International Workshop on New Opportunities for Particle Physics 2026

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Institute of High Energy Physics, Chinese Academy of Sciences
Asia/Shanghai timezone

Massive Amplitudes & Constructability

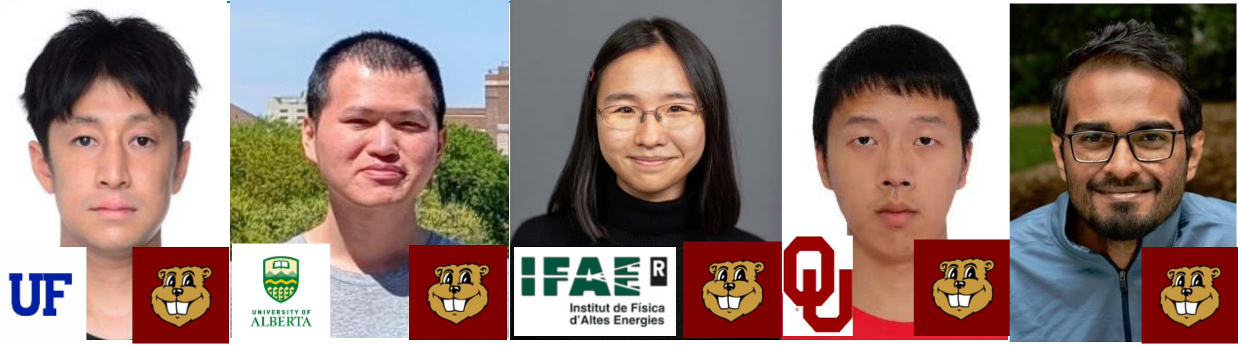
刘真

Zhen Liu

08/15/2026

*w Y. Ema, T. Gao, W. Ke, K. F. Lyu, **Ishmam Mahbub***

2403.15538, 2407.14587, 2506.02106 + ongoing studies.



Great Promise of the amplitude program

How many Feynman diagrams...?

n	2	3	4	5	6	7	8
# of diagrams	4	25	220	2485	34300	559405	10525900

Table 1: The number of Feynman diagrams contributing to the scattering process $gg \to \mathrm{n}\ g$

from hep-th/0509223

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Parke, Taylor did it in 1985:
Results presented in a 25-page paper

FERMILAB-Pub-85/118-T
August, 1985

GLUONIC TWO GOES TO FOUR

Stephen J. Parke and T.R. Taylor
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, IL 60510
U.S.A.

One year later…

VOLUME 56, NUMBER 23 PHYSICAL REVIEW LETTERS 9 JUNE 1986

Amplitude for *n*-Gluon Scattering

Stephen J. Parke and T. R. Taylor
Fermi National Accelerator Laboratory, Batavia, Illinois 60510
(Received 17 March 1986)

$$|\mathcal{M}_n(+++++\cdots)|^2 = c_n(g,N)[0+O(g^4)],$$

$$|\mathcal{M}_n(-++++\cdots)|^2 = c_n(g,N)[0+O(g^4)],$$

$$|\mathcal{M}_n(--+++\cdots)|^2 = c_n(q,N)[(p_1\cdot p_2)^4 \times \textstyle\sum_P[(p_1\cdot p_2)(p_2\cdot p_3)(p_3\cdot p_4)\cdots(p_n\cdot p_1)]^{-1} + O(N^{-2}) + O(g^2)],$$

…….

June, 1987

DUALITY AND MULTI-GLUON SCATTERING

MICHELANGELO MANGANO, STEPHEN PARKE and ZHAN XU [1]

$$m_{3+2-}(1,2,3,4,5) \;=\; 4\sqrt{2}ig^3\frac{\langle IJ\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}.$$

The Great Promise of Helicity Amplitudes

- On-shell recursion relation can be used to remarkably simplify calculation

6-pt MHV gluon squared amplitude →

$$\boxed{|A_6|^2 \sim \frac{(p_1 \cdot p_2)^3}{(p_2 \cdot p_3)(p_3 \cdot p_4)(p_4 \cdot p_5)(p_5 \cdot p_6)(p_6 \cdot p_1)}}$$

Parke, Taylor
PhysRevLett.56.2459

MHV: Maximal Helicity Violation

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R. Britto, F. Cachazo, B. Feng, and E. Witten (arXiv: 0501052)

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Parke, Taylor PhysRevLett.56.2459

- A powerful calculation tool for recursive construction
- A powerful tool to study structures of quantum field theory (BCJ, double-copy, etc.)

$$A_n[1^+ \cdots i^- \cdots j^- \cdots n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} .$$

MHV: Maximal Helicity Violation

Massive Helicity Amplitudes Just Inherit the Promise(?)

- On-shell methods well developed for massless particles

- Most Particles are Massive

N. Arkani-Hamed, T. C. Huang, Y. Huang, (1709.04891)

- Systematic description was introduced by Arkani-Hamed, Huang, and Huang.

Massive Spinor Variables

SU(2) little group index

$$[\mathbf{i}|^{I}_{\dot{a}} \qquad |\mathbf{i}\rangle^{I}_{a}$$

Chiral Index

Spin index $I = 1{,}2$ represents the little group freedom (choice of spin-axis)

We work in the *helicity basis*

$$|\mathbf{i}\rangle^{I}_{a} = |i\rangle_{a}\delta^{I}_{-} + |\eta_i\rangle_{a}\delta^{I}_{+}$$

$$[\mathbf{i}|^{I}_{\dot{a}} = [i|_{\dot{a}}\delta^{I}_{+} + [\eta_i|_{\dot{a}}\delta^{I}_{-}$$

On-shell condition

$$\langle i\eta_i\rangle = [i\eta_i] = m_i$$

In the HE limit $\eta \to 0$

Massive Spinor Variables

SU(2) little group index

$$[\mathbf{i}|_{\dot{a}}^{I} \qquad |\mathbf{i}\rangle_a^{I}$$

So, Nima et al cleared the way. *Just boldface* the spinors.
Everything is settled, why don't we just use them for calculations?
Helicity amplitudes should have dominated the field of particle physics calculations, in particular for higher points, and allow us to explore many other properties of QFT and particle physics. Have you heard about their full usage in REAL calculations? (Blackhat, Madgraph/HELAS…)

$$|\mathbf{i}\rangle_a^I = |i\rangle_a \delta_-^I + |\eta_i\rangle_a \delta_+^I$$

$$[\mathbf{i}|_{\dot{a}}^I = [i|_{\dot{a}} \delta_+^I + [\eta_i|_{\dot{a}} \delta_-^I$$

$$\langle i\eta_i\rangle = [i\eta_i] = m_i$$

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Confusion in QED

- Gluing results in discrepancies for massive $ee \to \mu\mu$ calculations

$$A_4|_{\mathrm{AHH}} = \tilde{e}^2 \frac{\langle \mathbf{12}\rangle\langle \mathbf{34}\rangle}{ms}(p_1 - p_2)\cdot p_3,$$

N. Arkani-Hamed, T. C. Huang, Y. Huang, arXiv:1709.04891

$$A_4|_{\mathrm{CDFHM}} = \frac{\tilde{e}^2}{2m_e m_\mu s}\left[\left(u - t + 2m_e^2 + 2m_\mu^2\right)[\mathbf{12}][\mathbf{34}] + 2\left([\mathbf{12}][\mathbf{3}|\mathbf{21}|\mathbf{4}] + [\mathbf{34}][\mathbf{1}|\mathbf{43}|\mathbf{2}]\right)\right].$$

N. Christensen, H. Diaz-Quiroz, B. Field, J. Miles arXiv:2209.15018 Pheno 2023 Talk

- Correct amplitude (Validated by Feynman Rule Calculation):

$$A_4 = \frac{\tilde{e}^2}{p_{12}^2}\left[\langle\mathbf{13}\rangle[\mathbf{24}] + \langle\mathbf{14}\rangle[\mathbf{23}] + \langle\mathbf{23}\rangle[\mathbf{14}] + \langle\mathbf{24}\rangle[\mathbf{13}]\right],$$

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So using massive on-shell amplitude, we couldn't even get your kindergarten QFT calculation correct.

In fact, it is a common pattern in amplitudes…

- Write down a 3-point
- Put them together (naively or more consistently, e.g., BCFW) to get a 4-point;
- But it doesn’t stop here at 4-point… next steps are highly non-satisfying…

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- But it doesn't stop here at 4-point… next steps are highly non-satisfying…

- (Secretly behind the scenes) contrast against your target results/properties (sometimes obtained via Feynman Rules)
- Find some physical arguments (good high-energy behavior, consistent low-energy theory, soft theorem, Adler zeros, etc.) to further justify the missing terms.
- Add the needed additional terms
- Same for higher points and even quickly lose track (in particular, if naïve gluing).

Resolving the Discrepancy

(It is the so called “contact term ambiguities”)

- (In the complex momentum plane) Many different forms of 3-point interactions (minimal amplitudes) are equivalent on-shell.
- These equivalent on-shell forms lead to different 4-point amplitudes when naively glued together.

Resolving the Discrepancy

(It is the so called "contact term ambiguities")

- (In the complex momentum plane) Many different forms of 3-point interactions (minimal amplitudes) are equivalent on-shell.
- These equivalent on-shell forms lead to different 4-point amplitudes when naively glued together.
- Resolution: One should perform the analytic continuation (momentum shift) consistently to be free from discrepancy.

All expressions agree when internal photon is on-shell (an unphysical configuration) (but differ for off-shell photon—the real physics configuration).

We **proved** this point in our work, fully resolving the confusion.

Recursion Relations (on-shell construction)

- **Deform** the momentum by complex variable z: $p_i \to \hat{p}_i = p_i + z\, q_i$

$$\boxed{A_n \to \hat{A}_n(z)} \qquad \hat{A}_n(0) = A_n$$

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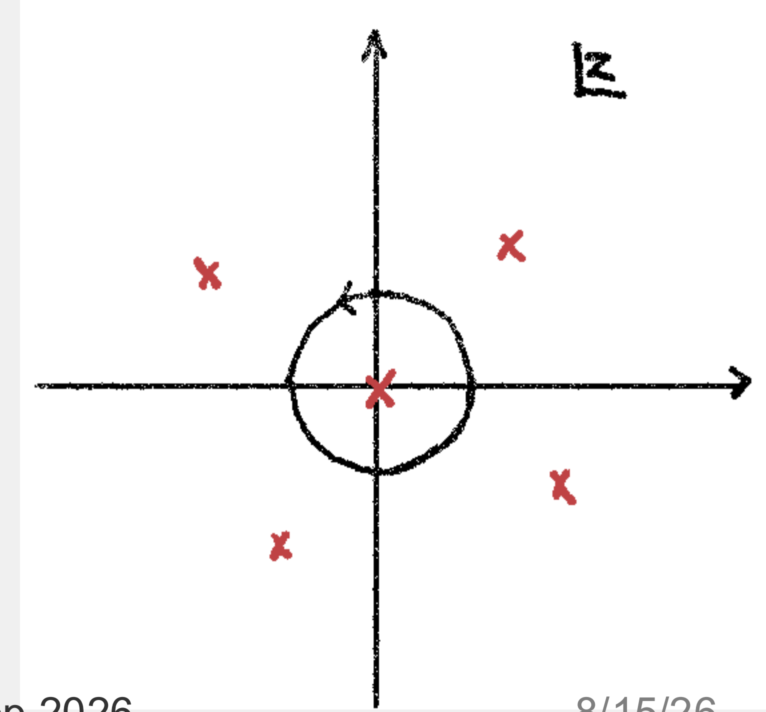
- **Locality**: physical poles of $\hat{A}_n(z)$ = intermediate particles on shell

- **Cauchy theorem (analytic function):**

$$A_n = \oint_{z=0} dz \, \frac{\hat{A}_n}{2\pi i z}$$



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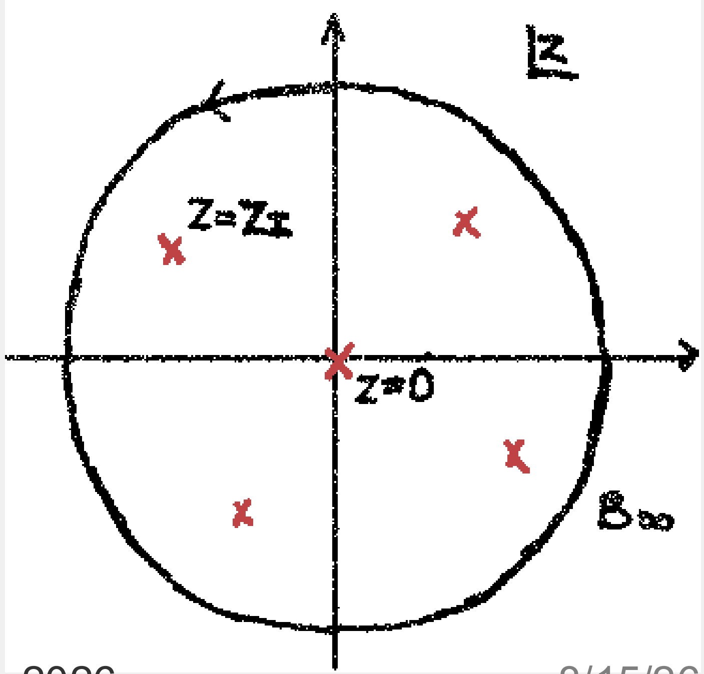
$$A_n = \oint_{z=0} dz\, \frac{\hat{A}_n}{2\pi i z}$$

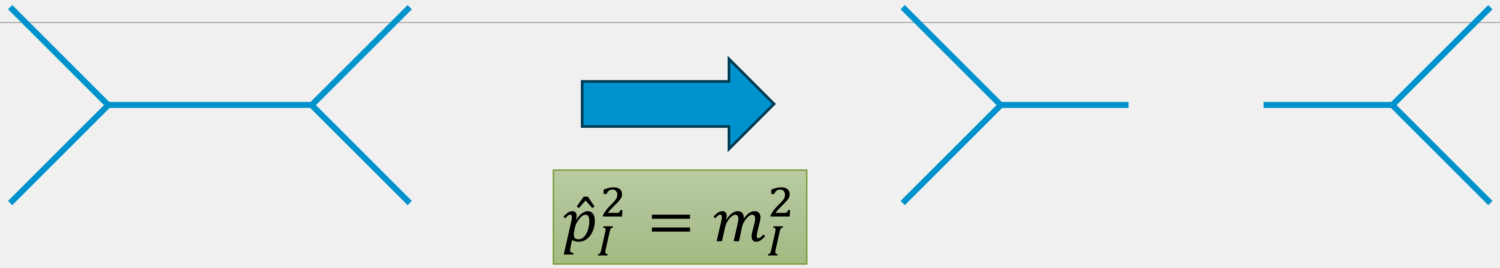
$$= -\sum_{z_I} Res_{z=z_I}\left[\frac{\hat{A}_n}{z}\right] + B_\infty$$

Intermediate particle on shell

Boundary term (no accessible pole structures)



❖ At the pole: the residue **factorizes** into **sub**-amplitudes



$$A_n = -\sum_{z_I} Res_{z=z_I}\left[\frac{\hat{A}_n}{z}\right] + B_\infty$$

$$= -\sum_{z_I}\sum_{\lambda} Res_{z=z_I}\left[\frac{1}{z}\hat{A}_{n-m+2}^{(\lambda)}\frac{1}{\hat{p}_I - m_I^2}\hat{A}_m^{(-\lambda)}\right] + B_\infty$$

Intermediate particle helicity

subamplitudes

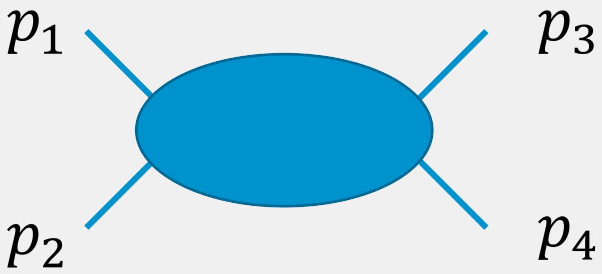
❖ If $B_\infty = 0$: combine lower point to get higher point amplitude

The theory is **'constructible'**

Boundary Terms B_∞

$$\lim_{z\to\infty} \hat{A}(z) = 0 \Leftrightarrow B_\infty = 0 \Leftrightarrow \text{Constructible}$$

"shift" → choose a **good shift** $p_i \to \hat{p}_i = p_i + z\, q_i$



- Which momentum p_i do we shift?
- What is the shifted momentum q_i?

- All "dependent" contact terms will be automatically generated through this process.
- One shall include additional contact term contributions in the construction if they are "independent", e.g., higher-dim operators.

In other words, four-point contact interactions are automatically encoded in the three-point interactions through a consistent amplitude process, such as those with gauge symmetries or SUSY. I.E, almost all SM contact interactions are not inputs, except for the Higgs self-interaction.

Recap of the picture

- Just putting multiplying two amplitudes (naïve gluing)→gets part of the total amplitudes, with all the relevant pole structures, with additional, unfixed non-pole terms (depending on your input lower point on-shell amplitudes); you never know whether you get the full expression using locality, analyticity, and locality.
- Performing analytic continuation consistently will allow one to get an unambiguous amplitude(this is invariant under on-shell identities at the lower point level)
- One needs to perform additional analysis on its constructability under a given continuation to know if there are boundary terms needed. (If needed, we still cannot fix it without additional inputs or a better continuation.)

Recap of the picture

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A side remark: was Nima et al wrong? The question is dodged by the following footnote:

[9] Of course the amplitude cannot be uniquely determined in this way, since we can always simply have contact terms that are simply polynomials with no poles at all (corresponding to piece in N that cancels all the poles). To avoid clutter, we will suppress the possible contact terms in what follows.

- One needs to perform additional analysis on its constructability under a given continuation to know if there are boundary terms needed. (If needed, we still cannot fix it without additional inputs or a better continuation.)

We understood the problem. Let’s go and calculate!

But a proper shift for general massive amplitude did not fully exist*!

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But a proper shift for general massive amplitude did not fully exist*!

- BCFW is for massless theory
- Many other shifts for the massless theory
- *(Partial) generalization of BCFW exists for massive theory (depending on exactly which leg is massive and massless→ constructability analysis quite hard.)
- We decided to just be smart and create our own. (We first independently discovered massive generalization of BCFW and were dissatisfied with its properties; and also invented a new solution, that is, one formulation applies for all massive, massive-massless, and massless theories!)

Our proposed Momentum Shift: ALT

*The ALT shift defined in vector Rep is easiest for us to see many of its properties, but the true amplitude definition is done in the spinors. It is also expressing time-like vector by two lightcone vectors that are P related.

- We introduce an All-Line-Transverse ("ALT") shift

$$\boxed{\hat{p}_i = p_i + z\, c_i m_i \epsilon_i^{(\pm)}}$$

  - Here we are in circular polarization

- In helicity basis:

For an external particle with helicity

$+\frac{1}{2}, +1, +\frac{3}{2} \ldots$

$$\begin{cases} +: & |\lambda_i\rangle \to |\lambda_i\rangle + z c_i\, |\eta_i\rangle \\ -: & |\tilde{\lambda}_i] \to |\tilde{\lambda}_i] + z c_i |\tilde{\eta}_i] \end{cases}$$

*There exists (recently) a massive BCFW shift for certain helicities and configurations.
[R. Franken, C. Schwinn, arXiv:1910.13407]
[C. Wu, S. H. Zhu, arXiv: 2112.12312]
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Features of the ALT shift

After the shift:

(concerning the property of A(z)/z)

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✅ On-shell condition preserved: $\hat{p}_i^2 = m_i^2$

→ because $p_i \cdot \epsilon_i^{(\pm)} = 0$ and $(\epsilon_i^{(\pm)})^2 = 0$

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  $\rightarrow$ so far show the **best** applicability for amplitude construction.
- Universal for Massive and Massless theories

  $\rightarrow$ A smooth continuation to the massless all-line shift, with additional d.o.f.
- High-z is automatically in a form applicable for Ward Identities

  $\rightarrow$ Additional analysis power for the underlying theory

Features of the ALT shift: Constructability Analysis

$$A_n \sim \left(\sum_{diag} g \times F\right) \times \prod_{vector} \epsilon \times \prod_{fermion} u$$

propagators, momenta

couplings

- n-point amplitude has mass dimension $4-n$, so

$$4-n=[g]+[F]+\frac{N_F}{2}$$

N_F: # of fermions

- ALT: 1) momentum *linearly* shifted, 2) ϵ, u not shifted

$$\hat{A}_n(z) \sim z^\gamma \;,\; \gamma \le [F] = 4-n-[g]-\frac{N_F}{2}$$

- *Renormalizable* theory: $[g] \ge 0 \Rightarrow \gamma \le 4-n-\frac{N_F}{2}$

For 5-pt amplitude and 4-pt with $N_F > 0$: $\gamma < 0$

$\lim_{z\to\infty} \hat{A}(z) = 0$ $\Longleftrightarrow$ $B_\infty = 0$ $\Longleftrightarrow$ amplitude constructible

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$$A_n \sim \left(\sum_{diag} g \times F\right) \times \prod_{vector} \epsilon \times \prod_{fermion} u$$

propagators, momenta

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- ❖ n-point amplitude has mass dimension $4-n$, so

$\Rightarrow$ $4-n=[g]+[F]+\frac{N_F}{\ldots}$

We prove renormalizable theory are constructable under ALT shift. (Ward identity is needed for spontaneously broken gauge theory during the proof; this is not extra requirement; expressions give us this property directly.)

In other words, one can define their own shifts, it might converge better or worse (most likely worse) so whatever one obtain won't be the full result as the theory is not constructable under an arbitrary shift.

$\Rightarrow$ For 5-pt amplitude and 4-pt with $N_F > 0 : \gamma < 0$

$\lim_{z\to\infty} \hat{A}(z) = 0 \Longleftrightarrow B_\infty = 0 \Longleftrightarrow$ amplitude constructible

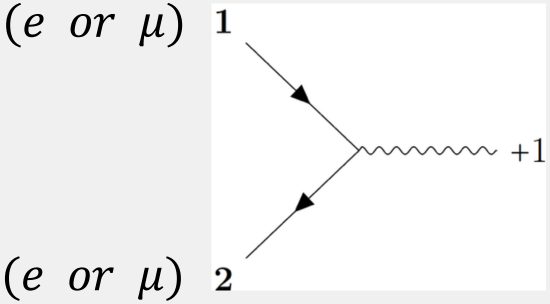
Solving QED confusion

Example: $e^+e^- \to \mu^+\mu^-$ amplitude using ALT shift

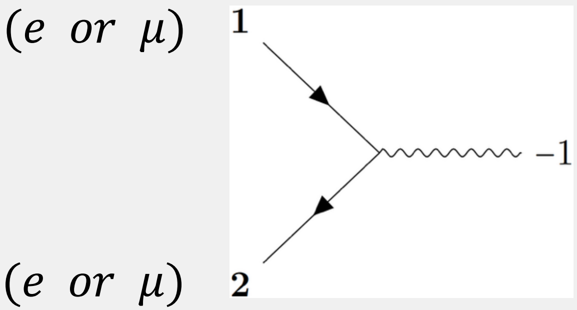
❖ 4-pt, external fermion, renormalizable ⇒ constructible

❖ Start with 3-pt building blocks, *fixed* by little-group covariance:

$$\hat{A}_3(\mathbf{12}I^+) = -e\frac{\langle\hat{\mathbf{1}}\xi\rangle[\hat{I}\hat{\mathbf{2}}]+\langle\hat{\mathbf{2}}\xi\rangle[\hat{I}\hat{\mathbf{1}}]}{\langle\xi\hat{I}\rangle}, \qquad \hat{A}_3(\mathbf{12}I^-) = -e\frac{[\hat{\mathbf{1}}\xi]\langle\hat{I}\hat{\mathbf{2}}\rangle+[\hat{\mathbf{2}}\xi]\langle\hat{I}\hat{\mathbf{1}}\rangle}{[\xi\hat{I}]}$$



(photon)



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In fact, many different on-shell vertices are equivalence due to on-shell identities, as but differ off-shell and correspond to different Lagrangian terms. How can they lead to the same result?
There are wrong statements in the literature such as one should only use SM form of vertices (freeze the off-shell info) to get SM results. This is not the amplitude logic.

$(e \;\; or \;\; \mu)$ $\mathbf{2}$

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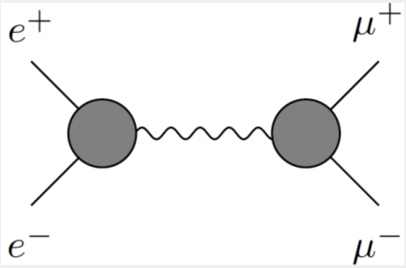
❖ 4-pt amplitude

$$A_4 =$$ 

$$\frac{1}{\hat{p}_I^2(z)} = \frac{1}{p_I^2}\frac{z^+ z^-}{(z-z^+)(z-z^-)}$$

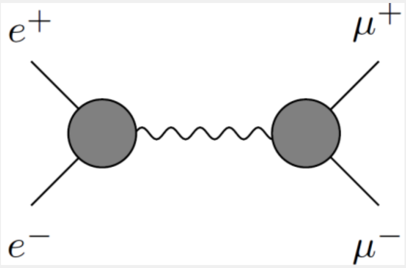
$$= -\sum_{z=z^\pm} Res_{z=z^\pm}\left[\hat{A}_3 \frac{1}{z}\frac{1}{\hat{p}_I^2}\hat{A}_3\right]$$

pole at $\hat{p}_I^2(z^\pm) = 0$
(internal photon on-shell)

$$= \frac{1}{p_I^2}\frac{1}{z^+ - z^-}\sum_{\lambda=\pm}\left[\boxed{\left[z^- \hat{A}_3\left(\mathbf{12}I^\lambda\right)\times \hat{A}_3\left(\mathbf{34}I^{-\lambda}\right)\right]}_{z^+} - (z^+ \leftrightarrow z^-)\right]$$

4-pt amplitude from the *product* of 3-pt building blocks

❖ 4-pt amplitude

$A_4 =$ 

$$\frac{1}{\hat{p}_I^2(z)} = \frac{1}{p_I^2}\frac{z^+z^-}{(z-z^+)(z-z^-)}$$

$$= -\sum_{z=z^\pm} Res_{z=z^\pm}\left[\hat{A}_3 \frac{1}{z}\frac{1}{\hat{p}_I^2}\hat{A}_3\right]$$

pole at $\hat{p}_I^2(z^\pm) = 0$
(internal photon on-shell)

$$\hat{A}_3(\mathbf{12}I^\lambda) \times \hat{A}_3(\mathbf{34}I^{-\lambda}) = e^2(\langle\hat{\mathbf{1}}\hat{\mathbf{3}}\rangle[\hat{\mathbf{2}}\hat{\mathbf{4}}] + \langle\hat{\mathbf{1}}\hat{\mathbf{4}}\rangle[\hat{\mathbf{2}}\hat{\mathbf{3}}] + (\hat{\mathbf{1}} \leftrightarrow \hat{\mathbf{2}}))$$

ALT shift does not shift external wave functions

$$= e^2(\langle\mathbf{13}\rangle[\mathbf{24}] + \langle\mathbf{14}\rangle[\mathbf{23}] + (\mathbf{1} \leftrightarrow \mathbf{2}))$$

The product is independent of z

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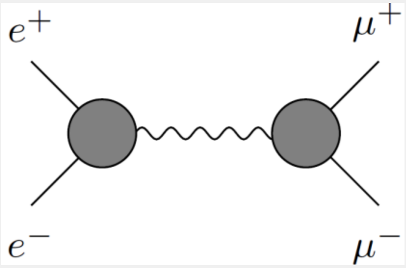

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The product is independent of z

Naïve gluing seems to be working (and hence the origin of some confusion in the literature), but it is only true upon proper shift with a proper form of 3-point interactions, and it will lead to the same result as any on-shell equivalent vertices!

❖ 4-pt amplitude

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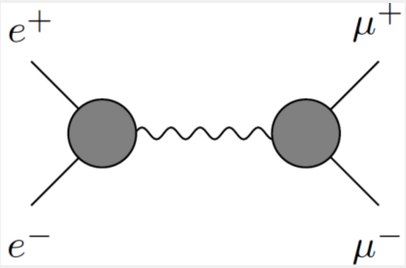
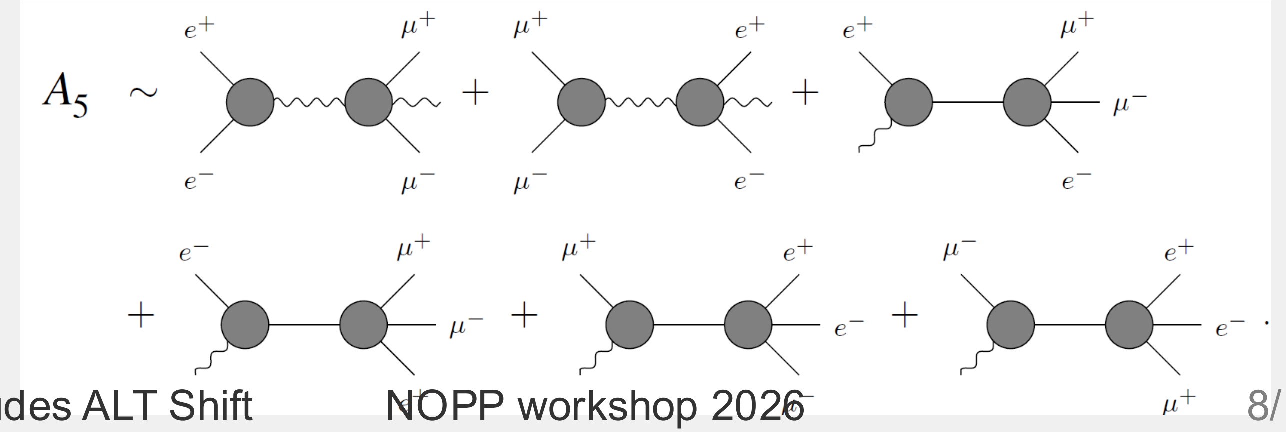


$$= e^2 \frac{\langle \mathbf{13}\rangle[\mathbf{24}]+\langle \mathbf{14}\rangle[\mathbf{23}]+(\mathbf{1}\leftrightarrow\mathbf{2})}{p_I^2}$$

- ✅ *Agrees with Feynman diagram result*
- ✅ *Beautiful high-energy behavior (just unbold the spinors)*

Similarly, we can do 5-pt amplitudes and higher…



Electroweak Amplitudes

Again, literature couldn't get the correct one with massive amplitudes; one often had to invoke other inputs to fix it (by secretly knowing the SM Feynman result).

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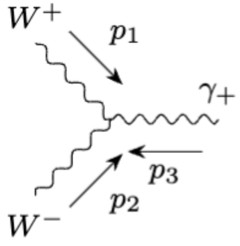
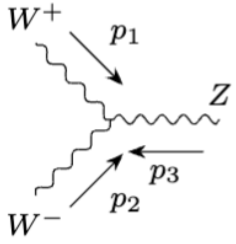
- Gauge boson amplitude can be build using only three-point interaction

3-point input:



$$= \frac{g_{WW\gamma}}{m_W} x_{12}\langle \mathbf{12}\rangle^2 = \frac{g_{WW\gamma}}{\sqrt{2}\, m_W^2 \langle 3\xi\rangle}\left[\langle \mathbf{12}\rangle[\mathbf{21}]\langle \mathbf{3}|p_1 - p_2|\mathbf{3}] + \text{cycl.}\right]$$



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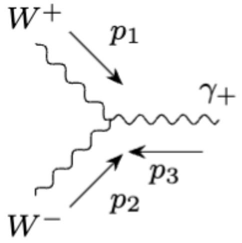

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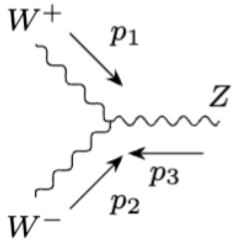

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- We can all-line transverse shift to construct 4-point amplitude.

Four-point contact interaction is not needed to write down the write amplitude.

Higgs interaction is not required to construct amplitude

Amplitude growth E^2 instead of E^4

Electroweak Amplitudes

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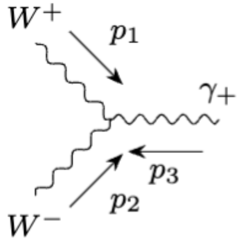
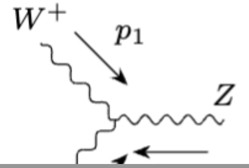


$$= \frac{g_{WW\gamma}}{m_W} x_{12} \langle \mathbf{12} \rangle^2 = \frac{g_{WW\gamma}}{\sqrt{2} m_W^2 \langle 3 \xi \rangle} [\langle \mathbf{12} \rangle [\mathbf{21}] \langle \mathbf{3} | p_1 - p_2 | \mathbf{3}] + \text{cycl.}]$$



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Again, this fulfills the promise of the amplitude program that we only showed in massless YM theory: three-points + constructability gives the full four-point amplitudes (and hence higher points) without additional inputs. We showed this for massive amplitudes for the first time (to our knowledge).

Constructability ≠ Renoramlizability

Theory can be finite and constructible at finite E (e.g., EFTs); Constructability only demands large z behavior.

Four-point contact interaction is not needed to write down the write amplitude.

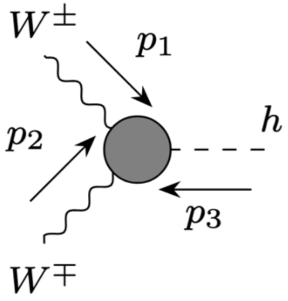
Higgs interaction is not required to construct amplitude

Amplitude growth E^2 instead of E^4

Unitarization (Independent from Constructability)

- These amplitudes has bad-high energy behavior.
- $W_L W_L W_L W_L$, $W_L W_L Z_L Z_L$ scale as $\mathcal{O}(E^2)$
- In the simplest model, they can be controlled by introducing a neutral scalar

$$A^{(\lambda_1\lambda_2)}_{WWh} =$$



$$= \frac{g_{WWh}}{m_W^2}\langle \mathbf{12}\rangle[\mathbf{21}]$$

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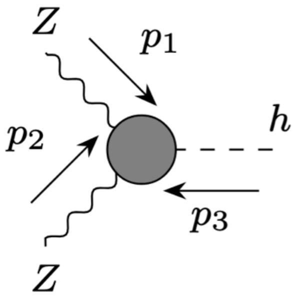

$$= \frac{g_{ZZh}}{m_Z^2}\langle \mathbf{12}\rangle[\mathbf{21}].$$

- Demanding smooth massless limit imposes constraints on the couplings

$$g_{WWh} = g m_W, \quad g_{ZZh} = \frac{g m_Z^2}{m_W}, \quad \rho = \frac{g^2 m_W^2}{g_{WWZ}^2 m_Z^2} = 1.$$

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- It is *just* tree-level recursion
  - (We need to start with tree-level) We also cleared the way for loop-level calculation; otherwise, there will always be unknown pieces during the generalized unitarity methods. Work in progress.

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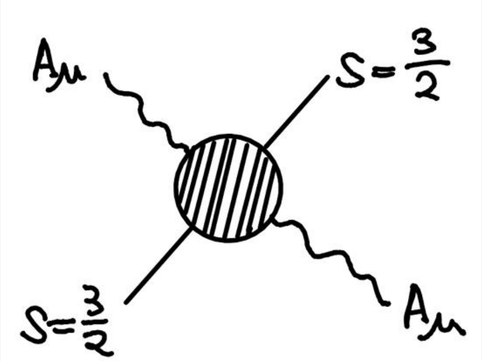
- It is *just* tree-level recursion
  - (We need to start with tree-level) We also cleared the way for loop-level calculation; otherwise, there will always be unknown pieces during the generalized unitarity methods. Work in progress.
- Higher multiplicity, from a practical (LHC) physics point of view, seems to be fine (MadGraph is great). We also already have Berends-Giele recursion.
  - We further completed formulation, provided a universal shift for massive and massless theories, and extended the usability of massive amplitudes and constructability.
  - We are interested in understanding different facets of QFT through the amplitude angle. Can we discover new relations and consistent amplitudes?

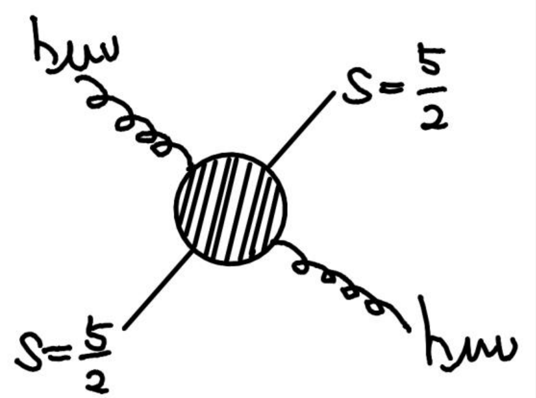
Higher Spin Scattering

Even if the (decoupled) Lagrangian is unknown,
we can still calculate some amplitudes.

Higher Spin Scattering

These massive helicity amplitudes form a basis for **black-hole mergers** calculation. They can be of pheno relevance beyond postulating new particles.





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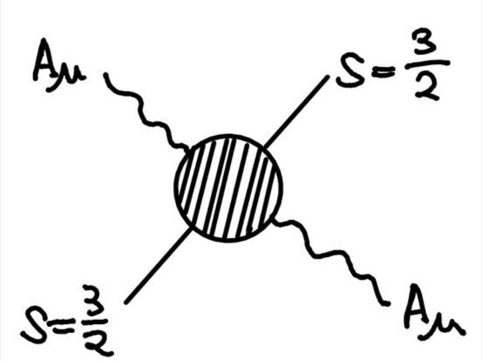
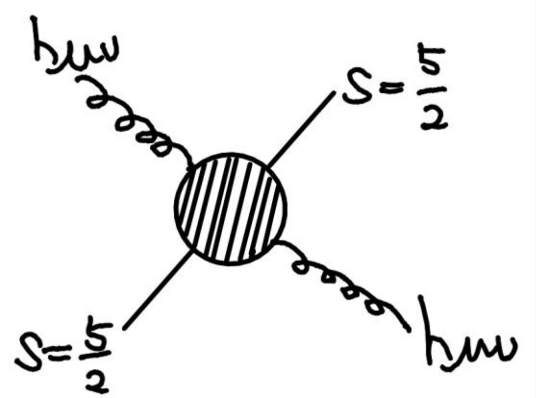


It's constructible if:

1) No more than 1 derivative (minimal coupling)
2) Current constraint satisfied



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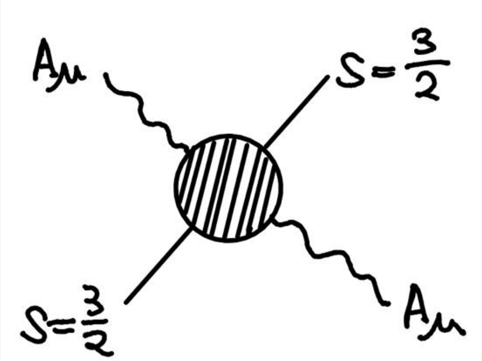

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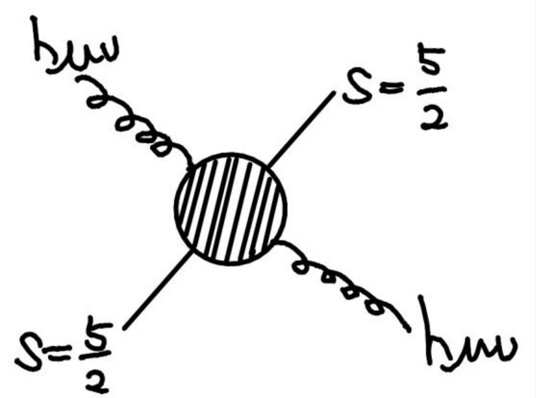

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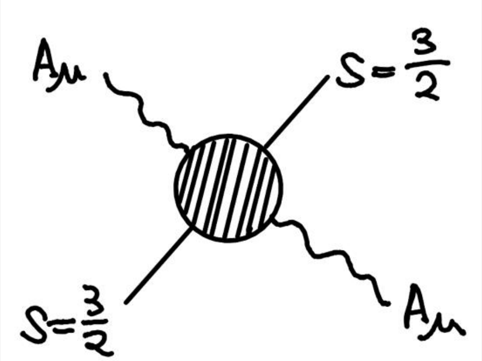

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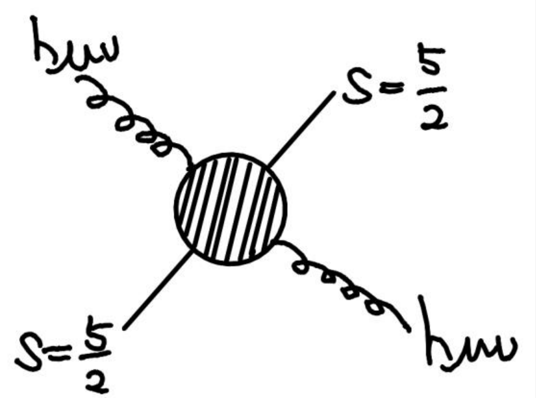

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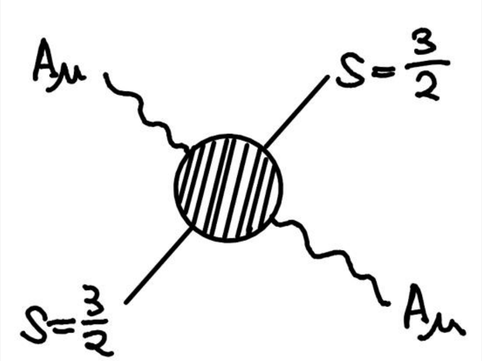

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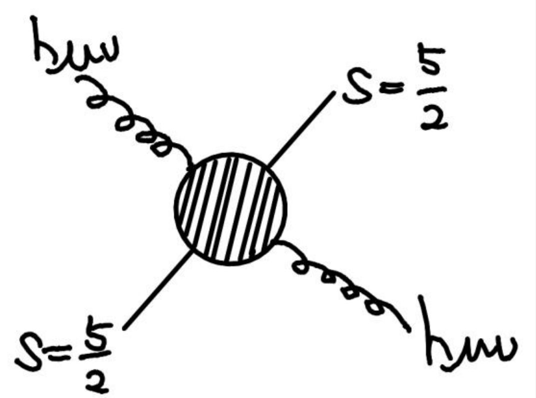

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$$A^{(\lambda_1\lambda_2-+)}_{\psi_{3/2}\bar{\psi}_{3/2}\gamma\gamma} = \frac{\left([\mathbf{1}|p_{13}|\mathbf{2}\rangle + \langle\mathbf{1}|p_{14}|\mathbf{2}]\right)\left([\mathbf{1}4]\langle\mathbf{2}3\rangle + [\mathbf{2}4]\langle\mathbf{1}3\rangle\right)^2}{2m^2 t_{13}t_{14}} - \frac{\left([\mathbf{1}4]\langle\mathbf{2}3\rangle + [\mathbf{2}4]\langle\mathbf{1}3\rangle\right)}{2m^4}\left[\frac{[\mathbf{2}4]\langle\mathbf{1}3\rangle[\mathbf{1}|p_{13}|\mathbf{2}\rangle}{t_{13}} + \frac{[\mathbf{1}4]\langle\mathbf{2}3\rangle\langle\mathbf{1}|p_{14}|\mathbf{2}]}{t_{14}}\right]$$

Within the class of theories satisfying 1) 2): this amplitude is *unique*

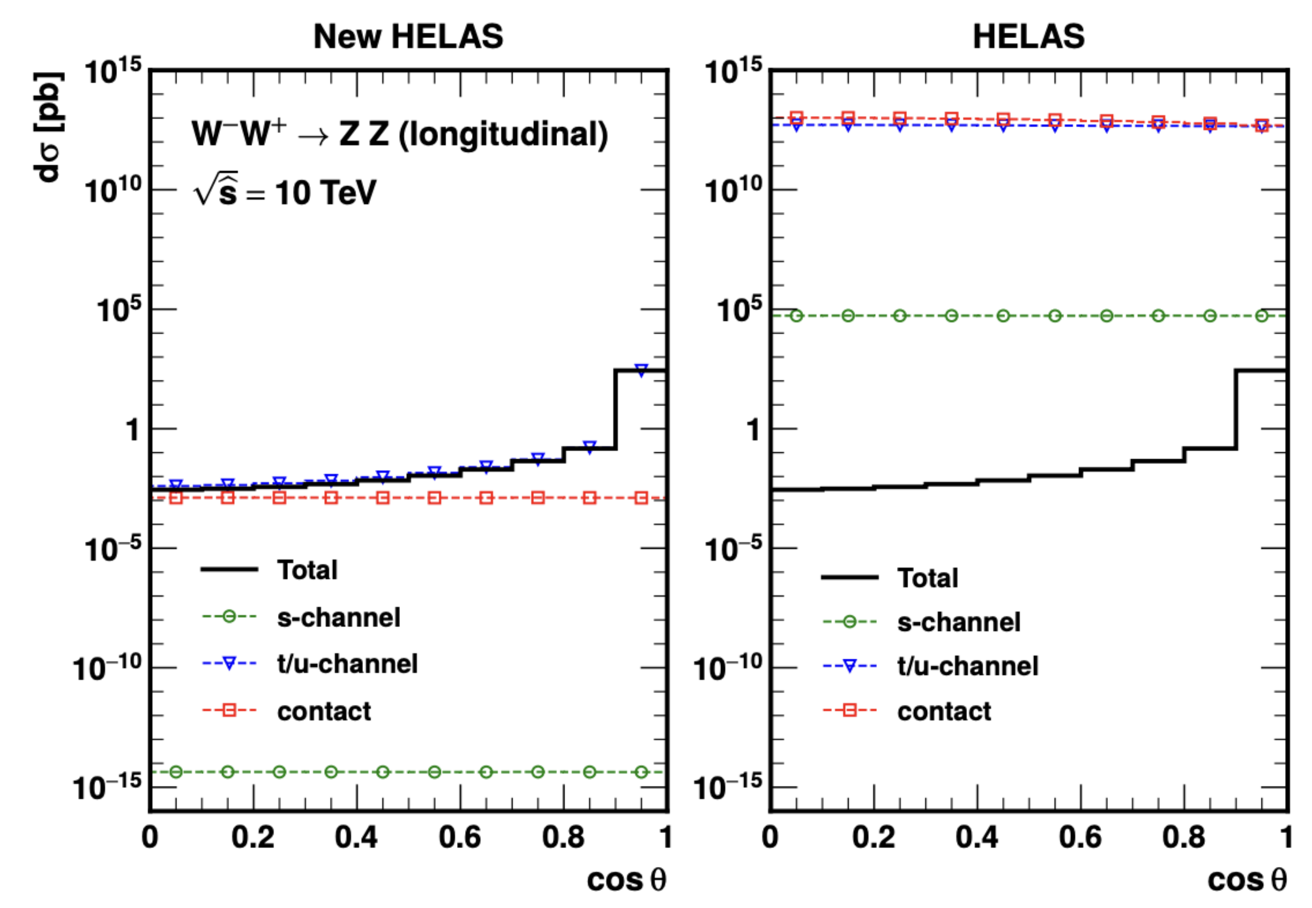
Amplitude w ALT shift v.s. FD gauge

[Helicity amplitudes without gauge cancellation for electroweak processes](), Hagiwara et al, [2203.10440]() Already in Madgraph

So amplitude promised no-gauge cancellation needed, will it automatically generate subamplitudes without bad high energy behavior, e.g., in the WW scattering, channel-by-channel finite?

$$\tilde{P}_{\mu\nu} = \sum_{\lambda=\pm 1} \epsilon^{\mu}(q,\lambda)^{*}\epsilon^{\nu}(q,\lambda) + \mathrm{sgn}(q^2)\tilde{\epsilon}^{\mu}(q,0)\tilde{\epsilon}^{\nu}(q,0)$$
$$= -\tilde{g}_{\mu\nu},$$

$$P^{U}_{V\,\mu\nu} = -\tilde{g}_{\mu\nu} + \tilde{\epsilon}_{\mu}(q,0)^{*}\frac{q_\nu}{Q} + \frac{q_\mu}{Q}\tilde{\epsilon}_{\nu}(q,0) + \frac{q_\mu q_\nu}{m_V^2}$$



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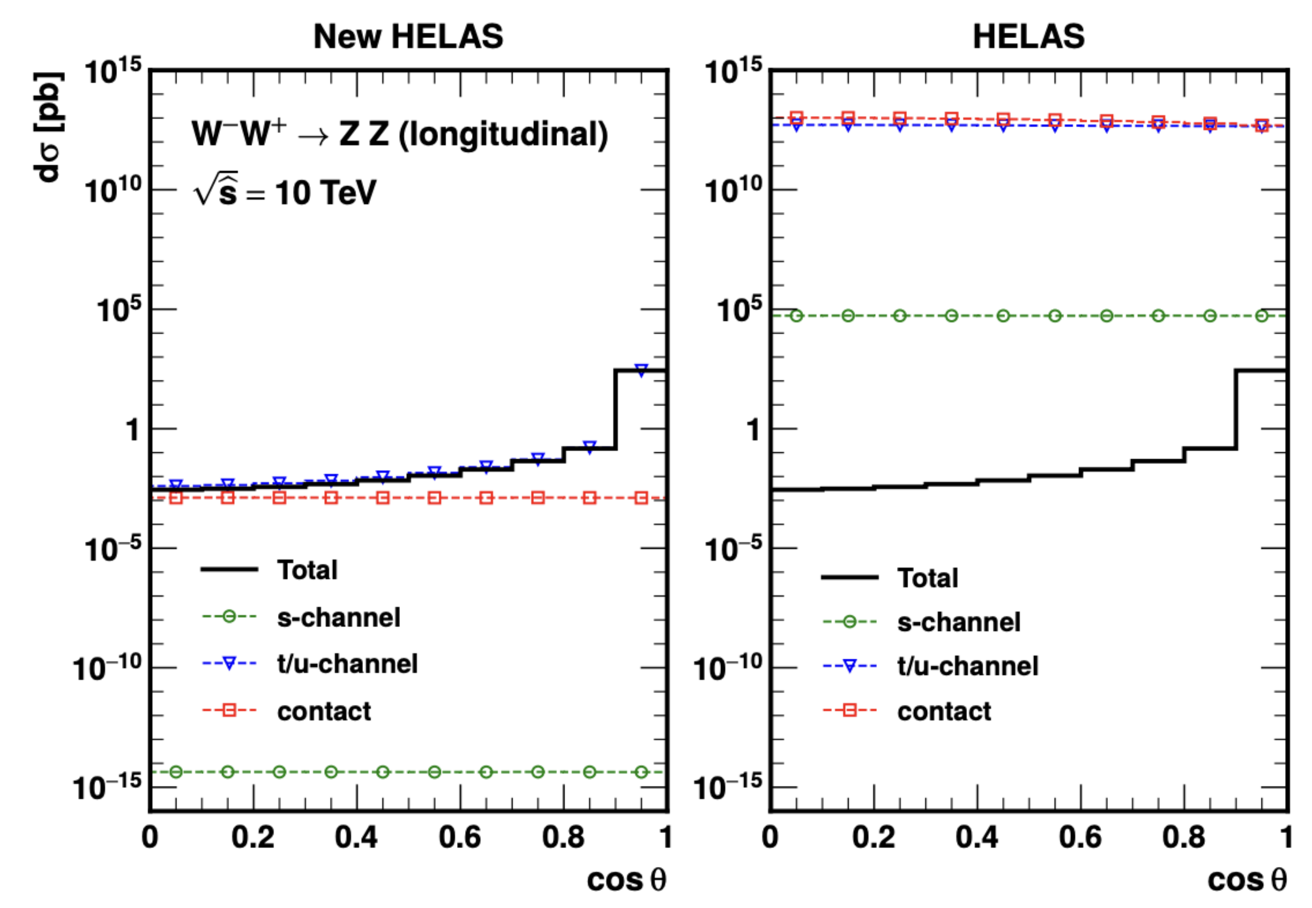

Since amplitudes have all good properties without invoking gauge redundancies, will it have this feature as well?
(partially) Yes! Work in progress.

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- Not necessarily. Work/thinking in progress.
- Massless theories are actually disjoint between + and -.
- Massive theory is a continuous deformation between + and -.
- Massive theory can also be viewed as a (precursor of) smooth deformation from on-shell to off-shell.
- Maybe the math will be simpler and more unique if one moves from massive→massless?

Summary and Outlook

Recursive construction of amplitudes requires consistent analytical continuation (momentum shift).

Our work:

- We found a powerful **massive helicity amplitude** shift**: ALT shift**
- On-shell construction of QED, EW, supergravity and higher spins
- Analyzed UV completions of the SM EW amplitudes to unitarize
- Proved various Ansatz (people invoke when dealing with contact term ambiguities)
- Constructability analysis for various theories (fully proved that SM is 3-point constructible under ALT shift, modulo 4h input)
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- Constructability is certainly Amplitude-oriented operable criteria and we see its connection (but not just) with renormalizability, gauge symmetry/ward identity, current constraints, etc. Will it allow us to discover a new class of theory? (e.g., higher spin theories whose Lagrangian construction are tedious or non-existent)
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- Are there better shift scheme to allow us explore QFT further?

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Backup

[R. Franken, C. Schwinn, arXiv:1910.13407]

[C. Wu, S. H. Zhu, arXiv: 2112.12312]

Massive BCFW-type Shift

Decompose momentum p_i, p_j as linear combination of two null-momentum

$$p_i = l_i + \frac{m_i^2}{2l_j \cdot l_i} l_j, \quad p_j = l_j + \frac{m_j^2}{2l_j \cdot l_i} l_i$$

Shift the spinors as

$$[\mathbf{i}, \mathbf{j}\rangle \text{ shift : } \begin{cases} [\hat{\mathbf{i}}|^2 = [\mathbf{i}|^2 - z[\mathbf{j}|^2, \\ |\hat{\mathbf{j}}\rangle^1 = |\mathbf{j}\rangle^1 + z|\mathbf{i}\rangle^1 \end{cases}$$

Good shift only exists for spin-projection $(s_i, s_j) = (1,2)$ or $(2,1)$

Need to use spin raising or lowering operator to construct amplitude

Note that (we are not aware of) constructability analysis for BCFW-like shift is lacking and hence one cannot obtain the correct full amplitudes without additional inputs regarding the contact terms for general amplitudes. In fact, one often obtain spurious poles in the calculation, in contrast to our results with ALT shift.

All-line Transverse (ALT) Shift

- Shift external momentum by respective transverse polarization for spin-1/2 and spin-1

$$p_i \to \hat{p}_i = p_i + z\frac{c_i m_i}{\sqrt{2}}\epsilon_i^{(I_i)},$$

- Define shift for spin-1/2 and spin-1 transverse modes

$$|i\rangle \to |\hat{i}\rangle = |i\rangle + z c_i |\eta_i\rangle \quad \text{for} \quad I_i = +,$$
$$|i] \to |\hat{i}] = |i] + z c_i |\eta_i] \quad \text{for} \quad I_i = -,$$

On-shell condition satisfied

$$\langle \hat{i}\eta_i\rangle = [\hat{i}\eta_i] = m_i$$

All-line Transverse Shift

- The shift is defined for spin-1 longitudinal modes

$$\begin{cases} |i\rangle \to |\hat{i}\rangle = |i\rangle + z\dfrac{c_i}{2}|\eta_i\rangle, \\ [\eta_i| \to [\hat{\eta}_i| = [\eta_i| - z\dfrac{c_i}{2}[i|, \end{cases} \quad \text{or} \quad \begin{cases} [i| \to [\hat{i}| = [i| + z\dfrac{c_i}{2}[\eta_i|, \\ |\eta_i\rangle \to |\hat{\eta}_i\rangle = |\eta_i\rangle - z\dfrac{c_i}{2}|i\rangle, \end{cases} \quad \text{for} \quad I_i = L,$$

- The definition can be extended for massless legs; more degree of freedom of choices there.
- Shift defined in helicity basis and does not require specific spin-axis choice
- The shift exists for all possible spin configurations

Polarization tensors

Massless:

$$\epsilon^{(+)}_{a\dot{a}} = \sqrt{2}\frac{|\xi\rangle_a [i|_{\dot{a}}}{\langle i\xi\rangle}, \quad \epsilon^{(-)}_{a\dot{a}} = \sqrt{2}\frac{|i\rangle_a [\xi|_{\dot{a}}}{[i\xi]}$$

Massive:

$$\epsilon^{(+)}_{a\dot{a}} = \sqrt{2}\frac{|\eta_i\rangle_a [i|_{\dot{a}}}{m_i}, \quad \epsilon^{(-)}_{a\dot{a}} = \sqrt{2}\frac{|i\rangle_a [\eta_i|_{\dot{a}}}{m_i}, \quad \epsilon^{(L)}_{a\dot{a}} = \frac{|i\rangle_a [i|_{\dot{a}} + |\eta_i\rangle_a [\eta_i|_{\dot{a}}}{m_i}$$

Momentum:

$$(p_i)_{a\dot{a}} = |i\rangle_a [i|_{\dot{a}} - |\eta_i\rangle_a [\eta_i|_{\dot{a}}$$

Explicit form of spinors
In helicity basis:

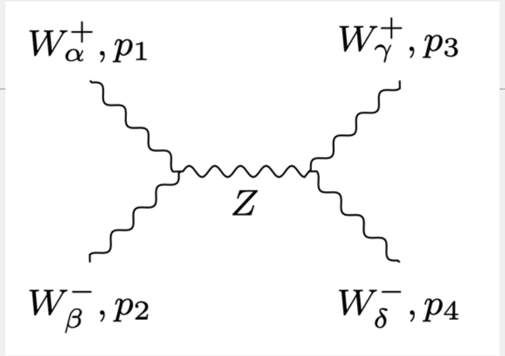
$$|i\rangle_a = \sqrt{E_i + p_i}\begin{pmatrix} -s_i^* \\ c_i \end{pmatrix}, \quad [i|_{\dot{a}} = \sqrt{E_i + p_i}\begin{pmatrix} -s_i & c_i \end{pmatrix}$$

$$|\eta_i\rangle_a = \sqrt{E_i - p_i}\begin{pmatrix} c_i \\ s_i \end{pmatrix}, \quad [\eta_i|_{\dot{a}} = -\sqrt{E_i - p_i}\begin{pmatrix} c_i & s_i^* \end{pmatrix}$$

3-point QED:

$$A_3(\mathbf{1}\mathbf{2}3^+) = \tilde{e}x_{12}\langle\mathbf{1}\mathbf{2}\rangle$$
$$= -\tilde{e}\frac{\langle\mathbf{1}\xi\rangle[3\mathbf{2}] + \langle\mathbf{2}\xi\rangle[3\mathbf{1}]}{\langle\xi 3\rangle}$$

EW calculation: WWWW



Example calculation:

$$
\begin{aligned}
A_s^Z(z)\Big|_{\hat{p}_{12}^2=m_Z^2} = \frac{g_{WWZ}^2}{m_W^4}\Big(& 2(\hat{p}_1-\hat{p}_2)\cdot(\hat{p}_3-\hat{p}_4)\langle\hat{\mathbf{1}}\hat{\mathbf{2}}\rangle[\hat{\mathbf{2}}\hat{\mathbf{1}}]\langle\hat{\mathbf{3}}\hat{\mathbf{4}}\rangle[\hat{\mathbf{4}}\hat{\mathbf{3}}] \\
& - 2\langle\hat{\mathbf{1}}\hat{\mathbf{2}}\rangle[\hat{\mathbf{2}}\hat{\mathbf{1}}]\langle\hat{\mathbf{3}}|\hat{p}_1-\hat{p}_2|\hat{\mathbf{3}}]\langle\hat{\mathbf{4}}|\hat{p}_3|\hat{\mathbf{4}}] + 2\langle\hat{\mathbf{1}}\hat{\mathbf{2}}\rangle[\hat{\mathbf{2}}\hat{\mathbf{1}}]\langle\hat{\mathbf{4}}|\hat{p}_1-\hat{p}_2|\hat{\mathbf{4}}]\langle\hat{\mathbf{3}}|\hat{p}_4|\hat{\mathbf{3}}] + (\hat{1},\hat{2})\leftrightarrow(\hat{3},\hat{4}) \\
& + 4\langle\hat{\mathbf{1}}\hat{\mathbf{3}}\rangle[\hat{\mathbf{3}}\hat{\mathbf{1}}]\langle\hat{\mathbf{2}}|\hat{p}_1|\hat{\mathbf{2}}]\langle\hat{\mathbf{4}}|\hat{p}_3|\hat{\mathbf{4}}] - 4\langle\hat{\mathbf{1}}\hat{\mathbf{4}}\rangle[\hat{\mathbf{4}}\hat{\mathbf{1}}]\langle\hat{\mathbf{2}}|\hat{p}_1|\hat{\mathbf{2}}]\langle\hat{\mathbf{3}}|\hat{p}_4|\hat{\mathbf{3}}] - (\hat{1}\leftrightarrow\hat{2})\Big)
\end{aligned}
\qquad (3.4)
$$

Need to sum over two poles at this factorization channel

Ward identity and simplification

$$A_4 = \epsilon_{1-}^{\mu}\epsilon_{2L}^{\nu}\epsilon_{3L}^{\lambda}\epsilon_{4L}^{\sigma}F_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)$$

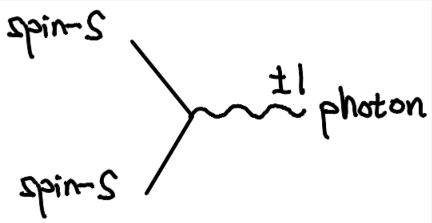
As $z \to \infty$, for ALT shift

$$r_1^{\mu}F_{\mu\nu\lambda\sigma}(r_1,r_2,r_3,r_4)$$

Goes to zero due to **Ward identiy.**
Thus leading z behavior vanishes

What can we learn from on-shell methods?

On shell, the three-point interaction is *fixed*



$$M^{\min,+1} = x\frac{\langle \mathbf{12}\rangle^{2S}}{m^{2S-1}}, \quad M^{\min,-1} = \frac{1}{x}\frac{[\mathbf{12}]^{2S}}{m^{2S-1}}$$

(They also give $g = 2$)

The four-point can be constructed under *assumptions*

- No higher-derivative couplings

  ≤ 1 derivative (coupling to photon)

  ≤ 2 derivatives (coupling to graviton)

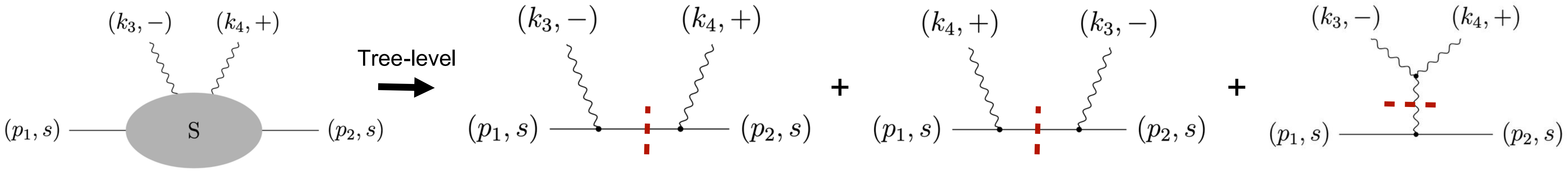
- The higher-spin 'current constraint' is satisfied

$$\partial^\mu J_\mu = O(m)$$

Gauge invariance when $m \to 0$

The spin-s propagator doesn't diverge when $m \to 0$

Higher-spin Compton Amplitude
Spurious Pole



On-shell techniques: gluing, BCFW recursion relation

$$\mathcal{A}_4\left(\Psi_1^s\,\bar{\Psi}_2^s\,h_3^-,h_4^+\right)=\frac{\left([4\mathbf{1}]\langle 3\mathbf{2}\rangle+[4\mathbf{2}]\langle 3\mathbf{1}\rangle\right)^{2s}}{s_{12}\,t_{13}\,t_{14}\underline{[4|\,p_1\,|3\rangle^{2s-4}}}\,.$$

[Arkani-Hamed, Huang, Huang ’17]
[Johansson, Ochirov ’19]

$\sqrt{s} \leq 1$ (external photon)
$\sqrt{s} \leq 2$ (external graviton)

Spurious pole for $s>2$ for graviton Compton amplitude

Higher-spin Compton Amplitude

ALT Shift

- Off-shell current constraint
- Photon/Graviton Ward identity

ALT shift

✓Photon Compton amplitude up-to $s = 3/2$
✓Graviton Compton amplitude upto $s = 5/2$

No spurious poles

- The shift **uniquely** fixes these Compton amplitude
- Uniquely fix Compton amplitude with **multiple Photon/Graviton**
- For even higher-spin, there is still ambiguity

Complexity Analysis

It can be done for ϕ^3 theory (to our analysis):

- Feynman diagram: $(2n-5)!! \rightarrow O(2^n \mathrm{n}!)$
- ALT shift: $2^{n-1} - \ n \ - 1 \rightarrow O(2^n)$
- BCFW: $2^{n-2} - 2 \rightarrow O(2^n)$