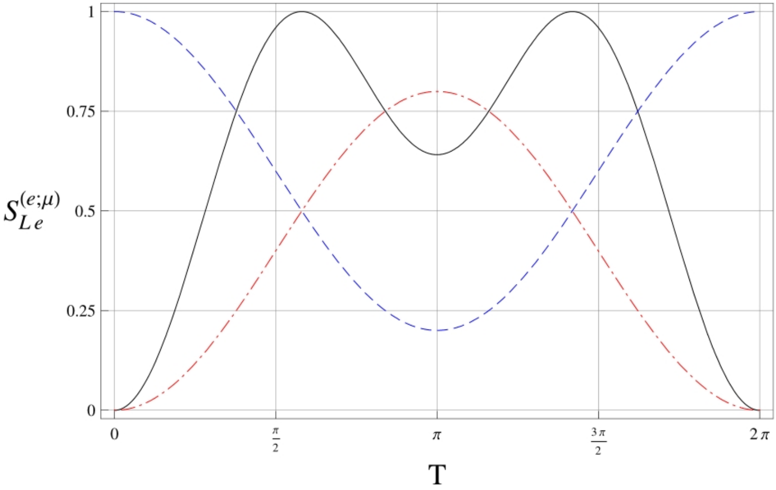
$$|\nu_\alpha(t)\rangle \;=\; A_{\alpha e}(t)|10\rangle_f + A_{\alpha\mu}(t)|01\rangle_f, \quad \alpha = e, \mu.$$

Take density matrix for electron neutrino state $\rho^{(e)} = |\nu_e(t)\rangle\langle\nu_e(t)|$, and trace over mode $\mu \;\Rightarrow\; \rho_e^{(e)}$.

- The associated linear entropy is :

$$S_L^{(e;\mu)}(\rho^{(e)}) \;=\; 4\,|A_{e\mu}(t)|^2\,|A_{ee}(t)|^2 \;=\; 4\,P_{\nu_e\rightarrow\nu_e}(t)\,P_{\nu_e\rightarrow\nu_\mu}(t)$$

- Linear entropy given by product of transition probabilities



Linear entropy $S_{Le}^{(e;\mu)}$ (full) as a function of the scaled time $T = \frac{2Et}{\Delta m_{12}^2}$, with $\sin^2\theta = 0.314$. Transition probabilities $P_{\nu_e \to \nu_e}$ (dashed) and $P_{\nu_e \to \nu_\mu}$ (dot-dashed) are reported for comparison.

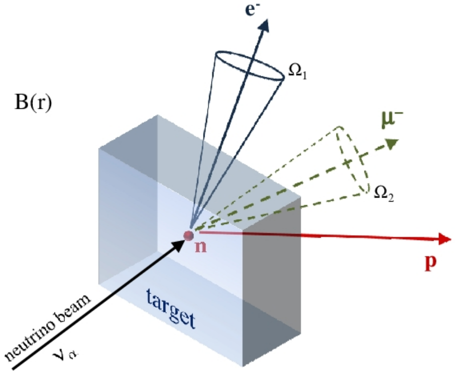
ν–oscillations as a resource for quantum information

• Single-particle entanglement encoded in flavor states $|\underline{\nu}^{(f)}(t)\rangle$ is a real physical resource that can be used, at least in principle, for protocols of quantum information.

– Experimental scheme for the transfer of the flavor entanglement of a neutrino beam into a single-particle system with *spatially separated modes.*

Charged-current interaction between a neutrino ν_α with flavor α and a nucleon N gives a lepton α^- and a baryon X:

$$\nu_\alpha + N \longrightarrow \alpha^- + X \,.$$



Generation of a single-particle entangled lepton state (two flavors):

In the target the charged-current interaction occurs: $\nu_\alpha + n \longrightarrow \alpha^- + p$ with $\alpha = e\,,\mu$. A spatially nonuniform magnetic field $\mathbf{B}(\mathbf{r})$ constraints the momentum of the outgoing lepton within a solid angle Ω_i, and ensures spatial separation between lepton paths. The reaction produces a superposition of electronic and muonic spatially separated states.

• Given the initial Bell-like superposition $|\nu_\alpha(t)\rangle$ the unitary process associated with the weak interaction leads to the superposition

$$|\alpha(t)\rangle \,=\, \Lambda_e|1\rangle_e|0\rangle_\mu + \Lambda_\mu|0\rangle_e|1\rangle_\mu \,,$$

where $|\Lambda_e|^2 + |\Lambda_\mu|^2 = 1$, and $|k\rangle_\alpha$, with $k = 0, 1$, is the lepton qubit.

The coefficients Λ_α are proportional to $A_{\alpha\beta}(t)$ and to the cross sections associated with the creation of an electron or a muon.

• Analogy with single-photon system: quantum uncertainty on the "*which path*" of the photon at the output of an unbalanced beam splitter $\Leftrightarrow$ uncertainty on the "*which flavor*" of the produced lepton.

The coefficients Λ_α plays the role of the transmissivity and of the reflectivity of the beam splitter.

Other results

- Generalization to three flavors. Extension to wave packets;[¶]

- Flavor entanglement in Quantum Field Theory.[‖]

¶ M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2015).

‖ M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2014).

Quantum Field Theory of neutrino mixing and oscillations

Neutrino mixing in QFT

- Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\,\nu_1(x) + \sin\theta\,\nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\,\nu_1(x) + \cos\theta\,\nu_2(x)$$

can be written as*

$$\nu_e^\alpha(x) = G_\theta^{-1}(t)\,\nu_1^\alpha(x)\,G_\theta(t)$$

$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t)\,\nu_2^\alpha(x)\,G_\theta(t)$$

– Mixing generator:

$$G_\theta(t) = \exp\left[\theta\int d^3\mathbf{x}\,\left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

For ν_e, we get $\frac{d^2}{d\theta^2}\,\nu_e^\alpha = -\nu_e^\alpha$ with i.c. $\nu_e^\alpha\big|_{\theta=0} = \nu_1^\alpha$, $\frac{d}{d\theta}\nu_e^\alpha\big|_{\theta=0} = \nu_2^\alpha$.

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*M.B. and G.Vitiello, Annals Phys. (1995)

• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\;|0\rangle_{1,2}$$

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V\to\infty}\;{}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty}\; e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\;\ln\left(1-\sin^2\theta\,|V_{\mathbf{k}}|^2\right)^2} = 0$$

with

$$|V_{\mathbf{k}}|^2 \;\;\equiv\;\; \sum_{r,s} |\; v^{r\dagger}_{-\mathbf{k},1} u^{s}_{\mathbf{k},2}\;|^2 \neq 0 \quad for \quad m_1 \neq m_2$$

.

- The "flavor vacuum" $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state[†]:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[(1-\sin^2\theta\,|V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\,\cos\theta\,|V_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\,\epsilon^r \sin^2\theta\,|V_{\mathbf{k}}||U_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_{\mathbf{k}}|^2\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2}$$

- Condensation density:

$${}_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; {}_{e,\mu}\langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; \sin^2\theta\;|V_{\mathbf{k}}|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing).

– Condensate structure as in systems with SSB (e.g. superconductors)

– Exotic condensate: mixed pairs

– Note that $|0\rangle_{e\,\mu} \neq |a\rangle_1 \otimes |b\rangle_2 \Rightarrow$ entanglement.

[†] A. Perelomov, *Generalized Coherent States,* (Springer V., 1986)

• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)$$

• Mixing transformation = Rotation + Bogoliubov transformation .

– Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) \quad = \quad u^{r\dagger}_{\mathbf{k},2} u^r_{\mathbf{k},1}\; e^{i(\omega_{k,2}-\omega_{k,1})t} \quad ; \qquad V_{\mathbf{k}}(t) = \epsilon^r\; u^{r\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2}\; e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

Decomposition of mixing generator *

The mixing generator can be expressed in terms of a rotation and a Bogoliubov transformation:

$$G_\theta \; = \; B(\Theta_1, \Theta_2) \;\; R(\theta) \;\; B^{-1}(\Theta_1, \Theta_2)$$

- However, Bogoliubov and Pontecorvo do not commute!

[ , ] ≠ 0

As a result, flavor vacuum gets a non-trivial term:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}|0\rangle_{1,2} \; = \; |0\rangle_{1,2} \; + \; \Big[B(m_1, m_2) \; , \; R^{-1}(\theta)\Big] \;\; |\widetilde{0}\rangle_{1,2}$$

with $|\widetilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2}$.

*M. B., M.V. Gargiulo and G. Vitiello, Phys. Lett. B (2017)

Currents and charges for mixed fermions *

- Lagrangian in the mass basis:

$$\mathcal{L} = \bar{\Psi}_m \left(i \not{\partial} - M_d\right) \Psi_m$$

where $\Psi_m^T = (\nu_1, \nu_2)$ and $M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$.

– Lagrangian in the flavor basis:

$$\mathcal{L} = \bar{\Psi}_f \left(i \not{\partial} - M\right) \Psi_f$$

where $\Psi_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

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*M. B., P. Jizba and G. Vitiello, Phys. Lett. B (2001)

– Two sets of charges:

$$
\begin{aligned}
Q_i &= \int d^3\mathbf{x}\, \nu_i^\dagger(x)\, \nu_i(x)\ ; \qquad i = 1,2 \\
Q_\sigma(t) &= \int d^3\mathbf{x}\, \nu_\sigma^\dagger(x)\, \nu_\sigma(x)\ ; \qquad \sigma = e, \mu
\end{aligned}
$$

– In presence of mixing, neutrino flavor charges not conserved charges $\Rightarrow$ flavor oscillations.

– They are still (approximately) conserved in the vertex $\Rightarrow$ define flavor neutrinos as their eigenstates.

- Problem: find the eigenstates of the above charges.

- Flavor charge operators are diagonal in the flavor ladder operators:

$$:: Q_\sigma(t) :: \;\equiv\; \int d^3\mathbf{x} \, :: \nu_\sigma^\dagger(x)\, \nu_\sigma(x) ::$$

$$= \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^{r}_{\mathbf{k},\sigma}(t) \,-\, \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^{r}_{-\mathbf{k},\sigma}(t) \right) , \quad \sigma = e, \mu.$$

Here :: ... :: denotes normal ordering w.r.t. flavor vacuum:

$$:: A :: \equiv A \,-\, {}_{e,\mu}\langle 0|A|0\rangle_{e,\mu}$$

- Define flavor neutrino states with definite momentum and helicity:

$$|\nu^r_{\mathbf{k},\sigma}\rangle \,\equiv\, \alpha^{r\dagger}_{\mathbf{k},\sigma}(0)\,|0\rangle_{e,\mu}$$

– Such states are eigenstates of the flavor charges (at t=0):

$$:: Q_\sigma :: |\nu^r_{\mathbf{k},\sigma}\rangle \,=\, |\nu^r_{\mathbf{k},\sigma}\rangle$$

Neutrino oscillation formula (QFT)

– We have, for an electron neutrino state:

$$\mathcal{Q}_{\mathbf{k},\sigma}(t) \equiv \langle \nu^r_{\mathbf{k},e}| :: Q_\sigma(t) :: |\nu^r_{\mathbf{k},e}\rangle$$
$$= \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2$$

with $Q_\sigma(t) \equiv \int d^3\mathbf{x}\, \nu^\dagger_\sigma(x)\, \nu_\sigma(x)$.

• Neutrino oscillation formula (exact result)*:

$$\mathcal{Q}_{\mathbf{k},e}(t) = 1 - |U_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\, t\right) - |V_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}\, t\right)$$

- For $k \gg \sqrt{m_1 m_2}$, $|U_\mathbf{k}|^2 \to 1$ and $|V_\mathbf{k}|^2 \to 0 \Rightarrow$ Pontecorvo formula is recovered.

*M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999).

Lepton charge violation for Pontecorvo states[†]

– Pontecorvo states:

$$|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos\theta\, |\nu^r_{\mathbf{k},1}\rangle \;+\; \sin\theta\, |\nu^r_{\mathbf{k},2}\rangle$$

$$|\nu^r_{\mathbf{k},\mu}\rangle_P \;=\; -\sin\theta\, |\nu^r_{\mathbf{k},1}\rangle \;+\; \cos\theta\, |\nu^r_{\mathbf{k},2}\rangle\;,$$

are *not* eigenstates of the flavor charges.

$\Rightarrow$ *violation of lepton charge conservation* in the production/detection vertices, at tree level:

$${}_P\langle\nu^r_{\mathbf{k},e}| :Q_e(0): |\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos^4\theta + \sin^4\theta + 2|U_{\mathbf{k}}|\, \sin^2\theta\cos^2\theta \;<\; 1,$$

for any $\theta \neq 0$, $\mathbf{k} \neq 0$ and for $m_1 \neq m_2$.

[†] M. B., A. Capolupo, F. Terranova and G. Vitiello, Phys. Rev. **D** (2005)
C. C. Nishi, Phys. Rev. **D** (2008).

Entanglement for flavor neutrino states in QFT

– Entanglement for flavor neutrino states in QFT can be expressed by means of the variances* of the neutrino charges:† Q_i, $Q_\sigma(t)$

– Variance of $Q_i \to$ static entanglement:

$$\begin{aligned}\Delta Q_i(\nu_e) &= \langle \nu^r_{\mathbf{k},e}|Q_i^2(t)|\nu^r_{\mathbf{k},e}\rangle - \langle \nu^r_{\mathbf{k},e}|Q_i|\nu^r_{\mathbf{k},e}\rangle^2 \\ &= \cos^2\theta\,\sin^2\theta\end{aligned}$$

– Variance of $Q_\sigma \to$ flavor (dynamical) entanglement:

$$\begin{aligned}\Delta Q_\sigma(\nu_e)(t) &= \langle \nu^r_{\mathbf{k},e}|Q_\sigma^2(t)|\nu^r_{\mathbf{k},e}\rangle - \langle \nu^r_{\mathbf{k},e}|Q_\sigma|\nu^r_{\mathbf{k},e}\rangle^2 \\ &= \mathcal{Q}^{\mathbf{k}}_{e\to e}(t)\,\mathcal{Q}^{\mathbf{k}}_{e\to\mu}(t)\end{aligned}$$

in formal agreement with results obtained in QM.

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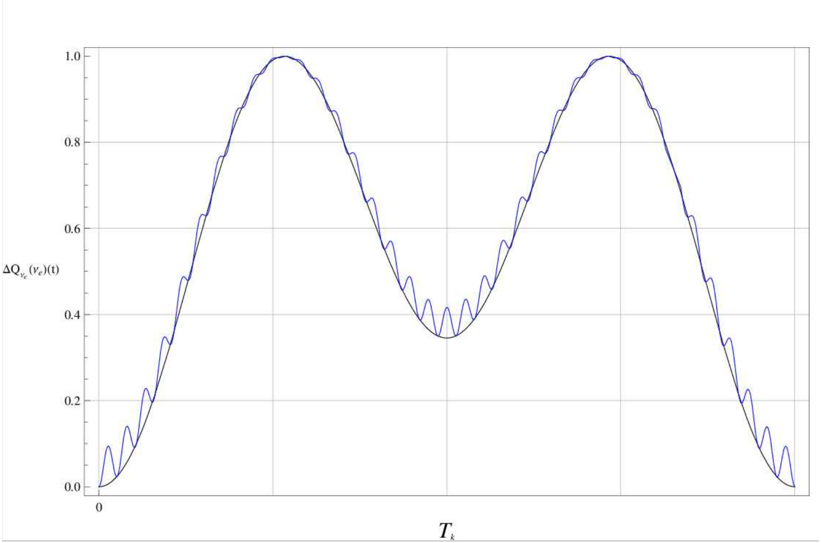
* A. A. Klyachko, B. Öztop, and A. S. Shumovsky, *Phys. Rev.* **A** (2007).
† M.Blasone, F. Dell'Anno, S.De Siena and F. Illuminati, EPL (2014)

QFT flavor entanglement



QM vs. QFT flavor entanglement for $|\nu_e(t)\rangle$.

Neutrino ontology: flavor or mass?

- In view of the unitary inequivalence of mass and flavor representations, we have the problem of the fundamental (ontological) nature of neutrino.

Flavor or mass, that is the question...

Neutrino ontology: research directions

- How to verify the fundamental nature of neutrino states?

Two directions:

- Investigate the phenomenology of flavor neutrinos, with corrections expected in the non-relativistic regime: oscillations, beta decay endpoint, quantum correlations, ...
- Use the formal consistency of QFT, by comparing neutrino processes in two different frames (inertial and comoving) for accelerated particle: Unruh effect.*

*M. B., G. Lambiase, G. Luciano and L.Petruzziello, Phys. Rev. D (2018); G.Cozzella, S.Fulling, A.Landulfo, G.Matsas and D.Vanzella, Phys.Rev.(2018) M. B., G.Lambiase, G. Luciano and L.Petruzziello, Phys. Lett. B (2020)

- Dynamical generation of fermion mixing[*].
- Flavor-energy uncertainty relations for mixed states[†].
- Poincaré invariance for flavor neutrinos[‡].
- Violation of equivalence principle[§].

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[*] M.B., P.Jizba, N.E.Mavromatos and L.Smaldone, Phys. Rev. D (2019)
[†] M. B., P. Jizba and L. Smaldone, Phys. Rev. D (2019)
[‡] M.B., P.Jizba, N.E.Mavromatos and L.Smaldone, Phys. Rev. D (2020) ; A. E. Lobanov, Ann. Phys. (2019)
[§] M.B., P.Jizba, G.Lambiase and L.Petruzziello, Phys. Lett. B (2020)

Quantum correlations & nonlocality in neutrino oscillations

Quantum Resource Theory

Resource theories are a versatile set of tools developed in quantum information theory[¶].

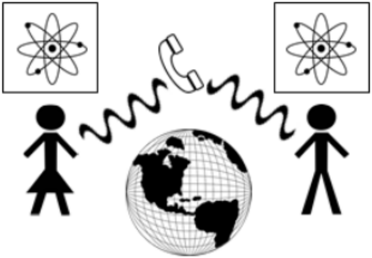
The basic idea of a quantum resource theory is to study quantum information processing under a restricted set of physical operations, called *free* operations.

These allow as to prepare only certain physical states, called *free* states. The other are called **resource** states.

[¶]E. Chitambar, G. Gour, Rev. Mod. Phys. (2019)
A. Streltsov, G. Adesso and M. B. Plenio, Rev. Mod. Phys. (2017),

Entanglement in QRT

Alice and Bob work in their laboratory separated by a large distance. They can communicate only by telephone.



The free operations consist in local operations and classical communication (LOCC). But an entangled state cannot be generated using LOCC $\Rightarrow$ Entanglement is a (quantum) resource.

Quantum Correlations

Quantum systems exhibit properties that are beyond our understanding of reality. They show correlations that have no classical counterpart.

Entanglement is the most known of these correlations. But the terminology *quantum correlations* refers to a broader concept:

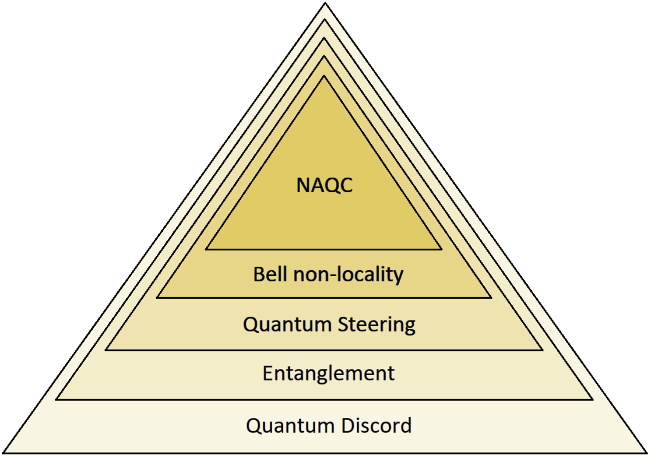
Quantum correlations related to entanglement:

- Bell non-locality
- Entanglement
- Quantum steering

Quantum correlations beyond entanglement:

- Quantum discord

Quantum Correlations



Hierarchy of quantum correlations (figure adapted from G.Adesso et al., J. Phys. A (2016))[‖];

[‖] Note: for pure states, the above hierarchy collapses to entanglement only, except NAQC due to its definition (see below).

Quantum correlations in neutrino oscillations

- Recently, quantum correlations in neutrino oscillations have been thoroughly investigated. Some of our papers:

– M.B., S.De Siena and C.Matrella, *Wave packet approach to quantum correlations in neutrino oscillations*, Eur. Phys. J. C (2021)

– V.Bittencourt, M.B., S.De Siena and C.Matrella, *Complete complementarity relations for quantum correlations in neutrino oscillations*, Eur. Phys. J. C (2022)

– V.Bittencourt, M.B., S.De Siena and C.Matrella, *Quantifying quantumness in three-flavor neutrino oscillations*, Eur. Phys. J. C (2024)

– A.K.Alok, M.B., T.J.Chall, N.R.S.Chundawat, G.Lambiase *Coherent dynamics of flavor mode entangled neutrinos*, Nucl. Phys. B (2026)

– B.W.Farooq, B.S.Koranga, M.B., work in progress

Quantification of quantumness in neutrino oscillations

- Quantumness in neutrino oscillations has been quantified through various correlation measures*: Non-local Advantage of Quantum Coherence (NAQC), quantum steering and Bell non-locality.
- The criterion for NAQC is:

$$N^{l_1}(\rho_{AB}) = \frac{1}{2}\sum_{i,j,b} p(\rho^b_{\Pi_{j\neq i}}) C^{\sigma_i}_{l_1}(\rho_{B|\Pi_{j\neq i}}) > \sqrt{6}.$$

- Bell non-locality (violation of CHSH inequality):

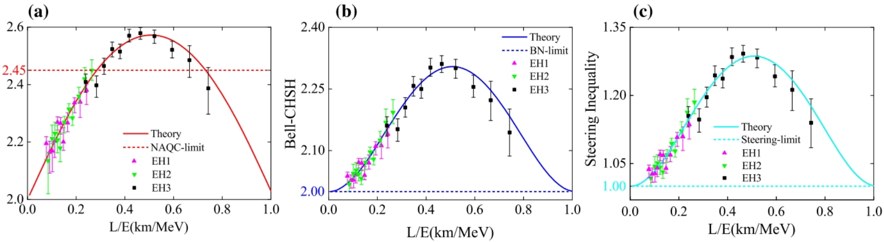
$$B(\rho_{AB}) = |\langle B_{CHSH}\rangle| \leq 2.$$

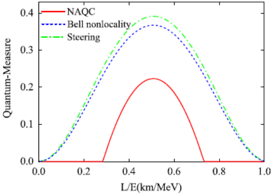
- Quantum steering:

$$F_n(\rho_{AB},\varsigma) = \frac{1}{\sqrt{n}}\left|\sum_{i=1}^{n} Tr(\rho_{AB} A_i \otimes B_i)\right| \leq 1.$$

*F. Ming, X-K. Song, D. Wang, Eur. Phys. J. C (2020)

Quantumness in neutrino oscillations (Daya Bay) †



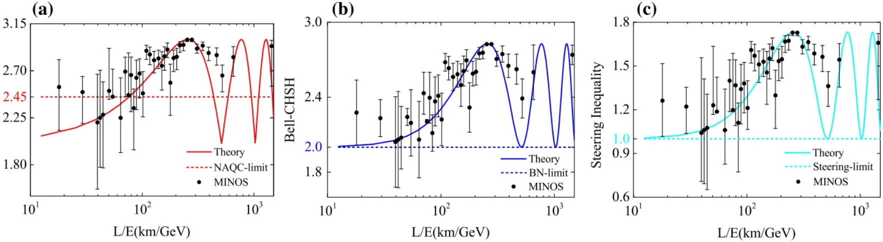


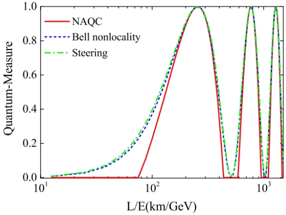
Daya Bay: $\sin^2 2\theta_{13} = 0.084$ and $\Delta m^2_{ee} = 2.42 \times 10^{-3} eV^2$

NAQC is a stronger nonclassical correlation than Bell non-locality and quantum steering.

† F. Ming, X-K. Song, D. Wang, Eur. Phys. J., (2020)

Quantumness in neutrino oscillations (MINOS) ‡





MINOS: $\sin^2 2\theta_{23} = 0.95$ and $\Delta m^2_{32} = 2.32 \times 10^{-3} eV^2$.

NAQC is a stronger nonclassical correlation than Bell non-locality and quantum steering.

‡ F. Ming, X-K. Song, D. Wang, Eur. Phys. J., (2020)

NAQC & Bell nonlocality in the wave packet approach[¶]

• We have extended the studies on quantumness of neutrino oscillations through NAQC using the wave packet approach.[§]

Neutrino with definite flavor:

$$|\nu_\alpha(x,t)\rangle = \sum_j U^*_{\alpha j}\psi_j(x,t)|\nu_j\rangle$$

where:

$$\psi_j(x,t) = \frac{1}{\sqrt{2\pi}}\int dp\ \psi_j(p)e^{ipx-iE_j(p)t}$$

with:

$$\psi_j(p) = \left(2\pi{\sigma_p^P}^2\right)^{-\frac{1}{4}}\exp-\frac{(p-p_j)^2}{4{\sigma_p^P}^2}$$

[§] C. Giunti, C.W. Kim, *Fundamentals of Neutrino Physics and Astrophysics*, Oxford University Press (2007)

[¶] M.B., S. De Siena and C. Matrella, Eur. Phys. J. C (2021)

Wave packet description of neutrino oscillations

Assume the condition $\sigma_p^P \ll E_j^2(p_j)/m_j$. Then we have:

$$E_j(p) \simeq E_j + v_j(p - p_j)$$

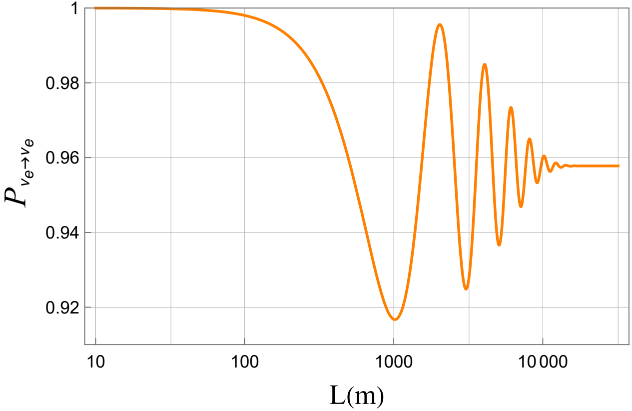
Integrating on p, one gets the wave packet in coordinate space:

$$\psi_j(x,t) = \left(2\pi\sigma_x^{P\,2}\right)^{-\frac{1}{4}} \exp\left[-iE_j t + ip_j x - \frac{(x - v_j t)^2}{4\sigma_x^{P\,2}}\right]$$

Write density matrix operator $\rho_\alpha(x,t) = |\nu_\alpha(x,t)\rangle\langle\nu_\alpha(x,t)|$. After time integration, one gets the oscillation formula in space

$$P_{\alpha\beta}(L) = \sum_{j,k} U^*_{\alpha j} U_{\alpha k} U^*_{\beta j} U_{\beta k} \exp\left[-2\pi i \frac{L}{L^{osc}_{jk}} - \left(\frac{L}{L^{coh}_{jk}}\right)^2 - 2\pi^2(1-\xi)^2\left(\frac{\sigma_x}{L^{osc}_{jk}}\right)^2\right]$$

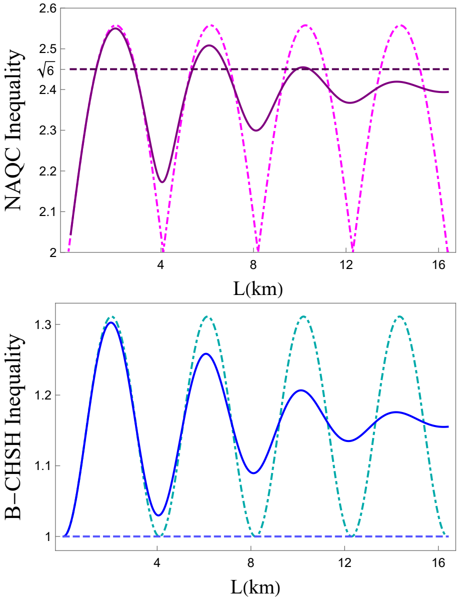
Wave packet description of neutrino oscillations‖



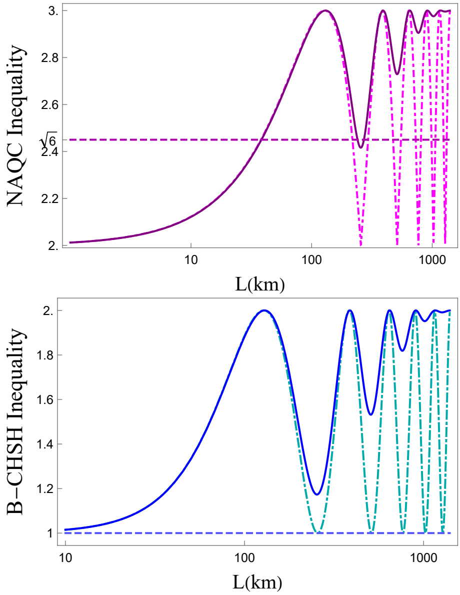
• Survival probability in the wave packet approach. $E = 2\,MeV$, $\xi = 0$, $\sin^2 2\theta_{13} = 0.084$ and $\Delta m^2_{ee} = 2.42 \times 10^{-3} eV^2$ and $\sigma_x = 3.3 \times 10^{-6} m$.

‖C. Giunti, C.W. Kim, *Fundamentals of Neutrino Physics and Astrophysics*, Oxford University Press (2007)

NAQC in the wave packet approach (Daya Bay)



NAQC in the wave packet approach (MINOS)



Results

- Our treatment based on wave packets leads to a improved agreement with experimental data in the case of MINOS.*
- NAQC has a different long-distance behaviour for the two experiments, due to the different values of the mixing angle.
- Existence of a “critical” angle for which NAQC exceeds the bound.

* M.B., S. De Siena and C. Matrella, Eur. Phys. J. C (2021)

Complete Complementarity Relations†

To better understand the above results, we resort to complete complementarity relations (CCR).

- Complementarity: a quantum system may possess properties which are equally real but mutually exclusive.

It is often associated with wave-particle duality, the complementarity aspect between propagation and detection.

In the double-slit interferometer, the wave aspect is characterized by the *interference fringes visibility*, while the particle nature is given by the *which-way information* of the path along the interferometer.

† N.Bohr, Nature (1928)

Quantitative wave-particle duality

• Wootters and Zurek *: first quantitative version of the wave-particle duality. A path-detecting device can give incomplete which-way information and a sharply interference pattern can still be retained.

Their work was then extended and formulated in terms of a complementarity relation†

$$P^2 + V^2 \leq 1$$

where P is the predictability and V is the visibility.

• A "**quanton**"‡ may behave partially as a wave or as a particle at the same time.

---

*W.K.Wootters and W.H.Zurek, Phys. Rev. D (1979)

†D.M.Greenberger and A.Yasin, Phys.Lett. A (1988); B.-G. Englert, PRL (1996).

‡J.-M.Lévy-Leblond, Physica (1988)

Triality relation for bipartite pure states

- For bipartite systems a complete complementarity relation (CCR) can be obtained by including the correlations between A and B subsystems[§]:

$$V_k^2 + P_k^2 + C^2 = 1$$

V_k and P_k, $k = 1, 2$, generate *local* single-partite realities which can be related to wave-particle duality.

C is the entanglement measure **concurrence** which generate an exclusive bipartite *nonlocal* reality.

[§] M.Jakob and J.A.Bergou, Opt. Comm. (2010)

Concurrence[¶]

The concurrence for a generic qubit system described by the density matrix ρ is given by

$$C(\rho) \equiv \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

where the λ_i are the square root of the eigenvalues λ_i^2 of the operator $\rho\tilde{\rho}$ in decreasing order, with

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$$

[¶] S. A. Hill, W. K. Wootters, Phys. Rev. Lett. (1997)

Triality relation

Consider the most general bipartite state of two qubits:

$$|\Theta\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$$

One obtains:

$$C = |\langle\Theta|\tilde{\Theta}\rangle| = 2|ad - bc|$$

$$V_k = 2|\langle\Theta|\sigma_k^\dagger|\Theta\rangle| \to \begin{cases} V_1 = 2|ac^* + bd^*| \\ V_2 = 2|ab^* + cd^*| \end{cases}$$

$$P_k = |\langle\Theta|\sigma_{z,k}|\Theta\rangle| \to \begin{cases} P_1 = |(|c|^2 + |d|^2) - (|a|^2 + |b|^2)| \\ P_2 = |(|b|^2 + |d|^2) - (|a|^2 + |c|^2)| \end{cases}$$

where: $|\tilde{\Theta}\rangle = (\sigma_y \otimes \sigma_y)\,|\Theta^*\rangle, \quad \sigma_k^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_{z,k} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

The CCR is satisfied, since the l.h.s. is just the square norm of the general *pure* bipartite state $|\Theta\rangle$: $(|a|^2 + |b|^2 + |c|^2 + |d|^2)^2 = 1.$

Examples

- Bell states (maximally entangled states)

$$\Phi^{\pm} = \frac{1}{\sqrt{2}} \left(|00\rangle \pm |11\rangle \right), \quad \Psi^{\pm} = \frac{1}{\sqrt{2}} \left(|01\rangle \pm |10\rangle \right)$$

We have $C = 1, \;\; V_1 = \; V_2 = \; P_1 = \; P_2 = 0.$

- Separable state

$$|\Theta_1\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |01\rangle \right) \; = \; \frac{1}{\sqrt{2}} |0\rangle \left(|0\rangle + |1\rangle \right)$$

In this case $C = 0, \;\; V_1 = \; P_2 = 0, \;\; V_2 = \; P_1 = 1.$

- Unbalanced state with all four terms

$$|\Theta_4\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle.$$

In this case we have $C = \frac{\sqrt{3}-1}{2\sqrt{2}}, V_1 = \frac{\sqrt{3}+1}{2\sqrt{2}}, \;\; V_2 = \frac{\sqrt{3}+2}{4}, \;\; P_1 = 0, \;\; P_2 = \frac{1}{4}.$

CCR for multipartite pure states

Alternative form of CCR for multipartite states*.

Consider a bipartite pure state in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$\rho_{A,B} = \sum_{i,k=0}^{d_A-1} \sum_{j,l=0}^{d_B-1} \rho_{ij,kl} \left|i, j\right\rangle \left\langle k, l\right| .$$

If the state of subsystem A is mixed:

$$P_{hs}(\rho_A) + C_{hs}(\rho_A) < \frac{d_A - 1}{d_A}.$$

where $P_{hs}(\rho_A)$ and $C_{hs}(\rho_A)$ are the predictability and the Hilbert-Schmidt quantum coherence (generalization of the visibility†).

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*M.L.W.Basso and J.Maziero, J. Phys. A (2020)

†T. Qureshi, Quanta (2019).

CCR for pure states

- The missing information about subsystem A is being shared via correlations with the subsystem B:

$$P_{hs}(\rho_A) + C_{hs}(\rho_A) + C^{nl}_{hs}(\rho_{A|B}) = \frac{d_A - 1}{d_A}$$

– Predictability

$$P_{hs}(\rho_A) \equiv \sum_{i=0}^{d_A-1} (\rho^A_{ii})^2 - \frac{1}{d_A}\,,$$

– Quantum coherence (visibility)

$$C_{hs}(\rho_A) \equiv \sum_{i \neq k}^{d_A-1} |\rho^A_{ik}|^2$$

– Non-local quantum coherence (entanglement)

$$C^{nl}_{hs}(\rho_{A|B}) = \sum_{i \neq k, j \neq l} |\rho_{ij,kl}|^2 - 2 \sum_{i \neq k, j < l} \Re(\rho_{ij,kj}\rho^*_{il,kl})$$

$C^{nl}_{hs}(\rho_{A|B})$ is equivalent to the linear entropy of subsystem A.

CCR for pure states - entropic formulation

- Another form of CCR can be obtained by defining the predictability and the coherence measures in terms of the von Neumann entropy:

$$C_{\rm re}(\rho_A) + P_{vn}(\rho_A) + S_{vn}(\rho_A) = \log_2 d_A$$

where

$$C_{\rm re}(\rho_A) = S_{vn}(\rho_{A\rm diag}) - S_{vn}(\rho_A)$$

$$P_{vn}(\rho_A) \equiv \log_2 d_A - S_{vn}(\rho_{A\rm diag})$$

For pure states $S_{vn}(\rho_A) = -\mathrm{Tr}\,(\rho_A \log_2 \rho_A)$ is a measure of entanglement between A and B.

CCR for mixed states*

• For mixed states, $S_{vn}(\rho_A)$ does not quantify entanglement, but it is just a measure of mixedness of A. CCR have to be modified:

$$P_{vn}(\rho_A) + C_{re}(\rho_A) + I_{A:B}(\rho_{AB}) + S_{A|B}(\rho_{AB}) = \log_2 d_A,$$

where:

– $P_{vn}(\rho_A) \equiv \ln d_A - S_{vn}(\rho_{Adiag})$ is the predictability;

– $C_{re}(\rho_A) = S_{vn}(\rho_{Adiag}) - S_{vn}(\rho_A)$ is the relative entropy of coherence;

– $I_{A:B}(\rho_{AB}) = S_{vn}(\rho_A) + S_{vn}(\rho_B) - S_{vn}(\rho_{AB})$ is the mutual information of A and B;

– $S_{A|B}(\rho_{AB}) = S_{vn}(\rho_{AB}) - S_{vn}(\rho_B)$ is the conditional entropy: It tells how much it is convenient knowing about subsystem A with respect to the whole system.

* M.L.W.Basso and J.Maziero, EPL (2021)

CCR for oscillating neutrinos[†]

- We now consider the CCR for neutrino oscillations, both for pure and mixed states.

Let us consider a two-flavor neutrino state:

$$|\nu_\alpha(t)\rangle = a_{\alpha\alpha}(t)\,|\nu_\alpha\rangle + a_{\alpha\beta}(t)\,|\nu_\beta\rangle$$

We can use the following correspondence:

$$|\nu_\alpha\rangle = |1\rangle_\alpha \otimes |0\rangle_\beta = |10\rangle$$
$$|\nu_\beta\rangle = |0\rangle_\alpha \otimes |1\rangle_\beta = |01\rangle$$

For an initial electronic neutrino, we have:

$$|\nu_e(t)\rangle = a_{ee}\,|10\rangle + a_{e\mu}\,|01\rangle$$

[†] V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C. (2022)

CCR for oscillating neutrinos - pure state (plane waves)

CCRs for pure states are verified in the case of neutrinos. We find:[‡]

$$
\begin{aligned}
P_{hs}(\rho_e) &= P_{ee}^2 + P_{e\mu}^2 - \frac{1}{2} \\
C_{hs}(\rho_e) &= 0 \\
C_{hs}^{nl}(\rho_{e\mu}) &= 2 P_{ee} P_{e\mu}
\end{aligned}
$$

where $|a_{ee}|^2 = P_{ee}$, $|a_{e\mu}|^2 = P_{e\mu}$ and $P_{ee} + P_{e\mu} = 1$.

Thus:

$$
P_{hs}(\rho_e) + C_{hs}(\rho_e) + C_{hs}^{nl}(\rho_{e\mu}) = \frac{1}{2}
$$

as expected.

[‡]V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C. (2022)

CCR for oscillating neutrinos - mixed state (wave packets)

- We consider the CCR in the case of a two-flavor neutrino oscillation, for an initial electron neutrino

$$P_{vn}(\rho_e) + C_{re}(\rho_e) + I_{A:B}(\rho_{e\mu}) + S_{e|\mu}(\rho_{e\mu}) = \log_2 d_e,$$

where:

$$\begin{aligned} P_{vn}(\rho_e) &= \log_{d_e} - S_{vn}(\rho_{e_{diag}}) \\ C_{re}(\rho_e) &= S_{vn}(\rho_{e_{diag}}) - S_{vn}(\rho_e) \\ I_{A:B}(\rho_{e\mu}) &= S_{vn}(\rho_e) + S_{vn}(\rho_\mu) - S_{vn}(\rho_{e\mu}) \\ S_{e|\mu}(\rho_{e\mu}) &= S_{vn}(\rho_{e\mu}) - S_{vn}(\rho_\mu) \end{aligned}$$

For a generic matrix ρ, the von Neumann entropy is defined as $S_{vn}(\rho) = -\sum_i \lambda_i \log_2 \lambda_i$, where λ_i are the eigenvalues of ρ.

CCR for neutrino mixed state

The starting density matrix is:

$$\rho_{e\mu}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & F^e_{ee} & F^e_{e\mu} & 0 \\ 0 & F^e_{\mu e} & F^e_{\mu\mu} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the reduced density matrices are:

$$\rho_e(x) = \begin{pmatrix} F^e_{ee} & 0 \\ 0 & F^e_{\mu\mu} \end{pmatrix} \qquad \rho_\mu(x) = \begin{pmatrix} F^e_{\mu\mu} & 0 \\ 0 & F^e_{ee} \end{pmatrix}$$

By evaluating the eigenvalues of these matrices, we obtain:

$$P_{vn}(\rho_e) = 1 + F^e_{ee} \log_2 F^e_{ee} + F^e_{\mu\mu} \log_2 F^e_{\mu\mu}$$

$$C_{re}(\rho_e) = 0$$

$$I_{e:\mu}(\rho_{e\mu}) + S_{e|\mu}(\rho_{e\mu}) = -F^e_{ee} \log_2 F^e_{ee} - F^e_{\mu\mu} \log_2 F^e_{\mu\mu}$$

By adding all the terms we find that the CCR for mixed states is satisfied for a neutrino state.

CCR for neutrino mixed state

The sum of the non-local terms of the CCR is equal to the Quantum Discord, defined as:

$$QD(\rho_{AB}) = I(\rho_{AB}) - CC(\rho_{AB}),$$

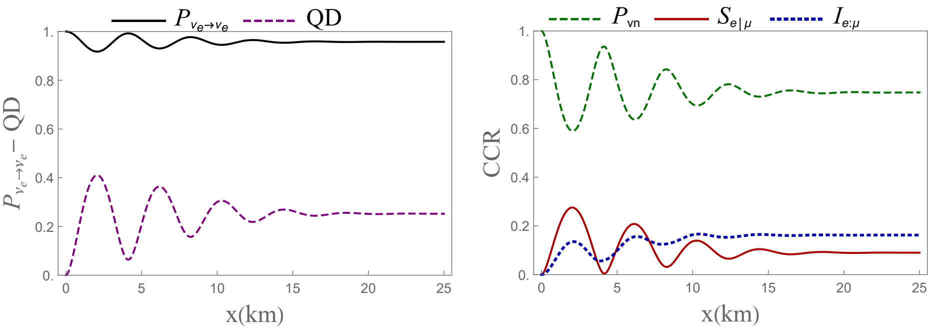
where $I(\rho_{AB})$ is the total correlations between the subsystems A and B; and $CC(\rho_{AB})$ quantifies the classical correlations. We have

$$QD(\rho_{AB}) = S_{\mathrm{vn}}(\rho_A) - S_{\mathrm{vn}}(\rho_{AB}) + \min_{\{\Pi_i^b\}} S_{\mathrm{vn},\{\Pi_i^b\}}(\rho_{A|B})$$

that, for the neutrino density matrix under consideration, gives

$$QD(\rho_{e\mu}) = -F^e_{ee}\log_2 F^e_{ee} - F^e_{\mu\mu}\log_2 F^e_{\mu\mu}$$

CCR for neutrino oscillations* - DAYA BAY

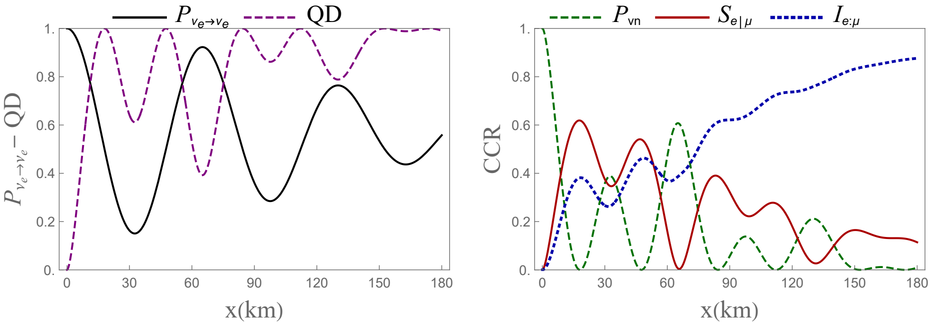


(a) *DAYA BAY* $(L \in [364m, 1912m])$

$$\Delta m^2_{ee} = 2.42 \times 10^{-3} eV^2, \; \sin^2 2\theta_{13} = 0.084, \; E = 4MeV$$

*V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C. (2022)

CCR for neutrino oscillations† - KamLAND

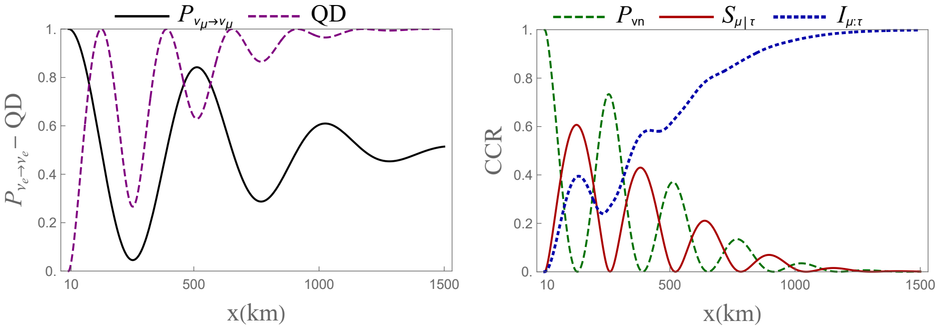


(b) *KamLAND (L = 180 Km)*

$$\Delta m^2_{12} = 7.49\times 10^{-5} eV^2,\ \tan^2 2\theta_{12} = 0.47,\ E = 2MeV$$

†V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C. (2022)

CCR for neutrino oscillations[‡] - MINOS



(c) *MINOS (L = 735 km)*

$$\Delta m^2_{32} = 2.32 \times 10^{-3} eV^2,\ \sin^2 2\theta_{23} = 0.95,\ E = 0.5 GeV$$

[‡]V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C. (2022)

Results

- We have studied CCR for the oscillating neutrino systems, both in the pure and in the mixed case.

- Complete characterization of quantum correlations in neutrino oscillations.

- Interesting long-distance behaviour of the correlations, depending on the mixing angle.

CCR for neutrino oscillations - 3 flavors*

Tripartite pure state:

$$\rho_{ABC} = \sum_{i,l=0}^{d_A-1} \sum_{j,m=0}^{d_B-1} \sum_{k,n=0}^{d_C-1} \rho_{ijk,lmn} \left| i,j,k \right\rangle_{ABC} \left\langle l,m,n \right| .$$

State of subsystem A:

$$\rho_A = \sum_{i,l=0}^{d_A-1} \left(\sum_{j=0}^{d_B-1} \sum_{k=0}^{d_C-1} \rho_{ijk,ljk} \right) \left| i \right\rangle_A \left\langle l \right| \equiv \sum_{i,l=0}^{d_A-1} \rho_{il}^A \left| i \right\rangle_A \left\langle l \right| ,$$

CCR

$$P_{\mathrm{hs}}(\rho_A) + C_{\mathrm{hs}}(\rho_A) + C_{\mathrm{hs}}^{nl}(\rho_{A|BC}) = \frac{d_A - 1}{d_A}$$

The non local coherence is given by:

$$C_{\mathrm{hs}}^{nl}(\rho_{A|BC}) = \sum_{i\neq l} \left(\sum_{\substack{j\neq m \\ k\neq n}} + \sum_{\substack{j=m \\ k\neq n}} + \sum_{\substack{j\neq m \\ k=n}} \right) \left| \rho_{ijk,lmn} \right|^2 - 2 \sum_{i\neq l} \left(\sum_{\substack{j=m \\ k<n}} + \sum_{\substack{j<m \\ k=n}} + \sum_{\substack{j<m \\ k\neq n}} \right) \Re(\rho_{ijk,ljk} \rho_{imn,lmn}^{*}).$$

*V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C (2024)

CCR for 3 flavor neutrino oscillations

- Entropic form of CCR still valid for the single-partite subsystems A, B and C. For the subsystem AB, we have:

$$C_{\rm re}(\rho_{AB}) + P_{\rm vn}(\rho_{AB}) + S_{\rm vn}(\rho_{AB}) = \log_2(d_A d_B),$$

and similar ones for AC and BC.

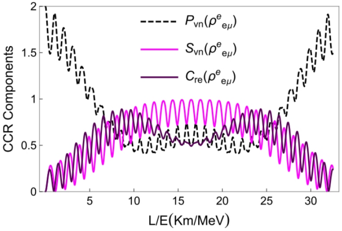
- For tripartite mixed states, CCR for subsystem AB takes the form:

$$P_{vn}(\rho_{AB}) + C_{re}(\rho_{AB}) + S_{AB|C}(\rho_{ABC}) + I_{AB:C}(\rho_{ABC}) = \log_2(d_A d_B),$$

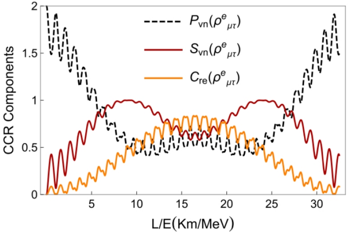
- The state for the subsystem C, on the other hand, satisfy the CCR:

$$P_{vn}(\rho_C) + C_{re}(\rho_C) + S_{C|AB}(\rho_{ABC}) + I_{C:AB}(\rho_{ABC}) = \log_2(d_C).$$

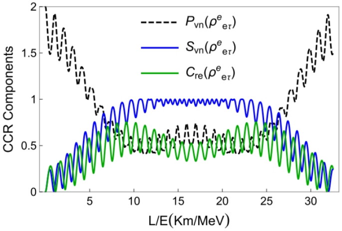
CCR 3 flavors – plane waves*



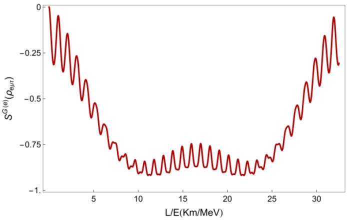
(a) *eμ subsystem*



(c) *μτ subsystem*

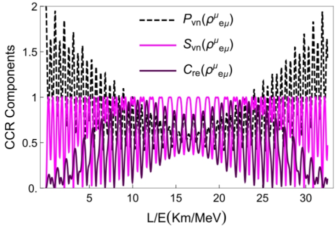


(b) *eτ subsystem*

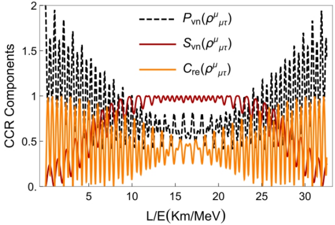


• CCR terms and tripartite entanglement for an initial electron neutrino state.

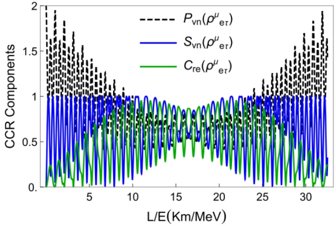
CCR 3 flavors – plane waves*



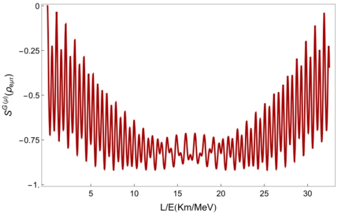
(a) *eμ subsystem*



(c) *μτ subsystem*



(b) *eτ subsystem*



- CCR terms and tripartite entanglement for an initial muon neutrino state.

*V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C (2024)

CCR 3 flavors – wave packets*

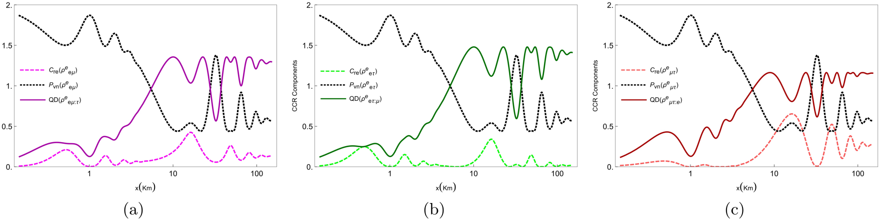


Figure 1: CCR terms for two flavor subsystems $e\mu$ (a), $e\tau$ (b) and $\mu\tau$ (c) as function of x - electron neutrino.

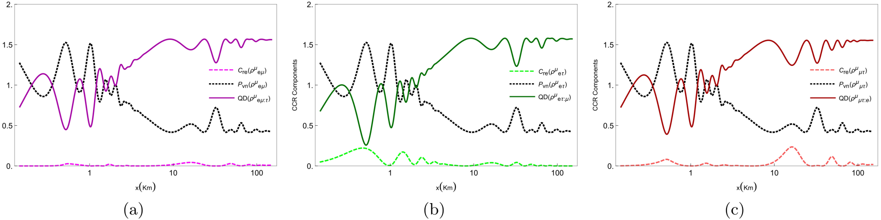


Figure 2: CCR terms for two flavor subsystems $e\mu$ (a), $e\tau$ (b) and $\mu\tau$ (c) as function of x - muon neutrino.

*V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C (2024)

CCR 3 flavors with CP violation*

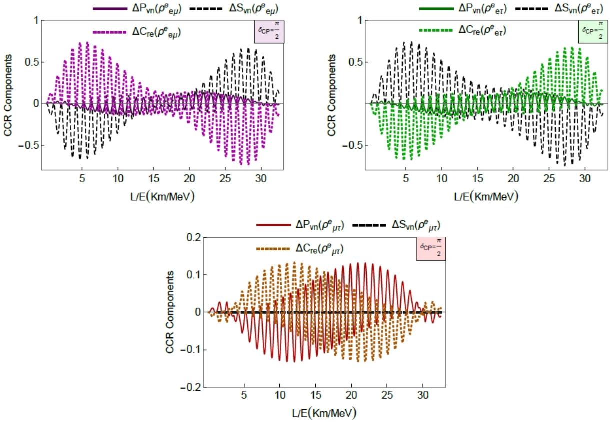


Fig. 6. Difference between neutrino and anti-neutrino CCR terms for bipartite subsystems $e\mu$, $e\tau$ and $\mu\tau$ as function of L/E in the case of an initial electronic neutrino.

- Recent review of neutrino correlations for three flavors:

G. Wang, X. Song, L. Ye and D. Wang, Acta Phys. Sin. (2025)

*M.B., S. De Siena and C. Matrella, Int. J. Quant. Inf. (2024).

Violations of macrorealism in neutrino oscillations

- The notion of Macroscopic realism (Macrorealism) tries to encode our intuition of macroscopic world[*].
- Violations of macrorealism tested by Leggett-Garg inequalities (LGIs): temporal analogue of Bell inequalities.
- Violations of LGIs in neutrino oscillations have been proved by using the MINOS data[†].
- Bell vs LGIs:
  - Bell inequalities: necessary and sufficient for local realism[‡],
  - LGIs: only necessary but not sufficient for macrorealism.

---

[*] A. J. Leggett and A. Garg, Phys. Rev. Lett. (1985)
[†] J. Formaggio, D. Kaiser, M. Murskyj, and T. Weiss, Phys. Rev. Lett. (2016)
[‡] A. Fine, Phys. Rev. Lett. (1982).

Violations of macrorealism in neutrino oscillations

- A necessary and sufficient condition can be formulated in terms of two set of *equalities*[*]: non-signaling in time (NSIT) and arrow of time (AoT).
- We computed NSIT/AoT in the case of two-flavor neutrino oscillations in the wave-packet formalism[†] and in the case of meson oscillations[‡]
- NSIT/AoT reveal violations of macrorealism hidden by LGIs.

[*] L. Clemente and J. Kofler, Phys. Rev. A (2015).
[†] M.B., F.Illuminati, L.Petruzziello, K.Simonov and L.Smaldone. EPJC (2023)
[‡] M.B., F.Illuminati, L.Petruzziello, K.Simonov and L.Smaldone. PRA (2024).

Macrorealism

Macrorealism[§]:

- Macrorealism per se: A macroscopic object is always in one of the available states, regardless of the measurement process
- Noninvasive measurability: It is in principle possible to measure the state of the system without affecting its dynamical evolution

We measure a dichotomic observable $O(t)$ at three equidistant times $t_n = nt$, $n = 0, 1, 2$.

[§] A. J. Leggett and A. Garg, Phys. Rev. Lett. (1985)

Macrorealism violation in neutrino oscillations

Leggett–Garg inequalities (LGIs):

$$\mathcal{L}_1(t_0, t_1, t_2) = 1 + C_{01} + C_{12} + C_{02} \geq 0$$
$$\mathcal{L}_2(t_0, t_1, t_2) = 1 - C_{01} - C_{12} + C_{02} \geq 0$$
$$\mathcal{L}_3(t_0, t_1, t_2) = 1 + C_{01} - C_{12} - C_{02} \geq 0$$
$$\mathcal{L}_4(t_0, t_1, t_2) = 1 - C_{01} + C_{12} - C_{02} \geq 0$$

with

$$C_{ij} = \langle O(t_i) O(t_j) \rangle$$

must be fulfilled in macrorealistic systems: only necessary but not sufficient for macrorealism![¶]

[¶] J. J. Halliwell, Phys. Rev. A (2017)

NSIT/AoT conditions

Necessary and sufficient condition is given by a combination of NSIT[‖]

$$\mathrm{NSIT}^{(1)}: \quad P(O_2) \ = \ \sum_{O_1} P(O_1, O_2)$$

$$\mathrm{NSIT}^{(2)}: \quad P(O_0, O_2) \ = \ \sum_{O_1} P(O_0, O_1, O_2)$$

$$\mathrm{NSIT}^{(3)}: \quad P(O_1, O_2) \ = \ \sum_{O_0} P(O_0, O_1, O_2)$$

and AoT

$$\mathrm{AoT}^{(1)}: \quad P(O_0, O_1) \ = \ \sum_{O_2} P(O_0, O_1, O_2)$$

$$\mathrm{AoT}^{(2)}: \quad P(O_0) \ = \ \sum_{O_1} P(O_0, O_1)$$

$$\mathrm{AoT}^{(3)}: \quad P(O_1) \ = \ \sum_{O_2} P(O_1, O_2)$$

[‖] L. Clemente and J. Kofler, Phys. Rev. A (2015)

NSIT/AoT for neutrinos

NSIT for neutrinos (two-flavor case)** ($O_0 = e$, $O_1 = O_2 = \mu$)

$$\mathrm{NSIT}^{(1)}: \quad P_{e\to\mu}(2t) \;=\; 2P_{e\to\mu}(t)P_{e\to e}(t)$$

$$\mathrm{NSIT}^{(2)}: \quad P_{e\to\mu}(2t) \;=\; 2P_{e\to\mu}(t)P_{e\to e}(t)$$

$$\mathrm{NSIT}^{(3)}: \quad P_{e\to\mu}(t)P_{\mu\to\mu}(t) \;=\; P_{e\to\mu}(t)P_{\mu\to\mu}(t)$$

Only one non-trivial relation

$$\mathcal{N}(t) \;\equiv\; P_{e\to\mu}(2t) - 2P_{e\to\mu}(t)\,P_{e\to e}(t) \;=\; 0$$

The AoT conditions are all trivially satisfied.

**M. B., F. Illuminati, L. Petruzziello, K. Simonov and L.Smaldone, EPJC (2023)

LGIs for neutrino oscillations

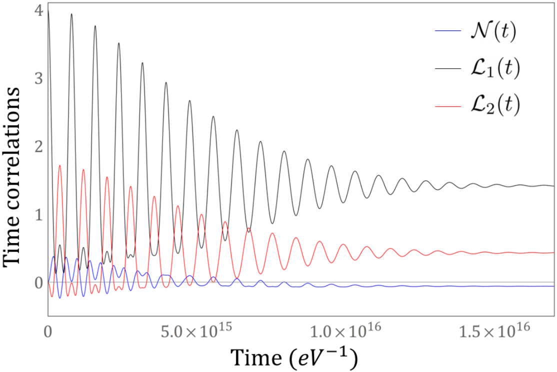
Correlators

$$
\begin{aligned}
C_{01} &= P_{e\to e}(t) - P_{e\to\mu}(t)\\
C_{12} &= P_{e\to e}(t) - P_{e\to\mu}(t)\\
C_{02} &= P_{e\to e}(2t) - P_{e\to\mu}(2t)
\end{aligned}
$$

LGIs

$$
\begin{aligned}
\mathcal{L}_1(t) &= 2P_{e\to e}(t) + 2P_{e\to e}(2t) - 2P_{e\to\mu}(t) \geq 0\\
\mathcal{L}_2(t) &= 2P_{e\to\mu}(t) - P_{e\to\mu}(2t) \geq 0\\
\mathcal{L}_3(t) &= 2P_{e\to\mu}(2t) \geq 0\\
\mathcal{L}_4(t) &= \mathcal{L}_3(t)
\end{aligned}
$$

NSIT vs LGIs



$\mathcal{N}(t)$ (blue) vs $\mathcal{L}_1(t)$ (black) and $\mathcal{L}_2(t)$ (red) as functions of time, with $\sin^2\theta = 0.314$, $\Delta m^2 = 7.92\times10^{-5}$ eV^2, $E = 10$ GeV and $\sigma_x = 0.5$ GeV^{-1}.

- At large times LGIs hide violation of macrorealism

Chiral oscillations & lepton-antineutrino entanglement

Chiral oscillations

- Taking into account (bi)spinorial nature of neutrinos and chiral nature of weak interaction, one naturally gets chiral oscillations [*]
- Interplay with flavor oscillations in the non-relativistic region[†]
- For CνB, chiral oscillations reduce detection by a factor of 2.[‡]

[*] A. Bernardini and S. De Leo, Phys. Rev. D (2005)

[†] V.A.Bittencourt, A.Bernardini and M.B.,Eur.Phys.J.C(2021);EPL Persp.(2022); M. W. Li, Z. L. Huang and X. G. He, Phys. Lett. B (2024); K. Kimura and A. Takamura, Annals Phys. (2025). T. Morozumi, and T. Tahara, Prog. Theor. Exp. Phys. (2025) V. Bittencourt, M. B. and G. Zanfardino, Phys. Lett. B (2025)

[‡] S.-F. Ge and P.Pasquini, Phys. Lett. B (2020)

Chiral oscillations

- Chiral oscillation formula

$$\mathcal{P}(t) = |\,\langle\psi_m(0)|\psi_m(t)\rangle\,|^2 = 1 - \frac{m^2}{E_{p,m}^2}\sin^2\left(E_{p,m}t\right),$$

Average value of the chiral operator $\langle\hat{\gamma}_5\rangle(t)$

$$\langle\hat{\gamma}_5\rangle(t) = \langle\psi_m(t)|\hat{\gamma}_5\,|\psi_m(t)\rangle = -1 + \frac{2m^2}{E_{p,m}^2}\sin^2\left(E_{p,m}t\right).$$

– Chiral oscillation period: $T_{ch} = \frac{2\pi}{E_{p,m}}$

– Chiral oscillation length: $L_{ch} = v\frac{2\pi}{E_{p,m}} = \frac{2\pi p}{E_{p,m}^2}$

Chiral and flavor oscillations

- State of a neutrino of flavor α at a given t:

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha,i}\,|\psi_{m_i}(t)\rangle \otimes |\nu_i\rangle\,,$$

where $|\psi_{m_i}(t)\rangle$ are bispinors.

- The state at $t = 0$ reads

$$|\nu_\alpha(0)\rangle = |\psi(0)\rangle \otimes \sum_i U_{\alpha,i}\,|\nu_i\rangle = |\psi(0)\rangle \otimes |\nu_\alpha\rangle\,,$$

where $|\psi(0)\rangle$ is a left handed bispinor.

- Survival probability:

$$\mathcal{P}_{\alpha\to\alpha} = |\langle\nu_\alpha(0)|\nu_\alpha(t)\rangle|^2 = \left|\sum_i |U_{\alpha,i}|^2\,\langle\psi(0)|\psi_{m_i}(t)\rangle\right|^2.$$

Two flavor mixing:

$$
\begin{aligned}
|\nu_e(t)\rangle &= \left[\cos^2\theta\,|\psi_{m_1}(t)\rangle + \sin^2\theta\,|\psi_{m_2}(t)\rangle\right] \otimes |\nu_e\rangle \\
&\quad + \sin\theta\cos\theta\left[|\psi_{m_1}(t)\rangle - |\psi_{m_2}(t)\rangle\right] \otimes |\nu_\mu\rangle\,,
\end{aligned}
$$

- The survival probability can be decomposed as

$$\mathcal{P}_{e\to e}(t) = \mathcal{P}^S_{e\to e}(t) + \mathcal{A}_e(t) + \mathcal{B}_e(t).$$

$\mathcal{P}^S_{e\to e}(t)$ is the standard flavor oscillation formula

$$\mathcal{P}^S_{e\to e}(t) = 1 - \sin^2 2\theta \sin^2\left(\frac{E_{p,m_2} - E_{p,m_1}}{2}t\right)$$

and

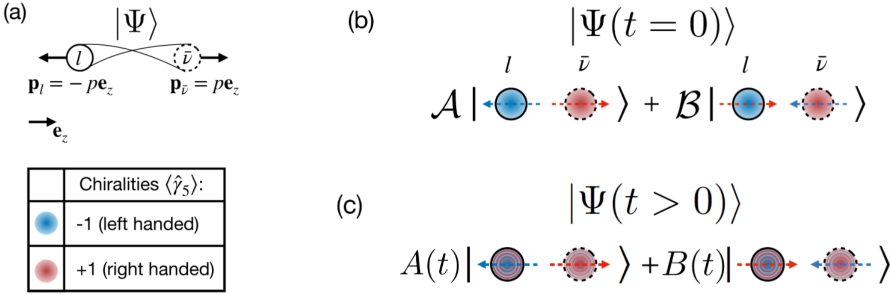
$$
\begin{aligned}
\mathcal{A}_e(t) &= -\left[\frac{m_1}{E_{p,m_1}}\cos^2\theta\sin\left(E_{p,m_1}t\right) + \frac{m_2}{E_{p,m_2}}\sin^2\theta\sin\left(E_{p,m_2}t\right)\right]^2, \\
\mathcal{B}_e(t) &= \frac{1}{2}\sin^2 2\theta\sin(E_{p,m_1}t)\sin(E_{p,m_2}t)\left(\frac{p^2 + m_1 m_2}{E_{p,m_1}E_{p,m_2}} - 1\right),
\end{aligned}
$$

are correction terms due to the bispinorial structure.

- Agreement with the QFT formula.

$l - \bar{\nu}$ entanglement and chiral oscillations*

Chiral oscillations, we consider induced spin correlations in pion decay products ($\pi \to l + \bar{\nu}$)



*V.A.Bittencourt, A.Bernardini and M.B., Universe (2021)

Spin entanglement at $t = 0$

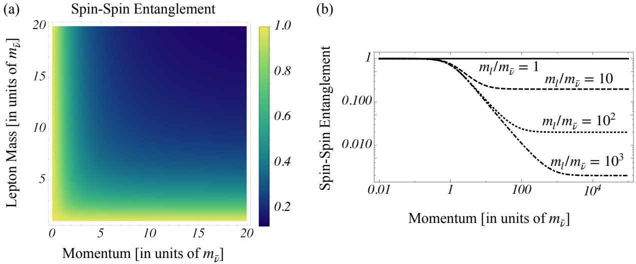
- The state of the lepton-antineutrino pair is then described in the composite Hilbert space $\mathcal{H}_{C_{\bar{\nu}}} \otimes \mathcal{H}_{S_{\bar{\nu}}} \otimes \mathcal{H}_{C_l} \otimes \mathcal{H}_{S_l}$.

- It is a 4-qubit entangled state.

- We can write $|\Psi(0)\rangle = |+_{C_{\bar{\nu}}}\rangle \otimes |-_{C_l}\rangle \otimes |\Psi_{S_{\bar{\nu}},S_l}\rangle$, with $|\pm_A\rangle$ denoting the positive (negative) chirality of $A = C_{\bar{\nu},l}$, and

$$|\Psi_{S_{\bar{\nu}},S_l}\rangle = \mathcal{A}(p, m_l, m_{\bar{\nu}})|\uparrow_{S_{\bar{\nu}}}\rangle \otimes |\downarrow_{S_l}\rangle - \mathcal{B}(p, m_l, m_{\bar{\nu}})|\downarrow_{S_{\bar{\nu}}}\rangle \otimes |\uparrow_{S_l}\rangle$$

is the joint spin state at $t = 0$.



Spin entanglement at $t \neq 0$

- The reduced matrix $\rho_{S_{\bar{\nu}},S_l}(t) = \mathrm{Tr}_{\mathrm{Chirality}}\left[|\Psi(t)\rangle\langle\Psi(t)|\right]$ describes a mixed state with entanglement dynamics directly affected by chiral oscillations.

- Entanglement between the spins at time t

$$\mathcal{N}_{S_{\bar{\nu}},S_l}(t) \equiv \mathcal{N}[\rho_{S_{\bar{\nu}},S_l}(t)] = ||\rho^{T}_{S_{\bar{\nu}},S_l}(t)|| - 1 = \mathcal{N}_{S_{\bar{\nu}},S_l}(0)\Gamma(t)$$

with

$$\Gamma(t) = \prod_{j=\bar{\nu},l}\left[1 - \frac{p^2}{m_j^2}\left(\langle\hat{\gamma}_5\rangle_j(t) - 1\right)^2\right]^{\frac{1}{2}}.$$

The average chiralities are given by $\langle\hat{\gamma}_5\rangle_A(t) = \mathrm{Tr}_A[\rho_A(t)]$ with $A = \bar{\nu}, l$:

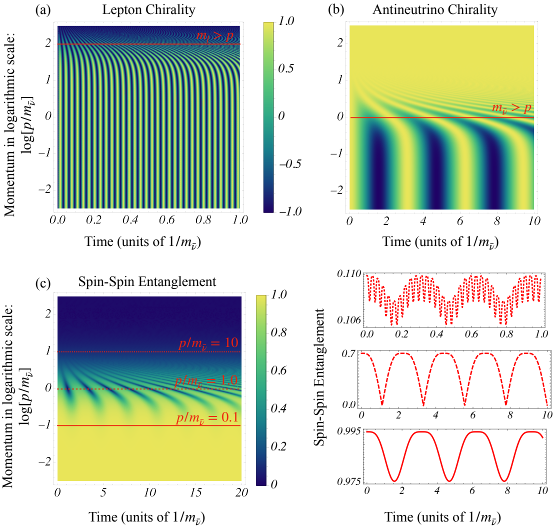
$$\begin{aligned}
\langle\hat{\gamma}_5\rangle_{\bar{\nu}}(t) &= 1 - \frac{m_{\bar{\nu}}^2}{E_{p,m_{\bar{\nu}}}^2}\left[1 - \cos\left(2E_{p,m_{\bar{\nu}}}t\right)\right], \\
\langle\hat{\gamma}_5\rangle_l(t) &= -1 + \frac{m_l^2}{E_{p,m_l}^2}\left[1 - \cos\left(2E_{p,m_l}t\right)\right].
\end{aligned}$$

Spin entanglement at $t \neq 0$

- $\mathrm{Tr}[\rho^2_{S_{\bar{\nu}},S_l}(t)] < 1 \Rightarrow$ entanglement initially encoded only in the spins redistributes into spin-chirality entanglement.

- Entanglement encoded in the bipartition $(C_{\bar{\nu}}, S_{\bar{\nu}}); (C_l, S_l)$ is conserved:

$$\begin{aligned} \mathrm{Tr}[\rho^2_{\bar{\nu}}(t)] = \mathrm{Tr}[\rho^2_l(t)] &= \mathcal{A}^4(p, m_l, m_{\bar{\nu}}) + \mathcal{B}^4(p, m_l, m_{\bar{\nu}}) \\ &= 1 - \frac{\mathcal{N}^2_{S_{\bar{\nu}},S_l}(0)}{2} < 1. \end{aligned}$$



(a) Average lepton chirality, (b) average antineutrino chirality and (c) spin-spin entanglement as a function of the momentum and of time.

Results

- We find that chiral oscillations do affect spin-spin correlations for the entangled lepton–antineutrino couple.

- Resonance of oscillation amplitude at neutrino mass: possibility of extracting fundamental information via quantum correlations.

- We have extended the study to the case in which flavor mixing is included: neutrino state is an *hyperentangled state* with three DoFs: chirality, spin and flavor.*

* V.A.Bittencourt, M.B. and G.Zanfardino, Physica Scripta (2024)

Entanglement dynamics in QED processes

Entanglement generation in QED processes*

How entanglement (in helicity) is generated in the processes:

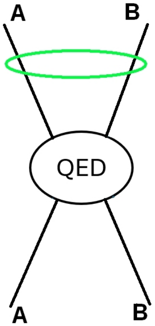
$$e^- e^+ \to \mu^- \mu^+$$

$$e^- e^+ \to e^- e^+$$

$$e^- \mu^- \to e^- \mu^-$$

$$e^- \gamma \to e^- \gamma$$

...



Initial and final helicity states:

$$|i\rangle = |R\rangle_A \otimes |L\rangle_B$$

$$|f\rangle = \sum_{r,s=R,L} \mathcal{M}(RL; rs)\, |r\rangle_A \, |s\rangle_B$$

* A. Cervera-Lierta et al., SciPost Phys. (2017); S. Fedida, A. Serafini, Phys. Rev. D (2023).

Unitarity

- S matrix is unitary: $\rho_f = S\rho_{in}S^\dagger$
- At tree level, unitarity is recovered by resorting by inclusion of 1-loop diagrams (optical theorem);†
- We consider a sharp momentum filtering on the final state ρ_f, corresponding to a POVM procedure: $\Pi_{\bar{p}} = \sum_\eta |\bar{p},\eta\rangle\langle\bar{p},\eta|$.
  Post-measurement state:
  $$\tilde{\rho}_f = \frac{\Pi_{\bar{p}}\rho_f\Pi_{\bar{p}}}{Tr(\Pi_{\bar{p}}\rho_f\Pi_{\bar{p}})}$$
  which can be expressed in terms of S-matrix elements.
- The process $\rho_{in} \to \tilde{\rho}_f$ is described by a non-unitary, non-linear map.

†K. Kowalska and E. M. Sessolo, JHEP (2024). S. Shivashankara and G. Gogliettino, Phys. Rev. D (2024)

Quantum process tomography[‡]

- The above scattering processes together with measurement of spin (or flavour) can be conveniently described in the framework of quantum channels and quantum instruments.

- Experimental reconstruction of Choi matrix is a powerful tool for precision tests of the Standard Model.

[‡] C. Altomonte, A. J. Barr, M. Eckstein, P. Horodecki, and K. Sakurai, Quantum Sci. Technol. (2025)

Quantum process tomography

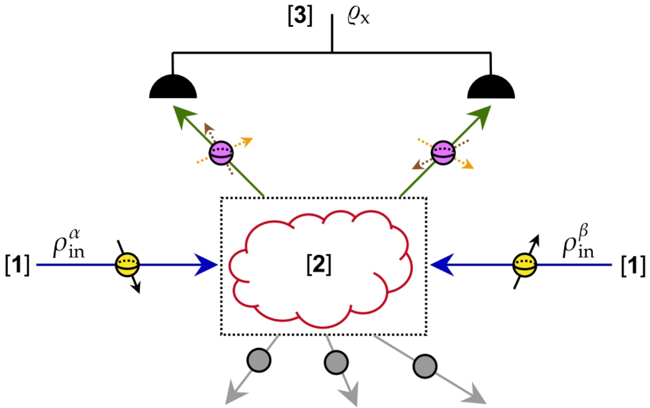


Figure 1. Experimental reconstruction of the Choi matrix associated with $2 \to 2$ scattering process. At step [**1**] we prepare the initial state, $\rho_{\rm in}^{\alpha} \otimes \rho_{\rm in}^{\beta}$, of the two beams. Step [**2**] is the scattering process, which is treated as a quantum-information-processing 'black-box'. At step [**3**] we reconstruct the relevant (unnormalised) states ϱ_x from the gathered data. The dotted coloured arrows of the outgoing particles illustrate the fact their joint spin (or flavour) state is entangled. The same procedure applies in the case of $2 \to n$ scattering process (see footnote 8), in which we trace-out the quantum degrees of freedom of the unobserved $n-2$ particles marked in gray colour in the figure.

Bhabha scattering in the COM reference frame

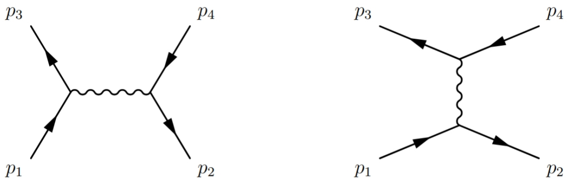


Figure 1: Feynman diagrams for Bhabha scattering.

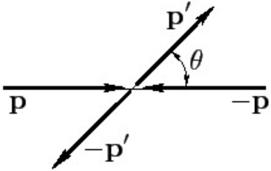


Figure 2: COM reference frame.

Bhabha scattering amplitudes (c.o.m. frame)

$$\mathcal{M}(RR;RR) = \mathcal{M}(LL;LL) = \frac{(2+11\mu^2+8\mu^4+2\cos\theta+\mu^2\cos 2\theta)\csc^2\left(\frac{\theta}{2}\right)}{4\mu^2(1+\mu^2)}$$

$$\mathcal{M}(RR;\genfrac{}{}{0pt}{}{RL}{LR}) = -\mathcal{M}(LL;\genfrac{}{}{0pt}{}{RL}{LR}) = -\frac{(1+\mu^2\cos\theta)\cot\left(\frac{\theta}{2}\right)}{\mu^2\sqrt{1+\mu^2}}$$

$$\mathcal{M}(RR;LL) = \mathcal{M}(LL;RR) = \frac{1+\mu^2(1+\cos\theta)}{\mu^2(1+\mu^2)}$$

$$\mathcal{M}(\genfrac{}{}{0pt}{}{RL}{LR};RR) = -\mathcal{M}(\genfrac{}{}{0pt}{}{RL}{LR};LL) = \frac{(1+\mu^2\cos\theta)\cot\left(\frac{\theta}{2}\right)}{\mu^2\sqrt{1+\mu^2}}$$

$$\mathcal{M}(RL;RL) = \mathcal{M}(LR;LR) = \frac{(1+\mu^2(1+\cos\theta))\cot^2\left(\frac{\theta}{2}\right)}{\mu^2}$$

$$\mathcal{M}(RL;LR) = \mathcal{M}(LR;RL) = 1-\cos\theta-\frac{1}{\mu^2}$$

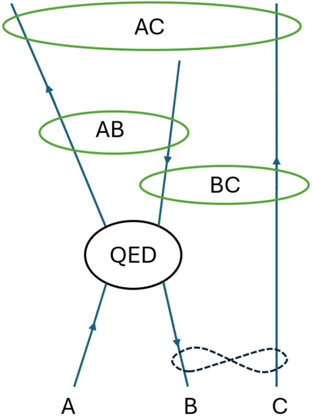
QED processes with spectator particles*

Initial and final states

$$|i\rangle = |R\rangle_A \otimes \Big(\cos\eta \, |R\rangle_B \, |R\rangle_C + e^{i\beta} \sin\eta \, |L\rangle_B \, |L\rangle_C \Big)$$

$$|f\rangle = \sum_{r,s=R,L} \big[\cos\eta \, \mathcal{M}(RR; rs) \, |r\rangle_A \, |s\rangle_B \, |R\rangle_C \; + \\ e^{i\beta} \sin\eta \, \mathcal{M}(RL; rs) \, |r\rangle_A \, |s\rangle_B \, |L\rangle_C \big]$$

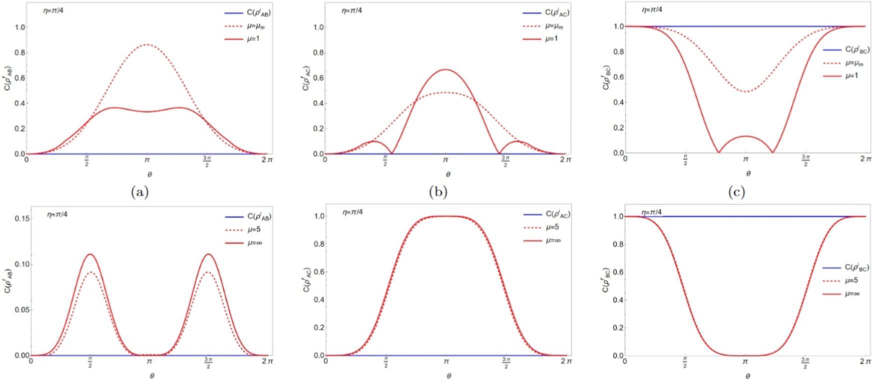
- We studied generation and distribution of entanglement in Bhabha scattering for the three bipartite channel AB, AC,BC.



*J.B. Araujo et al., Phys. Rev. D (2019).
M. B., G. Lambiase and B. Micciola, Phys. Rev. D (2024).

Entanglement distribution ($\eta = \pi/4$)

AB AC BC



(a) (b) (c)

(d) (e) (f)

Main results

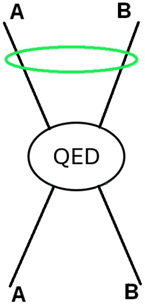
- The distribution and generation of entanglement tend to concentrate in some bipartitions.

- Complete transfer from BC channel to the AC channel in $\theta = \pi$ in relativistic regime: interaction as a *quantum gate* $\Rightarrow$ possible application in quantum information protocols.

- For $\frac{\pi}{4} < \eta < \frac{3\pi}{4}$ the interaction generates entanglement in the BC channel, where it was present before the interaction.

CCR for Bhabha scattering: initial states

We analyzed C, P, V before and after the QED interaction for three different states:



$$|i\rangle_I \quad = \quad |R\rangle_A \, |L\rangle_B$$

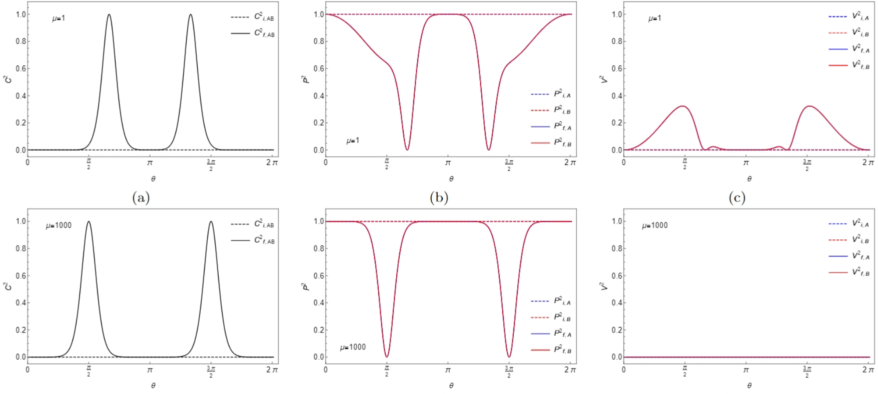
$$|i\rangle_{II} \quad = \quad \left(\cos\alpha \, |R\rangle_A + e^{i\xi} \sin\alpha \, |L\rangle_A\right) \otimes \left(\cos\beta \, |R\rangle_B + e^{i\eta} \sin\beta \, |L\rangle_B\right)$$

$$\begin{aligned} |i\rangle_{III} \quad = \quad & \cos\alpha \, |R\rangle_A \, |R\rangle_B + e^{i\xi} \sin\alpha \cos\beta \, |R\rangle_A \, |L\rangle_B \\ & + e^{i\eta} \sin\alpha \sin\beta \cos\chi \, |L\rangle_A \, |R\rangle_B + e^{i\tau} \sin\alpha \sin\beta \sin\chi \, |L\rangle_A \, |L\rangle_B \end{aligned}$$

* M.B., S.De Siena, G.Lambiase, C.Matrella and B.Micciola, Phys.Rev. D (2025).

Initial factorized state $|i\rangle_I$: $P_i \neq 1, \quad V_i \neq 0, \quad C = 0.$

C^2 $\qquad$ P_i^2 $\qquad$ V_i^2

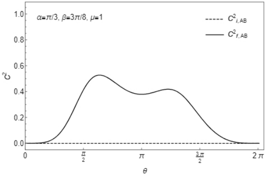


(a) (b) (c)

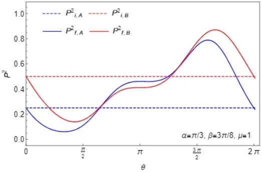
(d) (e) (f)

Initial factorized state $|i\rangle_{II}$: $P_i \neq 0, \quad V_i \neq 0, \quad C = 0.$

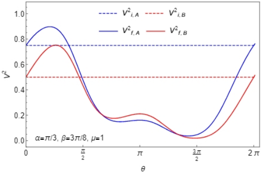
C^2 $\qquad$ P_i^2 $\qquad$ V_i^2



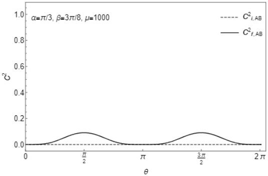
(a)



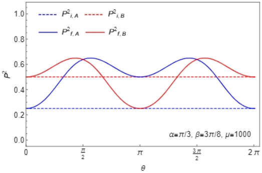
(b)



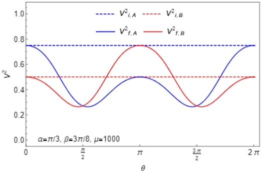
(c)



(d)

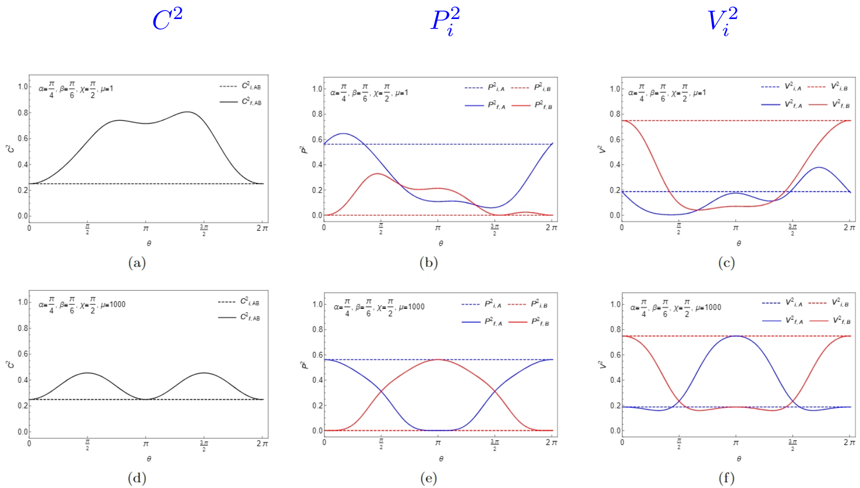


(e)



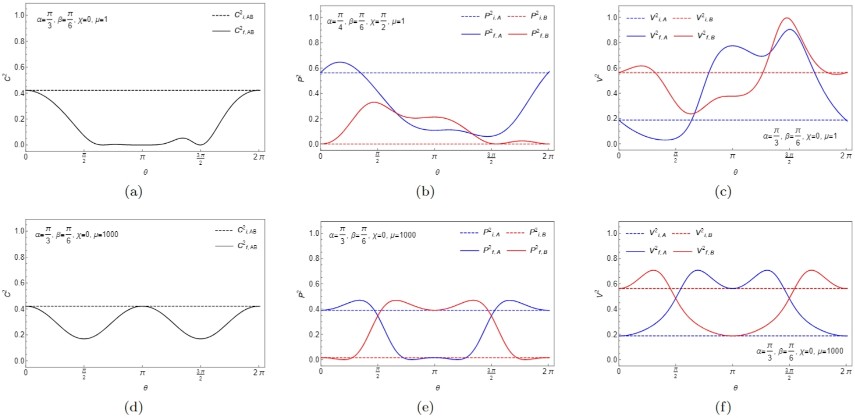
(f)

Initial entangled states $|i\rangle_{III}$: *Entanglophilus* regime



Initial entangled states $|i\rangle_{III}$: *Entanglophobous* regime

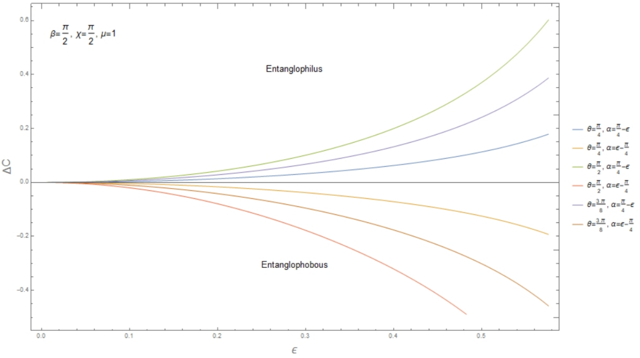
C^2 $\quad$ P_i^2 $\quad$ V_i^2



Entanglophilus, Entanglophobous and Mixed regimes

Entanglophilus, *Entanglophobous* states from Bell states

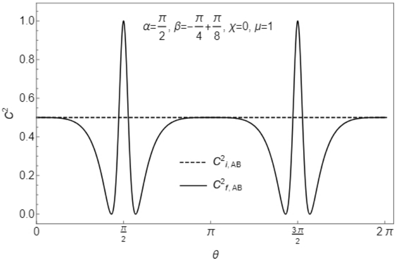
$$\Delta C \equiv (C_f - C_i)/C_i \qquad \Phi^{\pm}_{\alpha} = \cos\alpha\,|RR\rangle \pm \sin\alpha\,|LL\rangle \qquad \Phi^{\pm} = \frac{1}{\sqrt{2}}(|RR\rangle \pm |LL\rangle)$$



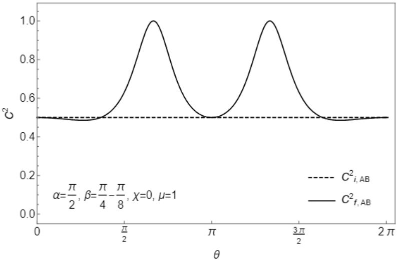
Mixed regime

Mixed regime from Bell states

$$\Psi^{\pm}_{\alpha} = \cos\alpha\,|RL\rangle \pm \sin\alpha\,|LR\rangle \qquad \Psi^{\pm} = \frac{1}{\sqrt{2}}(|RL\rangle \pm |LR\rangle)$$



(a)



(b)

Maximal entanglement conservation

$$P_i = 0 \qquad V_i = 0 \qquad C = 1$$

Processes preserving maximal entanglement:

- $e^- e^+ \to e^- e^+$
- $e^- e^- \to e^- e^-$
- $e^- e^+ \to \mu^- \mu^+$
- $e^- \mu^- \to e^- \mu^-$

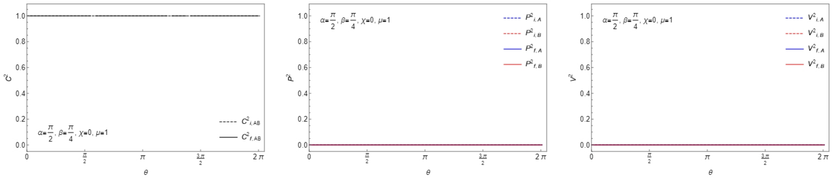
Example: Bhabha scattering

$$\Phi^+ \to \Phi^+$$

$$\Phi^- \to \cos s_1 \Phi^- + \sin s_1 \Psi^+$$

$$\Psi^+ \to \cos s_2 \Phi^- + \sin s_2 \Psi^+$$

$$\Psi^- \to \Psi^-$$



(a) (b) (c)

QED scattering with photons

- $e^- e^+ \to 2\gamma$: maximal entanglement is not conserved for all the initial maximally entangled states
- $e^- \gamma \to e^- \gamma$: maximal entanglement is never achieved

	$e^- e^+ \to \gamma\gamma$			Compton	
In. State	Fin. State	Conc.	In. State	Fin. State	Conc.
Φ^+	$\Phi-$	1	Φ^+	GC	< 1
Φ^-	$\cos r\, \Phi^+ + \sin r\, \Psi^+$	$\lvert \sin(2\,r) \rvert$	Φ^-	GC	< 1
Ψ^+	Ψ^+	1	Ψ^+	GC	< 1
Ψ^-	Ψ^-	1	Ψ^-	GC	< 1

Entanglement saturation

Quantum maps and invariant sets

We consider the matrices

$$\mathbf{M} = \begin{pmatrix} \mathcal{M}(RR;RR) & \mathcal{M}(RL;RR) & \mathcal{M}(LR;RR) & \mathcal{M}(LL;RR) \\ \mathcal{M}(RR;RL) & \mathcal{M}(RL;RL) & \mathcal{M}(LR;RL) & \mathcal{M}(LL;RL) \\ \mathcal{M}(RR;LR) & \mathcal{M}(RL;LR) & \mathcal{M}(LR;LR) & \mathcal{M}(LR;LL) \\ \mathcal{M}(RR;LL) & \mathcal{M}(RL;LL) & \mathcal{M}(LR;LL) & \mathcal{M}(LL;LL) \end{pmatrix}$$

$$\tilde{\rho}_f = \frac{\mathbf{M}\rho_{in}\mathbf{M}^T}{\mathrm{Tr}[\mathbf{M}\rho_{in}\mathbf{M}^T]},$$

The quantum map $\rho_{in} \to \tilde{\rho}_f$ is non-unitary and non-linear.

In Ref.† the invariant sets of the map, which include the set of the maximally entangled states for fermion-fermion processes, have been obtained.

† M. B., S. De Siena, G. Lambiase, C. Matrella and B. Micciola, Chaos, Solitons & Fractals (2025)

Spectral properties of M matrix ‡

For QED $2 \to 2$ scattering processes involving only fermions, the matrices $\mathbf{M}$ have the following properties:

- The set of maximally entangled states is invariant under the map;
- The eigenvectors of $\mathbf{M}$ are maximally entangled states, or special linear combinations of Bell states;
- For any n, the power $\mathbf{M}^n$ has the same form as $\mathbf{M}$.

These properties are related to invariance under parity.

‡ M. B., S. De Siena, G. Lambiase, B. Micciola and K.Simonov, [arXiv:2604.10136 [quant-ph]].

Entanglement saturation

By iterating this map, i.e. applying $\mathbf{M}^n$ to a general initial state (pure or mixed), we find that:

- Under infinite iteration of the map the process is driven by the largest eigenvalue, and always converges to an asymptotic state;

- Except for an extremely small set of degenerate cases, entanglement saturation is achieved both for pure and mixed initial states;

- The asymptotic rate of convergence is controlled by the spectral gap, which plays the role of an effective Lyapunov exponent for the induced dynamics.

Entanglement saturation in Bhabha scattering

The Bhabha process is described by the map

$$\mathbf{M} = \begin{pmatrix} A & -B & -B & D \\ B & E & F & -B \\ B & F & E & -B \\ D & B & B & A \end{pmatrix}.$$

with $A = \mathcal{M}(RR;RR), B = \mathcal{M}(RR;RL), D = \mathcal{M}(RR;LL), E = \mathcal{M}(RL;RL), F = \mathcal{M}(RL;LR)$.

After iteration, we obtain

$$\mathbf{M}^n = \begin{pmatrix} A_n & -B_n & -B_n & D_n \\ B_n & E_n & F_n & -B_n \\ B_n & F_n & E_n & -B_n \\ D_n & B_n & B_n & A_n \end{pmatrix},$$

where any element in this matrix is expressed in terms of those of $\mathbf{M}$.

Entanglement saturation mechanism (Bhabha scattering)

- The Bhabha matrix has four eigenvectors $|\lambda_i\rangle$ and eigenvalues λ_i.

– In a particular case, one obtains:

$|\lambda_1\rangle = \Phi^+, \;\; |\lambda_2\rangle = \Psi^-, \;\; |\lambda_3\rangle = \cos\delta_3\Phi^- + \sin\delta_3\Psi^+, \;\; |\lambda_4\rangle = \cos\delta_4\Phi^- + \sin\delta_4\Psi^+,$

δ_3, δ_4 functions of the matrix elements $A, ..., F$.

– Any initial state can be written in terms of the $|\lambda_i\rangle$. For example:

$$|RL\rangle = \frac{1}{\sqrt{2}}(\Psi^+ + \Psi^-) = \frac{1}{\sqrt{2}}[\Psi^- + c_3' \,|\lambda_3\rangle + c_4' \,|\lambda_4\rangle],$$

with $c_3' \cos\delta_3 + c_4' \cos\delta_4 = 0$ and $c_3' \sin\delta_3 + c_4' \sin\delta_4 = 1$.

– The action of $\mathbf{M}^n$ gives

$$\mathcal{N}_n^{-1}\mathbf{M}^n \,|RL\rangle \; = \; \frac{1}{\sqrt{2}}\mathcal{N}_n^{-1}\left[\lambda_2^n\Psi^- + \lambda_3^n c_3' \,|\lambda_3\rangle + \lambda_4^n c_4' \,|\lambda_4\rangle\right]$$

with $\mathcal{N}_n$ a normalization factor.

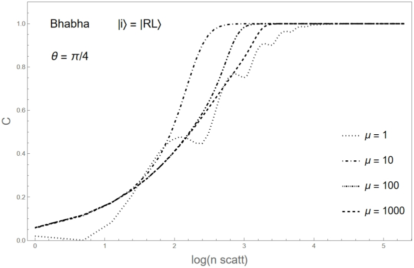
- For $n \to \infty$, the eigenvector with dominant eigenvalue will survive.

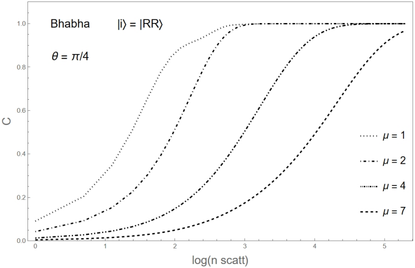
Entanglement saturation

SCATTERING PROCESS	INITIAL STATE(S)	REGIME	ASYMPTOTIC STATE
Bhabha	$\|RL\rangle$	u. r.	Ψ^+
Bhabha	$\|RL\rangle$	n. r.	$\cos s_1 \Phi^- + \sin s_1 \Psi^+$
Bhabha	$\|RR\rangle$	n. r.	Φ^+
Møller	$\|RL\rangle$	u. r.	Ψ^-
Møller	$\|RR\rangle$	n. r.	Φ^-
Møller	$\|RL\rangle$	n. r.	$\cos s_3 \Phi^+ + \sin s_3 \Psi^-$

Entanglement saturation: Bhabha scattering

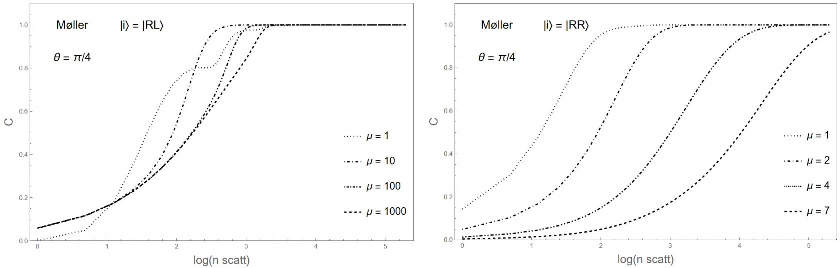
Entanglement saturation in Bhabha scattering at $\theta = \pi/4$ with initial states $|RR\rangle$ and $|LL\rangle$



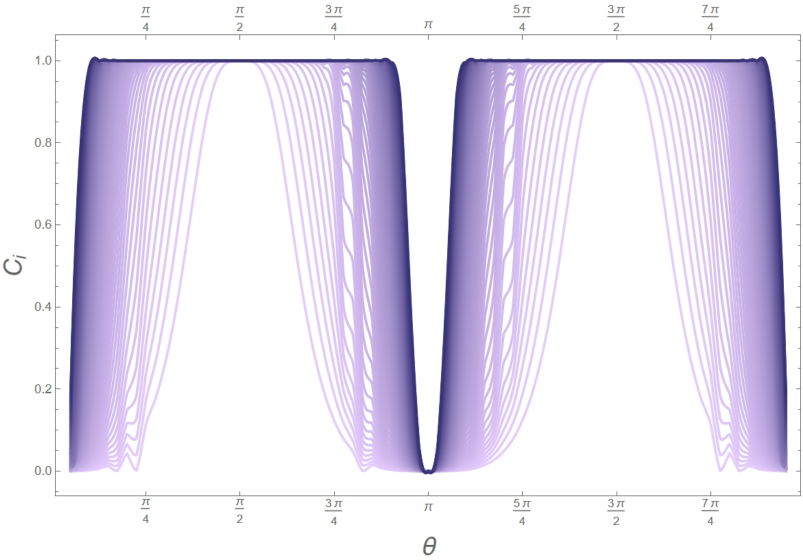


Entanglement saturation: Møller scattering

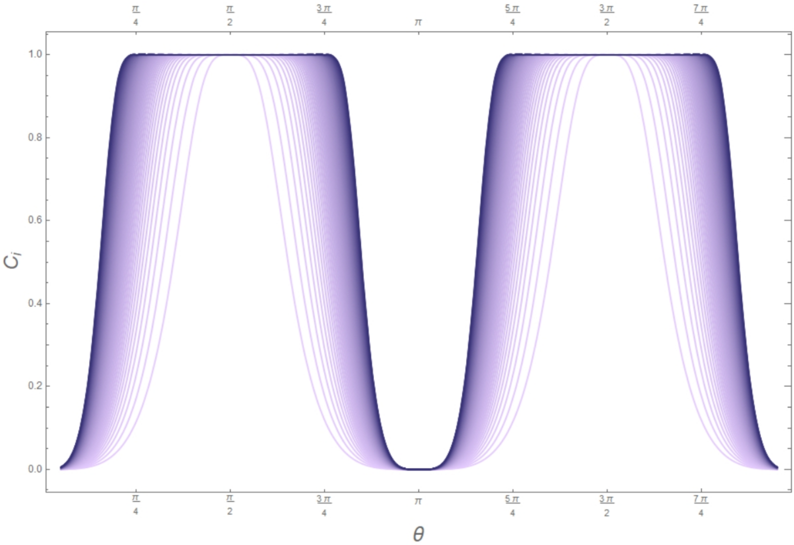
Entanglement saturation in Møller scattering at $\theta = \pi/4$ with initial statesv $|RR\rangle$ and $|LL\rangle$



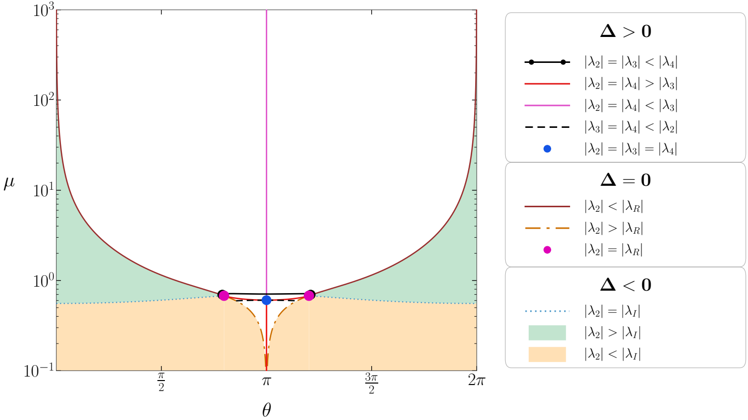
Entanglement saturation: Bhabha scattering low energy



Entanglement saturation: Bhabha scattering - high energy



Spectral analysis for Bhabha matrix $\mathbf{M}_B$



Regions in the plane (θ, μ) characterized by the spectral properties of the Bhabha matrix $\mathbf{M}_B$ for the case $c_1 = 0$ in the expansion $|v\rangle = \sum_s c_s |\lambda_s\rangle$.

Other work

In a recent paper, in the line of our work, tree-level Bhabha scattering process has been studied within the framework of quantum resource theory, exploring other observables like entropic uncertainty, quantum coherence, and Bell nonlocality.

Z.Cao, M.-L. Song, X.-K. Song, L.Ye, D. Wang,
Dynamical redistribution of quantum resources in tree-level Bhabha scattering, Physics Letters B (2026).

Conclusions

- CCR give a complete characterization of entanglement generation in QED scattering processes at tree level;

- We found regimes in which entanglement increases/decreases with respect to its initial value;

- Remarkably, maximal entanglement (Bell states) is totally preserved for processes that involve massive fermions;

- QED scattering processes are described by quantum maps whose iterations produces, for $2 \to 2$ fermions, a saturation of entanglement.

- Spectral properties of the maps characterizes entanglement saturation;

Outlook

- Extension of the analysis to other interactions;
- 1-loop corrections;
- Analysis of the same processes in other reference frames;
- Relation between conservation/saturation of entanglement and symmetries;
- Relevance for quantum information protocols;
- Possible identification of new physics BSM.

Entanglement in electroweak processes (preliminary results)

QEDtool

- **QEDtool**: object-oriented Python package for numerical QED calculations, with focus on entanglement quantification

Package Modules

- relativity
- qed
- qinfo
- standard_scattering

Computer Physics Communications
Available online 14 March 2026, 110127
In Press, Journal Pre-proof What's this?

Computer Programs in Physics

QEDtool: a Python package for numerical quantum information in quantum electrodynamics

Jesse Smeets [a], Preslav Asenov [b], Alessio Serafini [b]

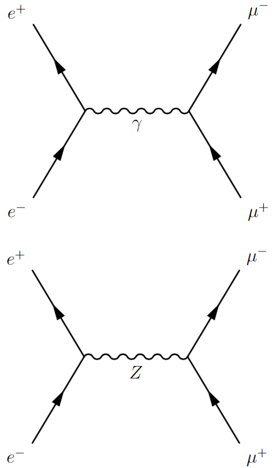
- By means of QED tools, previous results were confirmed; new modules added (partial trace, CCR, etc.)[§]

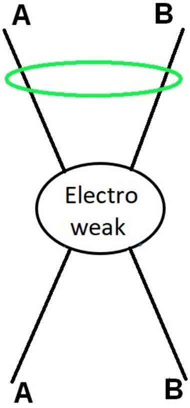
[§] J.Battich, *Entanglement Dynamics in Electroweak Scattering Processes*, Tesi di Laurea Magistrale in Fisica, Università di Salerno (2026).

Entanglement generation in electroweak processes[¶]

We consider the process:

$$e^- e^+ \to \mu^- \mu^+$$



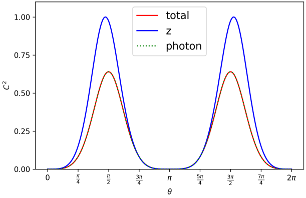


Initial states:

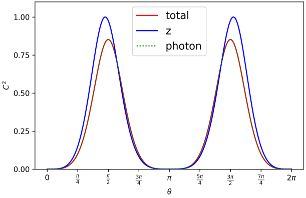
$$\{ |i\rangle_T \, , \; |i\rangle_I \, , \; |i\rangle_{II} \, , \; |i\rangle_{III} \}$$

[¶] J.Battich, M.B., S.De Siena, G.Lambiase and B.Micciola, work in progress.

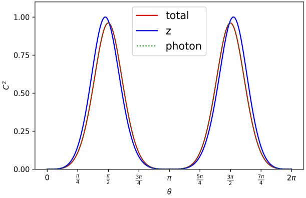
Entanglement generation for EW process $e^-e^+ \to \mu^-\mu^+$



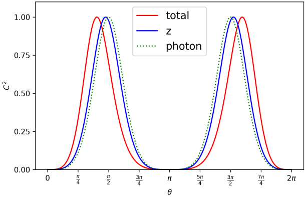
(e) $\sqrt{s} = 0.63\,\mathrm{GeV}$



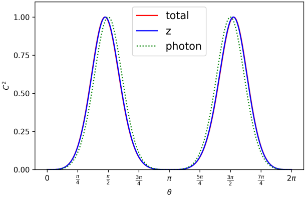
(f) $\sqrt{s} = 1.05\,\mathrm{GeV}$



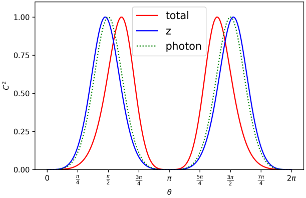
(g) $\sqrt{s} = 5.25\,\mathrm{GeV}$



(h) $\sqrt{s} = 63\,\mathrm{GeV}$



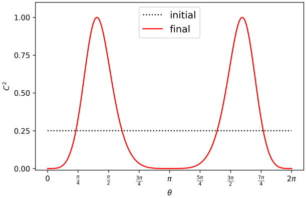
(i) $\sqrt{s} = 90.51\,\mathrm{GeV}$



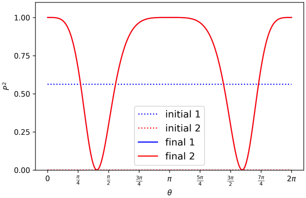
(j) $\sqrt{s} = 168\,\mathrm{GeV}$

CCR for $e^-e^+ \to \mu^-\mu^+$ at $\sqrt{s} = 63\text{GeV}$

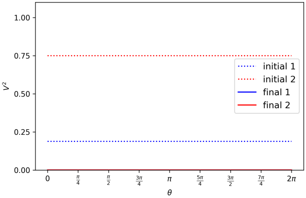
Previously Entanglophilus



(k) C^2

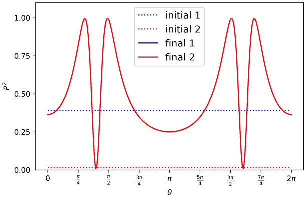


(l) P^2

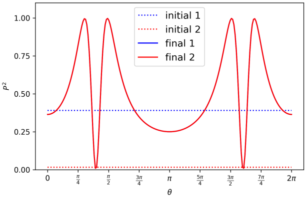


(m) V^2

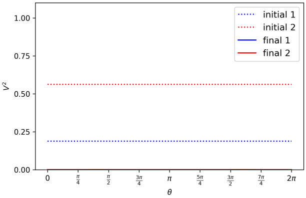
Previously Entanglophobous



(n) C^2

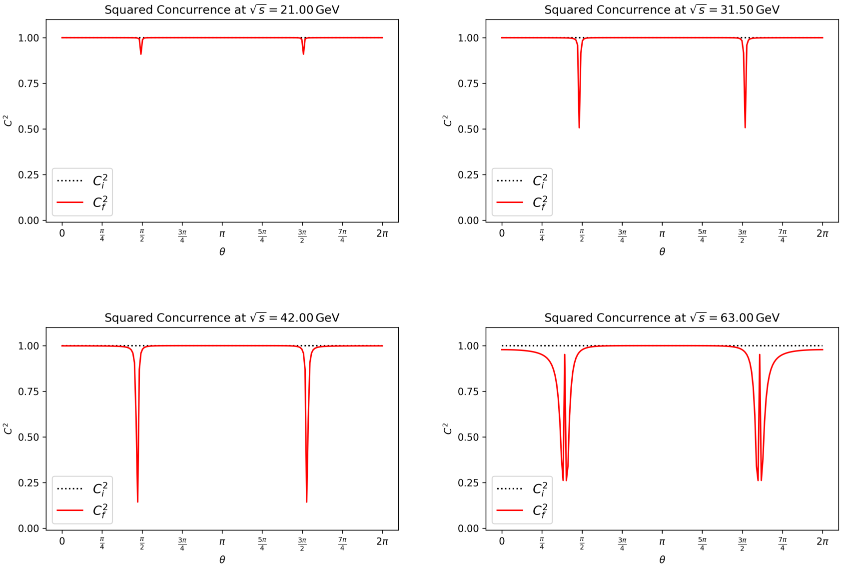


(o) P^2

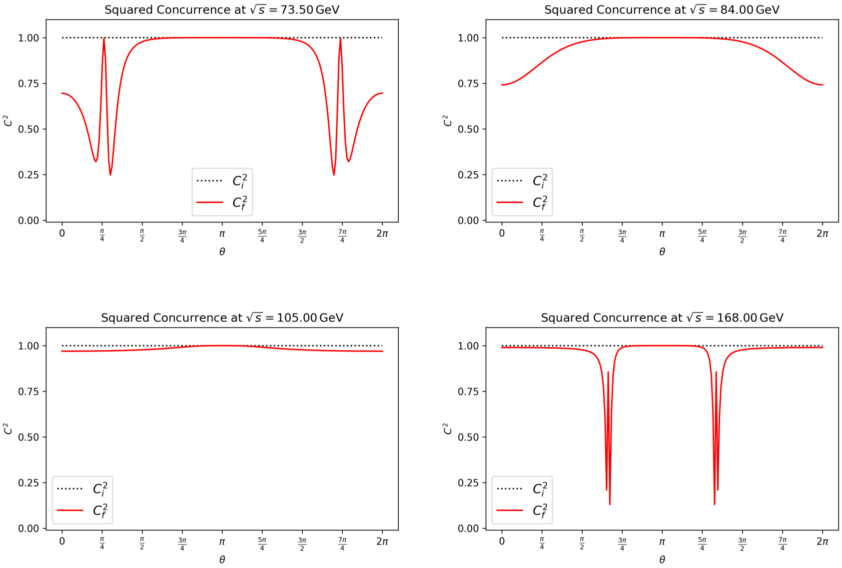


(p) V^2

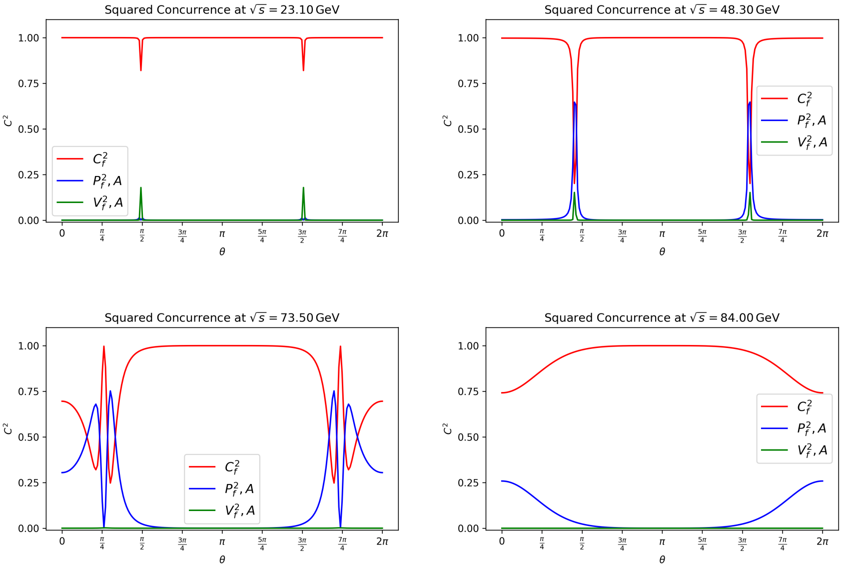
Maximal entanglement (non)-conservation



Maximal entanglement (non)-conservation

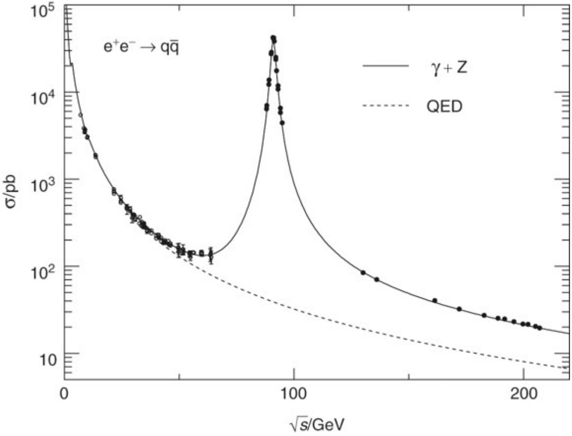


Maximal entanglement (non)-conservation



Electroweak total cross section for the prosess $e^{+}e \to q\bar{q}$

Measurements from LEP close to and above Z resonance



(q)

¶M.Thomson, *Modern Particle Physics* (Cambridge University Press, 2013).

(Preliminary) results

- QEDtool is an useful tool for the computation of quantum correlations in scattering processes;

- Some features of entanglement dynamics in QED processes, are not guaranteed to hold in the electroweak theory due to parity violation;

- Maximal entanglement non-conservation as a possible probe for new physics.