

A more complete description Of wave like dark matter

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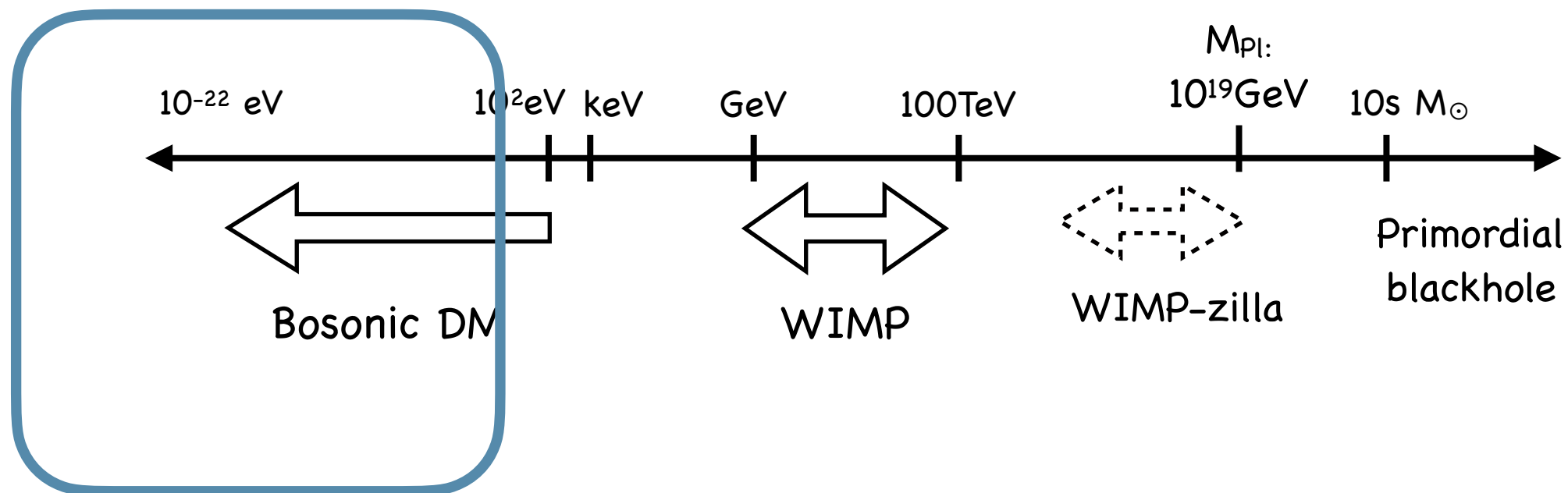
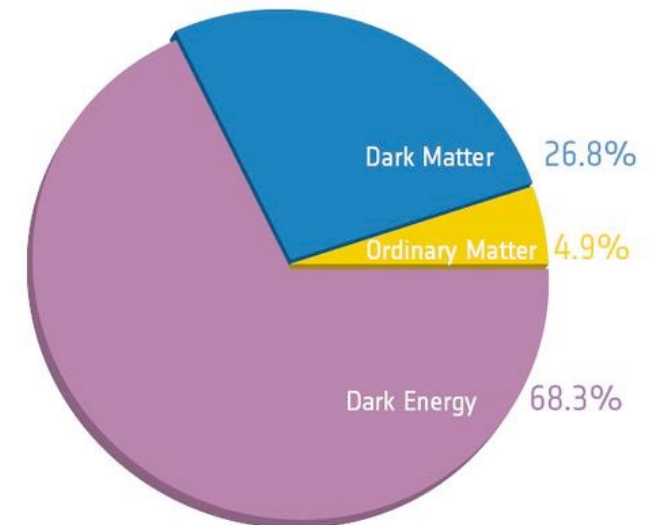
Based on

D. Y. Cheong and N. L. Rodd, LTW 2408.08696

Y. Bao, D. Y. Cheong, N. L. Rodd , J. Takach, K. Zhou and LTW, 2510.05198, 2607.27313,
2609.xxxxx

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Wave-like



Wave-like:

DM acts as a classical wave $\phi(t) \sim \frac{\sqrt{2\rho_{\text{DM}}}}{m_{\text{DM}}} \cos(m_{\text{DM}}t + \varphi)$

Quantum questions

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 - * A mixed state?
- * How “quantum” is it?

Quantum questions

- * What is the quantum state of dark matter?
 - * Lore: large occupation number $\Rightarrow$ classical
 - * Coherent state?
 - * Fock state $|N\rangle$, with $N \gg 1$?
 - * A mixed state?
- * How “quantum” is it?
 - * Difference between quantum and classical?

This talk

- * The likely quantum state of wave-like DM.
- * Emergence of classical limit.
- * Quantum effects of this state.

Mainly about understanding the quantum properties,
not new detection method

A number slides taken from my collaborator: Dhong Yeon Cheong

Density matrix

Using a coherent state $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ as a basis

$$\hat{\rho} = \int d^2\alpha P(\alpha) |\alpha\rangle \langle \alpha|$$

Coherent state basis overcomplete, diagonal decomposition of $\hat{\rho}$

[Glauber-Sudarshan P - distribution function]

[Glauber, 1963]
[Sudarshan 1963]

$P(\alpha)$ for wave-like DM

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For example, could be in coherent state from misalignment.

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Fortunately, there is an universal distribution for “very” random system

$P(\alpha)$ for wave-like DM

Gaussian state:

$$P(\alpha) = \frac{1}{\pi \langle N \rangle} \exp \left[-\frac{|\alpha|^2}{\langle N \rangle} \right]$$

Quantum central limit theorem (QCLT) [Glauber, 1963]:

Superposition of n similar yet independent fields gives this state as $n \rightarrow \infty$

Basically, for something which is “random enough”, it approaches Gaussian.

Example: adding Fock states

Fock state: $|n\rangle$

Consider equal mixture of $\mathcal{N}$ such states

Consider equal mixture $\alpha_{\text{eff}} = \sum_{p=1}^m \alpha_i / \sqrt{m}$.

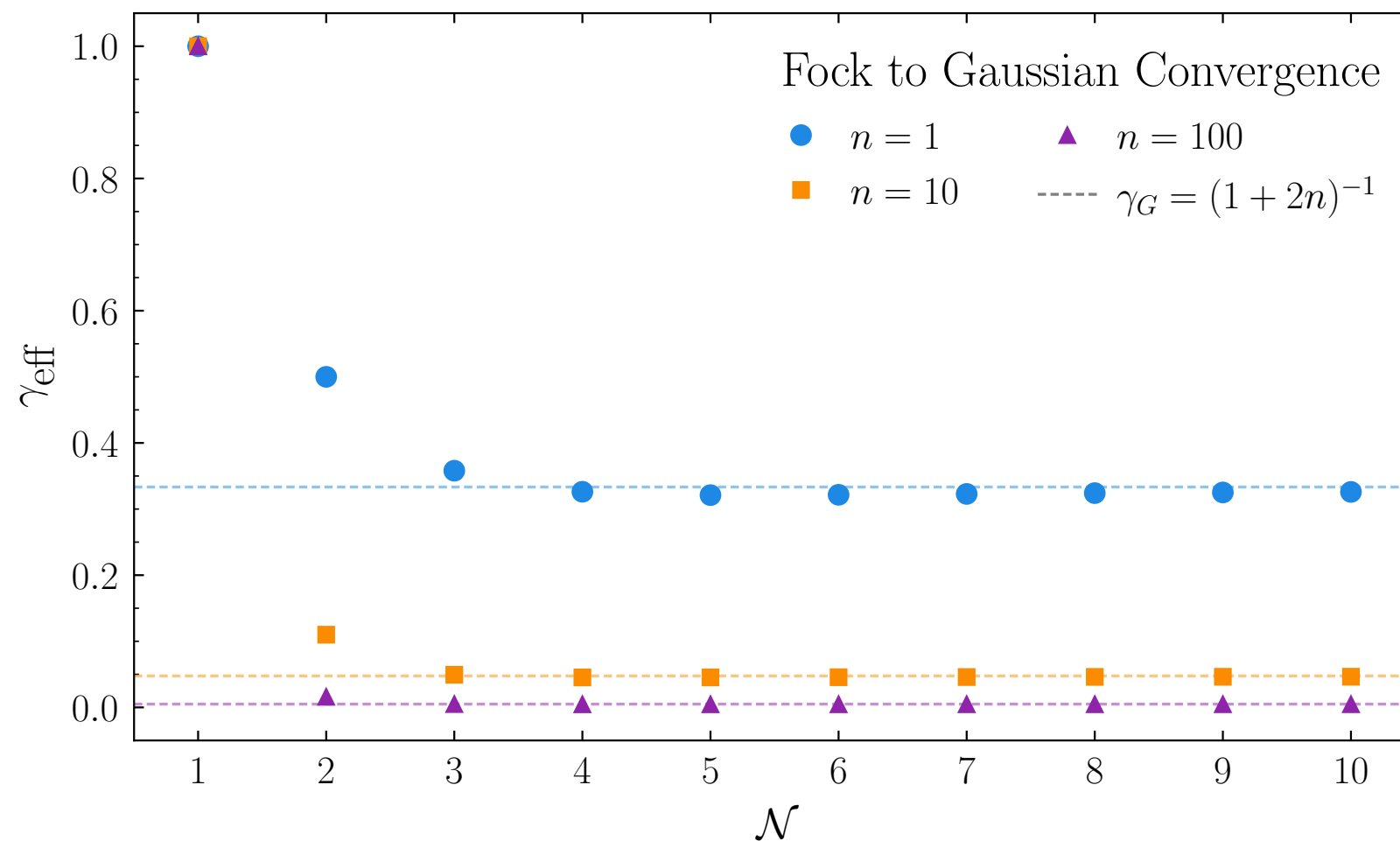
Using Fourier transform: $\tilde{P}(\lambda) = \int d\alpha P(\alpha) e^{\lambda\alpha^* - \lambda^*\alpha} = \text{tr} \left(e^{-\lambda^* a} \rho e^{\lambda a^\dagger} \right)$

$$\tilde{P}_{\text{eff}}(\lambda) = \prod_{k=1}^{\mathcal{N}} \tilde{P}_k(\lambda/\sqrt{\mathcal{N}})$$

$$\tilde{P}_{\text{eff}}(\lambda) = \left[L_n(|\lambda|^2/\mathcal{N}) \right]^{\mathcal{N}} = \left[1 - \frac{n}{\mathcal{N}} |\lambda|^2 + \mathcal{O}(\mathcal{N}^{-2}) \right]^{\mathcal{N}}$$

For large $\mathcal{N}$, this becomes $\tilde{P}_{\text{eff}} \rightarrow e^{-n|\lambda|^2}$ Gaussian!

Fast approach to Gaussian



$\mathcal{N}$: number of Fock state $|n\rangle$ in the sum

Applying this to DM halo

Dark matter field has many momentum modes

$$\hat{\rho} = \int d\{\alpha\} P(\{\alpha\}) |\{\alpha\}\rangle \langle \{\alpha\}| \quad |\{\alpha\}\rangle = \prod_{\mathbf{k}} |\alpha_{\mathbf{k}}\rangle$$

$$P(\{\alpha\}) = \prod_{\mathbf{k}} \frac{1}{\pi N_{\mathbf{k}}} e^{-|\alpha_{\mathbf{k}}|^2 / N_{\mathbf{k}}} \quad \mathbf{k}: \text{momentum of dark matter}$$

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$$\langle \hat{N}_{\mathbf{k}} \rangle = \text{Tr} \left[\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} \hat{\rho}_{\mathbf{k}} \right] \simeq \frac{(2\pi)^3}{g} \bar{n} p(\mathbf{k}) = N_{\mathbf{k}}$$

$\bar{n}$: average number density

$p(\mathbf{k})$: momentum space distribution, e.g. from a halo model

Wave-like: $\langle N_{\mathbf{k}} \rangle \gg 1$ Particle-like: $\langle N_{\mathbf{k}} \rangle \ll 1$

More on this later

Emergence of classical

Wave DM field

DM field operator:

$$\hat{\phi}(t, \mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\mathcal{V}\omega_{\mathbf{k}}}} \left(\hat{a}_{\mathbf{k}} e^{-ik \cdot x} + \hat{a}_{\mathbf{k}}^{\dagger} e^{ik \cdot x} \right)$$

General expectation value:

$$\langle \hat{\phi}^n(t, \mathbf{x}) \rangle = \int d^2\alpha P(\alpha) \langle \alpha | \left[\sum_{\mathbf{k}} \frac{1}{\sqrt{2\mathcal{V}\omega_{\mathbf{k}}}} \left(\hat{a}_{\mathbf{k}} e^{-ik \cdot x} + \hat{a}_{\mathbf{k}}^{\dagger} e^{ik \cdot x} \right) \right]^n | \alpha \rangle$$

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Note: this is not simple since $|\alpha\rangle$ is not an eigenstate of $\hat{a}^{\dagger}$

Quantum mechanics: $aa^{\dagger} = a^{\dagger}a + 1$

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Obstruction of being classical

Wave DM field

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Large occupation number limit: $\langle \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \rangle = \langle \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} \rangle + 1 \simeq N_{\mathbf{k}} \quad N_{\mathbf{k}} \gg 1$

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Field operator can be replaced by

$$\phi(t, \mathbf{x}) = \sum_{\mathbf{k}} \sqrt{\frac{2}{\nu \omega_{\mathbf{k}}}} \operatorname{Re} [\alpha_{\mathbf{k}} e^{-ik \cdot x}]$$

with each $\alpha_{\mathbf{k}}$ having a statistical weight $P(\alpha_{\mathbf{k}})$

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For $P(\alpha_{\mathbf{k}}) > 0$, $\phi(t, \mathbf{x})$ a classical random field.

For Gaussian $P(\alpha_{\mathbf{k}})$, $\phi(t, \mathbf{x})$ a Gaussian random field.

In particular, $\langle \phi(t, \mathbf{x}) \rangle = 0$, & $\langle \phi^2(t, \mathbf{x}) \rangle \simeq \frac{\rho}{m^2}$

Wave DM field

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We see it emerges in the classical limit of a quantum state.

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We see it emerges in the classical limit of a quantum state.

We can do the computation from the quantum state without classical approximation, and understands its applicability.

Quantum properties.

Correlation function

How correlated are the field in space and time?

$$\langle \Gamma(\tau, \mathbf{d}) \rangle = \langle \phi(t, \mathbf{x}) \phi(t + \tau, \mathbf{x} + \mathbf{d}) \rangle$$

Correlation function

How correlated are the field in space and time?

$$\langle \Gamma(\tau, \mathbf{d}) \rangle = \langle \phi(t, \mathbf{x}) \phi(t + \tau, \mathbf{x} + \mathbf{d}) \rangle$$

With Gaussian state:

$$\begin{aligned} \langle \Gamma(\tau, \mathbf{d}) \rangle &= \bar{n} \int d^3 \mathbf{k} \frac{p(\mathbf{k})}{\omega_{\mathbf{k}}} \cos(\omega_{\mathbf{k}} \tau - \mathbf{k} \cdot \mathbf{d}) \\ &\simeq \frac{\bar{n}}{m} \int d^3 \mathbf{v} f(\mathbf{v}) \cos \left(m \left(1 + \frac{v^2}{2} \right) \tau - m \mathbf{v} \cdot \mathbf{d} \right) \end{aligned}$$

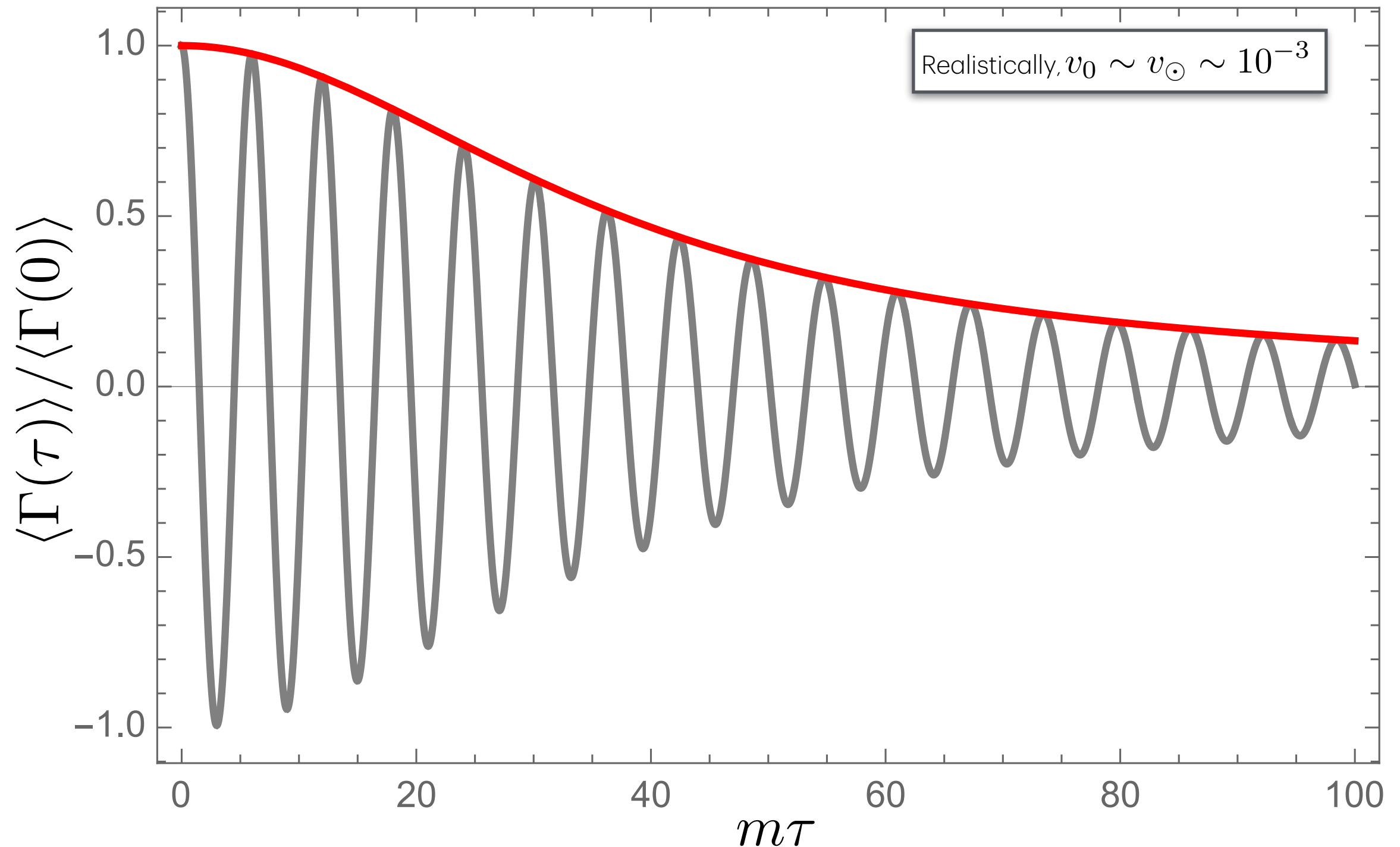
e.g. The Standard Halo Model (Maxwell-Boltzmann)

$$f(\mathbf{v}) = \frac{1}{\pi^{3/2} v_0^3} \exp \left[\frac{-(\mathbf{v} + \mathbf{v}_\odot)^2}{v_0^2} \right]$$

Wave-DM, temporal correlation

$$v_0 = v_{\odot} = 0.2$$

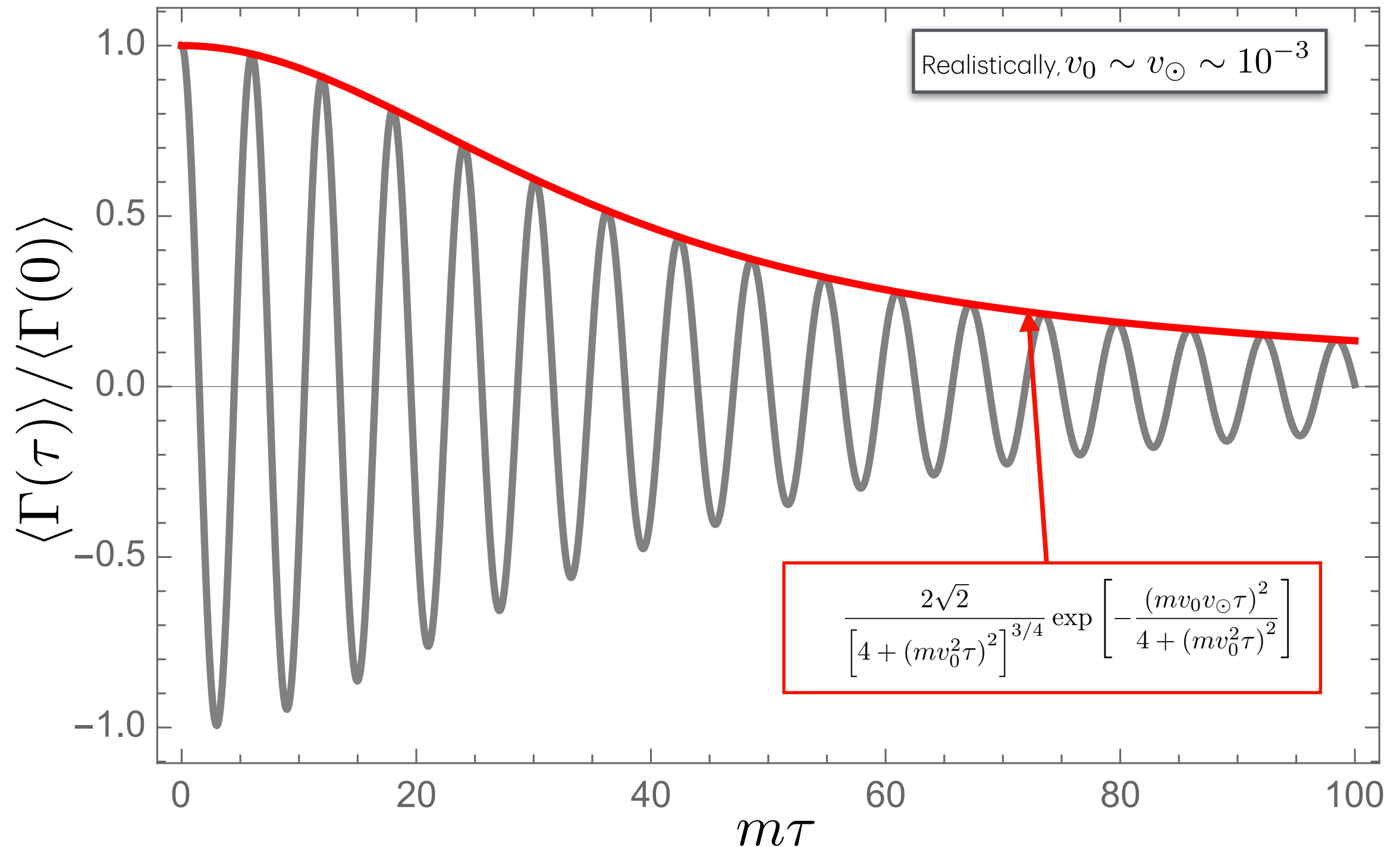
Realistically, $v_0 \sim v_{\odot} \sim 10^{-3}$



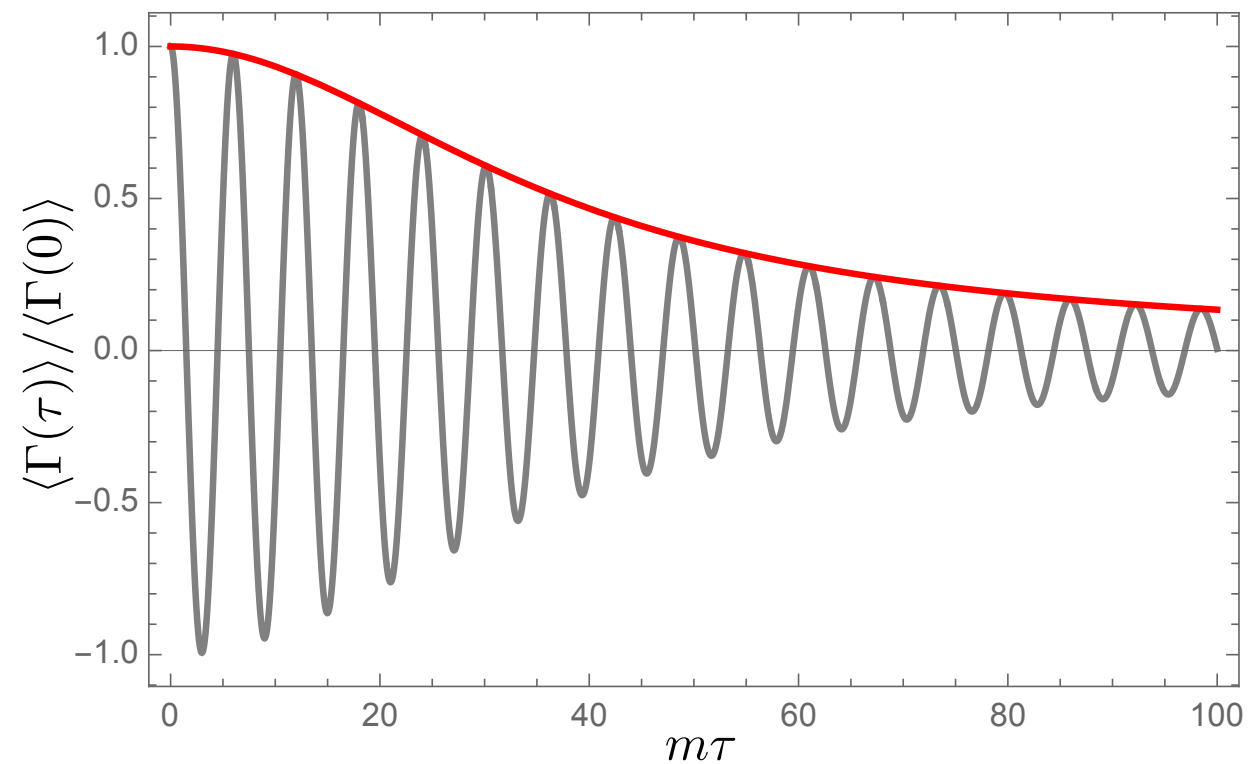
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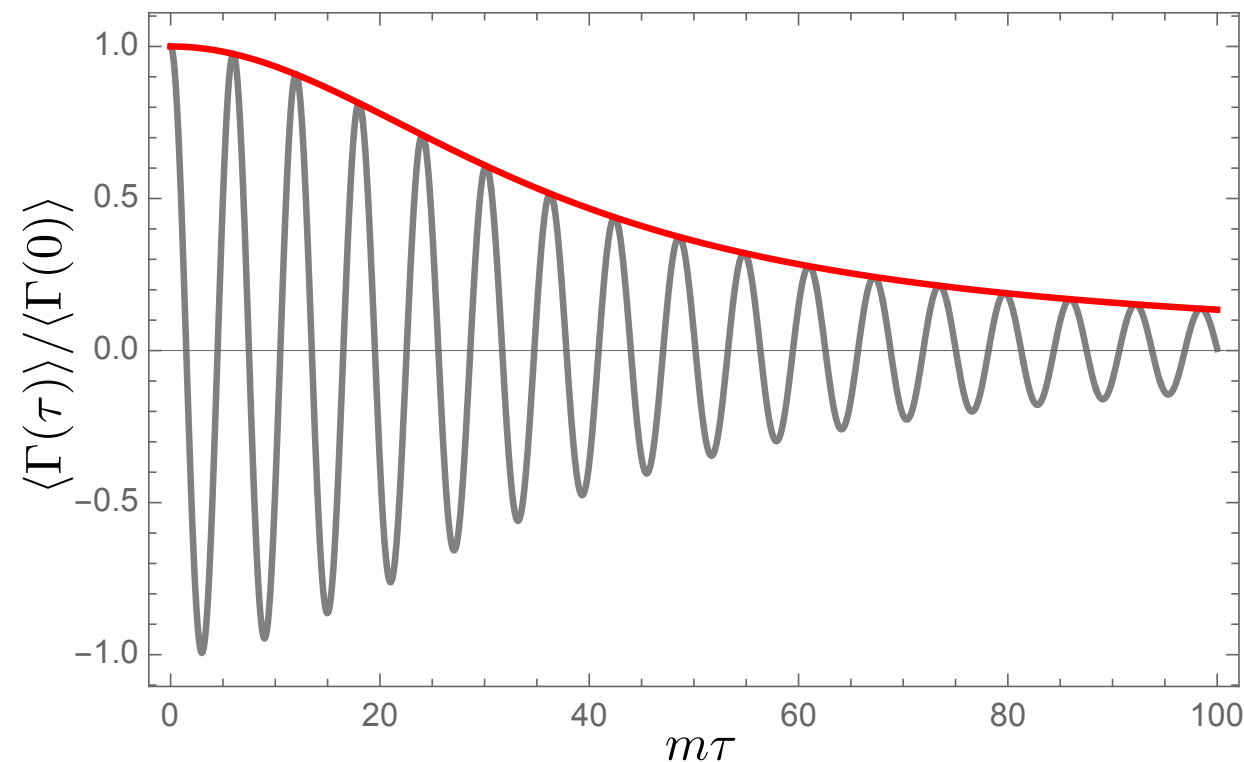


Coherence time



Exponential decay:
a time scale at which the field
is “coherent”.

Coherence time



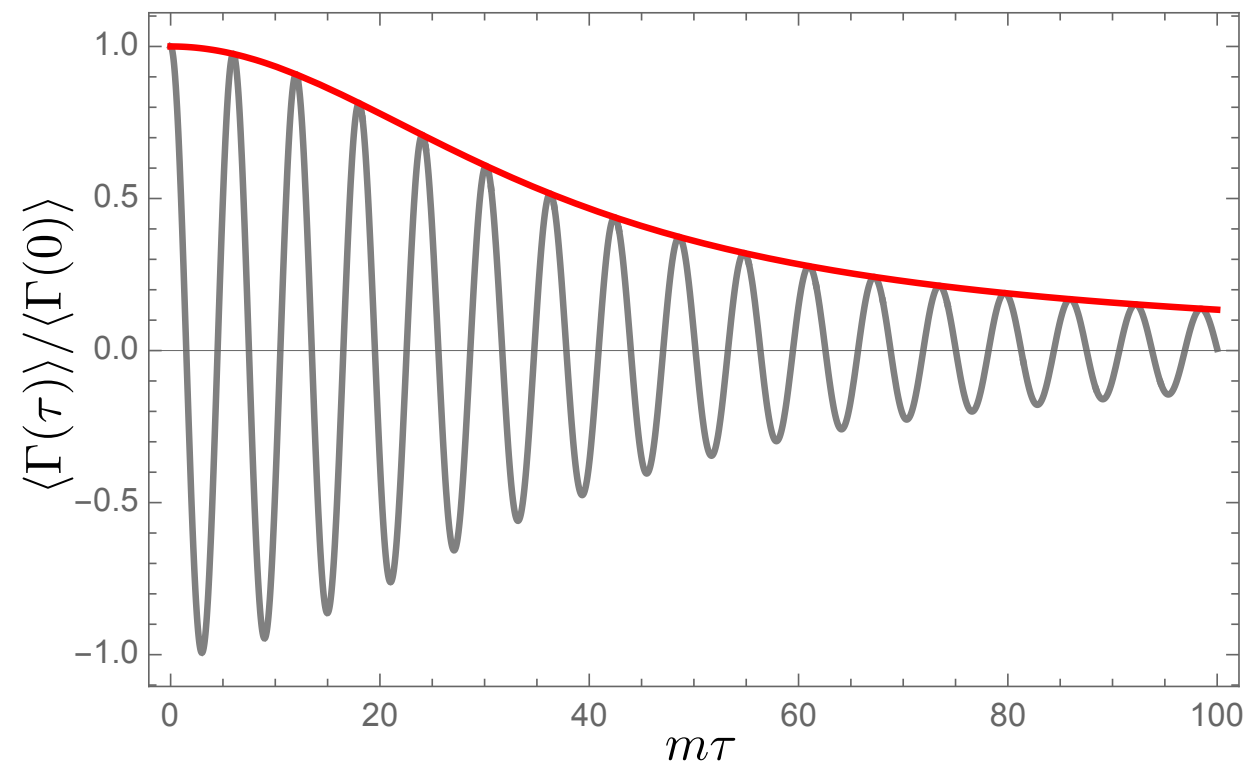
Exponential decay:
a time scale at which the field
is “coherent”.

Within this coherent time, the field can be approximated by

$$\phi(t) \sim \frac{\sqrt{2\rho_{\text{DM}}}}{m_{\text{DM}}} \cos(m_{\text{DM}}t + \varphi)$$

Beyond coherence time, DM field not correlated. Can be approximated by a random phase φ

Coherence time, a rigorous definition



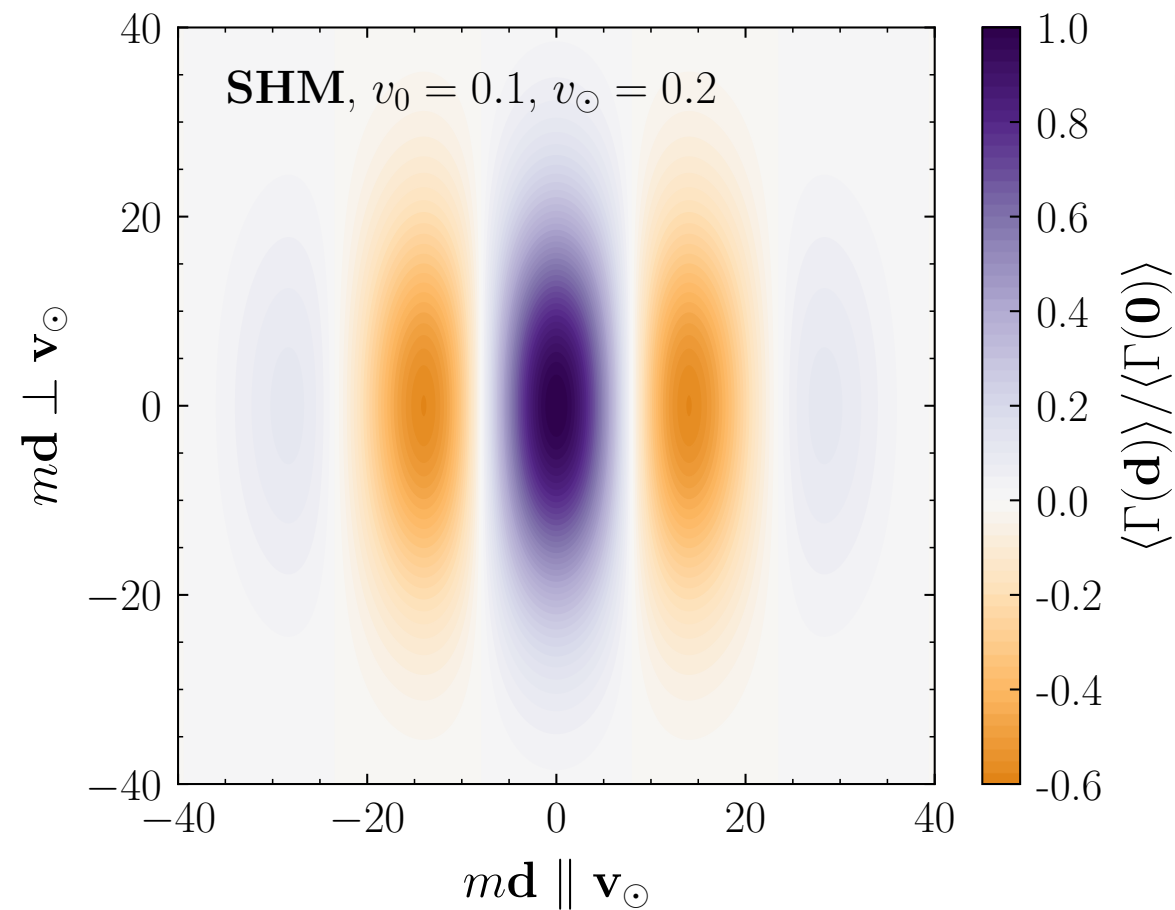
A time scale at which the field is “coherent”.

$$\tau_c = 2 \int d\tau \left[\frac{\langle \Gamma(\tau) \rangle}{\langle \Gamma(0) \rangle} \right]^2 = \frac{2\pi}{\langle 1/\omega \rangle^2} \int \frac{d\omega}{\omega^2} p(\omega)^2 = \frac{2\pi}{\omega_c}$$

DM - SHM

$$\tau_{\text{DM}} \simeq \frac{\sqrt{2\pi} \text{Erf} [\sqrt{2}v_{\odot}/v_0]}{mv_0v_{\odot}} [1 + \mathcal{O}(v^2)] \simeq 2.80\text{ms} \left(\frac{1\mu\text{eV}}{m} \right)$$

Coherence Volume / Length



$$\langle \Gamma(\mathbf{d}) \rangle = \langle \phi(\mathbf{x}) \phi(\mathbf{x} + \mathbf{d}) \rangle :$$

$$V_c \simeq 2 \int_{-\infty}^{\infty} d^3 \mathbf{d} \left[\frac{\langle \Gamma(\mathbf{d}) \rangle}{\langle \Gamma(\mathbf{0}) \rangle} \right]^2$$

DM - SHM

$$V_c = \frac{(2\pi)^3}{\langle 1/\omega \rangle^2} \int \frac{d^3 \mathbf{k}}{\omega_{\mathbf{k}}^2} p(\mathbf{k})^2 \simeq \left(\frac{\sqrt{2\pi}}{mv_0} \right)^3 d_{\text{DM}} \simeq \frac{\sqrt{2\pi}}{mv_0} \simeq 674 \text{ m} \left(\frac{1\mu\text{eV}}{m} \right)$$

Higher order coherence?

Coherence properties also important in higher order correlations.

Glauber correlation functions for higher order quantum coherence

[Glauber, 1963]

$$\hat{\phi}^+(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \hat{a}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$g^{(1)}(x_1, x_2) = \frac{\langle \hat{\phi}^-(x_1) \hat{\phi}^+(x_2) \rangle}{\sqrt{\langle \hat{\phi}^-(x_1) \hat{\phi}^+(x_1) \rangle \langle \hat{\phi}^-(x_2) \hat{\phi}^+(x_2) \rangle}} \quad g^{(2)}(x_1, x_2) = \frac{\langle \hat{\phi}^-(x_1) \hat{\phi}^-(x_2) \hat{\phi}^+(x_2) \hat{\phi}^+(x_1) \rangle}{\langle \hat{\phi}^-(x_1) \hat{\phi}^+(x_1) \rangle \langle \hat{\phi}^-(x_2) \hat{\phi}^+(x_2) \rangle}$$

Inserting the Gaussian $P(\alpha)$, and the occupation number $\langle N_{\mathbf{k}} \rangle$,

$$g^{(1)}(\tau, \mathbf{d}) = \frac{1}{\langle 1/\omega \rangle} \int d^3\mathbf{k} \frac{p(\mathbf{k})}{\omega_{\mathbf{k}}} e^{-i\mathbf{k}\cdot\mathbf{d}} \quad g^{(2)}(\tau, \mathbf{d}) = 1 + \left| g^{(1)}(\tau, \mathbf{d}) \right|^2$$

Quantum and classical

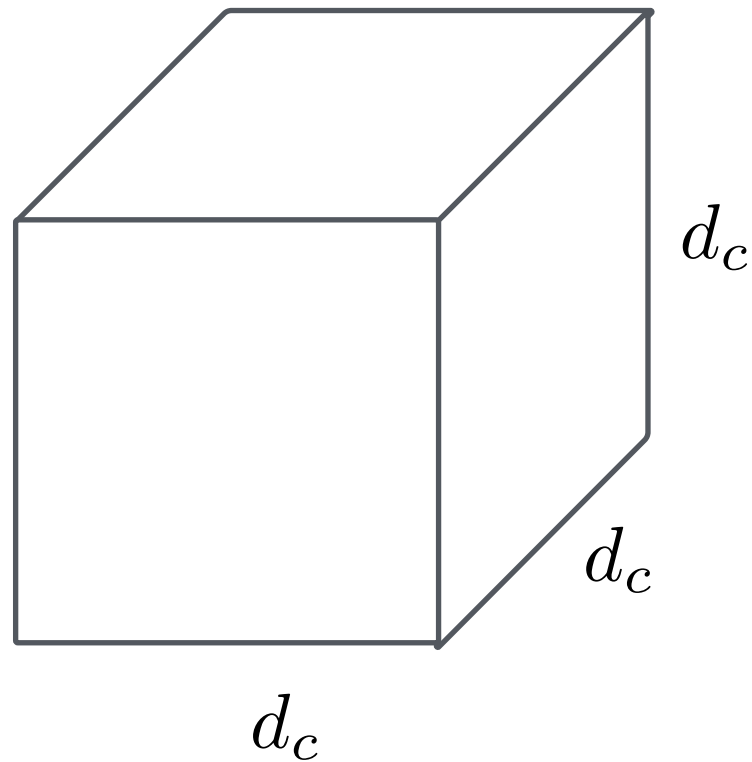
Boundary between wave and particle

Where is this boundary?

What's the transition look like?

For this, we focus on a specific calculation.

Number of axion in a box



Side = coherence length d_c

Volume $\equiv$ coherence volume V_c

For simplicity, consider a single **k** mode

Wave-particle boundary

Useful to rewrite the Gaussian state in Fock basis

$$\hat{\rho} = \int d^2\alpha \frac{e^{-|\alpha|^2/\langle N \rangle}}{\pi \langle N \rangle} |\alpha\rangle \langle \alpha| = \frac{1}{1 + \langle N \rangle} \sum_{k=0}^{\infty} \left(\frac{\langle N \rangle}{1 + \langle N \rangle} \right)^k |k\rangle \langle k|$$

$|k\rangle$: Fock state with occupation number k , and weight

$$p_k = \frac{\langle N \rangle^k}{(1 + \langle N \rangle)^{k+1}}$$

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Number statistics

$$\text{Mean:} \quad \mu_k = \langle k \rangle = \langle N \rangle$$

$$\text{Variance:} \quad \sigma_k^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle N \rangle (1 + \langle N \rangle)$$

Wave-particle boundary

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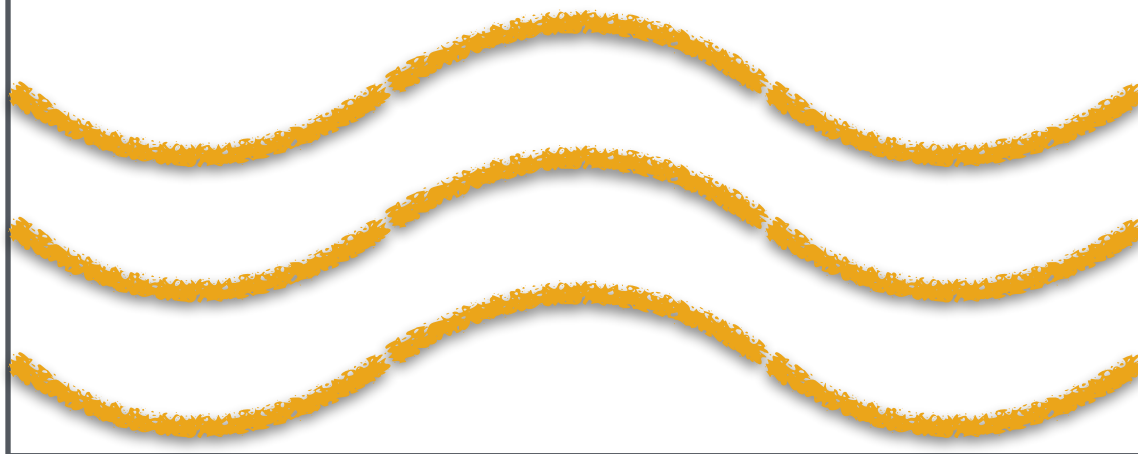
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$$\sigma_k^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle N \rangle(1 + \langle N \rangle)$$

$\langle N \rangle \gg 1$, $\sigma_k^2 \simeq N^2$, exponential



wave

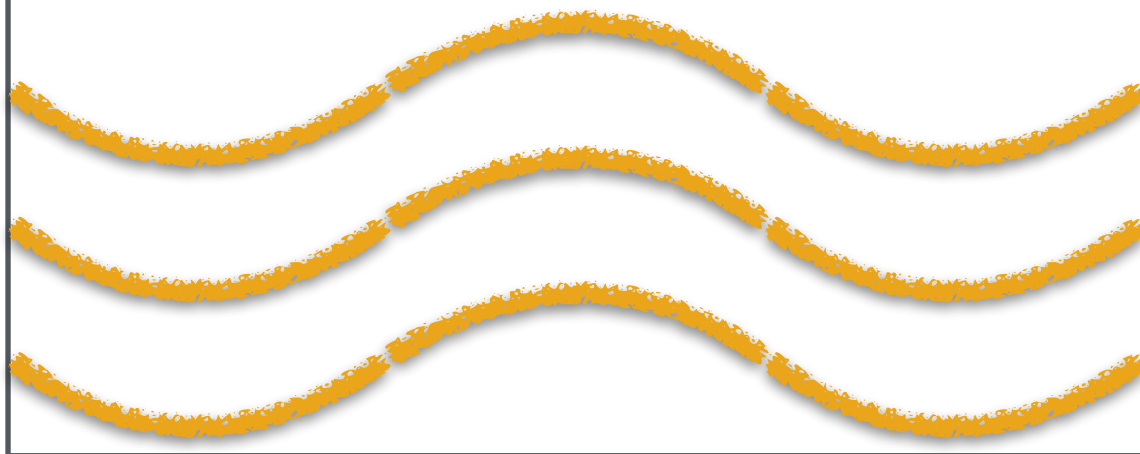
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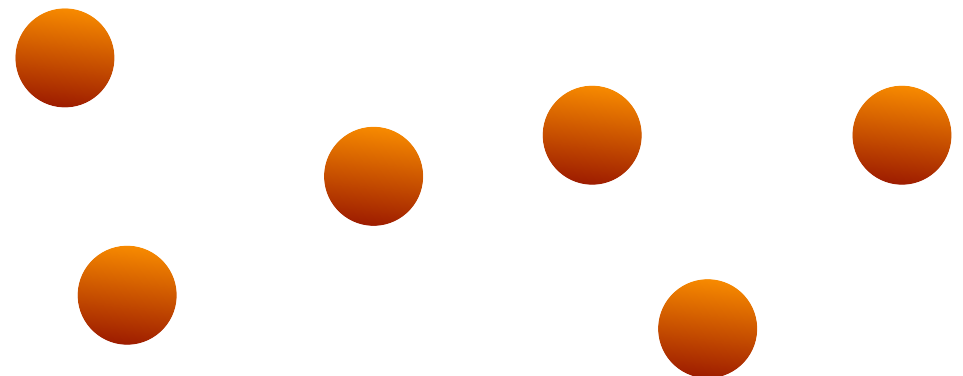
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wave

$\langle N \rangle \ll 1, \sigma_k^2 \simeq N = \mu_k$, Poisson



particle

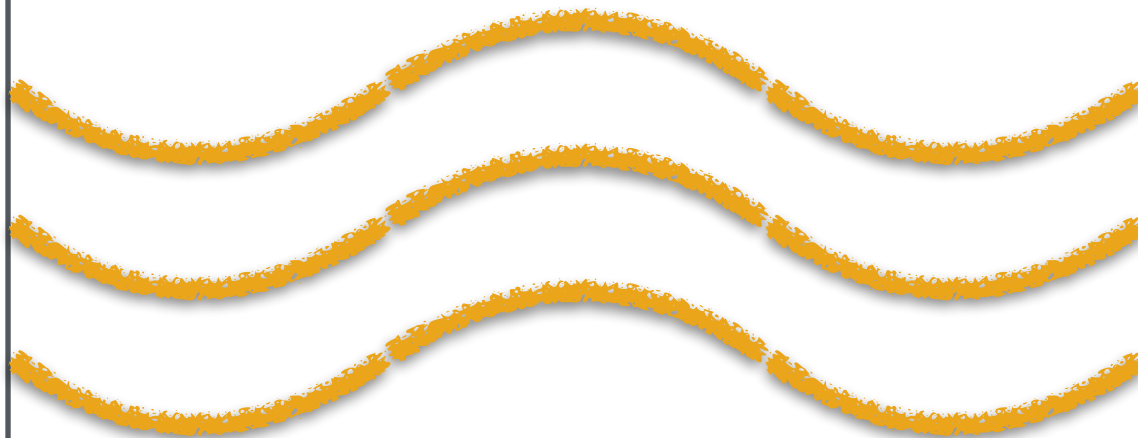
Wave-particle boundary

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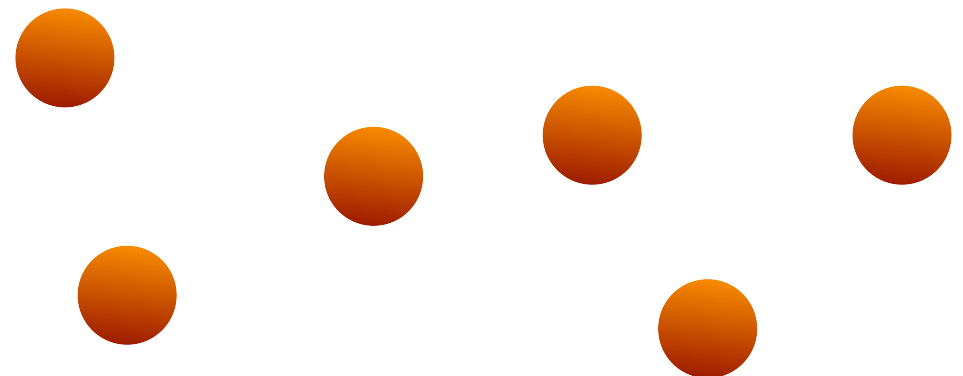
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$\langle N \rangle \gg 1$, $\sigma_k^2 \simeq N^2$, exponential



$\langle N \rangle \ll 1$, $\sigma_k^2 \simeq N = \mu_k$, Poisson



$\langle N \rangle \sim 1$ Neither Poisson or Exponential!

$$m \simeq \frac{18.7 \text{ eV}}{(g_s \langle N \rangle)^{1/4}}$$

Axion $\sim 18.7 \text{ eV}$

Dark Photon $\sim 14.2 \text{ eV}$

More general volume

$$V = L^3 \neq V_c \quad \boxed{N_c = V/V_c}$$

Statistics in the particle-wave boundary depends on volume!

$$V \ll V_c : \quad \mu_k = \bar{n}V = \langle N \rangle N_c, \quad \sigma_k^2 = \mu_k (\mu_k + 1)$$

Exponential when $NN_c \gg 1$, Poisson when $NN_c \ll 1$

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$$V \gg V_c : \quad \mu_k = \langle N \rangle N_c, \quad \sigma_k^2 = \mu_k [\langle N \rangle + 1]$$

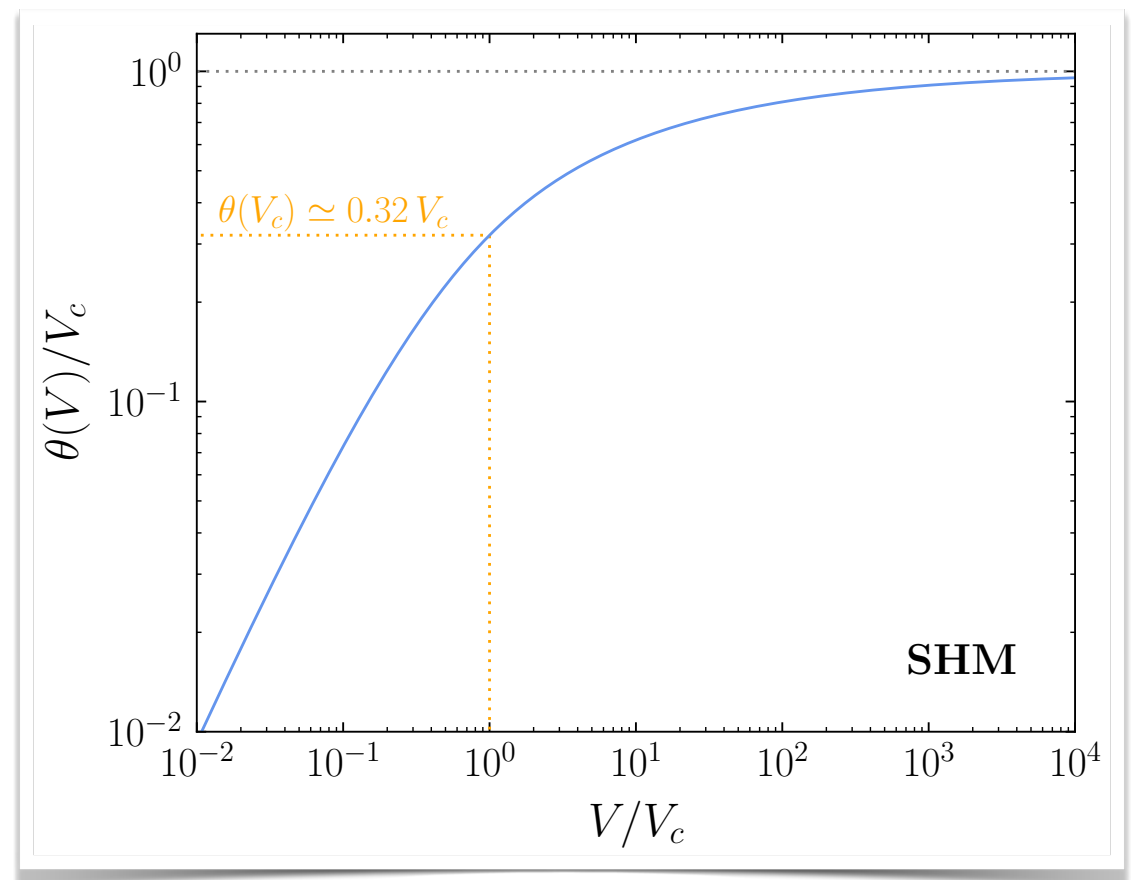
Poisson when $\langle N \rangle \ll 1$ Erlang when $\langle N \rangle \gg 1$

Different statistics

$$V = L^3 \neq V_c \quad \boxed{N_c = V/V_c}$$

$$\mu_k = \bar{n}V,$$

$$\sigma_k^2 = \bar{n}V[\bar{n}\theta(V) + 1]$$



Any observational implications?

Intrinsically quantum?

“quantum”: can not be mimicked by a stochastic classical signal
even if $n \gg 1$

For details, see talk by Dhong Yeon Chong yesterday

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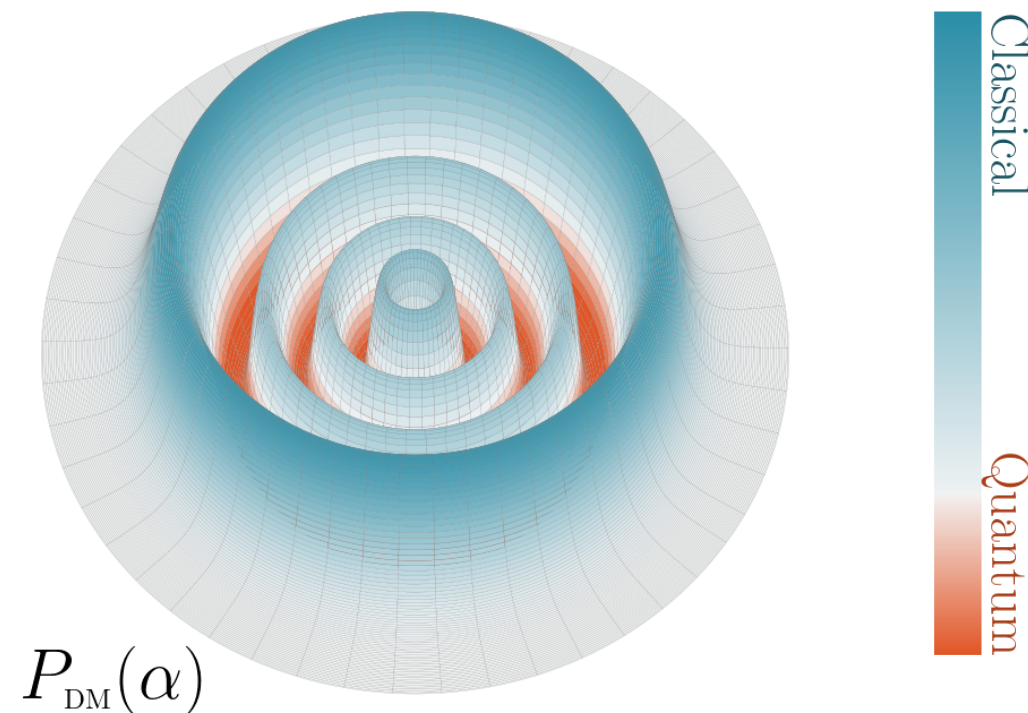
Similar question have been asked in “testing gravity is quantum”

For details, see talk by Dhong Yeon Chong yesterday

Seeing “quantum”

Need $P(\alpha) < 0$

e.g. 7 quanta added
thermal state

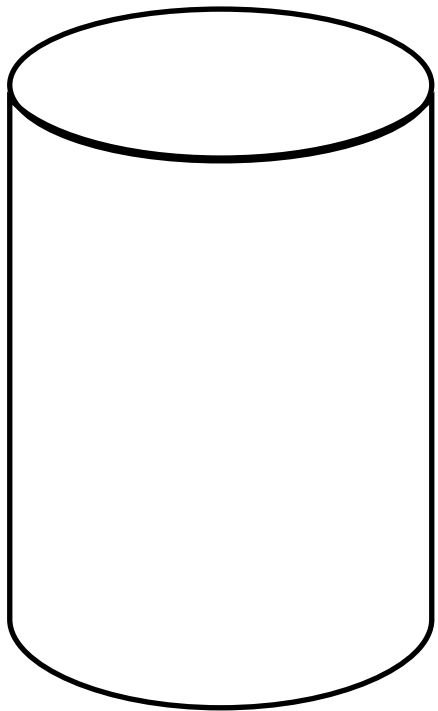


Q: can we be sensitive to the quantum $P(\alpha) < 0$ regions?

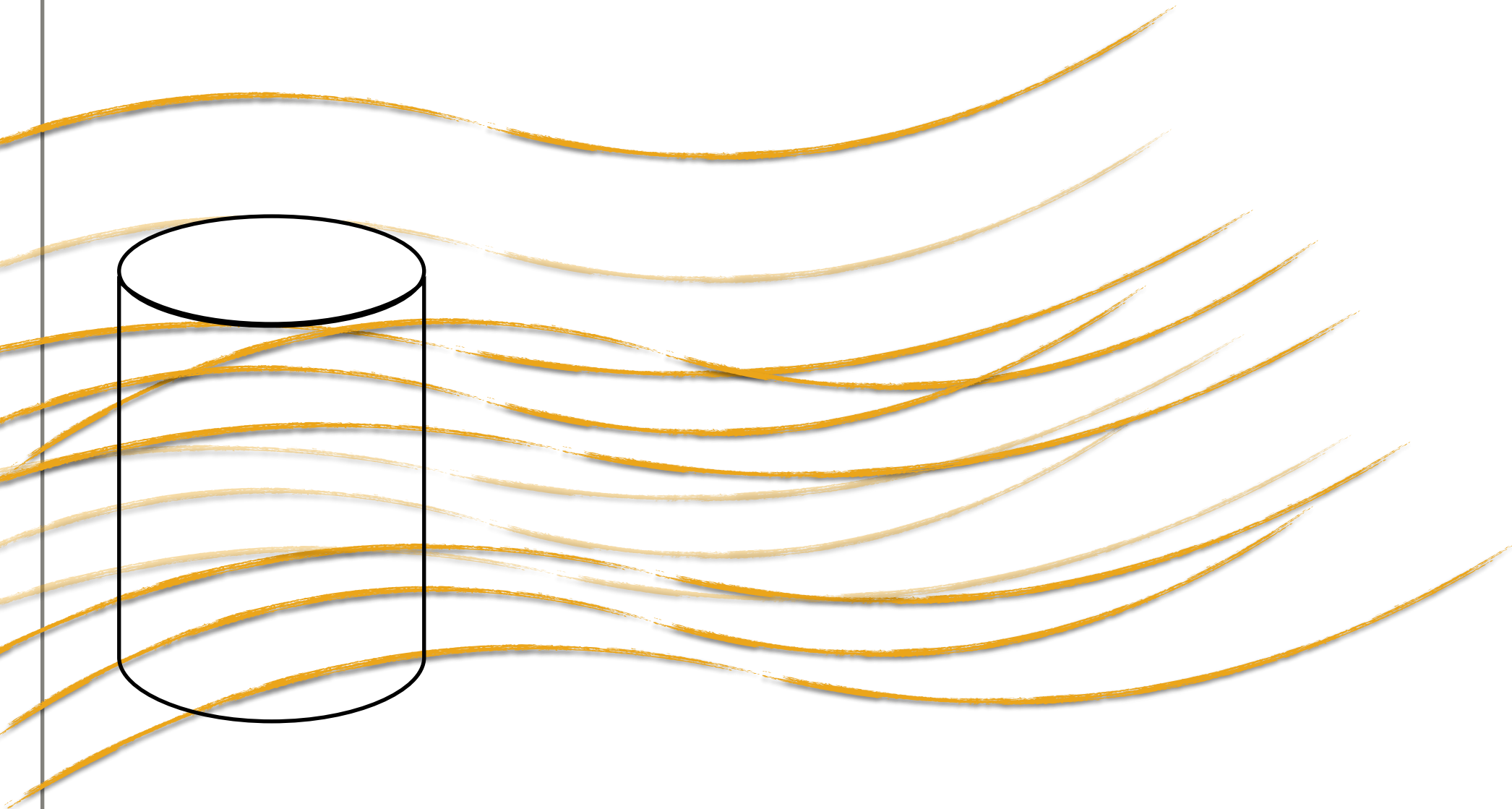
For details, see talk by Dhong Yeon Chong yesterday

An Axion Haloscope :

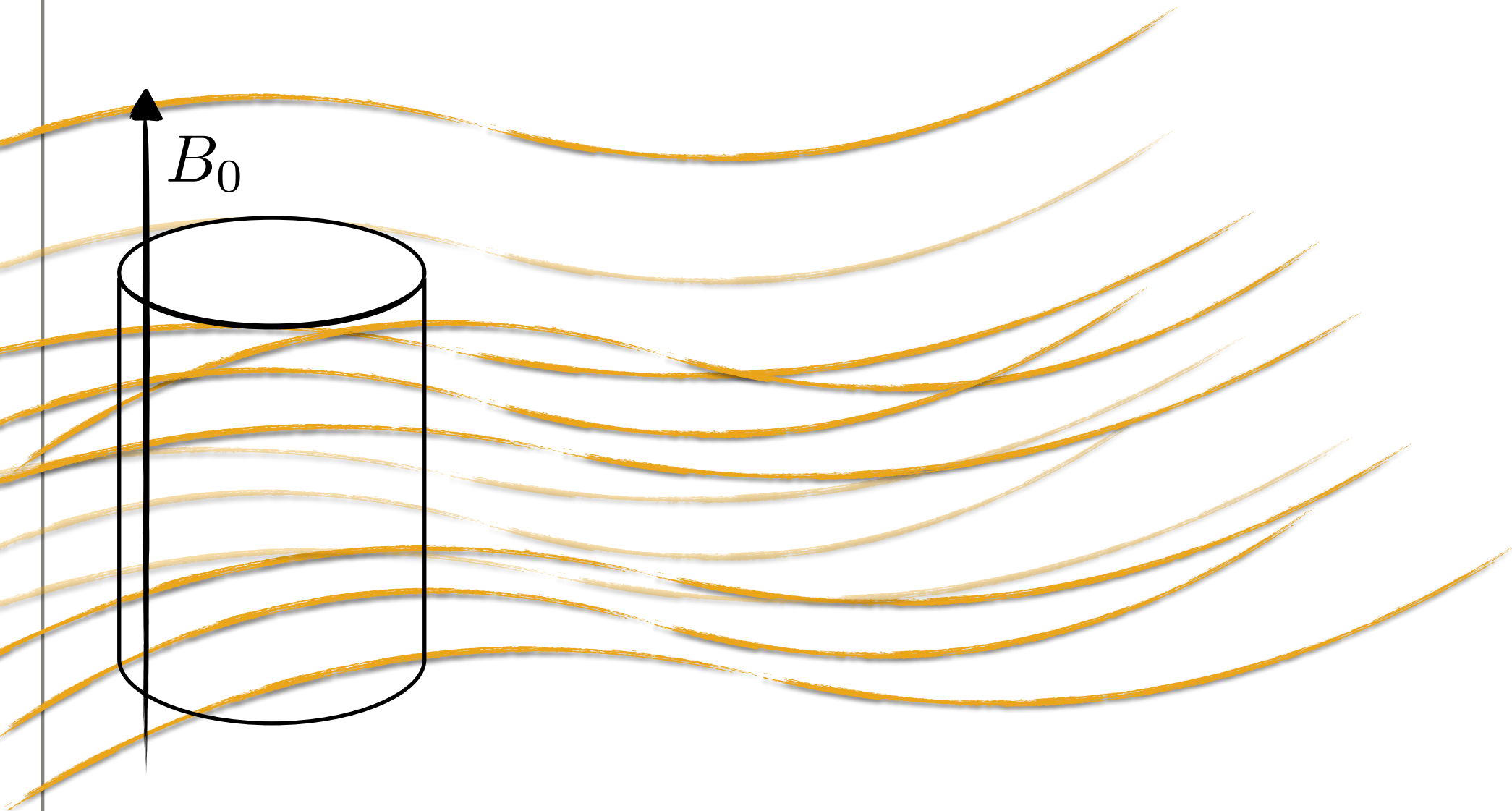
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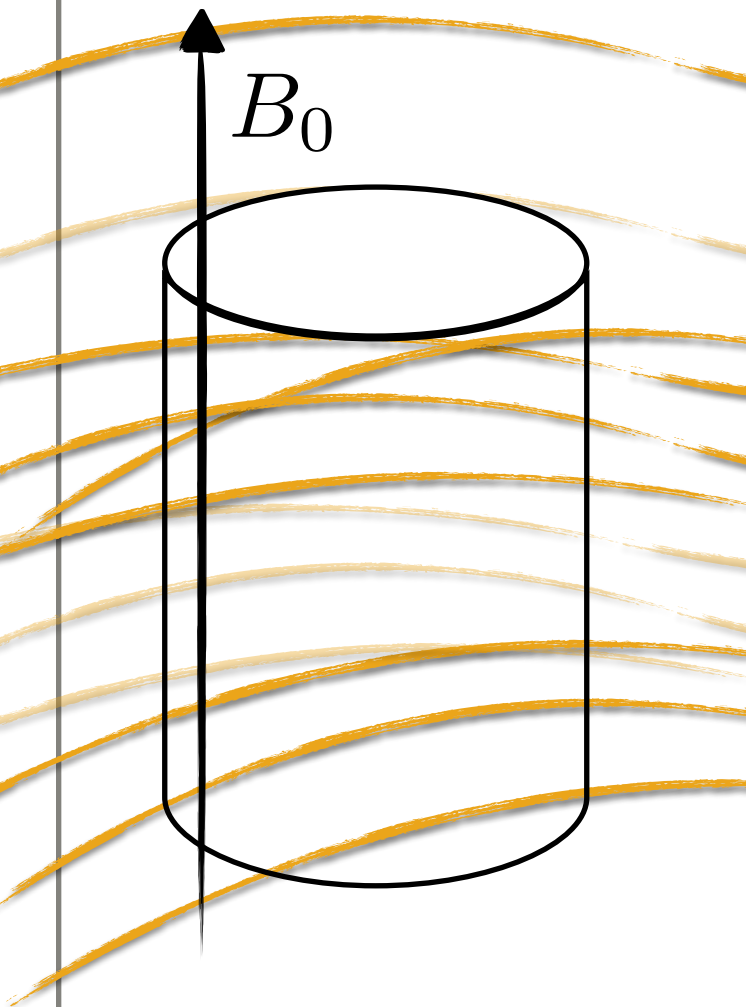
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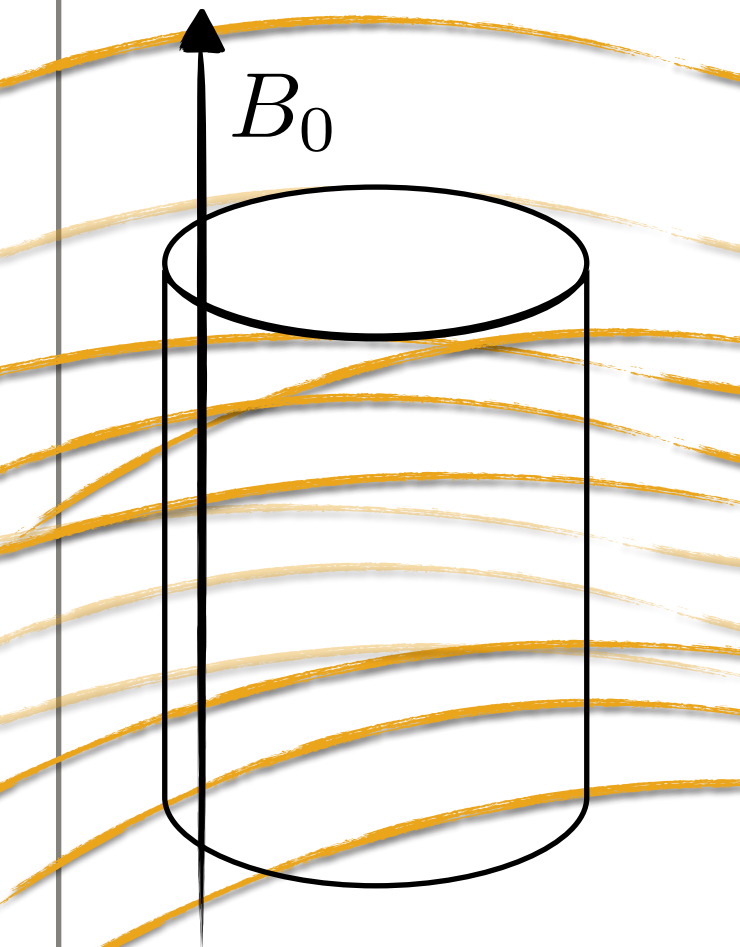
An Axion Haloscope :

Axion - Photon Interaction

$$\mathcal{L} = -\frac{1}{4}g_{a\gamma\gamma}\phi F_{\mu\nu}\tilde{F}^{\mu\nu} = g_{a\gamma\gamma}\phi\mathbf{E} \cdot \mathbf{B}$$



An Axion Haloscope :



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Interaction Hamiltonian

$$H_{\text{int}} = g_{a\gamma\gamma}B_0 \int_{V_c} d^3\mathbf{x}\phi(\mathbf{x})E_z(\mathbf{x})$$

$$E_z(\mathbf{x}) = \sum_{\ell} \sqrt{\frac{\omega_{\ell}}{2}} (ib_{\ell}\tilde{u}_{\ell,z}^*(\mathbf{x}) + \text{h.c.})$$

$$\phi(\mathbf{x}) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2\omega_p\mathcal{V}}} (a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + \text{h.c.})$$

Axion haloscope: effective mode

$$H_{\text{int}}(t) \simeq \frac{1}{2} g_{a\gamma\gamma} B_0 \sqrt{\frac{V_c}{\mathcal{V}}} \left(i c^\dagger \sum_{\mathbf{p}} C_{\mathbf{p}} e^{-iK_p t} a_{\mathbf{p}} + \text{h.c.} \right)$$

$$\omega_p = m_{\text{DM}} + K_p$$

$$C_{\mathbf{p}} = \sqrt{\frac{m_{\text{DM}}}{\omega_p V_c}} \int_{V_c} d^3\mathbf{x} e^{i\mathbf{p}\cdot\mathbf{x}} \tilde{E}_z(\mathbf{x})$$

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Effective (single) mode:

Measuring coherence time $\tau_{\text{DM}} \sim 1/\Delta\omega_{\text{DM}}$, $e^{-iK_p t} \sim \text{constant}$

We can't resolve these modes

Can only be sensitive to a coarse grained effective mode

$$a_{\text{eff}}(t) \equiv \frac{1}{2} \sqrt{\frac{V_c}{\Omega \mathcal{V}}} \sum_{\mathbf{p}} C_{\mathbf{p}} e^{-iK_p t} a_{\mathbf{p}}$$

Effect and effective mode

Consider effective mode loses information

If modes are independent

coarse graining + QCLT $\rightarrow$ Gaussian, quantum-ness washed out

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If, for some reason, QCLT does not apply:

Cavity state evolution

$$H_{\text{int}}(t) \simeq ig \left(c^\dagger a_{\text{eff}}(t) - c a_{\text{eff}}^\dagger(t) \right)$$

Solve evolution

$$e^{iH_{\text{int}}t} c e^{-iH_{\text{int}}t} = c \cos gt + a_{\text{eff}} \sin gt$$

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Evolve $\rho(t)$



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State of cavity: $P_{\text{cav}}^f(\alpha) \simeq \int d^2\beta \frac{P_{\text{eff}}(\beta/\sqrt{\eta})}{\eta} P_{\text{cav}}^i(\alpha - \beta)$

$$\text{Axion conversion probability } \eta \sim (gt)^2$$

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$$\eta \sim 10^{-22} \left(\frac{g_{a\gamma\gamma}}{10^{-15} \text{GeV}^{-1}} \frac{B_0}{10 \text{ T}} \frac{Q_c}{10^5} \frac{10^{-5} \text{eV}}{m_{\text{DM}}} \right)^2 \quad \text{Very small detection rate}$$

Non-classical Effects on $P(\alpha)$

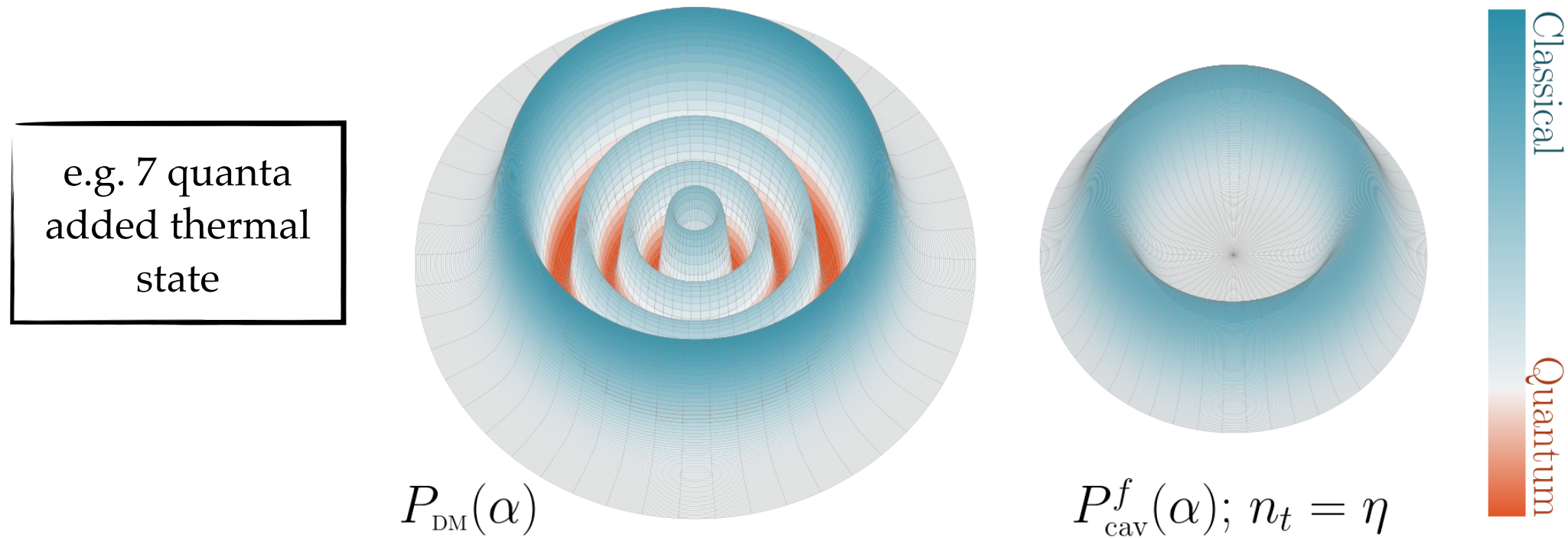
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Regions of $P_{\text{cav}}^f(\alpha) < 0$ restricted to $\mathcal{O}(\sqrt{\eta})$ regions in α space.

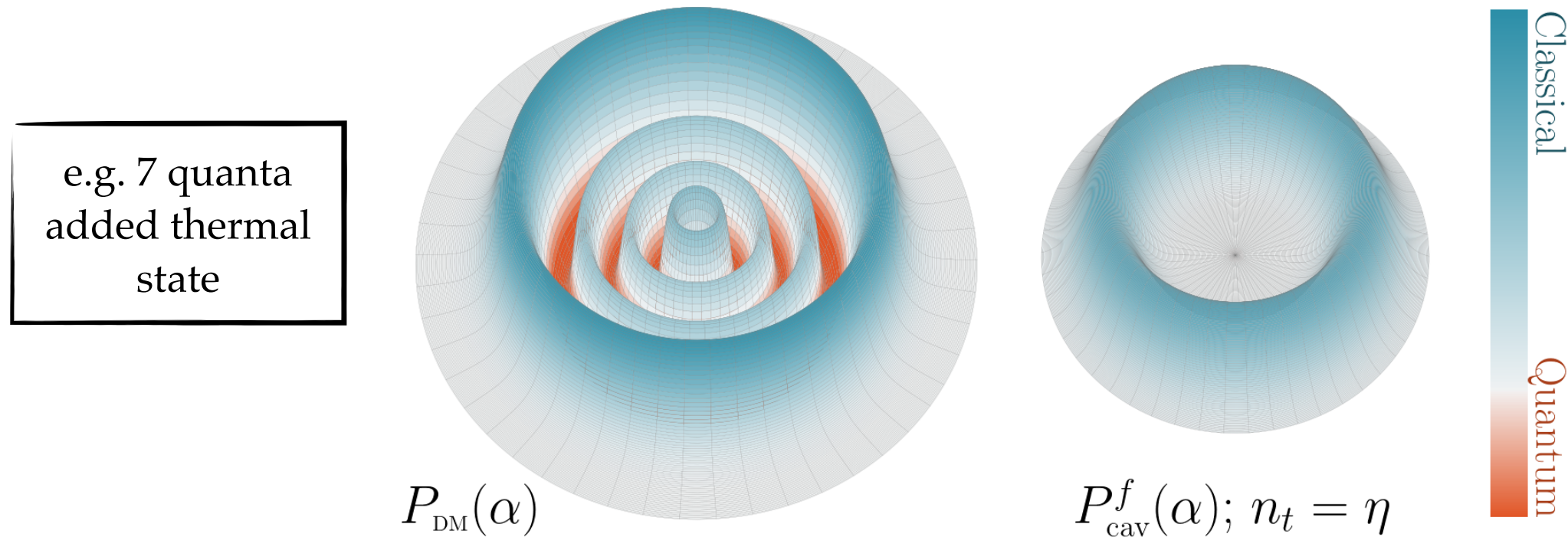
Typically very small. Need extremely good sensitivity.

Effect of (tiny) noise



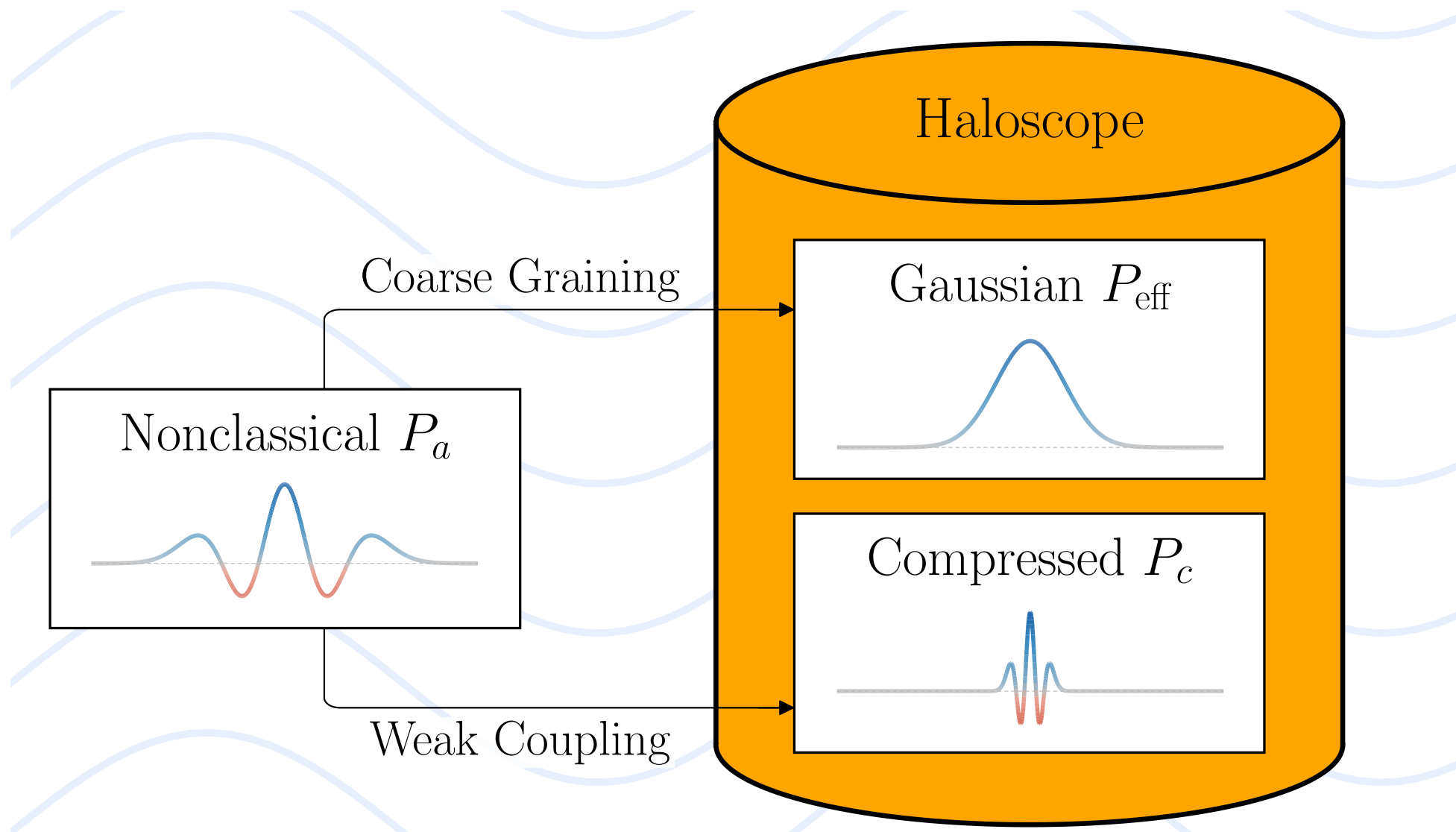
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Regions of $P_{\text{cav}}^f(\alpha) < 0$ restricted to $\mathcal{O}(\sqrt{\eta})$ regions in α space.

Effects washed out by e.g thermal noise $n_t \gtrsim \eta$ $T \gtrsim \frac{m_{\text{DM}}}{\log(1/\eta)} \simeq 2\text{mK} \left(\frac{m_{\text{DM}}}{10^{-5}\text{eV}} \right)$



Imprints of non-classicality

Quadrature $X = \frac{1}{\sqrt{2}} (b + b^\dagger)$

Number $n = b^\dagger b$

$$S(X) = \text{Var}(X) - \frac{1}{2}$$

$$Q = \frac{\text{Var}(n)}{\langle n \rangle} - 1,$$

Classical states : $Q, S > 0$,
Quantum states : $Q, S < 0$

$$S \geq -\frac{1}{2}$$

Saturated by Infinitely squeezed state

$$Q \geq -1$$

Saturated by number state

Detection of non-classicality

Quadrature	Number
$p(x) = \int d\alpha P_{\text{DM}}^{\text{eff}}(\alpha) \langle x \sqrt{\eta}\alpha \rangle ^2$ $S_{\text{cav}} = \eta S_{\text{DM}} \geq -\frac{\eta}{2}$	$p_n = \int d\alpha P_{\text{DM}}^{\text{eff}}(\alpha) \langle n \sqrt{\eta}\alpha \rangle ^2$ $Q_{\text{cav}} = \eta Q_{\text{DM}} \geq -\eta$
Negativities at most $\mathcal{O}(\eta)$, even in vacuum cavity	
Time required to observe $S_{\text{cav}}, Q_{\text{cav}} < 0$	$t_{\text{int}} \sim \frac{t_m}{\eta^2} \sim 10^{26} \text{ years}$ $g_{a\gamma\gamma} = 10^{-14} \text{ GeV}^{-1}$ $B_0 = 10 \text{ T}$ $Q_c = 10^5$ $m_a = 10^{-5} \text{ eV}$

What about distinguishing states?

In a post-discovery scenario : G : Gaussian C : Coherent

$$Q_{\text{DM}}^G \simeq N_{\text{eff}}$$

vs

$$Q_{\text{DM}}^C = 0$$



$$Q_{\text{cav}}^G \simeq \eta N_{\text{eff}} = n_s$$

vs

$$Q_{\text{cav}}^C = 0$$

Feasible*

Time required to distinguish : $t_{\text{int}} \sim \frac{t_m}{n_s^2} \sim 0.06 \left(\frac{g_{a\gamma\gamma}}{10^{-14} \text{ GeV}} \right)^{-4} \text{ yrs}$

Future directions

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We are currently investigating some of these questions.

extra

Density matrix

Using a coherent state $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ as a basis

$$\hat{\rho} = \int d^2\alpha P(\alpha) |\alpha\rangle \langle \alpha|$$

Coherent state basis overcomplete, diagonal decomposition of $\hat{\rho}$

[Glauber-Sudarshan P - distribution function]

[Glauber, 1963]
[Sudarshan 1963]

Unique $P(\alpha)$ for a generic $\hat{\rho}$ in coherent state basis

[Metha, 1967]

$$P(\alpha) = \int \frac{d^2\beta}{\pi^2} \langle -\beta | \hat{\rho} | \beta \rangle e^{|\alpha|^2 + |\beta|^2 + 2i \operatorname{Im}[\alpha\beta^*]}$$

Expectation values (ensemble averages now described with $P(\alpha)$)

$$\langle \hat{\mathcal{O}} \rangle = \operatorname{Tr} [\hat{\mathcal{O}} \hat{\rho}] = \int d^2\alpha P(\alpha) \operatorname{Tr} [\hat{\mathcal{O}} |\alpha\rangle \langle \alpha|] = \int d^2\alpha P(\alpha) \langle \alpha | \hat{\mathcal{O}} | \alpha \rangle$$

Energy density

The energy density with quadratures $\hat{x}_{\mathbf{k}} = \frac{1}{\sqrt{2\langle N_{\mathbf{k}} \rangle}} (\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}^\dagger)$, & $\hat{y}_{\mathbf{k}} = -\frac{i}{\sqrt{2\langle N_{\mathbf{k}} \rangle}} (\hat{a}_{\mathbf{k}} - \hat{a}_{\mathbf{k}}^\dagger)$

$$\begin{aligned} \rho &= \frac{1}{2} \left[(\partial_t \phi)^2 + (\nabla \phi)^2 + m^2 \phi^2 \right] \\ &= \sum_{\mathbf{k}, \mathbf{q}} \frac{1}{2V} \sqrt{\frac{N_{\mathbf{k}} N_{\mathbf{q}}}{\omega_{\mathbf{k}} \omega_{\mathbf{q}}}} \left[(\omega_{\mathbf{k}} \omega_{\mathbf{q}} + \mathbf{k} \cdot \mathbf{q}) y_{\mathbf{k}} y_{\mathbf{q}} + m^2 x_{\mathbf{k}} x_{\mathbf{q}} \right] \end{aligned}$$

See also.
e.g) Kim 2021, 2023

In the non-relativistic limit, sum of two χ^2 distributions with single dof.

$$\rho \simeq \sum_{\mathbf{k}, \mathbf{q}} \frac{m}{2V} \sqrt{N_{\mathbf{k}} N_{\mathbf{q}}} [y_{\mathbf{k}} y_{\mathbf{q}} + x_{\mathbf{k}} x_{\mathbf{q}}] \quad \text{Exponentially distributed!}$$

$$\langle \rho \rangle \simeq m \bar{n}$$

$$\langle \rho_{\phi}^n \rangle = n! \rho_0^n$$

Quantum central limit theorem

Glauber, 1963

$$\tilde{P}_{\text{eff}}(\lambda) = \int d\alpha P(\alpha) e^{\lambda \alpha_{\text{eff}}^* - \lambda^* \alpha_{\text{eff}}} = \int d\alpha P(\alpha) \prod_p \exp(\lambda c_p^* \alpha_p^* - \lambda^* c_p \alpha_p) \quad \tilde{P}_{\text{eff}}(\lambda) = \prod_p \tilde{P}_p(c_p^* \lambda)$$

$$\begin{aligned} \tilde{P}_p(c_p^* \lambda) &= \int d\alpha_p P_p(\alpha_p) \left[1 + (\lambda c_p^* \alpha_p^* - \lambda^* c_p \alpha_p) + \frac{1}{2} (\lambda c_p^* \alpha_p^* - \lambda^* c_p \alpha_p)^2 + \mathcal{O}((c_p \lambda)^3) \right] \quad (3.24) \\ &= 1 + \lambda c_p^* \langle a_p^\dagger \rangle - \lambda^* c_p \langle a_p \rangle + \frac{1}{2} \left(\lambda^2 c_p^{*2} \langle a_p^{\dagger 2} \rangle + \lambda^{*2} c_p^2 \langle a_p^2 \rangle \right) - |c_p \lambda|^2 \langle a_p^\dagger a_p \rangle + \mathcal{O}((c_p \lambda)^3) \end{aligned}$$

Many modes contributing: $c_p \ll 1$

$$\begin{aligned} \log \tilde{P}_{\text{eff}}(\lambda) &\simeq \lambda \mu^* - \lambda^* \mu + \frac{1}{2} [\lambda^2 \Delta^* + \text{c.c.}] - |\lambda|^2 N_{\text{eff}} \\ \mu &= \sum_p c_p \langle a_p \rangle, \\ \Delta &= \sum_p c_p^2 (\langle a_p^2 \rangle - \langle a_p \rangle^2), \\ N_{\text{eff}} &= \sum_p |c_p|^2 (\langle a_p^\dagger a_p \rangle - \langle a_p^\dagger \rangle \langle a_p \rangle) \end{aligned}$$

$$P_{\text{eff}}(\alpha) = \frac{1}{\pi \sqrt{N_{\text{eff}}^2 - |\Delta|^2}} \exp \left[-\frac{N_{\text{eff}} |\mu - \alpha|^2 - \text{Re}(\Delta^* (\mu - \alpha)^2)}{N_{\text{eff}}^2 - |\Delta|^2} \right]$$

Quantum central limit theorem

Glauber, 1963

Number of modes: $\mathcal{N} \sim (m_{\text{DM}} v_{\text{DM}} R)^3$ R: halo radius

Coarse graining Δp : $\mathcal{N} \sim (m_{\text{DM}} v_{\text{DM}} / \Delta p)^3$

Details require numerical simulation.

Characteristic functions

$$\tilde{P}^{(s)}(\lambda) = e^{s|\lambda|^2/2} \text{tr} \left[\rho e^{\lambda a^\dagger - \lambda^* a} \right] = \begin{cases} \text{tr} \left[e^{-\lambda^* a} \rho e^{\lambda a^\dagger} \right] & s = 1 \\ \text{tr} \left[\rho e^{\lambda a^\dagger - \lambda^* a} \right] & s = 0 \\ \text{tr} \left[e^{\lambda a^\dagger} \rho e^{-\lambda^* a} \right] & s = -1 \end{cases}$$

$$P^{(s)}(\alpha) = \int \frac{d\lambda}{\pi^2} \tilde{P}^{(s)}(\lambda) e^{-\lambda \alpha^* + \lambda^* \alpha} = \begin{cases} \text{Glauber-Sudarshan } P(\alpha) & s = 1 \\ \text{Wigner } W(\alpha) & s = 0 \\ \text{Husimi } Q(\alpha) & s = -1 \end{cases}$$

$$W(\alpha) = \frac{2}{\pi} \int d\beta P(\beta) e^{-2|\alpha - \beta|^2}, \quad Q(\alpha) = \frac{1}{\pi} \int d\beta P(\beta) e^{-|\alpha - \beta|^2}$$

State	$\tilde{P}^{(s)}(\lambda)$	$P(\alpha)$	$W(\alpha)$	$Q(\alpha)$
Coherent $ \alpha_0\rangle$	$e^{(s-1) \lambda ^2/2} e^{\lambda \alpha_0^* - \lambda^* \alpha_0}$	$\delta(\alpha - \alpha_0)$	$\frac{2}{\pi} e^{-2 \alpha - \alpha_0 ^2}$	$\frac{1}{\pi} e^{- \alpha - \alpha_0 ^2}$
Gaussian, $\langle n \rangle = N$	$e^{-[N+(1-s)/2] \lambda ^2}$	$\frac{1}{\pi N} e^{- \alpha ^2/N}$	$\frac{1}{\pi(N+1/2)} e^{- \alpha ^2/(N+1/2)}$	$\frac{1}{\pi(N+1)} e^{- \alpha ^2/(N+1)}$
Fock $ n\rangle$	$e^{(s-1) \lambda ^2/2} L_n(\lambda ^2)$	$L_n(-\partial_\alpha \partial_{\alpha^*}) \delta(\alpha)$	$\frac{2}{\pi} (-1)^n L_n(4 \alpha ^2) e^{-2 \alpha ^2}$	$\frac{1}{\pi} \frac{ \alpha ^{2n}}{n!} e^{- \alpha ^2}$

Gaussian and thermal

Gaussian state can be written as

$$\hat{\rho} = \frac{1}{1 + \langle n \rangle} \sum_n \left(\frac{\langle n \rangle}{\langle n \rangle + 1} \right)^n |n\rangle \langle n|$$

Same as a thermal ensemble of harmonic oscillators and

$$\hat{H} = \omega a^\dagger a \qquad \langle n \rangle = (e^{\beta\omega} - 1)^{-1}$$

Example: adding Fock states

Fock state: $|N\rangle$

Consider equal mixture of m such state $\alpha_{\text{eff}} = \sum_{p=1}^m \alpha_i / \sqrt{m}$.

Using Fourier transform: $\tilde{P}(\lambda) = \int d\alpha P(\alpha) e^{\lambda\alpha^* - \lambda^*\alpha} = \text{tr} \left(e^{-\lambda^* a} \rho e^{\lambda a^\dagger} \right)$

$$\tilde{P}_{\text{eff}}(\lambda) = \prod_{p=1}^m \tilde{P}_p(\lambda/\sqrt{m}) \quad \tilde{P}_p(\lambda) = L_N(|\lambda|^2)$$

$$\tilde{P}_{\text{eff}}(\lambda) = [L_N(|\lambda|^2/m)]^m = \left(1 - \frac{N}{m} |\lambda|^2 + \mathcal{O}(1/m^2) \right)^m$$

For large m , this becomes $\tilde{P}_G(\lambda) = e^{-N|\lambda|^2}$ **Gaussian!**