Revisiting Amplitude Construction: An Infrared Approach

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International Workshop on New Opportunities for Particle Physics 2026, IHEP CAS

Outline

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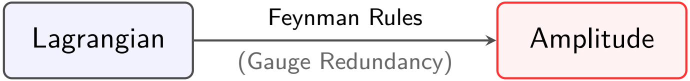
Beyond Feynman Rules: The Conceptual Shortcut

- The standard QFT approach: compute amplitudes via local fields



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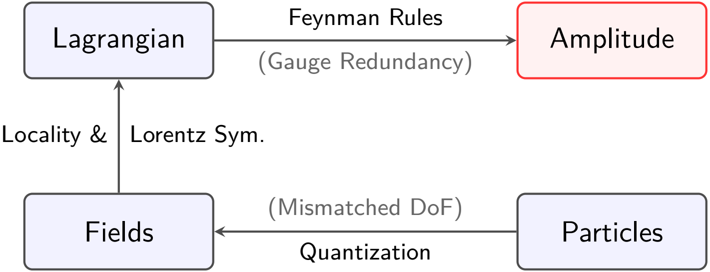

Beyond Feynman Rules: The Conceptual Shortcut

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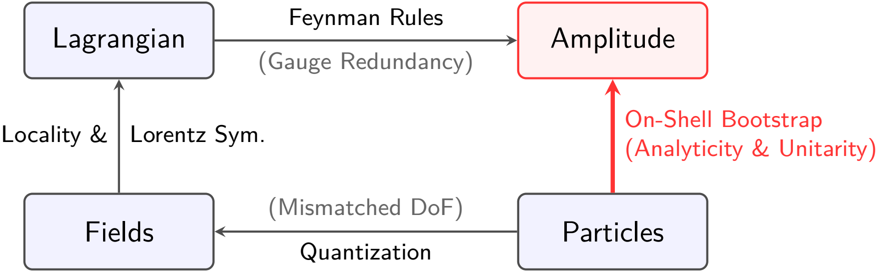


- The modern on-shell approach: bootstrap of physical scattering amplitudes
  - **First Principles:** Lorentz Invariance, Unitarity and Locality
  - **Theory Input:** Non-Factorizable Seed Amplitudes (3-pt couplings)

Spinor Helicity Variables

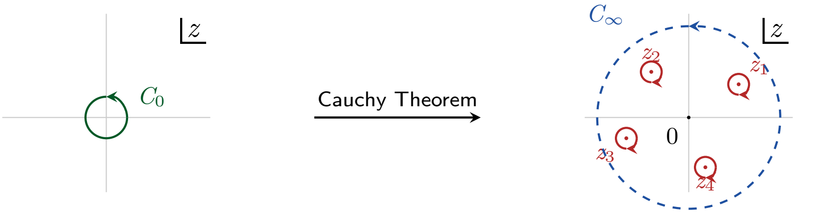
- **Massless kinematics** $(D = 4)$: $p^2 = 0 \;\Rightarrow\; p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \equiv |i\rangle[i|$

$$\left.\begin{array}{r} n\text{-pt amplitude } \mathcal{A}_n\big(\{p\};\,\{u(p),\,\varepsilon(p)\}\big) \\ p^2 = \not{p}u(p) = p\cdot\varepsilon(p) = 0 \\ \text{Ward Identity: } \varepsilon(p) \to \varepsilon(p) + \alpha p \end{array}\right\} \Longrightarrow \mathcal{A}_n(\lambda_i,\,\tilde{\lambda}_i)$$

- **Lorentz Invariance**: amplitudes only depend on the spinor products $\langle ij\rangle$, $[ij]$
  - Mandelstam variables: $s_{ij} = \langle ij\rangle[ji]$
- **Little group** scaling: $\mathcal{A}(\ldots,\, t\lambda_i,\, t^{-1}\tilde{\lambda}_i,\, \ldots) = t^{-2h_i}\mathcal{A}(\ldots)$
- **Massive Generalization**: little group $U(1) \to SU(2)$ — massive spinors $(\lambda_i^I, \tilde{\lambda}_i^I)$

On-Shell Recursion Relation

- **Complex momentum shift:** $\hat{p}_i = p_i + z r_i$
  - *e.g.* BCFW $[i, j\rangle$ shift: $r_i = -r_j = |i\rangle[j|$
- **Analytic structure:** Locality $\Rightarrow$ simple poles at z_I where $P_I^2(z_I) = 0$



$$\mathcal{A}(0) = \frac{1}{2\pi i}\oint_{C_0} \frac{\mathcal{A}(z)}{z}\, dz \qquad = \qquad -\sum_I \mathrm{Res}\left[\frac{\mathcal{A}(z)}{z}, z_I\right] + \mathcal{B}_\infty$$

- **Unitarity:** $\lim_{z \to z_I} \mathcal{A}(z) \sim \sum_h \mathcal{A}_L(z; h) \times \mathcal{A}_R(z; -h) / P_I^2(z)$

Benefits of On-Shell Approach

1. Simplify Calculation

Unphysical Polarizations/Ghosts due to **Gauge Symmetry**

↓

$\exp(n)$ Feynman Diag.

↓

On-Shell Recursion – $\mathrm{Poly}(n)$

e.g. **Parke–Taylor**

$$\mathcal{A}_n^{\mathrm{MHV}} = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

2. Non-Redundant Basis

Equivalent Lagrangian via **Field Redefinition**

↓

Redundant Operators

↓

Amp./Oper. Correspondence

Hilbert Series, Young Tensor

SMEFT, LEFT, ChPT, ...

3. Hidden Structures

- BCJ/KLT relations
- Double Copy
- CHY formalism
- Hidden Zeros
- Amplitudehedron

Challenges of On-Shell Approach

On-Shell Constructibility

Recursion relation works only when $\mathcal{B}_\infty = 0$, which requires $\mathcal{A}(z) \to 0$ as $z \to \infty$.

- Physical meaning of $\mathcal{B}_\infty$: non-factorizable (local) contributions = **new amplitude seeds**.
  - ϕ^4 coupling λ in **scalar QED**
  - Higher derivative operators in **effective field theories**
- Can we simply add the necessary new seeds to the recursion relation?
  - No **unique** way to split the contributions

$$\mathcal{A} = \underbrace{\sum_I \frac{\mathcal{A}_L(z_I) \cdot \mathcal{A}_R(z_I)}{P_I^2}}_{\text{unitarity}} + \underbrace{\mathcal{B}_\infty}_{\text{new seeds}}$$

Outline

Adler's Zero Condition

$\mathcal{B}_\infty$ may **not** be a problem if there are no **new independent parameters**.

Non-Linear Sigma Model:

$$\mathcal{L}^{(2)} = \frac{1}{2}(\partial\pi)^2 + \frac{\pi^2}{2f^2}(\partial\pi)^2$$

Shift Symmetry $\pi \to \pi + c$

PHYSICAL REVIEW D VOLUME 1, NUMBER 6 15 MARCH 1970

Algebraic Aspects of Pionic Duality Diagrams

LEONARD SUSSKIND* AND GRAHAM FRYE
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033

(Received 9 May 1969)

Certain algebraic aspects are abstracted from the duality principle and are incorporated in a simple model of pion n-point functions. An algorithm for constructing the n-point function in the tree-graph approximation is based on the duality assumption and the Adler condition which states that the amplitudes vanishes if any pion four-momentum vanishes, all others remaining on shell. The resulting amplitudes satisfy the constraints of current algebra and partial conservation of axial-vector current for $n=4$, 6, and 8, and (we conjecture) for all n. In addition, duality specifies a definite form for chiral symmetry breaking.

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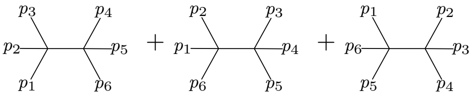
$\mathcal{A}_6 =$ 

$$= \underbrace{\frac{1}{f^4}\left(\frac{s_{13}s_{46}}{s_{123}} + \frac{s_{24}s_{51}}{s_{234}} + \frac{s_{35}s_{62}}{s_{345}}\right)}_{\text{violates Adler's Zero}}$$

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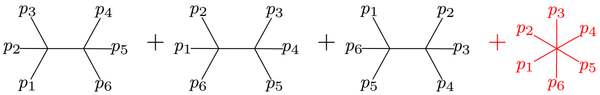
$\mathcal{A}_6 =$ 

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All-Line Shift and the Soft Poles

- To exploit the Adler's zero, shift **all** external momenta simultaneously:

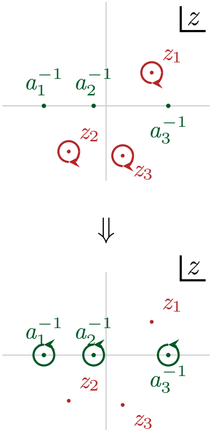
$$\hat{p}_i = p_i\,(1 - a_i z), \qquad \sum_i a_i p_i = 0$$

- **Soft Limits/Poles:** $\hat{p}_i \to 0$ at $z \to 1/a_i$
  - $n \geq 6$: non-trivial real solutions for $\{a_i\}$ exist in $D = 4$
  - Soft poles are **new IR data** that replace the UV boundary term
- Modified contour integral: $\mathcal{A}_{\text{phy}} = \text{Res}_{z=0}\, \dfrac{\hat{\mathcal{A}}(z)}{zF(z)}$
  - $F(z) = \prod_i (1 - a_i z)$ introduces the soft poles in the integrand.
  - They are **cancelled** by Adler's zero condition!

Soft Recursion Relation of NLSM

- **Soft recursion formula** (Cheung *et al.*, 2015)

$$\begin{aligned}\mathcal{A} &= -\sum_I \mathrm{Res}_{z_I^\pm} \frac{\hat{\mathcal{A}}_L(z)\times\hat{\mathcal{A}}_R(z)}{zF_n(z)\hat{P}_I^2(z)} \\ &= \underbrace{\sum_I \frac{\mathcal{A}_L\times\mathcal{A}_R}{P_I^2}}_{\text{hard poles: Unitarity}} + \underbrace{\sum_i \mathrm{Res}_{a_i^{-1}} \sum_I \frac{\hat{\mathcal{A}}_L(z)\times\hat{\mathcal{A}}_R(z)}{zF_n(z)\hat{P}_I^2(z)}}_{\text{soft poles: Adler's Zero}}\end{aligned}$$



- **Seed Amplitudes: Satisfy Adler's Zero**

Soft Blocks $\Leftrightarrow$ Operator Basis

  - Operator Basis for NLSM (Low, Yin, 2019; Low, Shu, **Xiao**, Zheng, 2023)
  - Operator Basis for ChPT/HEFT (Sun, **Xiao**, Yu, 2023)

Soft Degrees and Exceptional EFTs

- **Classification by soft degree** σ (enhanced $\mathcal{A} \sim p^\sigma$ behavior):

σ	Exceptional EFT	Symmetry
1	NLSM, ChPT	constant shift ($\delta\pi = c$)
2	Galileon/DBI	linear shift ($\delta\pi \supset b_\mu x^\mu$)
3	Special Galileon	quadratic shift ($\delta\pi \supset s_{\mu\nu} x^\mu x^\nu$)

  The suppressing factor can be enhanced to $F(z) = \prod_i (1 - a_i z)^\sigma$

- **Why "exceptional"?**
  - Lagrangian: symmetry **uniquely fixes** all Wilson coefficients (rigidity)
  - On-Shell: Introduce $F(z)$ to **suppress the boundary term** $\mathcal{B}_\infty$.

- Soft degree σ is a **new first principle** for amplitude construction.

Outline

Soft Behaviors of Gauge Bosons

Soft Photon Theorem (Weinberg 1965, Low 1958)

For a photon with momentum $q \to 0$:

$$\mathcal{A}_{n+1} \xrightarrow{q \to 0} \left(S^{(0)} + S^{(1)}\right) \mathcal{A}_n + O(q^1)$$

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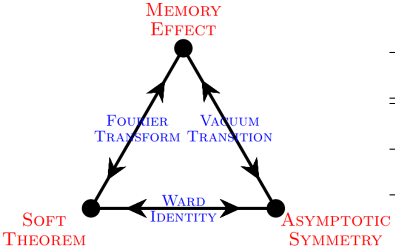
- **Leading:** $S^{(0)} = \sum_i \frac{e_i \; p_i \cdot \varepsilon}{p_i \cdot q} \sim q^{-1}$ (universal, fixed by gauge invariance)
- **Sub-leading:** $S^{(1)} = \sum_i \frac{e_i \; q_\mu \varepsilon_\nu J_i^{\mu\nu}}{p_i \cdot q} \sim q^0$ (angular momentum dependence)

The IR Triangle: Soft Theorem as First Principle

The soft theorem is the **Ward identity** of asymptotic symmetries (Strominger et al.)



Soft particles	**Symmetry**	**IR property**
Goldstone Bosons	Non-linear Sym.	Adler's Zero
Gauge Bosons	Asymp. Gauge Sym.	Soft Theorem

Both are hidden symmetries $\Rightarrow$ Both are first principles for amplitude construction.

Four Pillars of On-Shell Construction

1. **Global Symmetries** $\Rightarrow$ amplitude seeds (soft blocks)
2. **Locality + Causality** $\Rightarrow$ Cauchy's theorem $\rightarrow$ contour integral
3. **Unitarity** $\Rightarrow$ residues on **hard poles** (factorization)

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4. **Infrared** $\Rightarrow$ residues on **soft poles** $\begin{cases} \text{Goldstone Boson: Adler's Zero} \\ \\ \text{Gauge Boson: Soft Theorem} \end{cases}$

$$\mathrm{Res}_{a_s^{-1}} \frac{\hat{\mathcal{A}}_{n+1}(z)}{z(1-a_s z)} = \mathrm{Res}_{a_s^{-1}} \left[\frac{S^{(0)} \hat{\mathcal{A}}_n(z)}{z(1-a_s z)^2} + \frac{S^{(1)} \hat{\mathcal{A}}_n(z)}{z(1-a_s z)} \right]$$

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The 4th pillar is what enables on-shell EFT construction.

Gauge Interaction of Goldstone Bosons

Goldstone bosons $\pi^a \in G/H$ form a representation of the unbroken group H:

- G/H **gauged** $\Rightarrow$ **Higgs mechanism**: π's eaten by massive gauge bosons
- H (partially) **gauged** $\Rightarrow$ **pseudo-Goldstone bosons**, *e.g.* $\pi^\pm$ in ChPT

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Gauge interactions covariantize the shift symmetry:

$$\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu \quad \Longrightarrow \quad \lim_{p\to 0} \mathcal{M}\big(\ldots, \pi^\pm(p), \ldots; \gamma\big) \neq 0$$

Adler's Zero is lost — is the theory still on-shell constructible?

Example: $4\pi + 1\gamma$

$$\mathcal{M}_{4+1} = -\operatorname*{Res}_{z=1/a_k}\left[\frac{S^{(0)}\hat{\mathcal{M}}_4(z)}{z(1-a_k z)^2} + \frac{S^{(1)}\hat{\mathcal{M}}_4(z)}{z(1-a_k z)}\right] = \left(S^{(0)} + S^{(1)}\right)\mathcal{M}_4$$

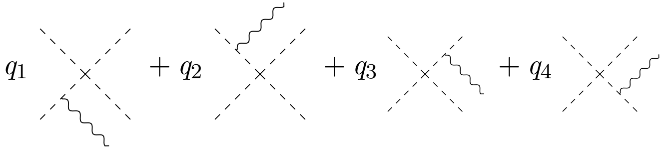
- Explicit result: $\mathcal{M}_{4+1}(\pi^+, \pi^-, \pi^0, \pi^0; \gamma^{(+)}) = \left(S^{(0)} + S^{(1)}\right)\mathcal{M}_4 = \frac{e\langle 12\rangle}{\langle 15\rangle\langle 25\rangle}\frac{s_{34}}{f^2}$
- $\lim_{p_3,p_4\to 0}\mathcal{M}_{4+1} = 0$ **neutral scalars $\Rightarrow$ Adler's Zero retained**
- $\lim_{p_1,p_2\to 0}\mathcal{M}_{4+1} \neq 0$ **charged scalars $\Rightarrow$ Adler's Zero broken**

Key Strategy: decompose the charge currents

Decompose $\mathcal{M}_{n+1}$ into components with a **single charged pair** $\{i^{+1}, (i{+}1)^{-1}\}$, so the other $n-2$ **scalars stay effectively neutral** and retain Adler's zero.

Feynman diagrams are not independent:

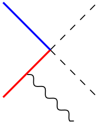
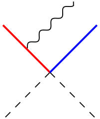
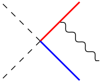
photon can attach to any scalar leg — but they are related by charge conservation



$\Rightarrow\ \ q_1 + q_2 + q_3 + q_4 = 0$ only 3 independent charge parameters

Charge Decomposition

Gauge-invariant combinations = Charge Decomposition:

Q_1  $+\ Q_2$  $+\ Q_3$ 

$$Q_1 = q_1\,,\ Q_2 = q_1 + q_2\,,\ Q_3 = q_1 + q_2 + q_3$$

Each Q_i combination is **gauge invariant** and **independent**

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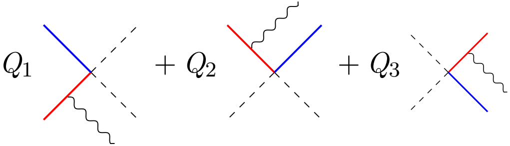

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Each Q_i combination is **gauge invariant** and **independent**

$$\mathcal{M}_{n+1}(1^{q_1},\, 2^{q_2}, \ldots, n^{q_n}; \gamma) = \sum_{i=1}^{n-1} Q_i\, \mathcal{A}^i(\ldots,\, i^{+1},\, (i{+}1)^{-1}, \ldots; \gamma)$$

- In each component $\mathcal{A}^i$, scalars not on the charge flow are **effectively neutral**
- Adler's zero is **partially restored** $\Rightarrow$ soft recursion applicable

Gauged Soft Recursion: The Solution

Multi-photon charge decomposition (Low–Xiao–Zheng, 2511.09403):

$$\mathcal{M}_{n+l} = \sum_{i_1,\ldots,i_l} Q_{i_1} \cdots Q_{i_l}\, \mathcal{A}^{i_1,\ldots,i_l}, \qquad \text{photon } s \text{ attaches to the current } i_s \text{ of charge } Q_{i_s}$$

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$\hat{S}=\hat{S}^{(0)}+\hat{S}^{(1)}$: soft photon theorem on the component with photon s removed

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Choice of suppressing soft factor $F(z)$:

- may contain $(1-a_s z)$ of gauge bosons — soft theorem
- may NOT contain $(1-a_i z)$ of charged scalars — Adler's zero broken
- constructible iff $F(z)$ has enough power of z to suppress $\mathcal{B}_\infty$ (gauged NLSM)

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Summary and Outlook

Take Away Messages:

1. **Infrared** structures are **first principles** for on-shell amplitude construction.
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Future Outlook:

- Amplitude construction from **Inverse Soft Limit**.
- On-shell bootstrap from the **Hidden Zeros**.
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Thank You

Backup: The Non-Abelian Version

Abelian $\to$ Non-Abelian: charges become color generators:

$$S^{(0)} = \sum_i \frac{e_i\, p_i\cdot\varepsilon}{p_i\cdot q} \qquad \longrightarrow \qquad S^{(0)} = \sum_i \frac{T_i^a\, p_i\cdot\varepsilon}{p_i\cdot q}$$

Charge decomposition $\to$ Color decomposition:

$$\sum_{i_1,\ldots,i_l} Q_{i_1}\cdots Q_{i_l}\,\mathcal{A}^{i_1,\ldots,i_l} \qquad \longrightarrow \qquad \sum_\sigma \mathrm{Tr}\big(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(l)}}\big)\,\mathcal{A}^{\sigma(1),\ldots,\sigma(l)}$$

	Abelian	**Non-Abelian**
soft factor	e_i (charge)	T_i^a (gen erator)
gauge self-interactions	none	3-pt seeds
partial amplitudes	charge component	color-ordered amplitude
constraint	charge conservation	Jacobi identity

Backup: On-Shell Constructibility

Large-z criterion: where the powers come from

A charge-decomposed component of $\mathcal{M}_{n+l}$ has $n^* = n - 2l$ **neutral** scalars.

- **Powers of $F(z)$** — soft factors of neutral scalars + gauge bosons:

$$F(z) = \prod_{\text{neutral } i} (1 - a_i z)^{\sigma_i} \prod_s (1 - a_s z) \;\sim\; z^{\,n^*\sigma + l}$$

- **Powers of $\hat{\mathcal{A}}(z)$** — soft blocks ($\sim p^m$) softened by each gauge boson soft theorem $S^{(0)} \sim p_s^{-1} \sim z^{-1}$:

$$\hat{\mathcal{A}}(z) \;\sim\; z^{\,m-l}$$

$$\mathcal{B}_\infty = 0 \iff (m - l) - (n^*\sigma + l) < 0 \iff m - 2l - n^*\sigma < 0$$

NLSM: $m = 2$, $\sigma = 1$ $\Rightarrow$ $m < n$ for all $n \geq 4$ $(l = 1)$ ✓