Determining the Spin-Analyzing Powers via Invariants and Probing the Quantum Entanglement and Bell Non-Locality at the Lepton Colliders

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Dianwei Wang, Xiqing Hao, Liwei Liu, Lina Wu, Tianjun Li, arXiv:2605.13250; 2606.16365 [hep-ph]; In preparations.

The Cornerstones for Modern Physics

- Quantum Mechanics.
- Special Relativity.

Two Most Genuine Features of Quantum Mechanics

- Quantum entanglement.
- Bell's theorem.

Quantum Entanglement

- EPR Paradox: in 1935, Einstein-Podolsky-Rosen demonstrated the conflict between local realism and quantum mechanics by considering quantum entangled states . They claimed that quantum mechanics is incomplete.
- In 1935, Schrödinger called it quantum entanglement, a characteristic of quantum mechanics.
- In 1964, Bell proposed the inequality which can be satisfied by local hidden-variable theory. However, it can be violated by quantum mechanics, which is called Bell non-locality.

Quantum Entanglement and Bell Non-Locality

- Over decades, quantum entanglement and Bell non-locality have been rigorously confirmed through various experiments violating Bell inequalities and demonstrations of quantum teleportation, primarily in low-energy systems such as photons, atoms, and solid-state qubits.
- During the last few years, the quantum entanglement and Bell non-locality have been studied extensively in the collider experiments.

Munchhausen Trilemma in Philosophy

- A circular argument.
- An infinite regression.
- Dogmatism

No-Go Theorem for Collider Physics

- No-Go Theorem: we cannot probe the quantum entanglement and Bell non-localtiy at the colliders [1].
- To evade the no-go theorem, we need to rule out the Local Hidden Variable Theories (LHVTs).
- We cannot use quantum mechanics (QM) and QFT since we test QM and QFT against the LHVTs.

[1] S. A. Abel, M. Dittmar and H. K. Dreiner, Phys. Lett. B **280**, 304-312 (1992); H. K. Dreiner, [arXiv:hep-ph/9211203 [hep-ph]]; S. Li, W. Shen and J. M. Yang, Eur. Phys. J. C **84**, no.11, 1195 (2024); M. Fabbrichesi, R. Floreanini and L. Marzola, [arXiv:2503.18535 [quant-ph]]; P. Bechtle, C. Breuning, H. K. Dreiner and C. Duhr, [arXiv:2507.15947 [hep-ph]]; S. A. Abel, H. K. Dreiner, R. Sengupta and L. Ubaldi, [arXiv:2507.15949 [hep-ph]]; M. Low, [arXiv:2508.10979 [hep-ph]].

No-Go Theorem for Collider Physics

- In the current collider experiments, we cannot conduct the direct spin measurements, and thus the spin correlations cannot be directly measured.
- The main idea for the collider studies is that the spin density matrix and spin correlations can be measured indirectly from the appropriate angular distributions and correlations between the final states from the intermediate particle decays.
- The bridges between the spin density matrix/spin correlations and angular distributions/correlations are the spin-analyzing powers of the intermediate particles.
- The great challenge is that we must determine the spin-analyzing powers without the assumptions of the QM and QFT.

The General Framework

- At the e^+e^- collider, we consider the production of a pair of fermions F_aF_b, *i.e.*, $e^+e^- \to F_aF_b$, which subsequently decay

$$F_{a/b} \to f_{a/b,1} + f_{a/b,2} + \dots + f_{a/b,N} \ .$$

- We choose the helicity rest frames for F_a and F_b, respectively, and define the same coordinate system for them

$$\hat{\mathbf{y}} \equiv \frac{\hat{\mathbf{p}}_{e^-} \times \hat{\mathbf{p}}_{F_a}}{|\hat{\mathbf{p}}_{e^-} \times \hat{\mathbf{p}}_{F_a}|}, \ \hat{\mathbf{z}} \equiv \hat{\mathbf{p}}_{F_a}, \ \hat{\mathbf{x}} \equiv \hat{\mathbf{y}} \times \hat{\mathbf{z}} \ .$$

- This coordinate system is convenient because the difference between the rest frames of F_a and F_b is a pure boost along their momentum directions.

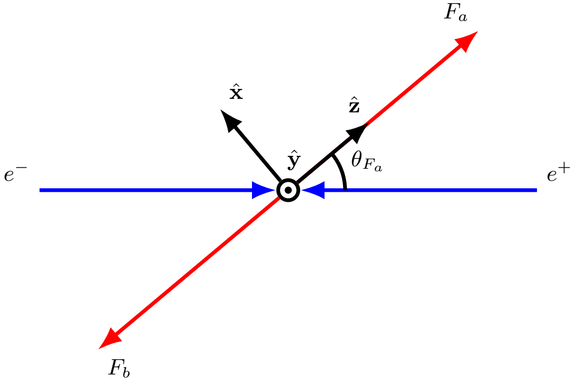

Figure: The coordinate system $(\hat{x}, \hat{y}, \hat{z})$ for the production process $e^+e^- \to F_a F_b$.

The General Framework

- Selecting two fermions $f_{a,1}$ and $f_{b,1}$ respectively from the decay products of F_a and F_b, we assume that the unit momentum directions for $f_{a,1}$ and $f_{b,1}$ in the rest frames of F_a and F_b are $\vec{q}_{a1}$ and $\vec{q}_{b1}$, respectively. Moreover, we formally define four-vectors q^μ_{a1} and q^μ_{b1} with $q^0_{a1} \equiv 1$ and $q^0_{b1} \equiv 1$, and the expectation values of their products

$$\mathcal{Q}_{\mu\nu} \equiv \langle q^\mu_{a1} q^\nu_{b1} \rangle .$$

  which can be measured in the collider experiments.
- In Quantum Information Science (QIS), we describe the spin polarization state for the quantum composite $F_a F_b$ system via a spin density matrix
- We study the relations between such angular distributions/correlations and the spin density matrix/spin correlation matrix for F_a and F_b.

Spin Density Matrix ρ

- The trace of ρ is unity: $\mathrm{Tr}\rho = 1$.
- ρ is a Hermitian matrix, and then $\rho^D = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$.
- All the diagonal elements are non-negative.
- ρ satisfies the Schwarz inequality: $|\rho_{ij}|^2 \leq \rho_{ii}\rho_{jj}$.
- $\mathrm{Tr}\rho^2 = \sum_{m=1}^{n} \lambda_m^2 \leq (\sum_{m=1}^{n} \lambda_m)^2 = (\mathrm{Tr}\rho)^2 = 1$.
- For pure and mixed states, we have $n = 1$ and $n > 1$, respectively.

Spin Density Matrix

- The spin polarization state for the composite quantum $F_a F_b$ system is described by a spin density matrix

$$\rho = \frac{I_4 + \sum_i \left(B_i^+ \sigma^i \otimes I_2 + B_i^- I_2 \otimes \sigma^i \right) + \sum_{i,j} C_{ij} \sigma^i \otimes \sigma^j}{4} .$$

- B_i^+ and B_i^- are respectively the spin polarizations of F_a and F_b, and C_{ij} is the spin correlation matrix. Under CP invariance, we obtain $B_i^+ = B_i^-$, and $C_{ij} = C_{ji}$.

Spin Density Matrix

$$\rho = \frac{1}{4}\begin{bmatrix} 1+B_3^+ + B_3^- + C_{33} & B_1^- + C_{31} - i(B_2^- + C_{32}) & B_1^+ + C_{13} - i(B_2^+ + C_{23}) & C_{11} - C_{22} - i(C_{12} + C_{21}) \\ B_1^- + C_{31} + i(B_2^- + C_{32}) & 1 + B_3^+ - B_3^- - C_{33} & C_{11} + C_{22} + i(C_{12} - C_{21}) & B_1^+ - C_{13} - i(B_2^+ - C_{23}) \\ B_1^+ + C_{13} + i(B_2^+ + C_{23}) & C_{11} + C_{22} + i(C_{21} - C_{12}) & 1 - B_3^+ + B_3^- - C_{33} & B_1^- - C_{31} - i(B_2^- - C_{32}) \\ C_{11} - C_{22} + i(C_{21} + C_{12}) & B_1^+ - C_{13} + i(B_2^+ - C_{23}) & B_1^- - C_{31} + i(B_2^- - C_{32}) & 1 - B_3^+ - B_3^- + C_{33} \end{bmatrix}$$

Spin Density Matrix ρ

- Defining $\sigma^\mu \equiv (\sigma^0, \sigma^1, \sigma^2, \sigma^3)$ with $\sigma^0 = I_2$, we can write the above spin density matrix as

$$\rho \equiv \frac{1}{4}\mathcal{T}_{\mu\nu}\sigma^\mu \otimes \sigma^\nu \ .$$

- $\mathcal{T}_{\mu\nu}$ are defined as

$$\mathcal{T}_{00} \ \equiv \ 1 \ , \ \mathcal{T}_{i0} \ \equiv \ B_i^+ \ , \ \mathcal{T}_{0i} \ \equiv \ B_i^- \ , \ \mathcal{T}_{ij} \ \equiv \ C_{ij} \ .$$

Spin Density Matrix ρ

- In the basis $(|\frac{1}{2}\rangle_{F_a} \otimes |\frac{1}{2}\rangle_{F_b}, |\frac{1}{2}\rangle_{F_a} \otimes |-\frac{1}{2}\rangle_{F_b}, |-\frac{1}{2}\rangle_{F_a} \otimes |\frac{1}{2}\rangle_{F_b}, |-\frac{1}{2}\rangle_{F_a} \otimes |-\frac{1}{2}\rangle_{F_b})$, we can write the spin density matrix as

$$\rho \equiv \sum_{i=1}^{n} p_i |\Psi_i\rangle\langle\Psi_i| \ , \ \sum_{i=1}^{n} p_i = 1 \ .$$

$n \leq 16$, and p_i is a positive real number.

- $|\Psi_i\rangle\langle\Psi_i|$ is a pure state, and we define its quantum state (polarization state) $|\Psi_i\rangle$ as

$$|\Psi_i\rangle \equiv \sum_{k,j=\pm\frac{1}{2}} \alpha^i_{k,j} |k\rangle_{F_a} \otimes |j\rangle_{F_b} \ .$$

- The normalization condition is

$$\sum_{k,j=\pm\frac{1}{2}} |\alpha^i_{k,j}|^2 = 1 \ .$$

The Correlation Relations in the QFT

- The relations between the spin density matrix's $\mathcal{T}_{\mu\nu}$ and angular distributions/correlations $\mathcal{Q}_{\mu\nu}$ in the QFT are [2]

$$\mathcal{T}_{\mu\nu} = \left(\frac{3}{\alpha_{F_a}}\right)^{\eta(\mu)} \left(\frac{3}{\alpha_{F_b}}\right)^{\eta(\nu)} \mathcal{Q}_{\mu\nu} \, .$$

$\eta(0) = 0$, and $\eta(1) = \eta(2) = \eta(3) = 1$.

- α_{F_a} and α_{F_a} are spin-analyzing powers for F_a and F_b, respectively. In particular, we have $-1 \leq \alpha_{F_{a/b}} \leq 1$.

[2] T. Han, M. Low and Y. Su, JHEP **10**, 217 (2025) [arXiv:2501.04801 [hep-ph]]; P. Bechtle, C. Breuning, H. K. Dreiner and C. Duhr, [arXiv:2507.15947 [hep-ph]].

The Correlation Relations in the LHVTs

In the LHVTs, to derive the relations between the spin correlations and angular correlations, we make the following assumptions [3]

1. A LHVT with a set of local hidden variables.
2. Special relativity and Poincaré-invariance hold.
3. The decays of F_a and F_b are independent of each other.
4. The spin for each particle is an element of reality in the EPR sense, *i.e.*, a vector with a definite orientation. And thus the spins play the role of hidden variables.
5. If the particles F_a and F_b have spins $\hat{s}_a$ and $\hat{s}_b$, the probability distributions (in their rest frames) of the momenta of the daughter particles $f_{a,1}$ and $f_{b,1}$ are always the same and depend only on $\vec{s}_a$ and $\vec{s}_b$, respectively.

[3] P. Bechtle, C. Breuning, H. K. Dreiner and C. Duhr, [arXiv:2507.15947 [hep-ph]]; J. A. Aguilar-Saavedra, J. A. Casas and J. M. Moreno, [arXiv:2603.19389 [hep-ph]].

- With these assumptions, we can obtain the same relations in the LHVTs as these in QFT.
- The key difference is that we now have $-3 \leq \alpha_{F_{a/b}} \leq 3$.
- With such ranges for the spin-analyzing powers, we can show that we can not probe the Bell non-locality at the collider.
- Therefore, the great challenge is how to determine the product of the spin-analyzing powers $\alpha_{F_a}\alpha_{F_b}$ without the QM and QFT assumptions.

Three Proposals to Evade the No-Go Theorem

- The absolute values of the spin-analyzing powers for $F_a F_b$ are equal to 1, *i.e.*, the maximal value for fermion in QFT, for example, $t\bar{t}$. ???
- Invariants of the spin correlation matrices.
- Invariants of the spin density matrices.

Invariants of the Spin Correlation Matrices

With the assumption that the spin is defined via the Lorentz symmetry or considering the implicit symmetry in the spin density matrix, we can show that we can determine the product of the spin-analyzing powers $\alpha_{F_a}\alpha_{F_b}$ by the invariants of the spin correlation matrices, and probe the Bell non-locality at the lepton collider.

Definition of Spin via the Lorentz Symmetry

- Poincaré symmetry is the basic symmetry for the special relativity, and contains the Lorentz symmetry and translation symmetry.
- There are two Casmir operators for Poincaré algebra: $P^\mu P_\mu$ and $W^\mu W_\mu$, where P_μ and W_μ are the four-momentum and Pauli–Lubanski pseudo-vector, respectively. These two Casmir operators define the rest mass and spin of a particle, respectively.
- We consider the proper orthochronous group $SO(3,1)$ for Lorentz symmetry. The complexified Lorentz algebra for $SO(3,1)$ is locally isomorphic to the direct sum of two $SU(2)$ algebra, and thus we can formally write it as $SO(3,1) \simeq SU(2)_1 \oplus SU(2)_2$.

Definition of Spin via the Lorentz Symmetry

- Every finite-dimensional representation of the Lorentz group can be written as two spins (j_1, j_2), where j_i is an integer or half-integer.
- We obtain the representations for the scalar, left-handed Weyl fermion, right-handed Weyl fermion, gauge boson as $(0, 0)$, $(1/2, 0)$, $(0, 1/2)$, $(1/2, 1/2)$, respectively.
- The spin $SU(2)_S$ group can be considered as the diagonal subgroup of $SU(2)_1 \oplus SU(2)_2$.

Assumptions: Strong to Weak

- The definition assumption is the solid and fundamental assumption to probe quantum entanglement and Bell non-locality at the colliders. However, it can be relaxed.
- The key point is the following. The Lorentz symmetry is not violated, and the spin density matrix describes this composite quantum system. And thus, Lorentz symmetry is an implicit symmetry in the spin density matrix.
- In fact, there exists a $SU(2)_a \times SU(2)_b$ symmetry in the spin density matrix. The generators for $SU(2)_a$ and $SU(2)_b$ are $\sigma^i \otimes I_2/4$ and $I_2 \otimes \sigma^i/4$, respectively.

Assumptions: Strong to Weak

- $(|\frac{1}{2}\rangle_{F_a}, |-\frac{1}{2}\rangle_{F_a})^T$ forms the fundamental representation of $SU(2)_a$, and $(|\frac{1}{2}\rangle_{F_b}, |-\frac{1}{2}\rangle_{F_b})^T$ forms the fundamental representation of $SU(2)_b$.
- The spin $SU(2)_S$ group can be considered as the diagonal subgroup of $SU(2)_a \times SU(2)_b$.

Invariants of the Spin Correlation Matrices

- Because both $(|\frac{1}{2}\rangle_{F_a}, |-\frac{1}{2}\rangle_{F_a})^T$ and $(|\frac{1}{2}\rangle_{F_b}, |-\frac{1}{2}\rangle_{F_b})^T$ belong to the fundamental representation of $SU(2)_S$, we can decompose the product of these two fundamental representations into the irreducible representations of $SU(2)_S$, *i.e.*, decompose the basis of spin density matrix into the irreducible representations of $SU(2)_S$. Such decomposition is

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3} \ .$$

- The singlet $\mathbf{1}$ and triplet $\mathbf{3}$ belong to the anti-symmetric and symmetric representations of $SU(2)_S$. In the collider experiments, $\mathbf{1}$ and $\mathbf{3}$ correspond to the CP-odd scalar and gauge boson, respectively, which can couple to both e^+e^- and F_aF_b.

Quantum States

By definition, the quantum state for the singlet in the anti-symmetric representation is

$$|\Psi_S\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2}\rangle_{F_a} \otimes |-\frac{1}{2}\rangle_{F_b} - |-\frac{1}{2}\rangle_{F_a} \otimes |\frac{1}{2}\rangle_{F_b}\right) .$$

And the quantum states for the triplet in the symmetric representation are

$$|\Psi_T^{(1,1)}\rangle = |\frac{1}{2}\rangle_{F_a} \otimes |\frac{1}{2}\rangle_{F_b} ,$$

$$|\Psi_T^{(1,0)}\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2}\rangle_{F_a} \otimes |-\frac{1}{2}\rangle_{F_b} + |-\frac{1}{2}\rangle_{F_a} \otimes |\frac{1}{2}\rangle_{F_b}\right) ,$$

$$|\Psi_T^{(1,-1)}\rangle = |-\frac{1}{2}\rangle_{F_a} \otimes |-\frac{1}{2}\rangle_{F_b} .$$

Quantum States

For the spin density matrix, we can obtain the non-zero components for the singlet and triplet

$$
\begin{aligned}
|\Psi_S\rangle : &\quad C_{11} = C_{22} = C_{33} = -1; \mathrm{Tr}[C] = -3 \ , \\
|\Psi_T^{(1,1)}\rangle : &\quad B_3^{\pm} = 1, C_{33} = 1; \mathrm{Tr}[C] = 1 \ , \\
|\Psi_T^{(1,0)}\rangle : &\quad C_{11} = C_{22} = -C_{33} = 1; \mathrm{Tr}[C] = 1 \ , \\
|\Psi_T^{(1,-1)}\rangle : &\quad B_3^{\pm} = -1, C_{33} = 1; \mathrm{Tr}[C] = 1 \ .
\end{aligned}
$$

Invariants of the Spin Correlation Matrices

- The quantum state for the CP-odd scalar exchange is $|\Psi_S\rangle$, and then we have $\mathrm{Tr}[C] = -3$.
- The quantum states for the gauge boson exchanges are the linear combinations of $|\Psi_T^{(1,1)}\rangle$, $|\Psi_T^{(1,0)}\rangle$, and $|\Psi_T^{(1,-1)}\rangle$, whose coefficients are functions of θ_{F_a}. And thus we have $\mathrm{Tr}[C] = 1$.
- The quantum state for the CP-even scalar exchange is $|\Psi_T^{(1,0)}\rangle$, so we have $\mathrm{Tr}[C] = 1$.

Prove $\mathrm{Tr}[C] = 1$ for Gauge Boson Exchanges

- Because $|\Psi_i\rangle$ and $|\Psi_S\rangle$ are orthogonal to each other, we obtain

$$\alpha^i_{1/2,-1/2} = \alpha^i_{-1/2,1/2} \ .$$

- And thus, we can prove that

$$\mathrm{Tr}[C] = 1 - 2\sum_{i=1}^{n} p_i |\alpha^i_{1/2,-1/2} - \alpha^i_{-1/2,1/2}|^2 = 1 \ .$$

Prove $\mathrm{Tr}[C] = 1$ for Gauge Boson Exchanges

- Because $|\Psi_i\rangle$ and $|\Psi_S\rangle$ are orthogonal to each other, we obtain

$$\rho |\Psi_S\rangle^T = 0 \ .$$

- And thus, we can prove that

$$\mathrm{Tr}[C] = 1 \ , \quad B_i^+ = B_i^- \ , \quad C_{ij} = C_{ji} \ .$$

- In short, $\mathrm{Tr}[C] = 1$, and CP is conserved.

Other Proofs

- If we can preform the calculations in QFT, $\mathrm{Tr}[C] = 1$ can be proved easily.
- If we can preform the calculations in QMs, $\mathrm{Tr}[C] = 1$ can be proved easily

$$\begin{aligned}\sigma_1 \cdot \sigma_2 &= 4S_1 \cdot S_2 = 2\left[(S_1 + S_2)^2 - S_1^2 - S_2^2\right] \\ &= 2\left[(S_1 + S_2)(S_1 + S_2 + 1) - S_1(S_1 + 1) - S_2(S_2 + 1)\right] \ .\end{aligned}$$

- For singlet and triplet, $\mathrm{Tr}[C]$ is equal to -3 and 1, respectively.

Invariants under Basis Rotations

- $\mathrm{Tr}[C]$ is invariant under the basis rotation of quantization axes.
- Assuming that $\vec{e}_i$ and $\vec{e}_i'$ with $i = 1, 2, 3$ are two different quantization axis choices, which relate to each other via a $SO(3)$ rotation R_{ij}, *i.e.*, $\vec{e}_i' = \vec{e}_j R_{ji}$, we obtain [4]

$$C'_{ij} = R^T_{ik} C_{kl} R_{lj} = \left(R^T C R \right)_{ij} \quad .$$

- Thus, we prove that

$$\mathrm{Tr}[C'] = \mathrm{Tr}[R^T C R] = \mathrm{Tr}[C] \ .$$

[4] K. Cheng, T. Han and M. Low, Phys. Rev. D **109**, no.11, 11 (2024) [arXiv:2311.09166 [hep-ph]]; K. Cheng, T. Han and M. Low, Phys. Rev. D **111**, no.3, 033004 (2025) [arXiv:2407.01672 [hep-ph]].

Summary

- For the two-fermion $F_a F_b$ productions and decays via one mediator exchange at the e^+e^- collider, we show that the trace $\mathrm{Tr}[C]$ of the spin correlation matrix is an invariant quantity, which is independent on the scattering angle.
- $\mathrm{Tr}[C]$ is equal to 1, 1, and -3 for the exchanges of gauge boson, CP-even scalar, and CP-odd scalar, respectively.
- Thus, for one mediator exchange, for example, photon, we can determine the product of the spin-analyzing powers for $F_a F_b$ via $\mathrm{Tr}[C]$, and reconstruct the spin correlation matrix.

Summary

- With the CHSH-Horodecki criterion [5], we can probe the Bell non-locality, and evade the no-go theorem.
- In particular, the invariant $\mathrm{Tr}[C]$ is a new physics observable to probe the new physics beyond the SM and study the SM precision measurements.
- For the CP-even scalar and CP-odd scalar changes, C_{11}, C_{22}, and C_{33} are all invariants as well, which are independent on the scattering angle. In particular, $C_{33} = -1$ for both CP-even and CP-odd scalars, and then we do not need to know the CP property of the scalar.

[5] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. **23**, 880-884 (1969); R. Horodecki, P. Horodecki and M. Horodecki, Phys. Lett. A **200**, no.5, 340-344 (1995).

Summary

- We can employ the observables C_{ii} and $\mathrm{Tr}[C]$ to determine the product of the spin-analyzing powers and study the Bell non-locality, and to probe the new physics beyond the SM and study the SM precision measurements.
- Similarly, we can study the Bell non-locality for the Higgs to $\tau^+\tau^-$ at the LHC.
- We can perform the calculations in the traditional helicity formalism [6]. And then the decomposition is $\mathbf{2} \otimes \overline{\mathbf{2}} = \mathbf{1} \oplus \mathbf{3}$, where the singlet $\mathbf{1}$ and triplet $\mathbf{3}$ belong to the trivial and adjoint representations of $SU(2)_S$. Although the expressions for the quantum states are different, the physics results are the same.

[6] H. Murayama, I. Watanabe and K. Hagiwara, KEK-91-11.

- We study the Bell non-locality for the $\Lambda\bar{\Lambda}$ pair production via the $e^+e^- \to J/\psi \to \Lambda\bar{\Lambda}$ process at the BESIII experiment.
- The subsequent decays are: $\Lambda \to p + \pi^-$, and $\bar{\Lambda} \to \bar{p} + \pi^+$.
- So we have $F_a \equiv \Lambda$ and $F_b \equiv \bar{\Lambda}$, and choose $f_{a,1} \equiv p$ and $f_{b,1} \equiv \bar{p}$.

- With $\mathrm{Tr}[C] = 1$, we obtain

$$\langle q_p^k q_{\bar{p}}^k \rangle = \frac{1}{9}\alpha_\Lambda \alpha_{\bar{\Lambda}} \ .$$

- And thus we have the spin correlation matrix with

$$C_{ij} = \frac{1}{\langle q_p^k q_{\bar{p}}^k \rangle} \langle q_p^i q_{\bar{p}}^j \rangle \ .$$

- Thus, the spin correlation matrix can be determined by the angular correlation measurements.

- We denote the eigenvalues of $C^T C$ as M_1, M_2, M_3, which satisfy $M_1 \geq M_2 \geq M_3$.
- The Bell variable is defined as [7]

$$\mathcal{B} = 2\sqrt{M_1 + M_2} \ .$$

- If $2 < \mathcal{B} \leq 2\sqrt{2}$, we realize the Bell non-locality via the CHSH-Horodecki criterion.

[7] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. **23**, 880-884 (1969); R. Horodecki, P. Horodecki and M. Horodecki, Phys. Lett. A **200**, no.5, 340-344 (1995).

Bell Non-Locality at the BESIII Experiment

- Because we do not have the BESIII experimental data, we consider the spin correlation matrix from the QFT calculations [8].
- With $\alpha_\Lambda = 0.7519 \pm 0.0036 \pm 0.0024$, $\alpha_{\bar{\Lambda}} = -0.7559 \pm 0.0036 \pm 0.0030$, $\alpha_\psi = 0.4748 \pm 0.0022 \pm 0.0031$, and $\Delta\Phi = 0.7521 \pm 0.0042 \pm 0.0066$ [9], we present the numerical results in the following figure with solid line.
- Within 5σ uncertainty in $(\alpha_\psi, \Delta\Phi)$ parameter space, the Bell non-locality is realized in range $|\cos\theta_\Lambda| < 0.44150$

[8] E. Perotti, G. Fäldt, A. Kupsc, S. Leupold and J. J. Song, Phys. Rev. D **99**, no.5, 056008 (2019) [arXiv:1809.04038 [hep-ph]]; V. Batozskaya, A. Kupsc, N. Salone and J. Wiechnik, Phys. Rev. D **108**, no.1, 016011 (2023) [arXiv:2302.07665 [hep-ph]]; S. Wu, C. Qian, Q. Wang and X. R. Zhou, Phys. Rev. D **110**, no.5, 054012 (2024) [arXiv:2406.16298 [hep-ph]].

[9] M. Ablikim *et al.* [BESIII], Phys. Rev. Lett. **129**, no.13, 131801 (2022) [arXiv:2204.11058 [hep-ex]].

Spin Correlation Matrix from the QFT

$$C_{ij} = \frac{1}{1+\alpha_\psi \cos^2\theta_\Lambda} \times \begin{pmatrix} \sin^2\theta_\Lambda & 0 & \gamma_\psi \sin\theta_\Lambda \cos\theta_\Lambda \\ 0 & -\alpha_\psi \sin^2\theta_\Lambda & 0 \\ \gamma_\psi \sin\theta_\Lambda \cos\theta_\Lambda & 0 & \alpha_\psi + \cos^2\theta_\Lambda \end{pmatrix} .$$

Figure: Here, $\alpha_\psi \in [-1, +1]$ is the decay parameter of the vector charmonium ψ, and $\gamma_\psi \equiv \sqrt{1-\alpha_\psi^2}\cos(\Delta\Phi)$ with $\Delta\Phi \in (-\pi, +\pi]$ the relative form factor phase.

Bell Non-Locality at the BESIII Experiment

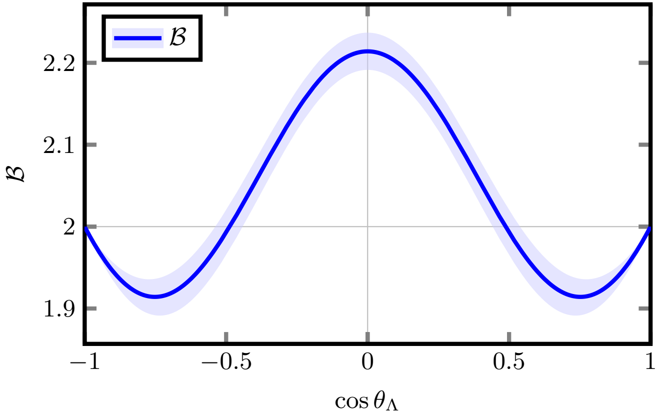

Figure: Bell variable vs $\cos\theta_\Lambda$. The shaded region represents the 5σ uncertainty in $(\alpha_\psi, \Delta\Phi)$ parameter space.

Spin Density Matrix

$$\mathcal{T}_{\mu\nu} = \frac{1}{1+\alpha_\psi \cos^2\theta_\Lambda} \begin{pmatrix} 1+\alpha_\psi \cos^2\theta_\Lambda & 0 & \beta_\psi \sin\theta_\Lambda \cos\theta_\Lambda & 0 \\ 0 & \sin^2\theta_\Lambda & 0 & \gamma_\psi \sin\theta_\Lambda \cos\theta_\Lambda \\ \beta_\psi \sin\theta_\Lambda \cos\theta_\Lambda & 0 & -\alpha_\psi \sin^2\theta_\Lambda & 0 \\ 0 & \gamma_\psi \sin\theta_\Lambda \cos\theta_\Lambda & 0 & \alpha_\psi + \cos^2\theta_\Lambda \end{pmatrix}.$$

Figure: $\alpha_\psi \in [-1, +1]$ is the decay parameter for the vector charmonium ψ, as well as $\beta_\psi \equiv \sqrt{1-\alpha_\psi^2}\sin\Delta\Phi$ and $\gamma_\psi \equiv \sqrt{1-\alpha_\psi^2}\cos(\Delta\Phi)$ with the relative form factor phase $\Delta\Phi \in (-\pi, +\pi]$.

Precision Measurements for Decay Parameters and CP Asymmetry

$$\alpha_\Lambda = \pm\left(\frac{9\mathcal{Q}_{20}\mathcal{Q}_{kk}}{\mathcal{Q}_{02}}\right)^{1/2}, \quad \alpha_{\bar{\Lambda}} = \mp\left(\frac{9\mathcal{Q}_{02}\mathcal{Q}_{kk}}{\mathcal{Q}_{02}}\right)^{1/2},$$

$$\alpha_\Psi = -\frac{\mathcal{Q}_{22}}{\mathcal{Q}_{11}}, \quad \tan\Delta\Phi = \frac{\mathcal{Q}_{13}}{\sqrt{\mathcal{Q}_{20}\mathcal{Q}_{02}\mathcal{Q}_{kk}}},$$

$$A_{CP} = \frac{\alpha_\Lambda + \alpha_{\bar{\Lambda}}}{\alpha_\Lambda + \alpha_{\bar{\Lambda}}} = \frac{\mathcal{Q}_{20} + \mathcal{Q}_{02}}{\mathcal{Q}_{20} - \mathcal{Q}_{02}}.$$

For the first time we find the concrete formulae to determine the decay parameters and CP asymmetry by experimental measurements, and provide a brand new method to measure the decay parameters and CP asymmetry.

Invariants of the Spin Density Matrices

- Inavriants of a matrix: Trace and Determinant. Because $\mathrm{Tr}[\rho] = 1$, we can only consider the determiants.
- We conjecture that the $N \times N$ spin density matrix with $r_\rho \leq N - 2$ can be reconstructed at the lepton colliders in general.
- We study two-fermion $F_a F_b$ productions and decays via one scalar or photon exchange at the e^+e^- collider whose ranks of spin density matrix are one and two, respectively.

Generic Scalar Mediator

- For one generic scalar mediator, because a generic scalar has one degree of freedom, the spin density matrix is rank one for a pure state.
- The spin polarization state for the quantum composite system $F_a F_b$ is

$$|\Psi_{GS}\rangle = x|\frac{1}{2}\rangle_{F_a} \otimes |-\frac{1}{2}\rangle_{F_b} + y|-\frac{1}{2}\rangle_{F_a} \otimes |\frac{1}{2}\rangle_{F_b}$$

with the normalization condition $|x|^2 + |y|^2 = 1$.

Spin Density Matrix

$$\rho_{GS} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |x|^2 & xy^* & 0 \\ 0 & x^*y & |y|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Figure: The spin density matrix for a generic scalar exchange.

Generic Scalar Mediator

- We obtain

$$\begin{aligned} B_1^{\pm} &= B_2^{\pm} = C_{13} = C_{23} = C_{31} = C_{32} = 0 \ , \\ B_3^{-} &= -B_3^{+}, C_{21} = -C_{12}, C_{22} = C_{11}, C_{33} = -1 \ . \end{aligned}$$

- From $C_{33} = -1$, we obtain

$$\alpha_{F_a}\alpha_{F_b} = -9\mathcal{Q}_{33} \ .$$

- And then we can determine the spin density matrix as

$$C_{ij} \ = \ -\mathcal{Q}_{ij}/\mathcal{Q}_{33} \ .$$

Thus, we can probe the Bell non-locality via the CHSH-Horodecki criterion.

Generic Scalar Mediator

- If $B_3^- = -B_3^+ \neq 0$, we get

$$\frac{\alpha_{F_a}}{\alpha_{F_b}} = -\frac{\mathcal{Q}_{30}}{\mathcal{Q}_{03}} \ .$$

- The spin analyzing powers are

$$\alpha_{F_a} = \pm 3\sqrt{\frac{\mathcal{Q}_{33}\mathcal{Q}_{30}}{\mathcal{Q}_{03}}} \ , \ \alpha_{F_b} = \mp 3\sqrt{\frac{\mathcal{Q}_{33}\mathcal{Q}_{03}}{\mathcal{Q}_{30}}} \ .$$

- From rank-one condition, we get

$$B_3^{+2} + C_{11}^2 + C_{12}^2 = 1 \ .$$

Generic Scalar Mediator

- Because the spin analyzing powers are determined up to a sign difference, we can reconstruct the spin density matrix up to an exchange $F_a \leftrightarrow F_b$.
- The condition for quantum entanglement from the Peres-Horodeki criteria or our discriminant criteria are

$$|B_3^{\pm}| \neq 1 \ , \ \text{or} \ \mathcal{Q}_{33} \neq \mathcal{Q}_{03}\mathcal{Q}_{30} \ .$$

- Because ρ_{GS} is a pure state, one can prove that the criteria for quantum entanglement are the same as the criterion for Bell non-locality via the Gisin theorem.

Generic Scalar Mediator

- If CP is conserved, we obtain $B_3^{\pm} = C_{12} = C_{21} = 0$. And then we obtain both C_{11} and C_{22} are equal to $+1$ and -1, which correspond to the exchanges of the CP-even and CP-odd scalars, respectively.
- We would like to emphasize that our studies for the generic scalar mediator can be applied to the $\eta_c \to \Lambda\bar{\Lambda}$ at the BESIII experiment and the Higgs $\to \tau\tau$ at the LHC.
- Comments: For the $\eta_c \to \Lambda\bar{\Lambda}$ at the BESIII experiment, we can rule out the local hidden variable theory from angular distributions.

Photon Mediator

- $\Lambda\bar{\Lambda}$ productions and decays
- $\tau^+\tau^-$ productions and decays.
- t^+t^- productions and decays.

- Spin density matrix along beam line: determine the product of the spin analyzing powers.

  This works only if F_aF_b are produced resonantly and the mass ratios $m_{f_{a,i}}/m_{F_a}$ and $m_{f_{b,i}}/m_{F_b}$ are small. For example, the threshold productions of $\Lambda_c^+\bar{\Lambda}_c^-$ and decays $\Lambda_c^+ \to \Lambda\pi^+/\bar{\Lambda}_c^- \to \bar{\Lambda}\pi^-$ at the BESIII experiment.
- Spin density matrix at $\theta_{F_a} = \pi/2$: determine the product of the spin analyzing powers.
- Principle minors of the spin density matrix: determine the spin analyzing powers.
- Here, we will study it in pure mathematics.

$T_{\mu\nu}$ in the Spin Density Matrix

$$\mathcal{T}_{\mu\nu} = \begin{pmatrix} \mathcal{T}_{00} & 0 & \mathcal{T}_{02} & 0 \\ 0 & \mathcal{T}_{11} & 0 & \mathcal{T}_{13} \\ \mathcal{T}_{20} & 0 & \mathcal{T}_{22} & 0 \\ 0 & \mathcal{T}_{31} & 0 & \mathcal{T}_{33} \end{pmatrix}.$$

Figure: $T_{\mu\nu}$ in the spin density matrix for a photon exchange.

Spin Density Matrix

$$\rho = \frac{1}{4}\begin{pmatrix} \mathcal{T}_{00}+\mathcal{T}_{33} & \mathcal{T}_{31}-i\mathcal{T}_{02} & \mathcal{T}_{13}-i\mathcal{T}_{20} & \mathcal{T}_{11}-\mathcal{T}_{22} \\ \mathcal{T}_{31}+i\mathcal{T}_{02} & \mathcal{T}_{00}-\mathcal{T}_{33} & \mathcal{T}_{11}+\mathcal{T}_{22} & -\mathcal{T}_{13}-i\mathcal{T}_{20} \\ \mathcal{T}_{13}+i\mathcal{T}_{20} & \mathcal{T}_{11}+\mathcal{T}_{22} & \mathcal{T}_{00}-\mathcal{T}_{33} & -\mathcal{T}_{31}-i\mathcal{T}_{02} \\ \mathcal{T}_{11}-\mathcal{T}_{22} & -\mathcal{T}_{13}+i\mathcal{T}_{20} & -\mathcal{T}_{31}+i\mathcal{T}_{02} & \mathcal{T}_{00}+\mathcal{T}_{33} \end{pmatrix}.$$

Figure: The spin density matrix for a photon exchange.

Photon Mediator

$$\frac{\alpha_\Lambda \alpha_{\bar{\Lambda}}}{9} = -\frac{\mathcal{Q}_{13}\mathcal{Q}_{31} - \mathcal{Q}_{11}\mathcal{Q}_{33}}{\mathcal{Q}_{02}\mathcal{Q}_{20} - \mathcal{Q}_{00}\mathcal{Q}_{22}} \equiv \mathcal{P} \ .$$

$$\begin{aligned}
\mathcal{D} &\equiv (\frac{\alpha_\Lambda}{3}\mathcal{Q}_{02} + \frac{\alpha_{\bar{\Lambda}}}{3}\mathcal{Q}_{20}) \\
&= \sqrt{(\mathcal{P}\mathcal{Q}_{00} + \mathcal{Q}_{22})^2 - (\mathcal{Q}_{13} + \mathcal{Q}_{31})^2 - (\mathcal{Q}_{11} - \mathcal{Q}_{33})^2}, \\
\mathcal{S} &\equiv (\frac{\alpha_\Lambda}{3}\mathcal{Q}_{02} - \frac{\alpha_{\bar{\Lambda}}}{3}\mathcal{Q}_{20}) \\
&= \sqrt{(\mathcal{P}\mathcal{Q}_{00} - \mathcal{Q}_{22})^2 - (\mathcal{Q}_{13} - \mathcal{Q}_{31})^2 - (\mathcal{Q}_{11} + \mathcal{Q}_{33})^2}.
\end{aligned}$$

$$\frac{\alpha_\Lambda}{3} = \frac{c_s\mathcal{S} + c_d\mathcal{D}}{2\mathcal{Q}_{02}}, \quad \frac{\alpha_{\bar{\Lambda}}}{3} = \frac{c_s\mathcal{S} - c_d\mathcal{D}}{2\mathcal{Q}_{20}},$$

where $c_s, c_d = \pm 1$ give 4 possible solutions.

Summary

- We prove that the trace Tr[C] of the spin correlation matrix C is an invariant quantity, and is invariant under basis rotations. Thus, for the exchanges of one mediator such as scalar and gauge boson, we can determine the product of the spin-analyzing powers for $F_a F_b$ via Tr[C], and reconstruct the spin correlation matrix. With the CHSH-Horodecki criterion, we can probe the Bell non-locality, and evade the no-go theorem.
- For the first time we find the concrete formulae to determine the decay parameters and CP asymmetry by experimental measurements, and provide a brand new method to measure the decay parameters and CP asymmetry.
- We conjecture that the $N \times N$ spin density matrix with $r_\rho \leq N - 2$ can be reconstructed at the lepton colliders in general. We prove it explicitly for the two fermion productions and decays at the lepton collider.

Thank You Very Much for Your Attention!