New Strucutes in Feynman Integrals

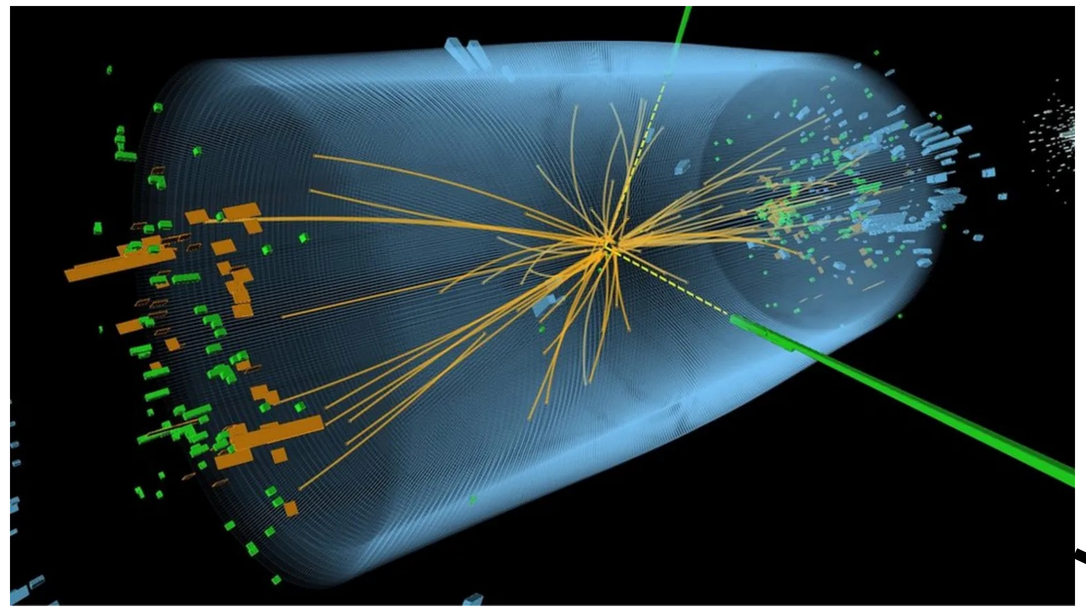
Xing Wang (王 星)

The Chinese University of Hong Kong, Shenzhen

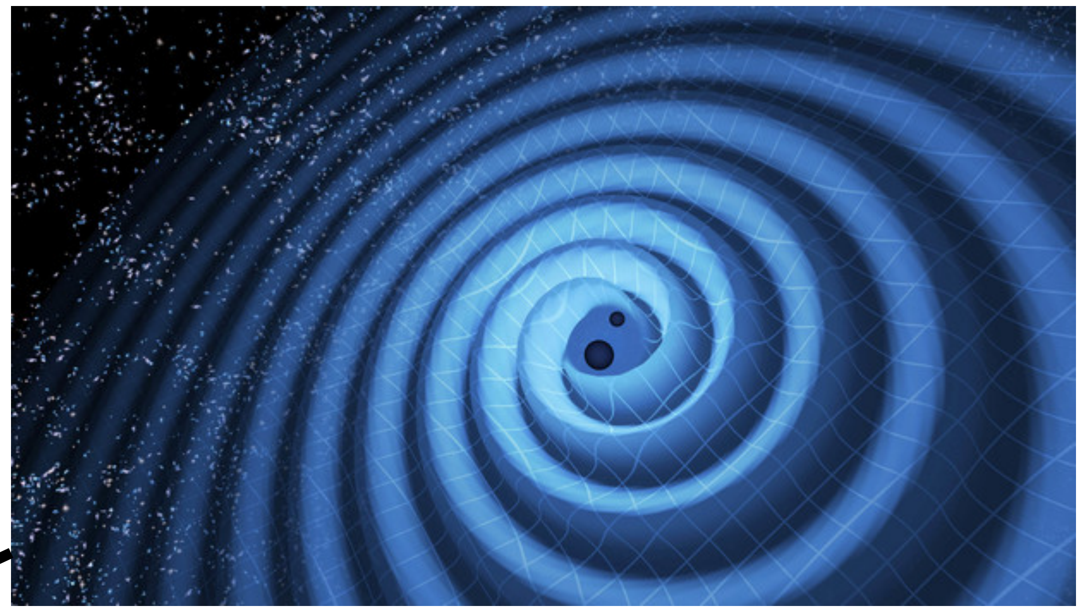
Based on 2506.09124 (PRL accepted), 2507.23594 (JHEP), 2511.15381 with ε-collaboration,
and 2603.18576, 2605.xxxxx, with Yefan Wang and Jian Wang

中科院高能所 @ 北京, 2026

Motivation of Feynman Integrals

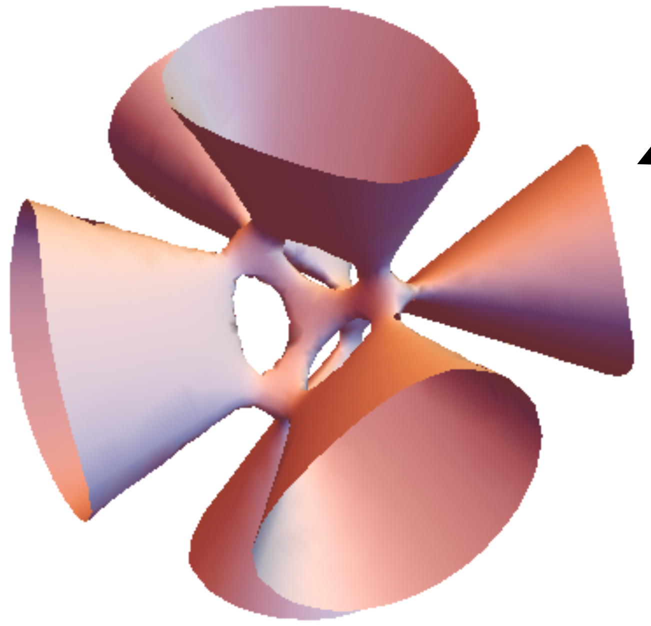


precision prediction
with QFTs



$$O\left(\alpha,\{x\}\right)=\sum_A c_A F_A$$

F_A: **Feynman Integrals**

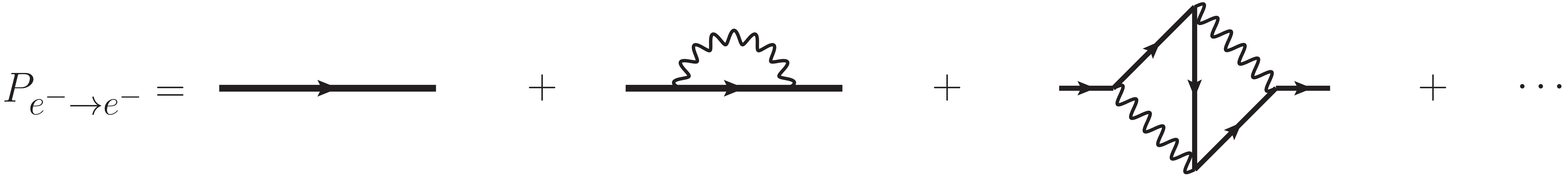


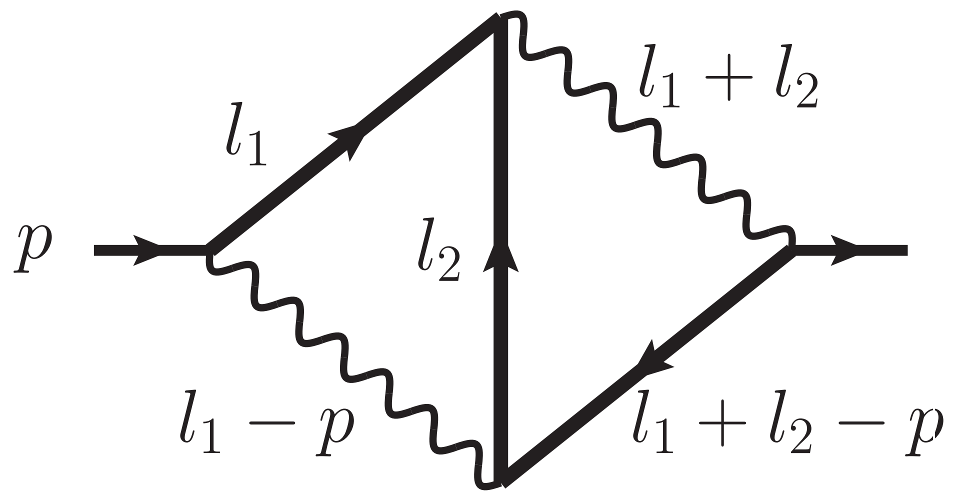
mathematics

$\mathcal{N}=4$ sYM,
string…

formal theory

Feynman Integrals are Inevitable in QFTs





$$= \underbrace{\int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \int \frac{\mathrm{d}^D l_2}{i\pi^{D/2}} \frac{e^{2\varepsilon\gamma_E} \cdot (-p^2)^{|\nu|-D} \cdot \mathrm{Num}(\{l\})}{\left[l_1^2-m^2\right]^{\nu_1}\left[l_2^2-m^2\right]^{\nu_2}\left[(l_1+l_2-p)^2-m^2\right]^{\nu_3}\left[(l_1+l_2)^2\right]^{\nu_4}\left[(l_1-p)^2-m^2\right]^{\nu_5}}}_{F_{\nu_1\nu_2\nu_3\nu_4\nu_5}}$$

Each Feynman diagram corresponds to a **Feynman Integral**, of which the complexity exploits **factorially**.

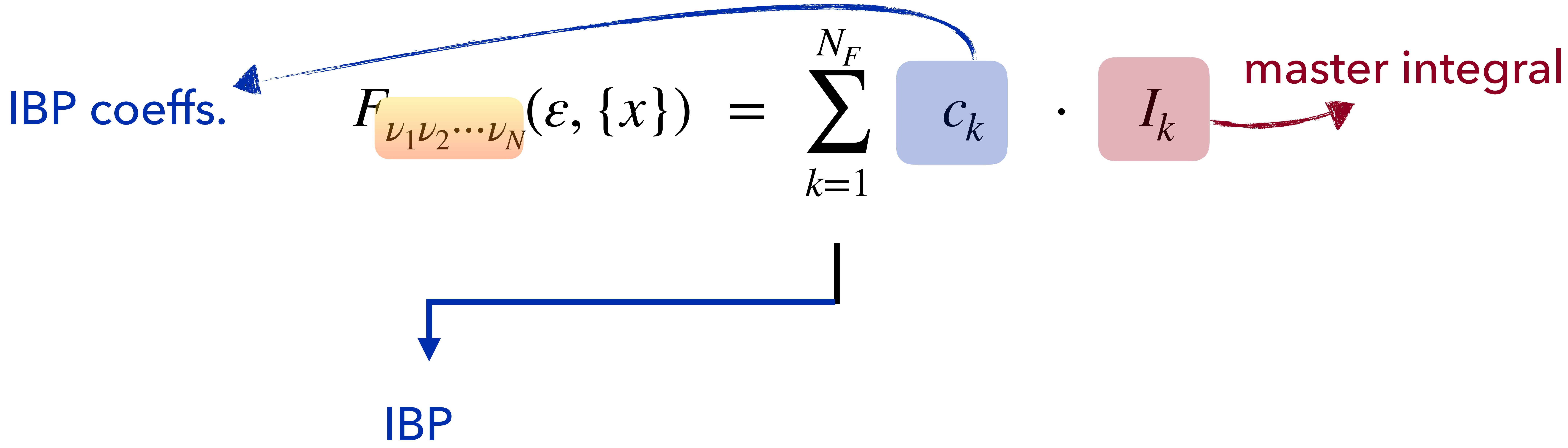
General Aspects of Feynman Integrals

IBP coeffs.

$$F_{\nu_1\nu_2\cdots\nu_N}(\varepsilon, \{x\}) \;=\; \sum_{k=1}^{N_F} \boxed{c_k} \cdot \boxed{I_k}$$

master integral

General Aspects of Feynman Integrals



IBP coeffs.

$$F_{\nu_1\nu_2\cdots\nu_N}(\varepsilon,\{x\}) = \sum_{k=1}^{N_F} c_k \cdot I_k$$

master integral

IBP

LiteRed, FIRE, Reduze, Kira, NeatIBP, Blade, Finiteflow…

IBP complexity explodes fast!

General Aspects of Feynman Integrals

IBP coeffs.

$$F_{\nu_1\nu_2\cdots\nu_N}(\varepsilon, \{x\}) = \sum_{k=1}^{N_F} c_k \cdot I_k$$

master integral

[Kotikov '91; Remiddi '97;

Gehrmann, Remiddi '00]

+ lots of analytic applications

+ AMFlow (numerics rev.)

IBP

LiteRed, FIRE, Reduze, Kira, NeatIBP, Blade, Finiteflow…

IBP complexity explodes fast!

MIs by DEs

$$\mathrm{d}\begin{pmatrix} I_1 \\ \vdots \\ I_{N_F} \end{pmatrix} = A_{N_F\times N_F}(\varepsilon, \{x\})\begin{pmatrix} I_1 \\ \vdots \\ I_{N_F} \end{pmatrix}$$

Solving this linearly coupled PDE system is non-trivial!

General Aspects of Feynman Integrals

IBP coeffs.

$$F_{\nu_1\nu_2\cdots\nu_N}(\varepsilon,\{x\}) = \sum_{k=1}^{N_F} \boxed{c_k} \cdot \boxed{I_k}$$

master integral

[Kotikov '91; Remiddi '97;

Gehrmann, Remiddi '00]

+ lots of analytic applications

+ AMFlow (numerics rev.)

IBP

bottlenecks

MIs by DEs

LiteRed, FIRE, Reduze, Kira, NeatIBP, Blade, Finiteflow…

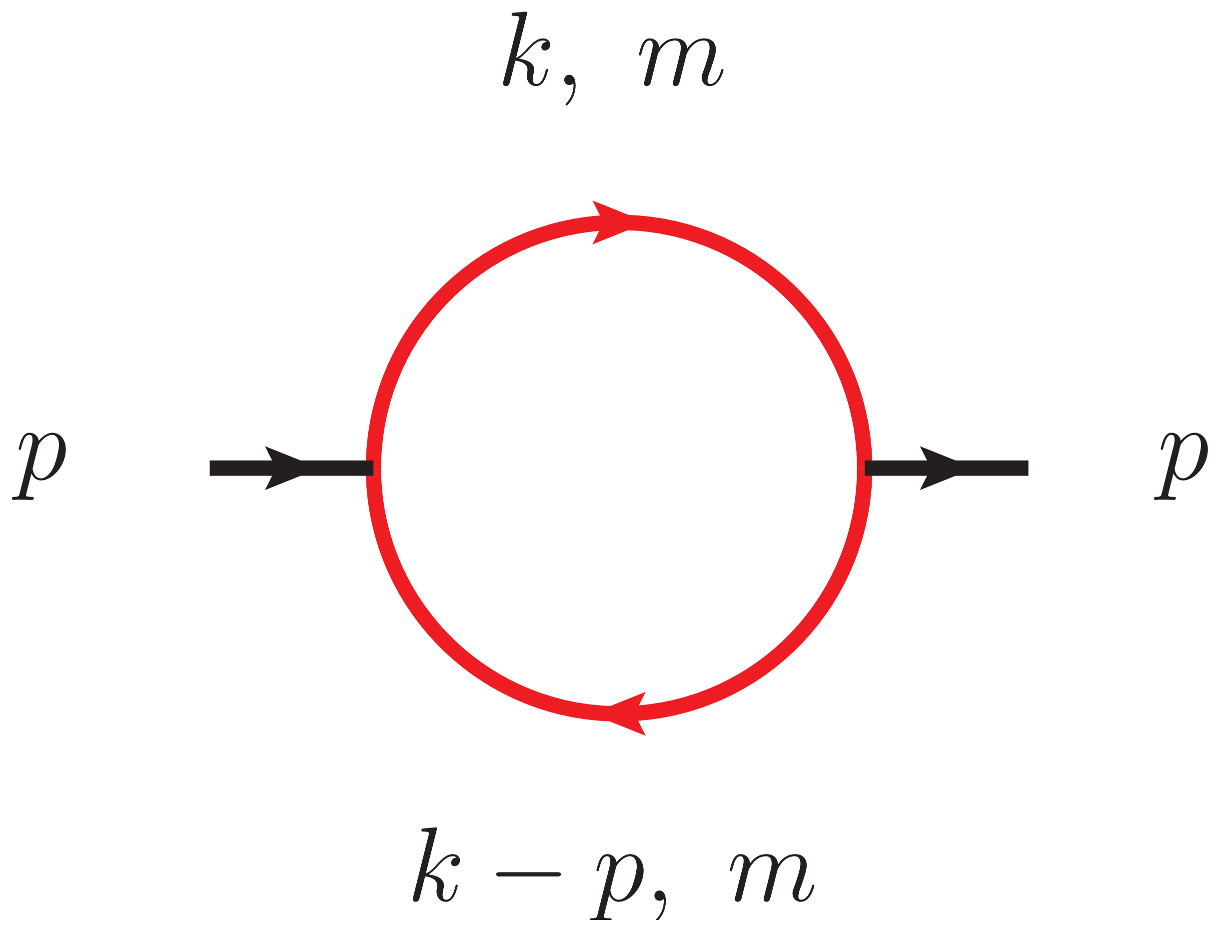
$$\mathrm{d}\begin{pmatrix} I_1 \\ \vdots \\ I_{N_F} \end{pmatrix} = A_{N_F\times N_F}(\varepsilon,\{x\})\begin{pmatrix} I_1 \\ \vdots \\ I_{N_F} \end{pmatrix}$$

IBP complexity explodes fast!

Solving this linearly coupled PDE system is non-trivial!

Integration-by-part Relations

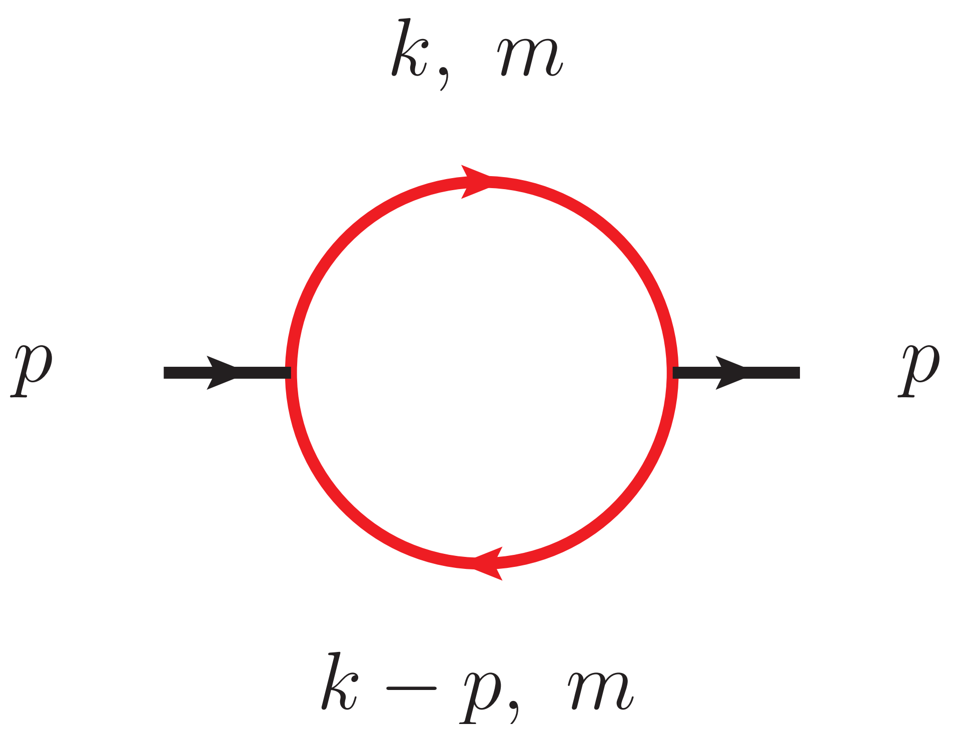
$$\int \frac{\mathrm{d}^D k}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k^\mu}\left[q^\mu \cdot f(k)\right] = 0 \quad \rightsquigarrow \quad \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k_i^\mu} q^\mu_{\mathrm{IBP}} \prod_{j=1}^{n_{\mathrm{int}}} \frac{1}{\left(q_j^2 - m_j^2\right)^{\nu_j}} = 0$$



$$I_{\nu_1\nu_2} = \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{e^{\varepsilon\gamma_E}\left(m^2\right)^{|\nu|-D/2}}{\left[-k^2+m^2\right]^{\nu_1}\left[-(k-p)^2+m^2\right]^{\nu_2}}$$

$$q_{\mathrm{IBP}} \in \{k, p\}$$

Integration-by-part Relations

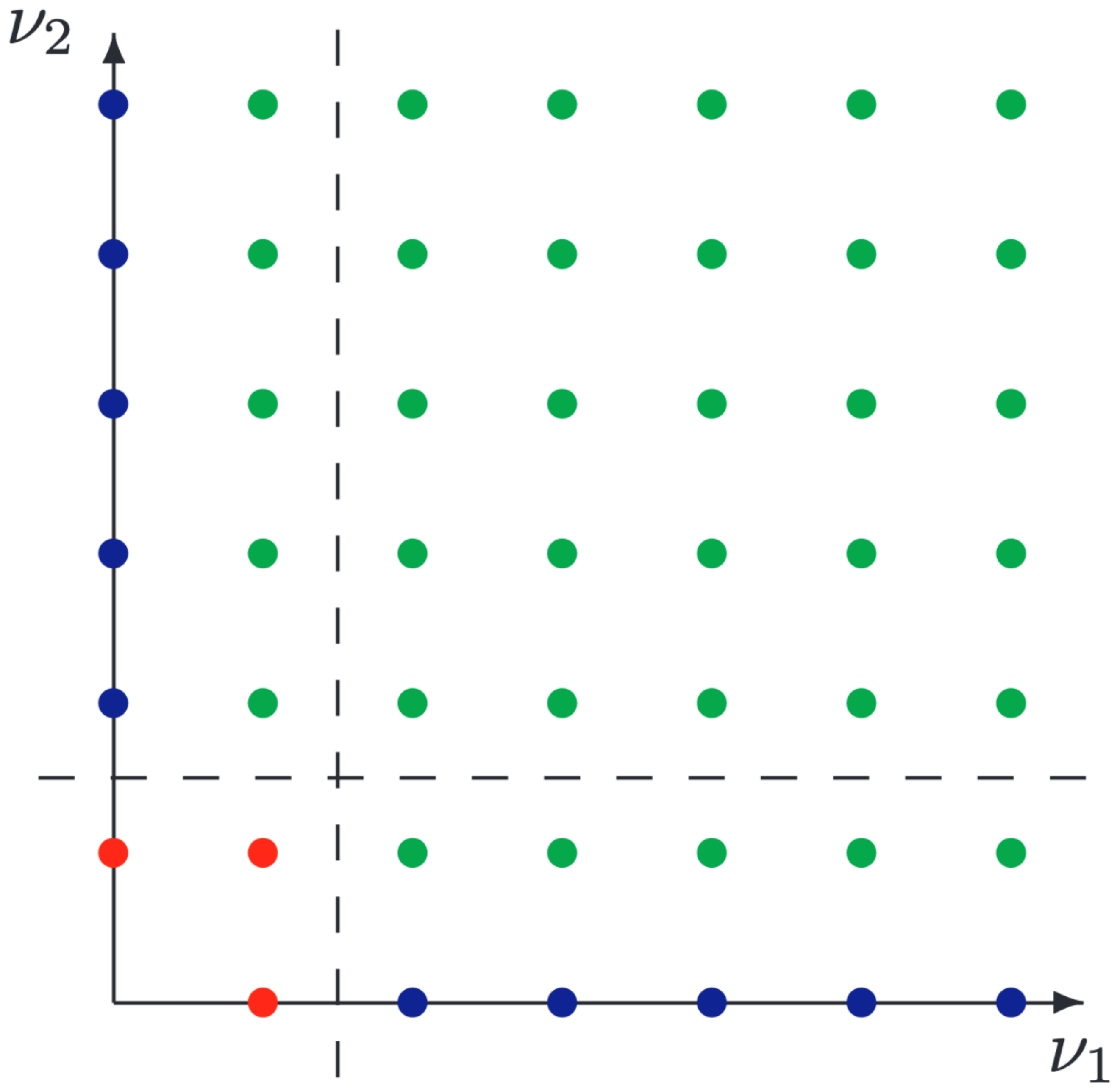


$$F_{\nu_1\nu_2} = \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{e^{\varepsilon\gamma_E}\left(m^2\right)^{|\nu|-D/2}}{\left[-k^2+m^2\right]^{\nu_1}\left[-(k-p)^2+m^2\right]^{\nu_2}}$$

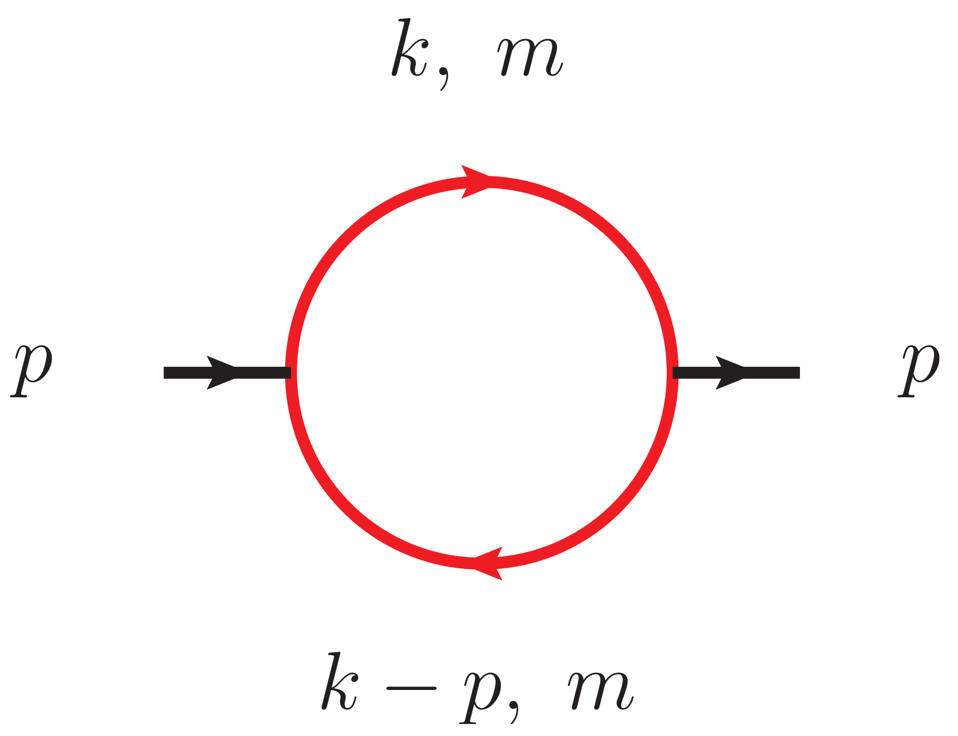
e.g., $q_{\text{IBP}} = p$

$$0 = p^\mu \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{\partial}{\partial k^\mu} \frac{e^{\varepsilon\gamma_E}\left(m^2\right)^{|\nu|-D/2}}{\left[-k^2+m^2\right]^{\nu_1}\left[-(k-p)^2+m^2\right]^{\nu_2}}$$

$$0 = \left(\nu_1-\nu_2\right) I_{\nu_1\nu_2} - \nu_1 I_{(\nu_1+1)(\nu_2-1)} + \nu_2 I_{(\nu_1-1)(\nu_2+1)} + \nu_1 x I_{(\nu_1+1)\nu_2} - \nu_2 x I_{\nu_1(\nu_2+1)}$$



Integration-by-part Relations

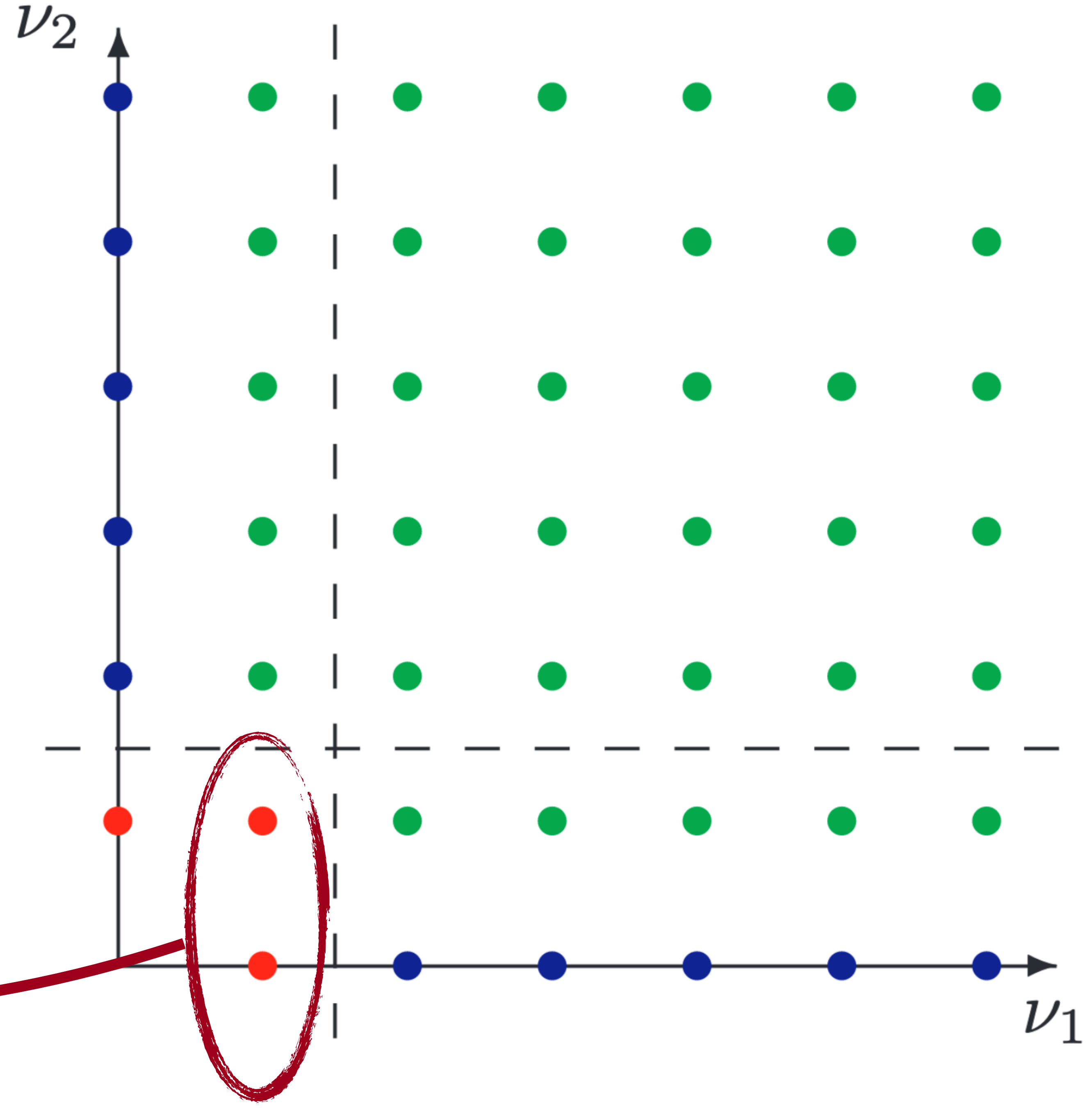


$$F_{\nu_1\nu_2} = \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{e^{\varepsilon\gamma_E}\left(m^2\right)^{|\nu|-D/2}}{\left[-k^2+m^2\right]^{\nu_1}\left[-(k-p)^2+m^2\right]^{\nu_2}}$$

e.g., $q_{\mathrm{IBP}} = p$

$$0 = p^\mu \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{\partial}{\partial k^\mu} \frac{e^{\varepsilon\gamma_E}\left(m^2\right)^{|\nu|-D/2}}{\left[-k^2+m^2\right]^{\nu_1}\left[-(k-p)^2+m^2\right]^{\nu_2}}$$

$$0 = \left(\nu_1-\nu_2\right) I_{\nu_1\nu_2} - \nu_1 I_{(\nu_1+1)(\nu_2-1)} + \nu_2 I_{(\nu_1-1)(\nu_2+1)} + \nu_1 x I_{(\nu_1+1)\nu_2} - \nu_2 x I_{\nu_1(\nu_2+1)}$$

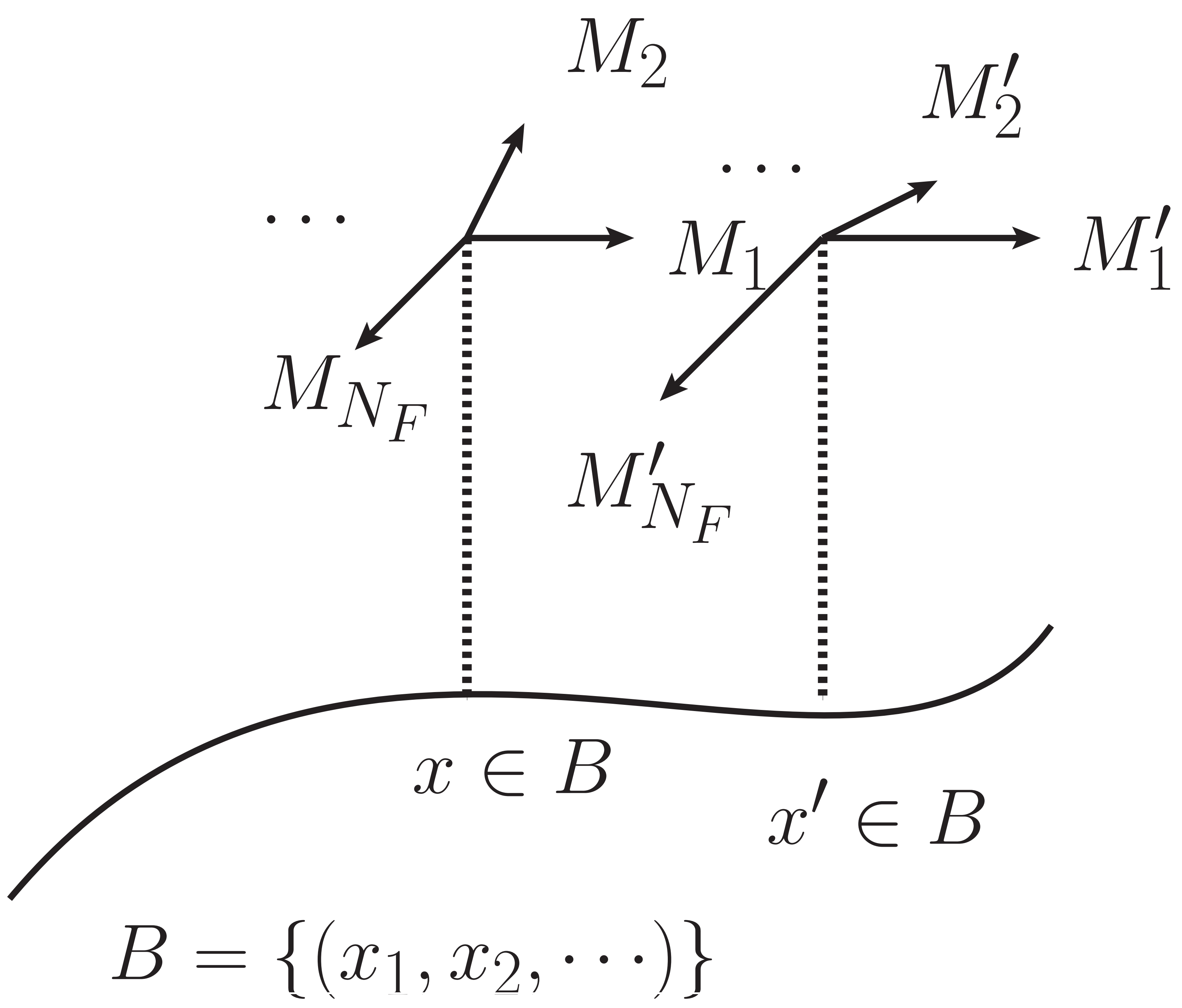


$I_1 = F_{10}^{\text{bubble}}, \quad I_2^{\text{bubble}} = F_{11}$

Kinematics Dependence

Kinematics vary⟶ **differential equations** of FIs (MIs).

The **primary** method for analytic and (semi-)numerical calculation of FIs.

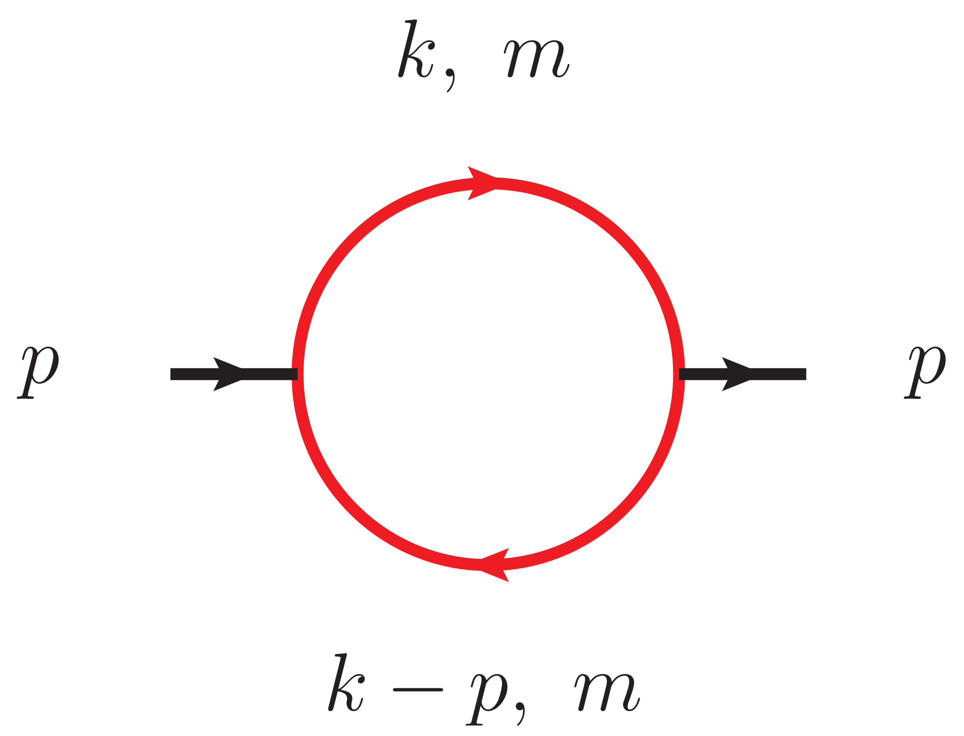


$$\mathrm{d}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{N_F} \end{pmatrix} = A_{N_F\times N_F}\big(\varepsilon, \{x\}\big)\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{N_F} \end{pmatrix}$$

[Kotikov '91; Remiddi '97; Gehrmann, Remiddi '00]

Solving this linearly coupled PDE system is non-trivial!

Kinematics Dependence



$$F_{\nu_1\nu_2} = \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{e^{\varepsilon\gamma_E}\left(m^2\right)^{|\nu|-D/2}}{\left[-k^2+m^2\right]^{\nu_1}\left[-(k-p)^2+m^2\right]^{\nu_2}}$$

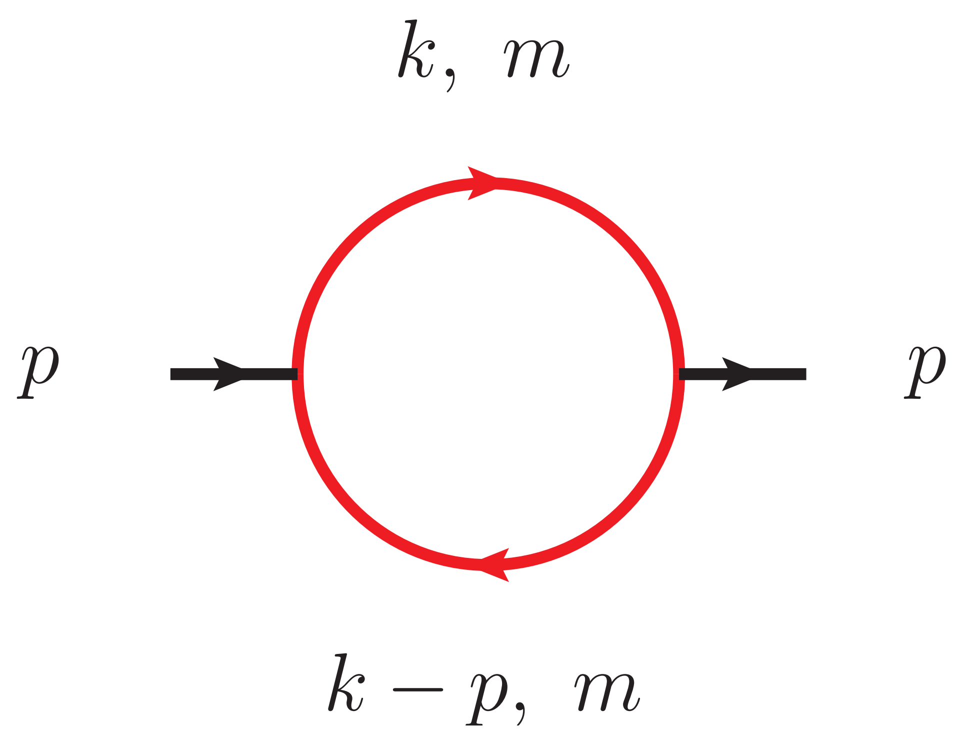
$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} I_{10}^{\text{bubble}} \\ I_{11}^{\text{bubble}} \end{pmatrix}$$

$$x = -p^2/m^2$$

1) Taking derivative: $\nu_i \to \nu_i + 1$; 2) IBPs map back onto (M_1, M_2)

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{2-D}{x(x+4)} & \frac{(D-4)x-4}{2x(x+4)} \end{bmatrix}}_{A_{2\times 2}(D=D_{\text{int}}-2\varepsilon,\, x)} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

Kinematics Dependence



$$F_{\nu_1\nu_2} = \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{e^{\varepsilon\gamma_E}\left(m^2\right)^{|\nu|-D/2}}{\left[-k^2+m^2\right]^{\nu_1}\left[-(k-p)^2+m^2\right]^{\nu_2}}$$

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} I_{10}^{\text{bubble}} \\ I_{11}^{\text{bubble}} \end{pmatrix}$$

$$x = -p^2/m^2$$

1) Taking derivative: $\nu_i \to \nu_i + 1$; 2) IBPs map back onto (M_1, M_2)

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{2-D}{x(x+4)} & \frac{(D-4)x-4}{2x(x+4)} \end{bmatrix}}_{A_{2\times2}\left(D=D_{\text{int}}-2\varepsilon,\, x\right)} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \xrightarrow{\text{brute-force}} \begin{aligned} &M_1 = \text{const}\,. \\ &M_2 \sim {}_2F_1\left(-\frac{1}{2}, \frac{5-D}{2}; \frac{1}{2}; -\frac{x}{4}\right) \end{aligned}$$

ε-factorisation (canonicalisation)

ε-factorisation:

Via **rotations** of basis (i.e., $\vec{M} \to \vec{K}$) and **variable change**, *ε-dependence factorises in the connection matrix, with suitable boundary conditions.* [Henn '13]

$$\mathrm{d}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} = \varepsilon\, B_{N_F\times N_F}\big(\{x\}\big)\begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} \Longrightarrow$$

ε-factorisation (canonicalisation)

ε-factorisation:

Via **rotations** of basis (i.e., $\vec{M} \to \vec{K}$) and **variable change**, *ε-dependence factorises in the connection matrix, with suitable boundary conditions.* [Henn '13]

$$\mathrm{d}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} = \varepsilon\, B_{N_F\times N_F}\big(\{x\}\big)\begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} \Longrightarrow$$

$$\vec{K} = \sum \varepsilon^n \vec{K}^{(n)} \longrightarrow \vec{K}^{(n+1)} = \int B_{N_F\times N_F}\vec{K}^{(n)} + \text{boundary}$$

$$\hookrightarrow\quad \vec{K} = \mathbf{P}\exp\left(\varepsilon \int B_{N_F\times N_F}\right)\vec{K}_{\text{boundary}}$$

MIs can be written as Chen's iterated integrals [Chen '77].

ε-factorisation (canonicalisation)

ε-factorisation:

Via **rotations** of basis (i.e., $\vec{M} \to \vec{K}$) and **variable change**, *ε-dependence factorises in the connection matrix, with suitable boundary conditions.* [Henn '13]

$$\mathrm{d}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} = \varepsilon\, B_{N_F\times N_F}(\{x\}) \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} \Longrightarrow$$

$$\vec{K} = \sum \varepsilon^n \vec{K}^{(n)} \longrightarrow \vec{K}^{(n+1)} = \int B_{N_F\times N_F} \vec{K}^{(n)} + \text{boundary}$$

$$\hookrightarrow \quad \vec{K} = \mathbf{P}\exp\left(\varepsilon \int B_{N_F\times N_F}\right) \vec{K}_{\text{boundary}}$$

MIs can be written as Chen's iterated integrals [Chen '77].

Once the ε-factorised (canonical) form is derived, FIs (MIs) are viewed as solved.

ε-factorisation (canonicalisation)

ε-factorisation:

Via **rotations** of basis (i.e., $\vec{M} \to \vec{K}$) and **variable change**, ***ε-dependence factorises in the connection matrix, with suitable boundary conditions.*** [Henn '13]

$$\mathrm{d}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} = \varepsilon\, B_{N_F\times N_F}(\{x\})\begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} \Longrightarrow$$

$$\vec{K} = \sum \varepsilon^n \vec{K}^{(n)} \longrightarrow \vec{K}^{(n+1)} = \int B_{N_F\times N_F}\vec{K}^{(n)} + \text{boundary}$$

$$\hookrightarrow \quad \vec{K} = \mathbf{P}\exp\left(\varepsilon\int B_{N_F\times N_F}\right)\vec{K}_{\text{boundary}}$$

MIs can be written as Chen's iterated integrals [Chen '77].

Once the ε-factorised (canonical) form is derived, FIs (MIs) are viewed as solved.

Very Hard! Lots of progress recently, e.g., Baune, Bönisch, Broedel, ε-collaboration, Dlapa, Duhr, Frellesvig, Görges, Henn, Klement, Jiang, Maggio, Nega, Porkert, Sauer, Sohnle, Stawinski, Tancredi, Wager, Wilhelm, Yan, Yang, Zhang, Zhu + many more…

ε-factorisation (canonicalisation)

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = 2\varepsilon(1-\varepsilon) \begin{bmatrix} 1 & 0 \\ \sqrt{\dfrac{x}{4+x}} & \sqrt{\dfrac{x}{4+x}} \end{bmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

↯

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \varepsilon \underbrace{\begin{bmatrix} 0 & 0 \\ \dfrac{1}{\sqrt{x(x+4)}} & \dfrac{1}{x+4} \end{bmatrix}}_{G_{2\times 2}(x)} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

ε-factorisation (canonicalisation)

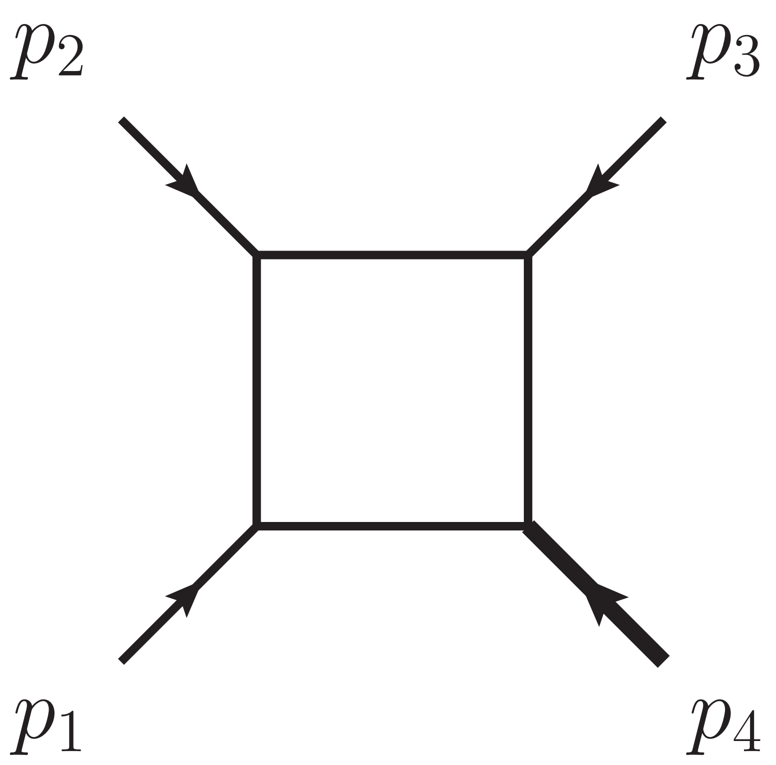
$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = 2\varepsilon(1-\varepsilon)\begin{bmatrix} 1 & 0 \\ \sqrt{\frac{x}{4+x}} & \sqrt{\frac{x}{4+x}} \end{bmatrix}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

$$\rightsquigarrow$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \varepsilon\underbrace{\begin{bmatrix} 0 & 0 \\ \frac{1}{\sqrt{x(x+4)}} & \frac{1}{x+4} \end{bmatrix}}_{G_{2\times 2}(x)}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

One can further do a variable change, s.t., the square-root is gone.
Then the MIs are easily written as iterated integrals to any orders of ε.

An Example on Iterated Integrals

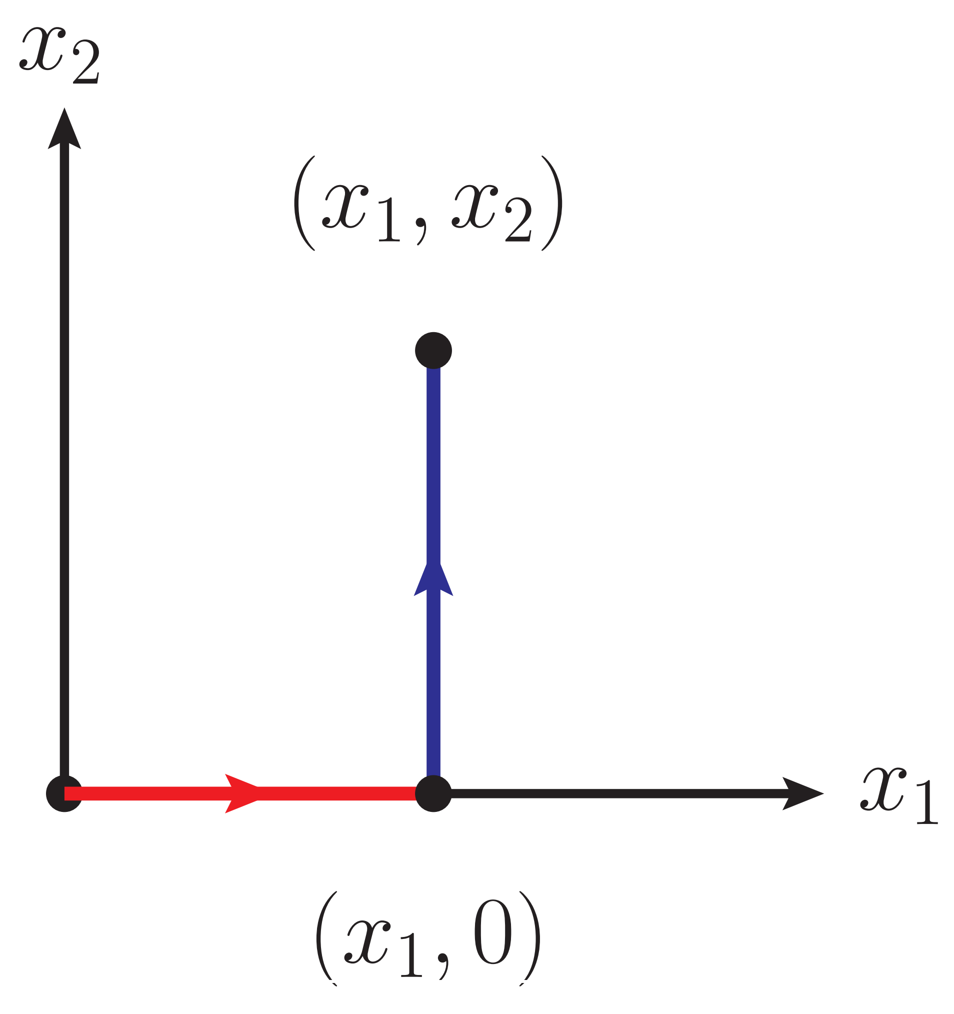


$$I_{\nu_1\nu_2\nu_3\nu_4}(D,x_1,x_2) = \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{e^{\varepsilon\gamma_E}(-p_4^2)^{|\nu|}}{\left(-q_1^2\right)^{\nu_1}\left(-q_2^2\right)^{\nu_2}\left(-q_3^2\right)^{\nu_3}\left(-q_4^2\right)^{\nu_4}}$$

$$x_1 = \frac{s}{p_4^2}, \qquad x_2 = \frac{t}{-p_4^2}$$

$$\mathrm{d}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{pmatrix} = \varepsilon \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\mathrm{d}x_1}{x_1} & 0 & 0 \\ 0 & 0 & \frac{\mathrm{d}x_2}{x_2} & 0 \\ \omega_1(x_1,x_2) & \omega_2(x_1,x_2) & \omega_3(x_1,x_2) & \omega_4(x_1,x_2) \end{bmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{pmatrix}$$

An Example on Iterated Integrals



integrating path

$$\mathrm{d}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{pmatrix} = \varepsilon \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\mathrm{d}x_1}{x_1} & 0 & 0 \\ 0 & 0 & \frac{\mathrm{d}x_2}{x_2} & 0 \\ \omega_1(x_1, x_2) & \omega_2(x_1, x_2) & \omega_3(x_1, x_2) & \omega_4(x_1, x_2) \end{bmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{pmatrix}$$

$$K_1 = K_1^{(0)}|_{\mathrm{BD}} + \varepsilon K_1^{(1)}|_{\mathrm{BD}} + \mathcal{O}(\varepsilon^2)$$

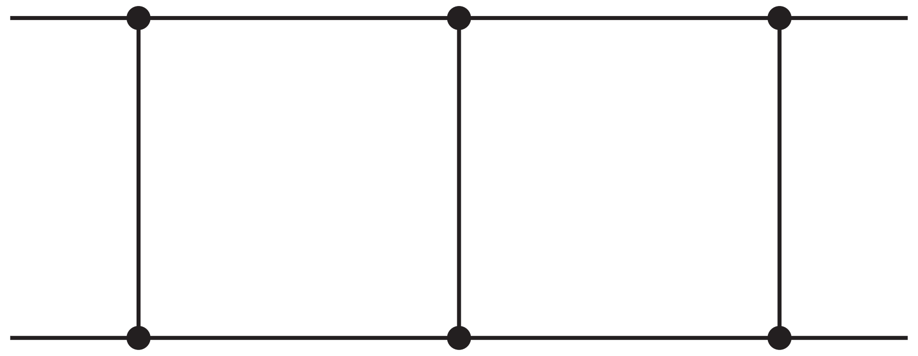
$$K_2 = K_2^{(0)}|_{\mathrm{BD}} + \varepsilon \left(K_2^{(1)}|_{\mathrm{BD}} - G(0; x_1) \right) + \mathcal{O}(\varepsilon^2)$$

$$K_3 = K_3^{(0)}|_{\mathrm{BD}} + \varepsilon \left(K_3^{(1)}|_{\mathrm{BD}} - G(0; x_2) \right) + \mathcal{O}(\varepsilon^2)$$

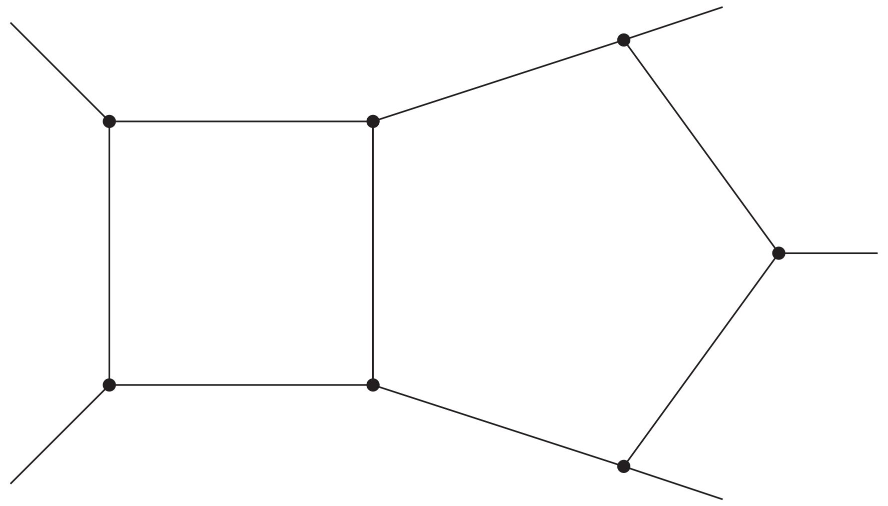
$$\vdots$$

$$G(0; x) = -\ln x$$

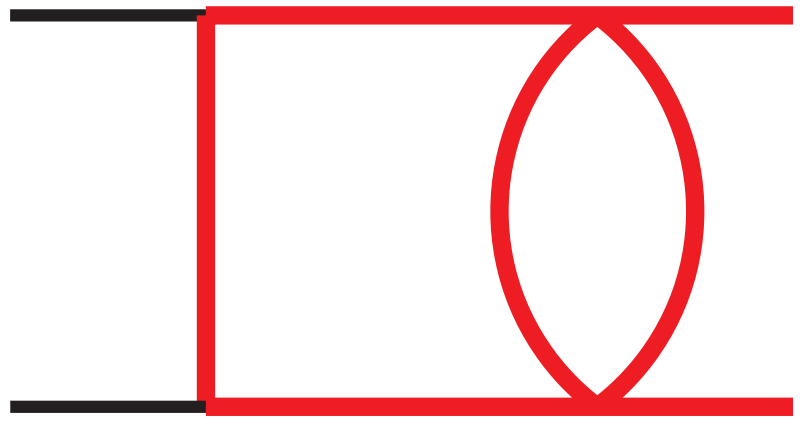
Feynman Integrals are Hard



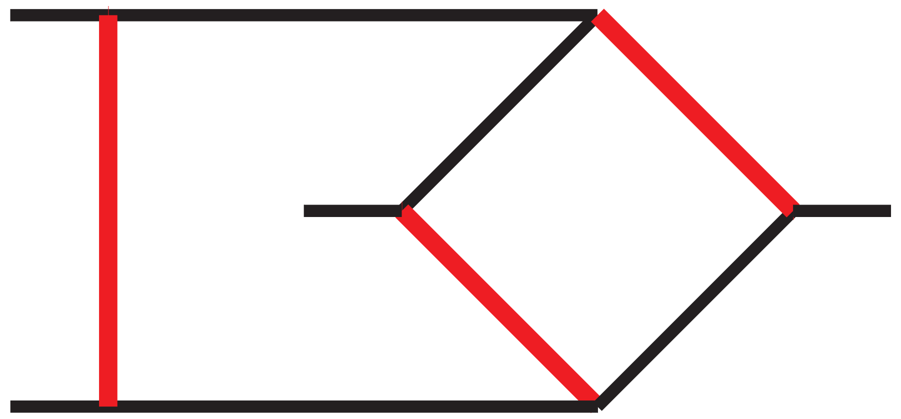
MPL type



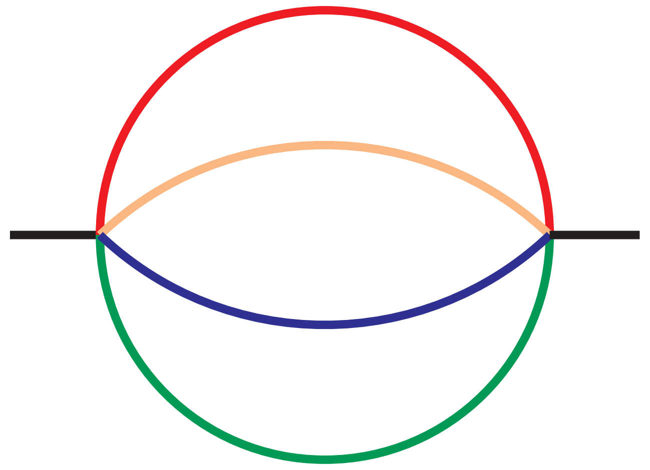
MPL type



elliptic type



hyper elliptic type



Calabi-Yau 2-folds type

Previously, canonicalization (典则化) was ad hoc and by experience. Now, we have found a **unified algorithm** for all Feynman integrals regardless of their geometries.

Outline and Takehome Message

It is well-known that FIs have **block structures**, specified by sectors.

They also have a **multi-layered (Hodge-like) structure** inside sectors.

☑ Improve IBP (with the Laporta algorithm), in general.

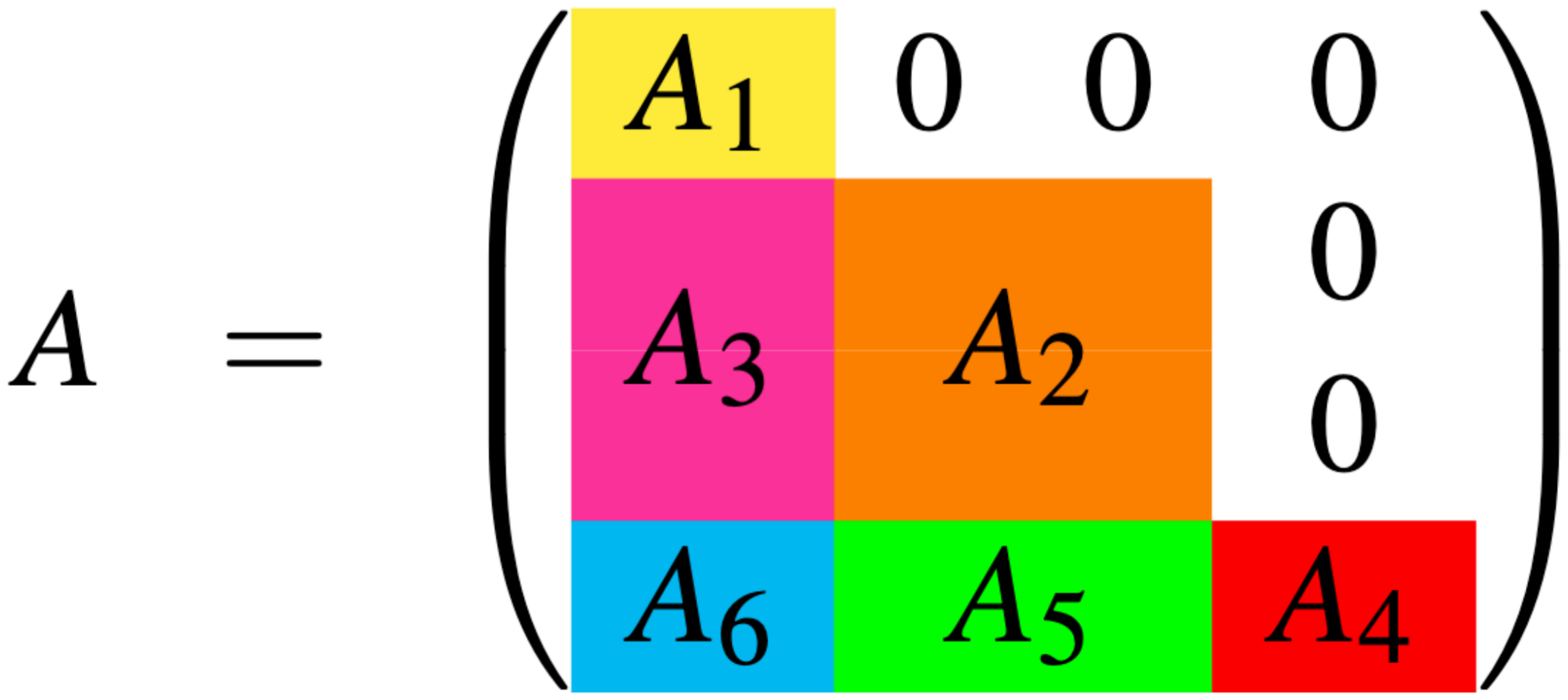
☑ An algorithm to derive ε-factorizing (canonical) DEs of MIs.

The structure is **universal**, as are two aspects of the method.

Sectors

- Integral family: given a set of propagators and <u>auxiliary scalar products</u> as a basis;

  $\hookrightarrow \quad F_{\nu_1\nu_2\cdots\nu_n} = c_1 M_1 + c_2 M_2 + \cdots + c_{N_F} M_{N_F}.$
- Some master integrals do not contain all propagators, e.g., $F^{\text{bubble}}_{\nu_1\nu_2} = c_1 F_{10} + c_2 F_{11}$.
- Masters with fewer propagators can be viewed as simpler ⇝ sectors.



$$A = \begin{pmatrix} A_1 & 0 \quad 0 & 0 \\ A_3 & A_2 & \begin{matrix} 0 \\ 0 \end{matrix} \\ A_6 & A_5 & A_4 \end{pmatrix}$$

A Toy Model of IBP

$$\underbrace{\begin{aligned} 14732\, I_1 - 2514\, I_2 - 5\, I_3 - 7\, I_4 &= 0, \\ 9872\, I_1 - 17294\, I_2 + 3\, I_3 - 11\, I_4 &= 0, \\ 5068\, I_1 - 49336\, I_2 + 18\, I_3 - 22\, I_4 &= 0. \end{aligned}}$$

$$\begin{aligned} I_1 &= \frac{1237}{3025750} I_3 + \frac{1229}{3025750} I_4, \\ I_2 &= \frac{1231}{3025750} I_3 - \frac{1223}{3025750} I_4 \end{aligned}$$

$$\begin{aligned} I_3 &= 1223 I_1 + 1229 I_2 \equiv 1223 J_1 + 1229 J_2, \\ I_4 &= 1231 I_1 - 1237 I_2 \equiv 1231 J_1 - 1237 J_2 \end{aligned}$$

A Toy Model of IBP

$$\underbrace{\begin{aligned} 14732\,I_1 - 2514\,I_2 - 5\,I_3 - 7\,I_4 &= 0, \\ 9872\,I_1 - 17294\,I_2 + 3\,I_3 - 11\,I_4 &= 0, \\ 5068\,I_1 - 49336\,I_2 + 18\,I_3 - 22\,I_4 &= 0. \end{aligned}}$$

$$\begin{aligned} I_1 &= \frac{1237}{3025750} I_3 + \frac{1229}{3025750} I_4, \\ I_2 &= \frac{1231}{3025750} I_3 - \frac{1223}{3025750} I_4 \end{aligned}$$

$$\begin{aligned} I_3 &= 1223 I_1 + 1229 I_2 \equiv 1223 J_1 + 1229 J_2, \\ I_4 &= 1231 I_1 - 1237 I_2 \equiv 1231 J_1 - 1237 J_2 \end{aligned}$$

This shows that I_1, I_2 as MIs are simpler than I_3, I_4, because no large denominators are involved.

A Practical Example of IBP

$$F_{\nu_1\nu_2\nu_3\nu_4} = \int \frac{\mathrm{d}^D q_1}{i\pi^{D/2}} \frac{\mathrm{d}^D q_2}{i\pi^{D/2}} \frac{1}{\left[(q_1+p-zk)^2-m^2\right]^{\nu_1}\left[q_1^2-m^2\right]^{\nu_2}\left[q_2^2-m^2\right]^{\nu_3}\left[(q_1+q_2+\bar{z}k-p)^2-m^2\right]^{\nu_4}}$$

$$F_{1311} = -\frac{3\varepsilon(3\varepsilon-1)(z\varepsilon+z-2\varepsilon-1)}{4(z-1)(2(z-2)\varepsilon-1)} \cdot F_{1111} + z\frac{-8z^2\varepsilon(\varepsilon+1)+2z\left(8\varepsilon^2+6\varepsilon-1\right)-2\varepsilon+1}{4(z-1)^2(2(z-2)\varepsilon-1)} \cdot F_{2111}$$

$$\boxed{F_{1311} = -\frac{3\varepsilon(\varepsilon+1)}{2(1+2\varepsilon)} \cdot \underbrace{F_{2111}}_{J_1} + \frac{2z\epsilon+2z-1}{2(2z-1)(1+2\varepsilon)} \cdot \underbrace{\left(-2\varepsilon \cdot F_{1311} - 2z \cdot F_{2211} + (z+\varepsilon) \cdot F_{1212}\right)}_{J_2}\,.}$$

A Practical Example of IBP

$$F_{\nu_1\nu_2\nu_3\nu_4} = \int \frac{\mathrm{d}^D q_1}{i\pi^{D/2}} \frac{\mathrm{d}^D q_2}{i\pi^{D/2}} \frac{1}{\left[(q_1+p-zk)^2-m^2\right]^{\nu_1}\left[q_1^2-m^2\right]^{\nu_2}\left[q_2^2-m^2\right]^{\nu_3}\left[(q_1+q_2+\bar{z}k-p)^2-m^2\right]^{\nu_4}}$$

$$F_{1311} = -\frac{3\varepsilon(3\varepsilon-1)(z\varepsilon+z-2\varepsilon-1)}{4(z-1)(2(z-2)\varepsilon-1)} \cdot F_{1111} + z\frac{-8z^2\varepsilon(\varepsilon+1)+2z\left(8\varepsilon^2+6\varepsilon-1\right)-2\varepsilon+1}{4(z-1)^2(2(z-2)\varepsilon-1)} \cdot F_{2111}$$

$$\boxed{F_{1311} = -\frac{3\varepsilon(\varepsilon+1)}{2(1+2\varepsilon)} \cdot \underbrace{F_{2111}}_{J_1} + \frac{2z\epsilon+2z-1}{2(2z-1)(1+2\varepsilon)} \cdot \underbrace{\Big(-2\varepsilon \cdot F_{1311} - 2z \cdot F_{2211} + (z+\varepsilon) \cdot F_{1212}\Big)}_{J_2}\ .}$$

This shows that we should choose $\{J_1, J_2\}$ as MIs. The new structure can find such a basis systematically. This pattern is inherited by DEs.

Sketch of the Algorithm: $\vec{I} \to \vec{J} \to \vec{K}$

Sketch of the Algorithm: $\vec{I} \to \vec{J} \to \vec{K}$

Step I: Given any pre-basis $\vec{I}$, derive a good (pre-) basis $\vec{J}$, compatible with filtrations (滤过), borrowed from Hodge theory.

$$\mathrm{d}\vec{J} = \left[\frac{1}{\varepsilon^{N_V}}\mathbf{B}^{(-N_V)}(x) + \frac{1}{\varepsilon^{N_V-1}}\mathbf{B}^{(-N_V+1)}(x) + \cdots + \mathbf{B}^{(0)}(x) + \varepsilon\,\mathbf{B}^{(1)}(x)\right]\vec{J}, \quad \vec{J} = \mathbf{R}_1 \cdot \vec{I}$$

Sketch of the Algorithm: $\vec{I} \to \vec{J} \to \vec{K}$

Step I: Given any pre-basis $\vec{I}$, derive a good (pre-) basis $\vec{J}$, compatible with filtrations (滤过), borrowed from Hodge theory.

$$\mathrm{d}\vec{J} = \left[\frac{1}{\varepsilon^{N_V}}\mathbf{B}^{(-N_V)}(x) + \frac{1}{\varepsilon^{N_V-1}}\mathbf{B}^{(-N_V+1)}(x) + \cdots + \mathbf{B}^{(0)}(x) + \varepsilon\,\mathbf{B}^{(1)}(x)\right]\vec{J}, \quad \vec{J} = \mathbf{R}_1 \cdot \vec{I}$$

$\mathbf{B}^{(-N_V)}(x), \cdots, \mathbf{B}^{(0)}(x)$ are in a **good block lower-triangular form**. Both $\mathbf{B}^{(i)}(x), \mathbf{R}_1(x)$ rational!

Sketch of the Algorithm: $\vec{I} \to \vec{J} \to \vec{K}$

Step I: Given any pre-basis $\vec{I}$, derive a good (pre-) basis $\vec{J}$, compatible with filtrations (滤过), borrowed from Hodge theory.

$$\mathrm{d}\,\vec{J} = \left[\frac{1}{\varepsilon^{N_V}}\mathbf{B}^{(-N_V)}(x) + \frac{1}{\varepsilon^{N_V-1}}\mathbf{B}^{(-N_V+1)}(x) + \cdots + \mathbf{B}^{(0)}(x) + \boxed{\varepsilon\,\mathbf{B}^{(1)}(x)}\right]\vec{J}, \quad \vec{J} = \mathbf{R}_1 \cdot \vec{I}$$

$\mathbf{B}^{(-N_V)}(x), \cdots, \mathbf{B}^{(0)}(x)$ are in a **good block lower-triangular form**. Both $\mathbf{B}^{(i)}(x), \mathbf{R}_1(x)$ rational!

Step II: $\vec{J}$ may be canonical. If not, one can always rotate it to be algorithmically.

$$\mathbf{R}_2^{(-N_V)}\mathbf{R}_2^{(-N_V+1)}\cdots\mathbf{R}_2^{(0)}\,\vec{K} = \vec{J}$$

Sketch of the Algorithm: $\vec{I} \to \vec{J} \to \vec{K}$

Step I: Given any pre-basis $\vec{I}$, derive a good (pre-) basis $\vec{J}$, compatible with filtrations (滤过), borrowed from Hodge theory.

$$\mathrm{d}\vec{J} = \left[\frac{1}{\varepsilon^{N_V}}\mathbf{B}^{(-N_V)}(x) + \frac{1}{\varepsilon^{N_V-1}}\mathbf{B}^{(-N_V+1)}(x) + \cdots + \mathbf{B}^{(0)}(x) + \boxed{\varepsilon\,\mathbf{B}^{(1)}(x)}\right]\vec{J}, \quad \vec{J} = \mathbf{R}_1 \cdot \vec{I}$$

$\mathbf{B}^{(-N_V)}(x), \cdots, \mathbf{B}^{(0)}(x)$ are in a **good block lower-triangular form**. Both $\mathbf{B}^{(i)}(x), \mathbf{R}_1(x)$ rational!

Step II: $\vec{J}$ may be canonical. If not, one can always rotate it to be algorithmically.

$$\mathbf{R}_2^{(-N_V)}\mathbf{R}_2^{(-N_V+1)}\cdots\mathbf{R}_2^{(0)}\,\vec{K} = \vec{J}$$

$\mathbf{R}_2^{(i)}(x)$ may be transcendental! The algorithm applies to FIs related to different geometries.

Multi-layered Structures of Feynman Integrals

Baikov Rep. : Integrals Hard, but Integrands Simple

$\mathrm{d}^D l \to \mathrm{d}P_1(l)\mathrm{d}P_2(l)\cdots$, i.e., propagators, P_i's, as integration variables: $z_i = P_i$

↪ non-trivial "Jacobian", **twist** $u(z) = \prod_{j\in\text{all}} \left[p_j(z)\right]^{\frac{1}{2}(a_i+b_i\varepsilon)}$, $a_i = 0{,}1\ ;\ b_i \in \mathbb{Z}$:

$$I_i = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} u\left(z\right) \underbrace{\frac{q_i(z)}{\prod_{j\in\text{all}} \left[p_j(z)\right]^{\mu_j}} \mathrm{d}z_{N_V} \wedge \cdots \wedge \mathrm{d}z_1,}_{\hat{\phi}} \qquad \mu_j \in \mathbb{Z}$$

Packages: [*Baikovletter*, Jiang, Yang; *BaikovPackage*, Frellesvig; *SOFIA*, Correia, Giroux, Mizera]

Baikov Rep. : Integrals Hard, but Integrands Simple

$\mathrm{d}^D l \to \mathrm{d}P_1(l)\mathrm{d}P_2(l)\cdots$, i.e., propagators, P_i's, as integration variables: $z_i = P_i$

↪ non-trivial "Jacobian", **twist** $u(z) = \prod_{j\in\text{all}} \left[p_j(z)\right]^{\frac{1}{2}(a_i + b_i\varepsilon)}$, $a_i = 0,1\ ;\ b_i \in \mathbb{Z}$:

$$I_i = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} u\left(z\right) \underbrace{\frac{q_i(z)}{\prod_{j\in\text{all}} \left[p_j(z)\right]^{\mu_j}} \mathrm{d}z_{N_V} \wedge \cdots \wedge \mathrm{d}z_1}_{\hat{\phi}}, \qquad \mu_j \in \mathbb{Z}$$

Packages: [*BaikovIetter*, Jiang, Yang; *BaikovPackage*, Frellesvig; *SOFIA*, Correia, Giroux, Mizera]

- ▸ Not for calculating FIs, but rather for studying the structures!
- ▸ It translates FIs to twisted cohomology language: $u\hat{\phi} \sim_u u\left(\hat{\phi} + \nabla_u \omega\right)$;
- ▸ Perfect for (onshell) propagator cuts: $\mathrm{cut}_i = \mathrm{res}_{z_i=0}$.

Setup: Twist

MIs in a given sector share the same (minimal) twist ($b_i, b_j \in \mathbb{Z}$):

$$u\left(z_1, z_2, \cdots, z_{N_V}\right) = \prod_{i \in I_{\text{odd}}} \left[p_i(z)\right]^{-\frac{1}{2}+\frac{1}{2}b_i\varepsilon} \prod_{j \in I_{\text{even}}} \left[p_j(z)\right]^{\frac{1}{2}b_j\varepsilon}$$

- ▸ Odd polynomials ⟶ geometry;
- ▸ Even polynomials ⟶ possible residues (punctures) to take;
- ▸ Different MIs ⟶ different rational parts.

Setup: Twist

MIs in a given sector share the same (minimal) twist ($b_i, b_j \in \mathbb{Z}$):

$$u(z_1, z_2, \cdots, z_{N_V}) = \prod_{i \in I_{\text{odd}}} [p_i(z)]^{-\frac{1}{2}+\frac{1}{2}b_i\varepsilon} \prod_{j \in I_{\text{even}}} [p_j(z)]^{\frac{1}{2}b_j\varepsilon}$$

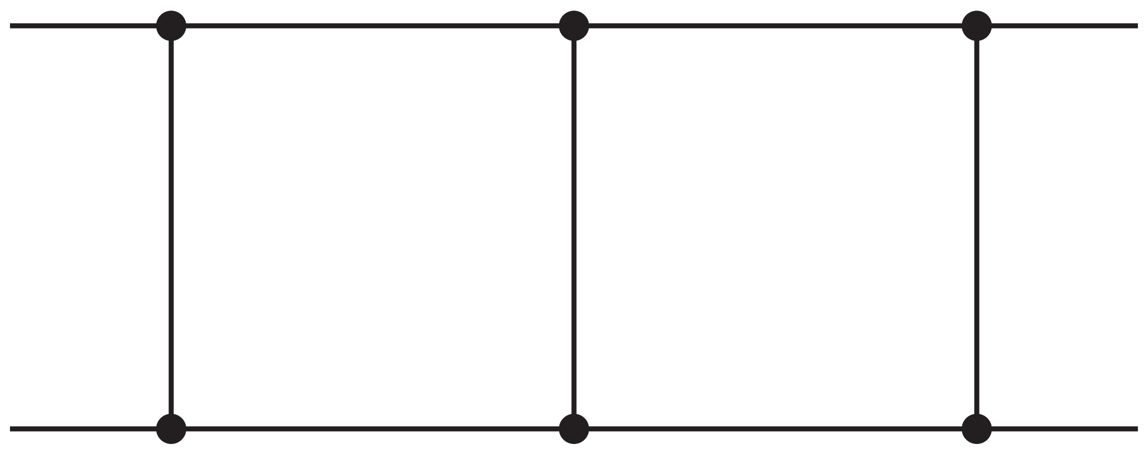
- Odd polynomials ⟶ geometry;
- Even polynomials ⟶ possible residues (punctures) to take;
- Different MIs ⟶ different rational parts.

$$I_i = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} u(z) \frac{q_i(z)}{\prod_{j \in \text{all}} [p_j(z)]^{\mu_j}} \mathrm{d}z_{N_V} \wedge \cdots \wedge \mathrm{d}z_1, \qquad \mu_j \in \mathbb{Z}$$

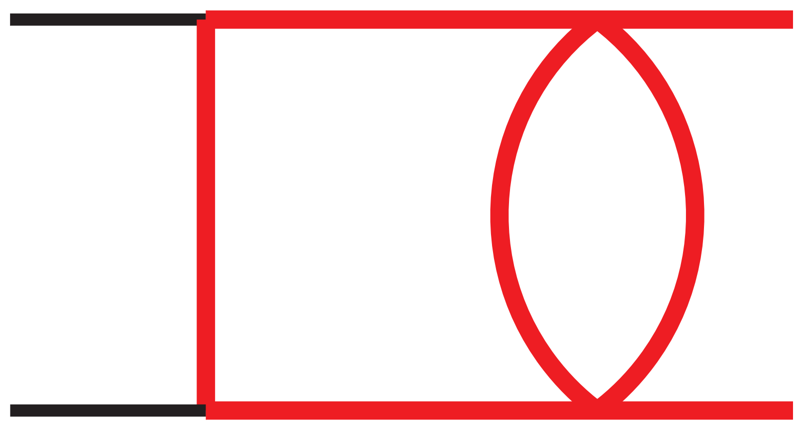
maximal cut in a sector

remaning variables after MC

Twist: Examples



$$I^{\text{MaxCut}}_{111111100} = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} \frac{\mathrm{d}z_1}{2\pi i}\, \overbrace{z_1^{-2\varepsilon}\left(z_1 - 1\right)^{-\varepsilon}\left(z_1 - x - 1\right)^{\varepsilon}}^{u(z_1)} \cdot 1$$



$$I^{\text{MaxCut}}_{111200100} = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} \frac{\mathrm{d}z_1}{2\pi i} \left[p_1(z_1)\right]^{-\frac{1}{2}} \left[p_2(z_1)\right]^{-\frac{1}{2}-\varepsilon} \left[p_3(z_1)\right]^{-\frac{1}{2}-\varepsilon} \cdot 1, \quad \text{w}\,.$$

$$p_1 = z_1 - x_2,\; p_2 = z_1 + 4 - x_2, \quad p_3 = \left(z_1 + 1\right)^2 - 4\left[x_2 + \frac{\left(1 - x_2\right)^2}{x_1}\right].$$

Setup: Twisted Cohomology

$$I_i = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} u(z) \underbrace{\frac{q_i(z)}{\prod_{j\in\text{all}}\left[p_j(z)\right]^{\mu_j}}\mathrm{d}z_{N_V} \wedge \cdots \wedge \mathrm{d}z_1}_{\hat{\phi}_i}, \qquad \mu_j \in \mathbb{Z}$$

Setup: Twisted Cohomology

$$I_i = C_{\mathrm{Baikov}} \int_{\mathscr{C}_{\mathrm{MC}}} u(z) \underbrace{\frac{q_i(z)}{\prod_{j \in \mathrm{all}} \left[p_j(z)\right]^{\mu_j}} \mathrm{d}z_{N_V} \wedge \cdots \wedge \mathrm{d}z_1}_{\hat{\phi}_i}, \qquad \mu_j \in \mathbb{Z}$$

- Given a master integral I_i, there is a differential form $\hat{\phi}_i$ ($H_\omega^{N_V} \twoheadrightarrow V^{N_V}$).

Setup: Twisted Cohomology

$$I_i = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} u(z) \underbrace{\frac{q_i(z)}{\prod_{j\in\text{all}} \left[p_j(z)\right]^{\mu_j}} \mathrm{d}z_{N_V} \wedge \cdots \wedge \mathrm{d}z_1,}_{\hat{\phi}_i} \qquad \mu_j \in \mathbb{Z}$$

- Given a master integral I_i, there is a differential form $\hat{\phi}_i$ ($H_\omega^{N_V} \twoheadrightarrow V^{N_V}$).
- One needs to mod out IBP relations $\hat{\phi}_i \sim \hat{\phi}_i + \nabla_u \hat{\eta}$, which leads us to the twisted cohomology [See e.g., Mastrolia and Mizera, '18].

Setup: Twisted Cohomology

$$I_i = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} u(z) \underbrace{\frac{q_i(z)}{\prod_{j\in\text{all}} \left[p_j(z)\right]^{\mu_j}} \mathrm{d}z_{N_V} \wedge \cdots \wedge \mathrm{d}z_1}_{\hat{\phi}_i}, \qquad \mu_j \in \mathbb{Z}$$

- Given a master integral I_i, there is a differential form $\hat{\phi}_i$ ($H_\omega^{N_V} \twoheadrightarrow V^{N_V}$).
- One needs to mod out IBP relations $\hat{\phi}_i \sim \hat{\phi}_i + \nabla_u \hat{\eta}$, which leads us to the twisted cohomology [See e.g., Mastrolia and Mizera, '18].
- Study the differential forms to represent the corresponding MIs.

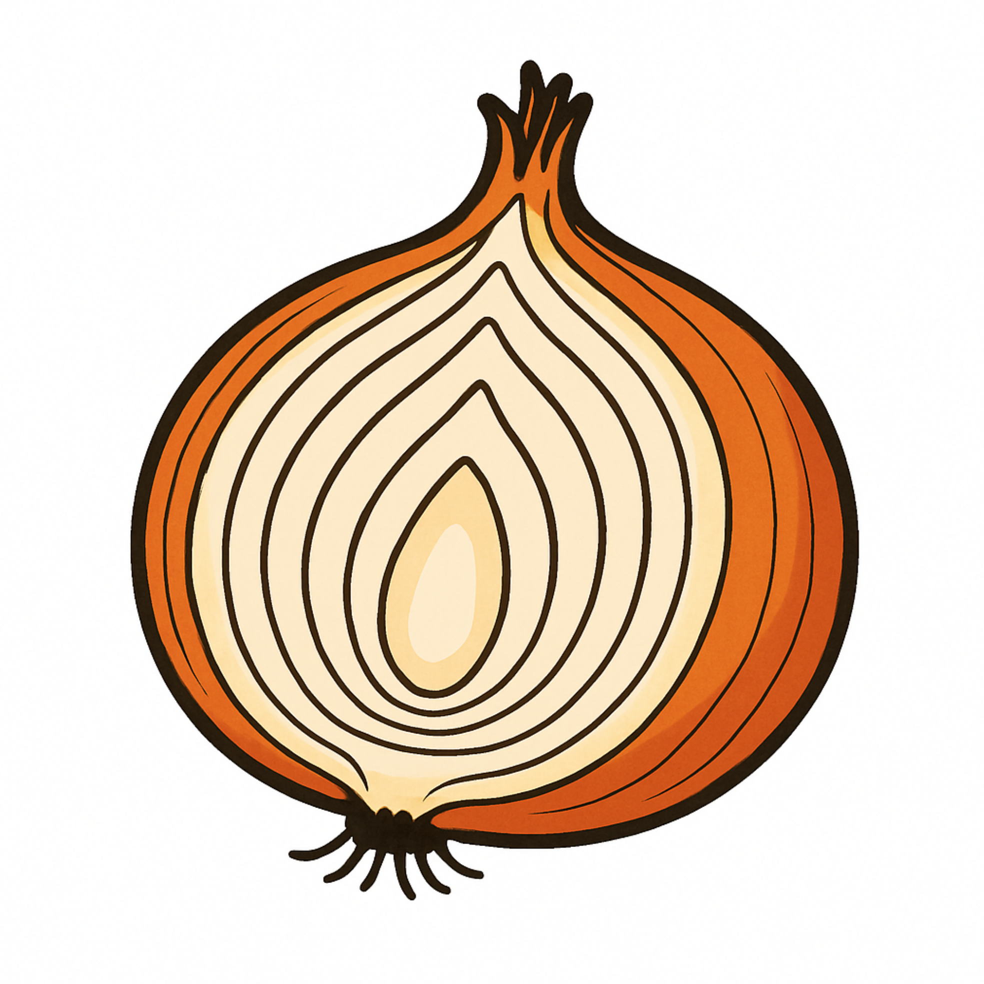
Setup: Twisted Cohomology

$$I_i = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} u(z) \underbrace{\frac{q_i(z)}{\prod_{j\in\text{all}} \left[p_j(z)\right]^{\mu_j}} \mathrm{d}z_{N_V} \wedge \cdots \wedge \mathrm{d}z_1}_{\hat{\phi}_i}, \qquad \mu_j \in \mathbb{Z}$$

- Given a master integral I_i, there is a differential form $\hat{\phi}_i$ ($H_\omega^{N_V} \twoheadrightarrow V^{N_V}$).
- One needs to mod out IBP relations $\hat{\phi}_i \sim \hat{\phi}_i + \nabla_u \hat{\eta}$, which leads us to the twisted cohomology [See e.g., Mastrolia and Mizera, '18].
- Study the differential forms to represent the corresponding MIs.
- Consider $\{\infty\}$ as well: $u \to U, \hat{\phi} \to \hat{\Phi}$ using homo. coord. $\left[z_0 : z_1 : \cdots : z_{N_V}\right]$.

Filtrations: a General Decomposition. of a Vector Space

"Layered" decomposition (**filtration, 滤**) of $H_\omega^{N_V} = \{\hat{\phi}\}/\sim_u$ into subspaces! （剥洋葱）



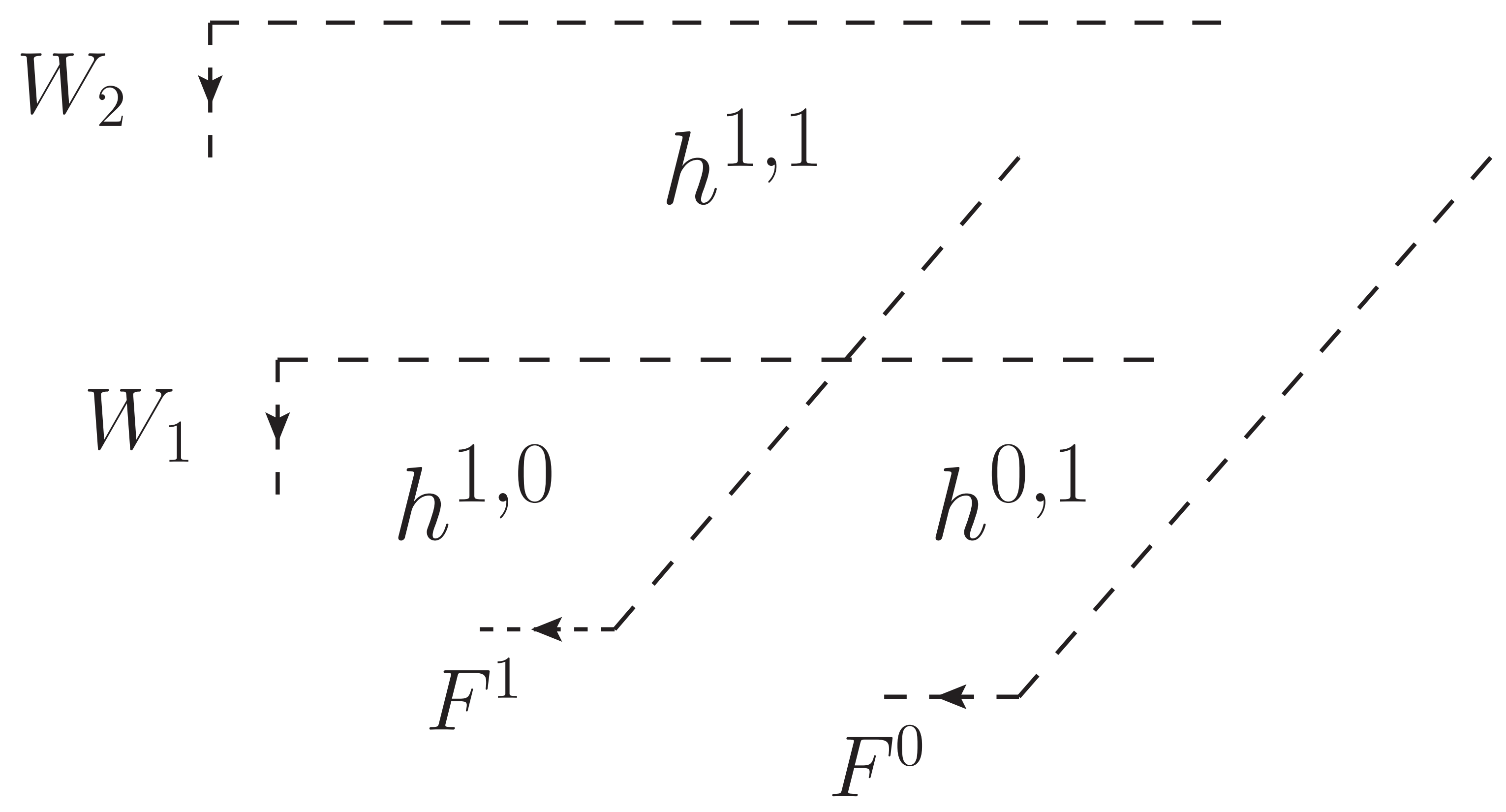
one filtration of onion

Filtrations: a General Decomposition. of a Vector Space

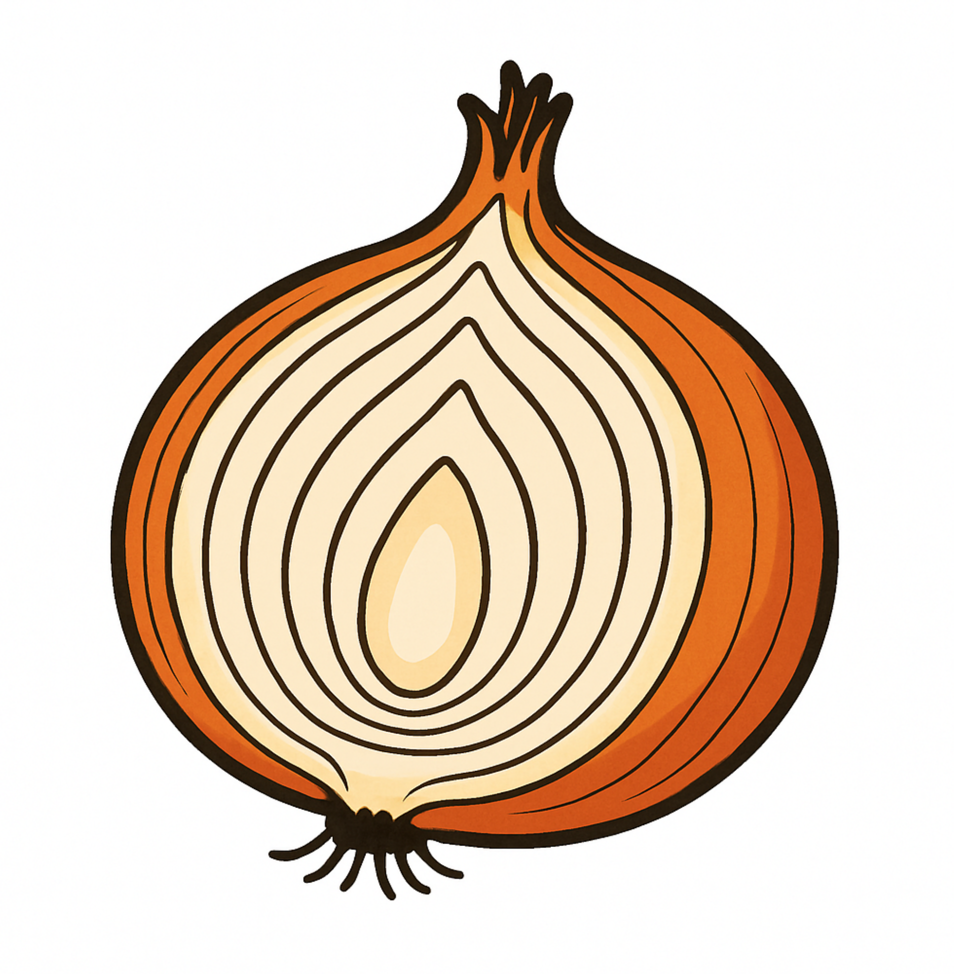
“Layered” decomposition (**filtration, 滤**) of $H_\omega^{N_V} = \{\hat{\phi}\}/\sim_u$ into subspaces! （剥洋葱）

$$\cdots \subseteq F^{p+1} H_\omega^{N_V} \subseteq F^{p} H_\omega^{N_V} \subseteq \cdots$$

$$\cdots \supseteq W_w H_\omega^{N_V} \supseteq W_{w-1} H_\omega^{N_V} \supseteq \cdots$$



two filtrations cut an “onion” into 3 subspaces in this example



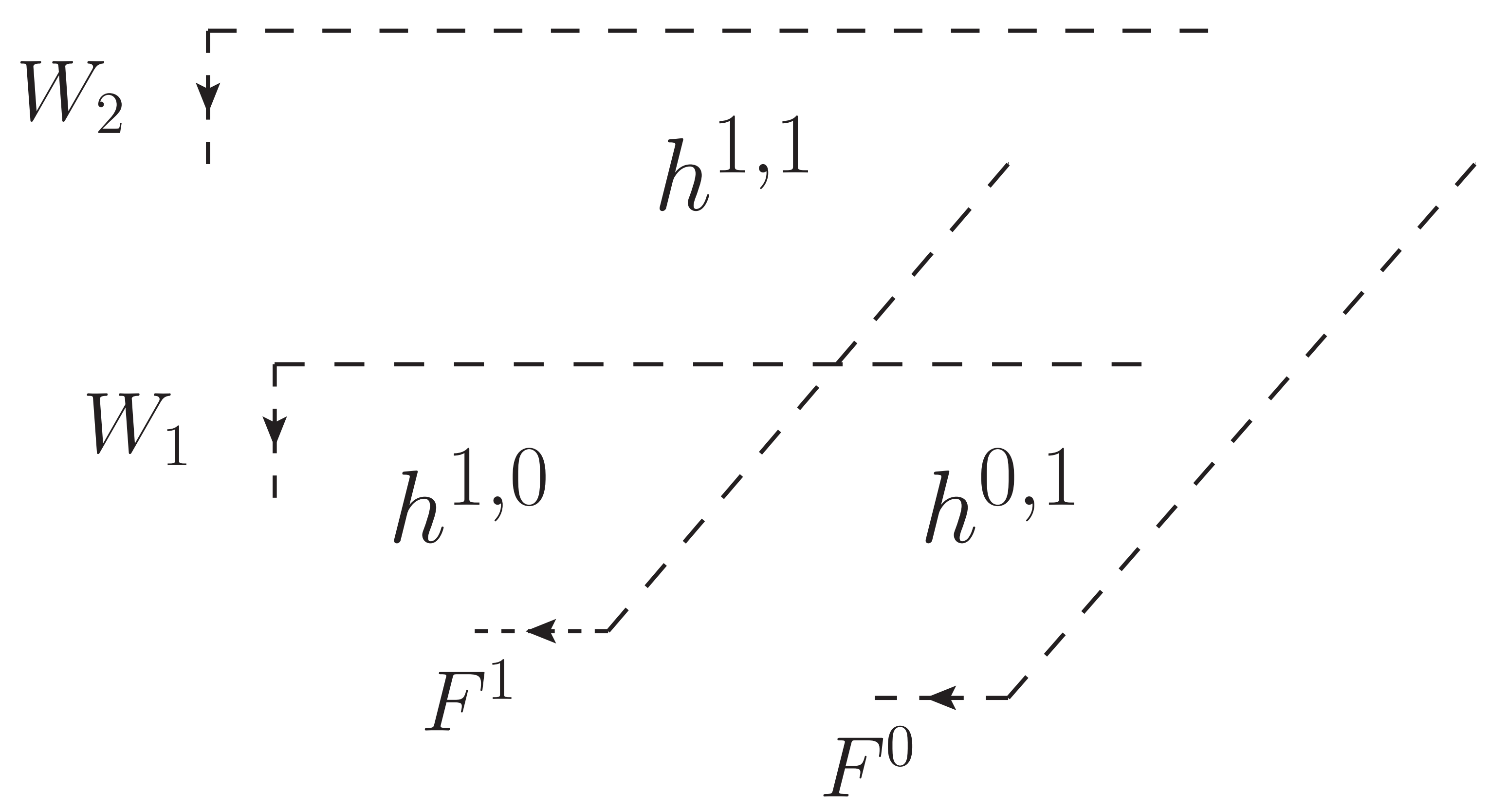
one filtration of onion

Filtrations: a General Decomposition. of a Vector Space

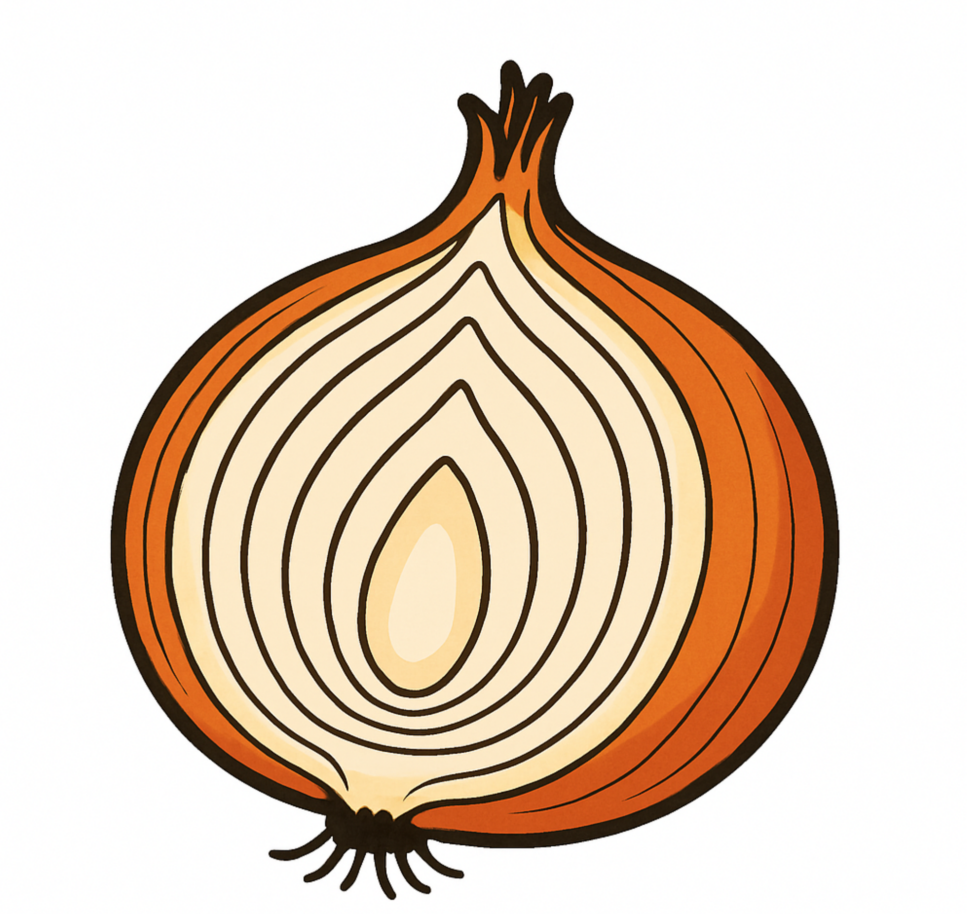
"Layered" decomposition (**filtration, 滤**) of $H_\omega^{N_V} = \{\hat{\phi}\}/\sim_u$ into subspaces! （剥洋葱）

$$\cdots \subseteq F^{p+1} H_\omega^{N_V} \subseteq F^{p} H_\omega^{N_V} \subseteq \cdots$$

$$\cdots \supseteq W_w H_\omega^{N_V} \supseteq W_{w-1} H_\omega^{N_V} \supseteq \cdots$$



two filtrations cut an "onion" into 3 subspaces in this example



one filtration of onion

Given $\Psi_{\mu_0\cdots\mu_{N_D}}[Q]$, two more ordering criteria (two **filtrations**): pole order o, and the number of non-zero residues r. *These two integers indicate "layer" numbers (两种剥洋葱的指标).*

$$p = N_V - o + r; \qquad q = o; \qquad w = p + q = N_V + r\,.$$

Step 1: Prefactor by Filtrations

For each element in $H_\omega^{N_V}$={differential forms} mod IBPs, i.e., **{integrands}**, we define a prefactor C_ε:

$$H_\omega^{N_V} = \left\{ \Psi_{\mu_0\ldots\mu_{N_D}}[Q] = C_\varepsilon(\{\mu\})\, C_{\text{Baikov}}\, U(z)\, \hat{\Phi}_{\mu_0\ldots\mu_{N_D}}[Q]\, \eta \right\}$$ modulo IBPs

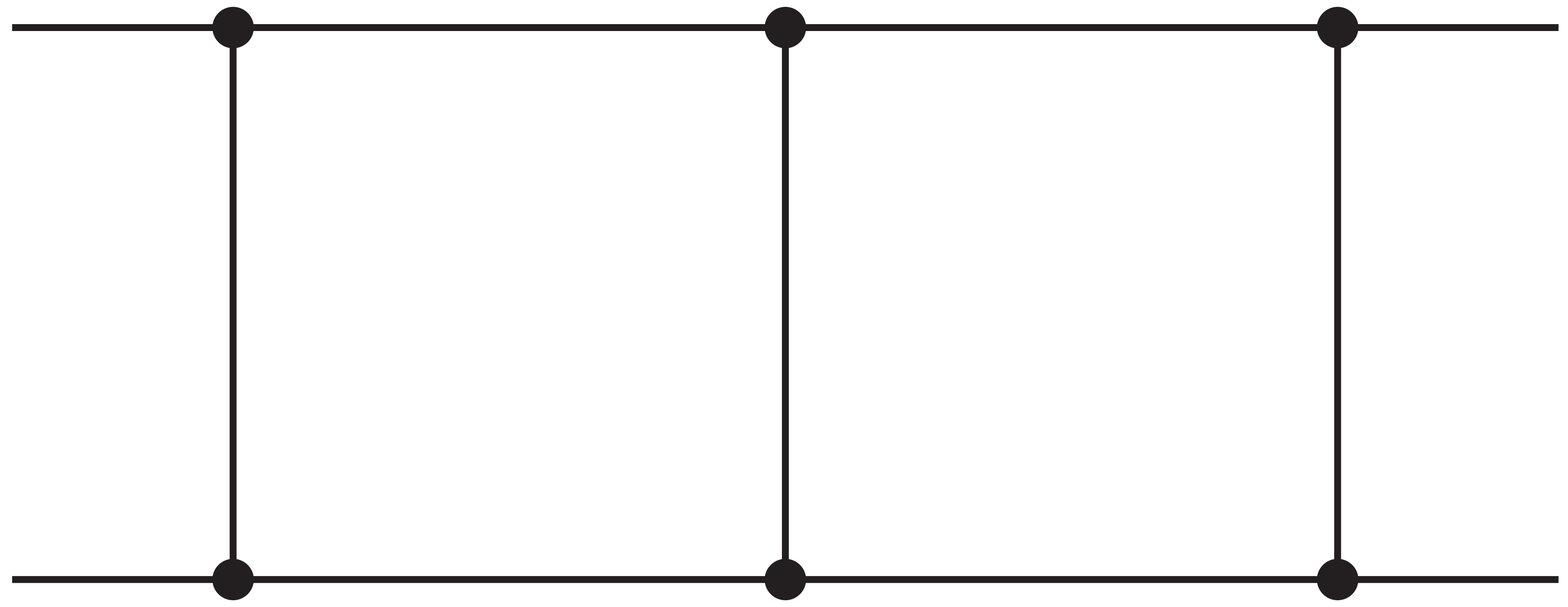
vector space
w./ fin. dim.

$$C_\varepsilon = C_{\text{abs}} \cdot \underbrace{\varepsilon^{-|\mu|}}_{C_{\text{clutch}}} \times \underbrace{\prod_{i\in I_{\text{odd}}} \left(-\frac{1}{2} + \frac{1}{2} b_i \varepsilon\right)_{\mu_i} \prod_{i\in I_{\text{even}}} \left(\frac{1}{2} b_i \varepsilon\right)_{\mu_i}}_{C_{\text{rel}}}$$

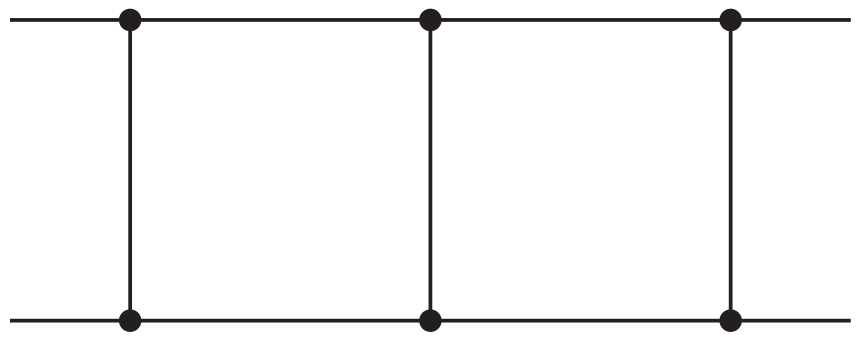
$$(a)_n = \frac{\Gamma(a+1)}{\Gamma(a+1-n)}$$

This pre-factor is entirely determined by the integrand, and it simplifies the ε-complexity.

Appetizers Example



Appetizers Example: Twist



$$I_{111111100} = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} \frac{\mathrm{d}z_1}{2\pi i} \overbrace{z_1^{-2\varepsilon} \left(z_1 - 1\right)^{-\varepsilon} \left(z_1 - x - 1\right)^{\varepsilon}}^{u(z_1)}$$

- Setting $C_{\text{abs}} = \varepsilon^4 x^2$ makes $C_{\text{abs}} \cdot C_{\text{Baikov}}$ pure.
- The minimal twist in the projective space reads: $U(z_0, z_1) = z_0^{2\varepsilon} \, z_1^{-2\varepsilon} \, (z_1 - z_0)^{-\varepsilon} \, \left[z_1 - (1+x) z_0\right]^{\varepsilon}$.
- All four polynomials are even $\Longrightarrow$ four possible (localisations) to take non-zero resides.

$$[z_0 : z_1] \in \left\{ [0:1], \quad [1:0], \quad [1:1], \quad [1:1+x] \right\}$$

Appetizers Example: Howto

$$H^1_\omega \ni \Psi_{\mu_0 \ldots \mu_3}[Q] = C_\varepsilon(\{\mu\})\, U(z)\, \hat{\Phi}_{\mu_0 \ldots \mu_3}[Q]\; \eta; \qquad \eta = z_0 \mathrm{d}z_1 - z_1 \mathrm{d}z_0\,.$$

Appetizers Example: Howto

$$H^1_\omega \ni \Psi_{\mu_0\ldots\mu_3}[Q] = C_\varepsilon(\{\mu\})\, U(z)\, \hat{\Phi}_{\mu_0\ldots\mu_3}[Q]\ \eta; \qquad \eta = z_0 \mathrm{d}z_1 - z_1 \mathrm{d}z_0\,.$$

$$U(z_0, z_1) = z_0^{2\varepsilon}\, z_1^{-2\varepsilon}\, (z_1 - z_0)^{-\varepsilon}\, \left[z_1 - (1+x) z_0\right]^{\varepsilon} \Longrightarrow [z_0 : z_1] \in \left\{ [0:1], \quad [1:0], \quad [1:1], \quad [1:1+x] \right\}$$

- ▸ $\deg U = 0, \deg \eta = 2 \Longrightarrow \deg \hat{\Phi} = -2.$
- ▸ Ψ should localise on those 4 points after taking one residue $\Longrightarrow$ 4 candidates.

Appetizers Example: Howto

$$H^1_\omega \ni \Psi_{\mu_0\ldots\mu_3}[Q] = C_\varepsilon(\{\mu\})\, U(z)\, \hat{\Phi}_{\mu_0\ldots\mu_3}[Q]\ \eta; \qquad \eta = z_0 \mathrm{d}z_1 - z_1 \mathrm{d}z_0\,.$$

$$U(z_0, z_1) = z_0^{2\varepsilon} z_1^{-2\varepsilon} (z_1 - z_0)^{-\varepsilon} \left[z_1 - (1+x) z_0\right]^{\varepsilon} \Longrightarrow [z_0 : z_1] \in \left\{ [0:1], \quad [1:0], \quad [1:1], \quad [1:1+x] \right\}$$

- ▸ $\deg U = 0, \deg \eta = 2 \Longrightarrow \deg \hat{\Phi} = -2.$
- ▸ Ψ should localise on those 4 points after taking one residue $\Longrightarrow$ 4 candidates.

$$\Psi_{1100}[1] = -4\varepsilon^4 x^2 C_{\text{Baikov}} U \frac{\eta}{z_0 z_1}, \qquad \Psi_{1010}[1] = -2\varepsilon^4 x^2 C_{\text{Baikov}} U \frac{\eta}{z_0 \left(z_1 - z_0\right)},$$

$$\Psi_{1001}[1] = 2\varepsilon^4 x^2 C_{\text{Baikov}} U \frac{\eta}{z_0 \left[z_1 - (1+x) z_0\right]}, \qquad \Psi_{0110}[1] = 2\varepsilon^4 x^2 C_{\text{Baikov}} U \frac{\eta}{z_1 \left(z_1 - z_0\right)}\,.$$

Appetizers Example: Howto

$$H^1_\omega \ni \Psi_{\mu_0\ldots\mu_3}[Q] = C_\varepsilon(\{\mu\})\,U(z)\,\hat{\Phi}_{\mu_0\ldots\mu_3}[Q]\,\eta; \qquad \eta = z_0\mathrm{d}z_1 - z_1\mathrm{d}z_0\,.$$

$$U(z_0,z_1) = z_0^{2\varepsilon}\,z_1^{-2\varepsilon}\,(z_1-z_0)^{-\varepsilon}\,\left[z_1-(1+x)z_0\right]^{\varepsilon} \Longrightarrow [z_0:z_1] \in \left\{[0:1],\quad [1:0],\quad [1:1],\quad [1:1+x]\right\}$$

- $\deg U = 0, \deg \eta = 2 \Longrightarrow \deg \hat{\Phi} = -2.$
- Ψ should localise on those 4 points after taking one residue $\Longrightarrow$ 4 candidates.

$$\Psi_{1100}[1] = -4\varepsilon^4x^2 C_{\mathrm{Baikov}}U\frac{\eta}{z_0z_1}, \qquad \Psi_{1010}[1] = -2\varepsilon^4x^2 C_{\mathrm{Baikov}}U\frac{\eta}{z_0\left(z_1-z_0\right)},$$

$$\Psi_{1001}[1] = 2\varepsilon^4x^2 C_{\mathrm{Baikov}}U\frac{\eta}{z_0\left[z_1-(1+x)z_0\right]}, \qquad \Psi_{0110}[1] = 2\varepsilon^4x^2 C_{\mathrm{Baikov}}U\frac{\eta}{z_1\left(z_1-z_0\right)}\,.$$

$$2\varepsilon^4x^2 = \underbrace{\varepsilon^4x^2}_{C_{\mathrm{abs}}} \cdot \underbrace{\varepsilon^{-2}}_{C_{\mathrm{clutch}}} \cdot \underbrace{(2\varepsilon\cdot\varepsilon)}_{C_{\mathrm{rel}}}$$

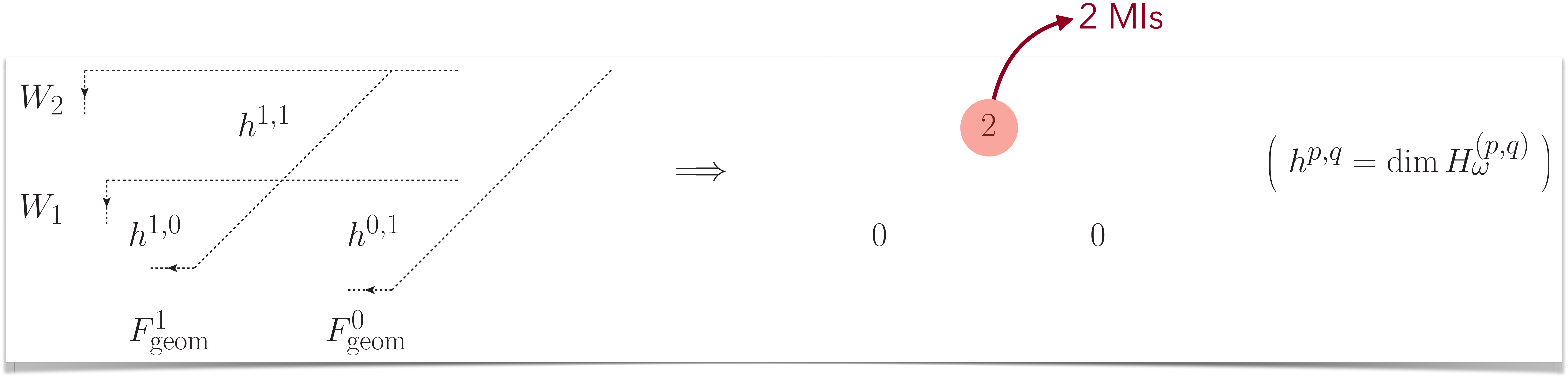
All have $(p,q) = (1,1)$.

Appetizers Example: MIs

There are IBP relations between these four candidates:

$$\{\,\Psi_{1100}\,[1], \Psi_{1010}\,[1], \Psi_{1001}\,[1], \Psi_{0110}\,[1]\,\} \Longrightarrow \{\,\Psi_{0110}\,[1], \Psi_{1010}\,[1]\,\}$$

$$\begin{aligned} \Psi_{0110}\,[1] \quad &\Longleftrightarrow \quad 2\varepsilon^4 x^2 I_{111111100} = K_1, \\ \Psi_{1010}\,[1] \quad &\Longleftrightarrow \quad -2\varepsilon^4 x^2 I_{1111111(-1)0} = K_2\,. \end{aligned} \quad \Longrightarrow \quad \dim V^1 = \dim H^1_\omega = 2.$$



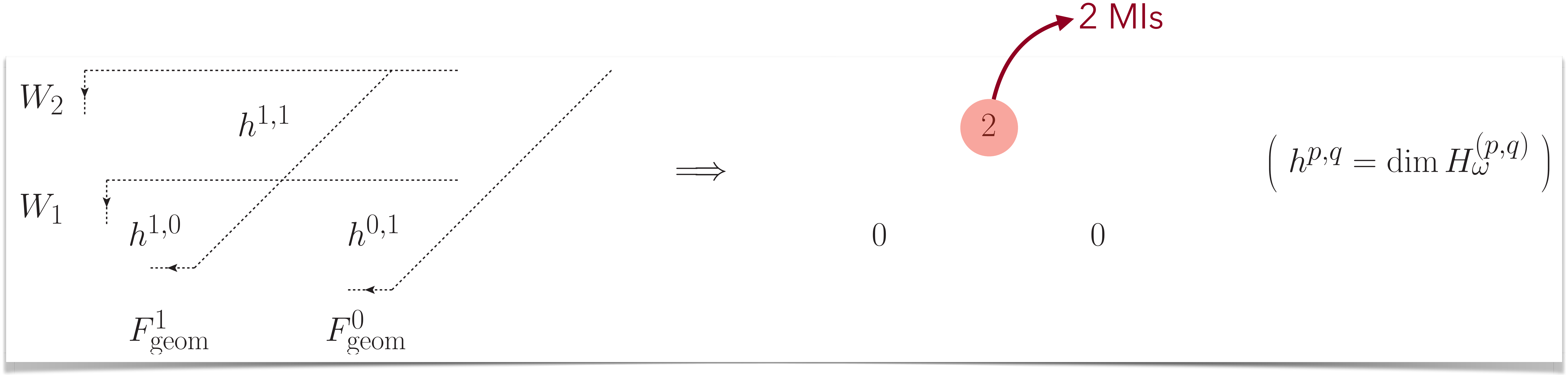
$$\left(\, h^{p,q} = \dim H_\omega^{(p,q)} \,\right)$$

Appetizers Example: MIs

There are IBP relations between these four candidates:

$$\left\{ \Psi_{1100}[1], \Psi_{1010}[1], \Psi_{1001}[1], \Psi_{0110}[1] \right\} \Longrightarrow \left\{ \Psi_{0110}[1], \Psi_{1010}[1] \right\}$$

$$\begin{aligned} \Psi_{0110}[1] \quad &\Longleftrightarrow \quad 2\varepsilon^4 x^2 I_{111111100} = K_1, \\ \Psi_{1010}[1] \quad &\Longleftrightarrow \quad -2\varepsilon^4 x^2 I_{1111111(-1)0} = K_2\,. \end{aligned} \quad \Longrightarrow \quad \dim V^1 = \dim H^1_\omega = 2.$$



2 MIs

This derived basis is already the ε-factorised: $\mathrm{d}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \varepsilon\,\mathbf{A}_{\mathrm{MC}}(x)\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$

Step 2 of the Algorithm

We pick elements in $H_{\omega}^{N_V}$ by the filtration criteria, translate back to MIs, defining a new basis $\vec{J}$:

$$\mathrm{d}\vec{J} = \left[\frac{1}{\varepsilon^{N_V}}\mathbf{B}^{(-N_V)}(x) + \frac{1}{\varepsilon^{N_V-1}}\mathbf{B}^{(-N_V+1)}(x) + \cdots + \mathbf{B}^{(0)}(x) + \varepsilon\,\mathbf{B}^{(1)}(x)\right]\vec{J}$$

$$\vec{M} = \mathbf{R}_1\,\vec{J}$$

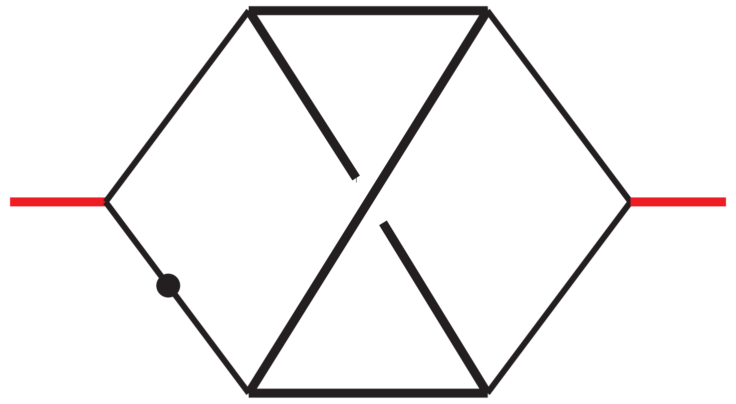
$\mathbf{B}^{(-N_V)}(x), \cdots, \mathbf{B}^{(0)}(x)$ are in a good block lower-triangular form!

$$\mathbf{R}_2^{(-N_V)}\mathbf{R}_2^{(-N_V+1)}\cdots\mathbf{R}_2^{(0)}\,\vec{K} = \vec{J}$$

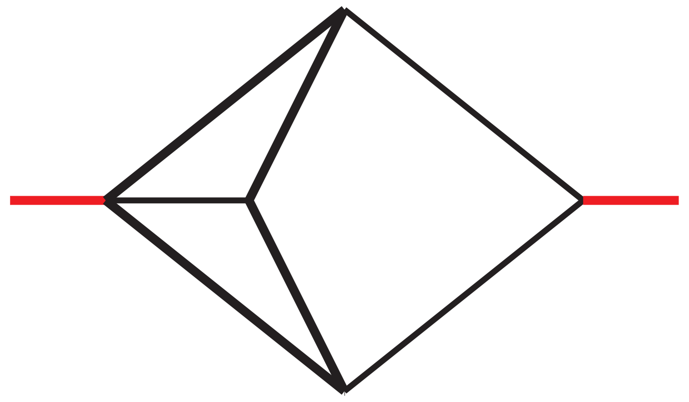
- Rotate away $\mathbf{B}^{(i)}$ step by step. It is systematic (the existence of ε-factorisation);
- $\mathbf{R}_2^{(i)}$ is determined by (*simpler*) PDEs. In particular, $\mathbf{R}_2^{(-N_V)}$ relates to periods of geometry.

An Example with All Sectors

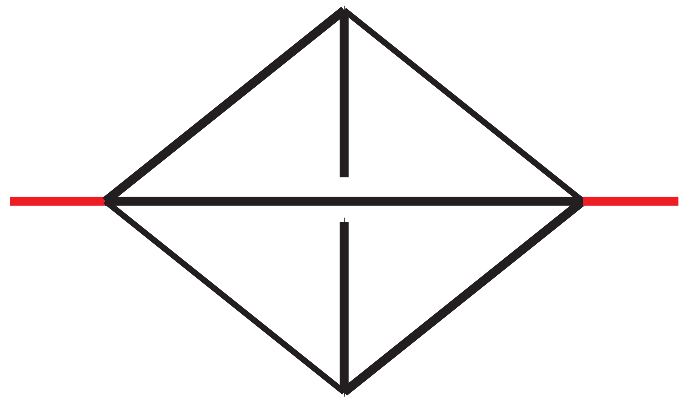
$N_{\rm id} = 255$



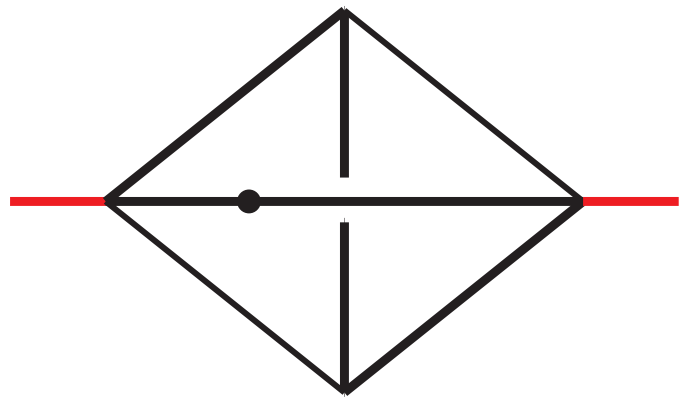
$N_{\rm id} = 255$



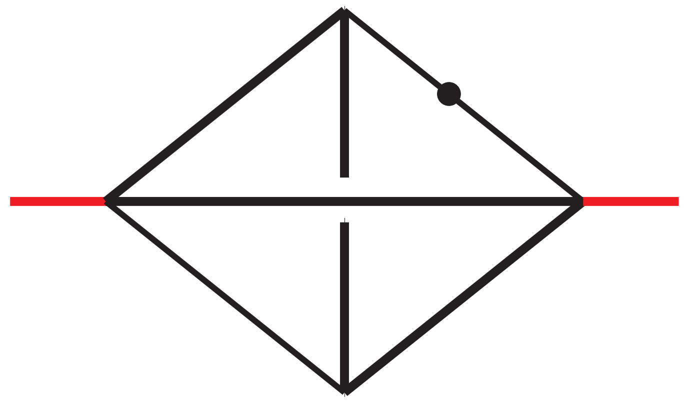
$N_{\rm id} = 254$



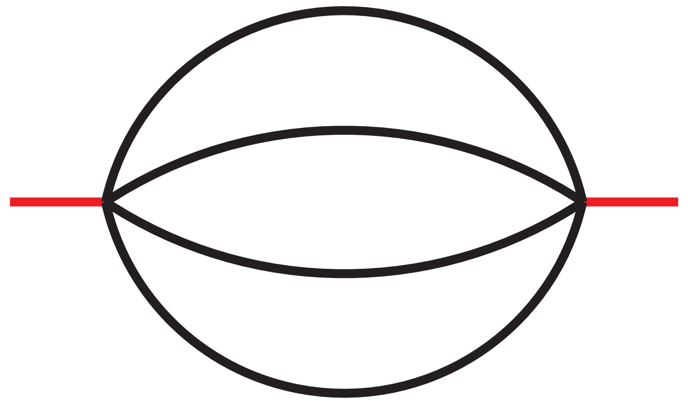
$N_{\rm id} = 159$



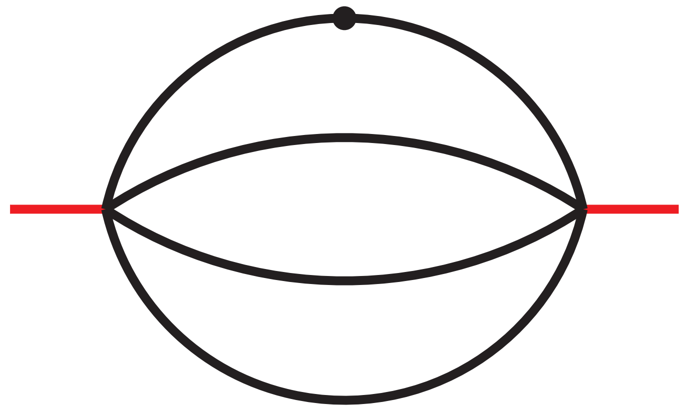
$N_{\rm id} = 159$



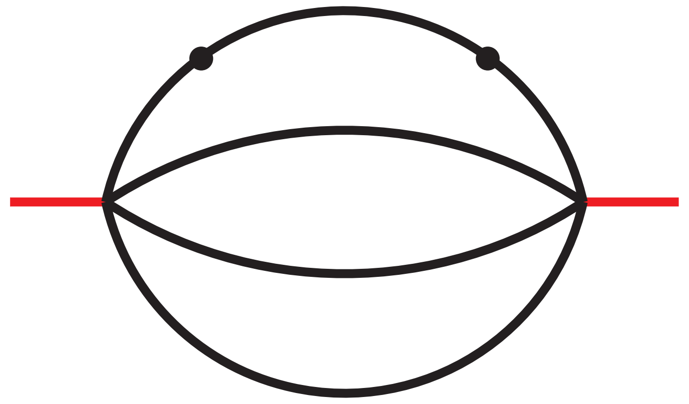
$N_{\rm id} = 159$



$N_{\rm id} = 150$



$N_{\rm id} = 150$



$N_{\rm id} = 150$

They contribute (with four massive cuts) to $\Gamma_{H\to b\bar{b}}$, see the talk by Yefan Wang.

An Example: Layered Structure

$$I_i^{(255)} = C_{\text{Baikov}}^{(255)} \cdot \int_{\mathscr{C}_{255}} u(z_1)\, \hat{\phi}_i^{(255)}(z_1)\,; \qquad u(z_1) = \Big[z_1\left(z_1+t\right)\Big(z_1\left(z_1+t\right)-4t\Big)\Big]^{-\frac{1}{2}-\varepsilon}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$C_{\text{Baikov}}^{(255)}$ defines required pre-factor $C_{\text{abs}}^{(255)}$; $U^{(255)}(z_0, z_1) = P_0^{2\varepsilon} \cdot P_1^{-1/2-\varepsilon} \cdot P_2^{-1/2-\varepsilon} \cdot P_3^{-1/2-\varepsilon}\,, \;\; (P_0 = z_0)\,.$

An Example: Layered Structure

$$I_i^{(255)} = C_{\text{Baikov}}^{(255)} \cdot \int_{\mathscr{C}_{255}} u(z_1)\, \hat{\phi}_i^{(255)}(z_1)\,; \qquad u(z_1) = \Big[z_1 \left(z_1 + t \right) \Big(z_1 \left(z_1 + t \right) - 4t \Big) \Big]^{-\frac{1}{2}-\varepsilon}$$

↓ ↓

$C_{\text{Baikov}}^{(255)}$ defines required pre-factor $C_{\text{abs}}^{(255)}$; $U^{(255)}(z_0, z_1) = P_0^{2\varepsilon} \cdot P_1^{-1/2-\varepsilon} \cdot P_2^{-1/2-\varepsilon} \cdot P_3^{-1/2-\varepsilon}\,, \;\; (P_0 = z_0)\,.$

- ▸ $w = N_V + r = 1 + 1$: only at $P_0 = z_0 = 0 \;\rightarrow\; \hat{\phi}_3^{(255)} = C_{\text{abs}}^{(255)} \cdot 2 \cdot \dfrac{z_1}{P_0}$;
- ▸ $w = N_V + r = 1 + 0$: two choices. $\hat{\phi}_1^{(255)} = C_{\text{abs}}^{(255)} \cdot 1,\; \hat{\phi}_2^{(255)} = C_{\text{abs}}^{(255)} \cdot \left(-\dfrac{1}{2} - \varepsilon \right) \cdot \varepsilon^{-1} \cdot \dfrac{z_0 z_1}{P_3}.$

An Example: Layered Structure in Top Sector

- $w = N_V + r = 1 + 1$: only at $P_0 = z_0 = 0 \ \rightarrow \ \hat{\phi}_3^{(255)} = C_{\text{abs}}^{(255)} \cdot 2 \cdot \frac{z_1}{P_0}$;
- $w = N_V + r = 1 + 0$: two choices: $\hat{\phi}_1^{(255)} = C_{\text{abs}}^{(255)} \cdot 1,\ \hat{\phi}_2^{(255)} = C_{\text{abs}}^{(255)} \cdot \left(-\frac{1}{2} - \varepsilon\right) \cdot \varepsilon^{-1} \cdot \frac{z_0 z_1}{P_3}$.

  $\hat{\phi}_1^{(255)}$ and $\hat{\phi}_2^{(255)}$ are specified by the other layer (Hodge filtration).

  - ☑ $\hat{\phi}_1^{(255)}$: holomorphic, $p = N_V - o + r = 1 - 0 + 0 = 1$;
  - ☑ $\hat{\phi}_2^{(255)}$: $o = \lfloor 3/2 \rfloor = 1$, no residue, $p = 1 - 1 + 0 = 0$.
- We can't proceed, and have exhausted all possibilities.

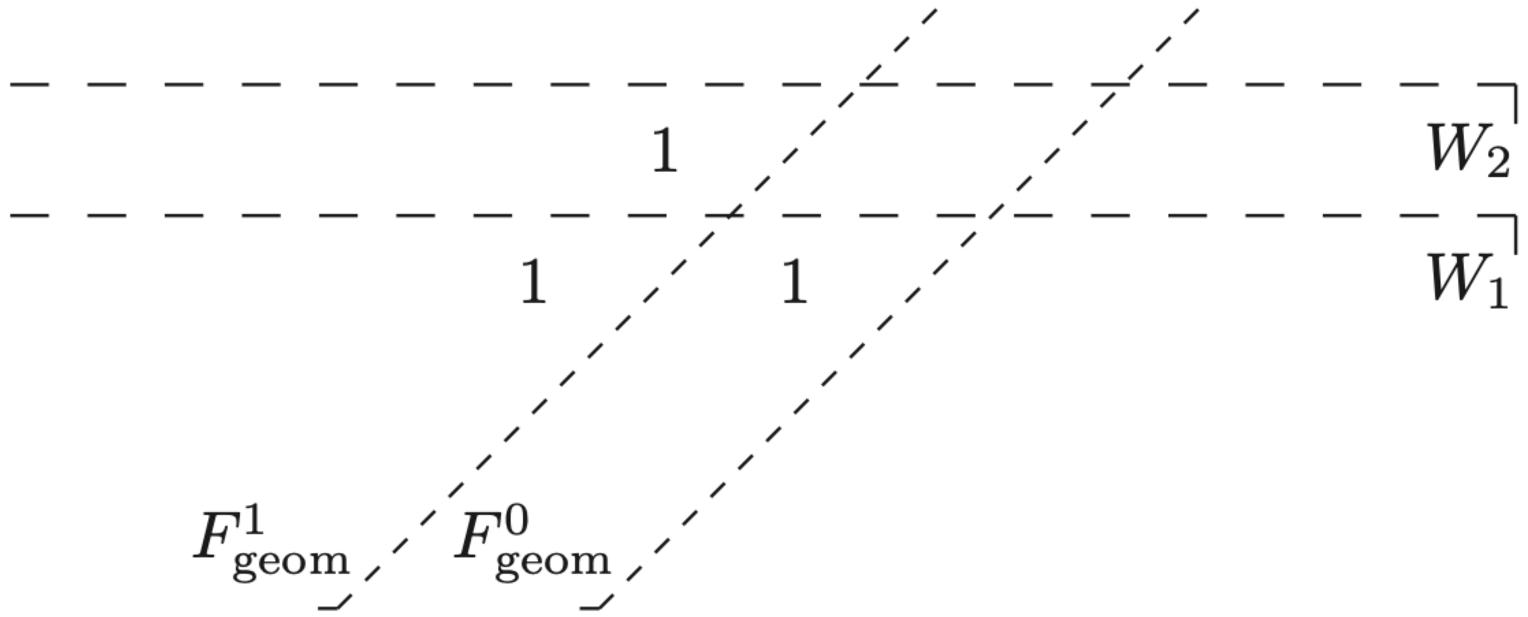
An Example: Layered Structure in Top Sector

- $w = N_V + r = 1 + 1$: only at $P_0 = z_0 = 0$ $\rightarrow$ $\hat{\phi}_3^{(255)} = C_{\text{abs}}^{(255)} \cdot 2 \cdot \dfrac{z_1}{P_0}$;
- $w = N_V + r = 1 + 0$: two choices: $\hat{\phi}_1^{(255)} = C_{\text{abs}}^{(255)} \cdot 1$, $\hat{\phi}_2^{(255)} = C_{\text{abs}}^{(255)} \cdot \left(-\dfrac{1}{2} - \varepsilon\right) \cdot \varepsilon^{-1} \cdot \dfrac{z_0 z_1}{P_3}$.

$\hat{\phi}_1^{(255)}$ and $\hat{\phi}_2^{(255)}$ are specified by the other layer (Hodge filtration).

☑ $\hat{\phi}_1^{(255)}$: holomorphic, $p = N_V - o + r = 1 - 0 + 0 = 1$;

☑ $\hat{\phi}_2^{(255)}$: $o = \lfloor 3/2 \rfloor = 1$, no residue, $p = 1 - 1 + 0 = 0$.



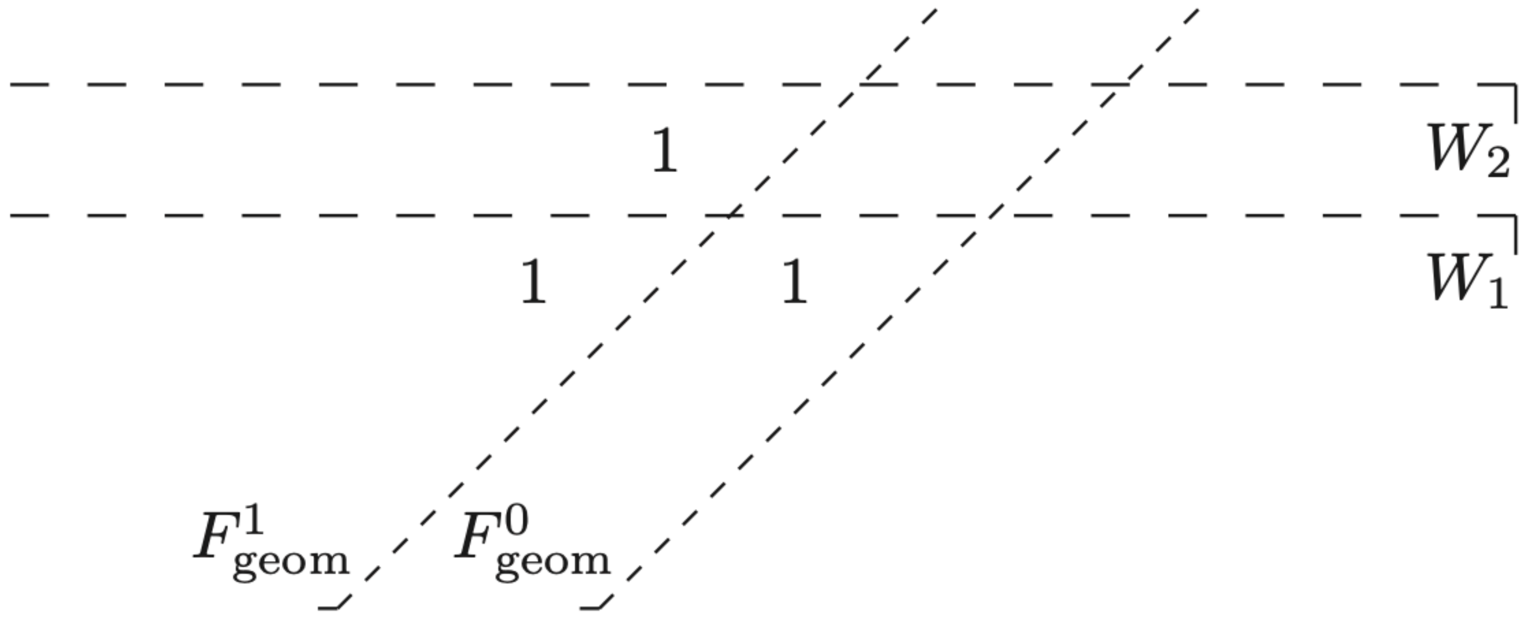
- We can't proceed, and have exhausted all possibilities.

An Example: Layered Structure in Top Sector

- $w = N_V + r = 1 + 1$: only at $P_0 = z_0 = 0$ → $\hat{\phi}_3^{(255)} = C_{\text{abs}}^{(255)} \cdot 2 \cdot \frac{z_1}{P_0}$;
- $w = N_V + r = 1 + 0$: two choices: $\hat{\phi}_1^{(255)} = C_{\text{abs}}^{(255)} \cdot 1$, $\hat{\phi}_2^{(255)} = C_{\text{abs}}^{(255)} \cdot \left(-\frac{1}{2} - \varepsilon\right) \cdot \varepsilon^{-1} \cdot \frac{z_0 z_1}{P_3}$.

$\hat{\phi}_1^{(255)}$ and $\hat{\phi}_2^{(255)}$ are specified by the other layer (Hodge filtration).

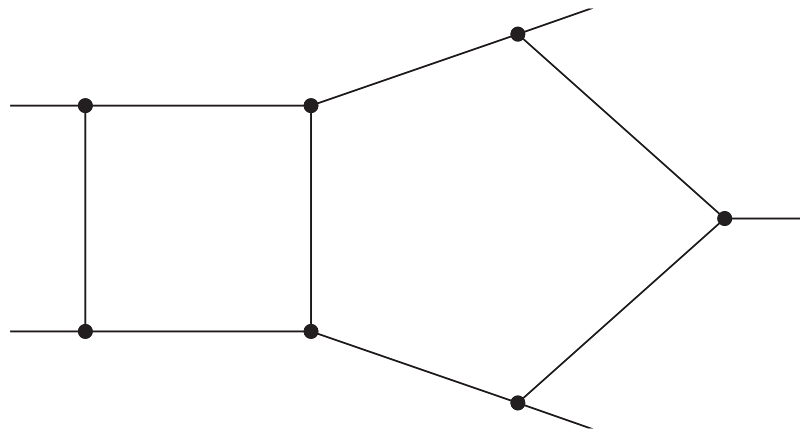
  - ☑ $\hat{\phi}_1^{(255)}$: holomorphic, $p = N_V - o + r = 1 - 0 + 0 = 1$;
  - ☑ $\hat{\phi}_2^{(255)}$: $o = \lfloor 3/2 \rfloor = 1$, no residue, $p = 1 - 1 + 0 = 0$.

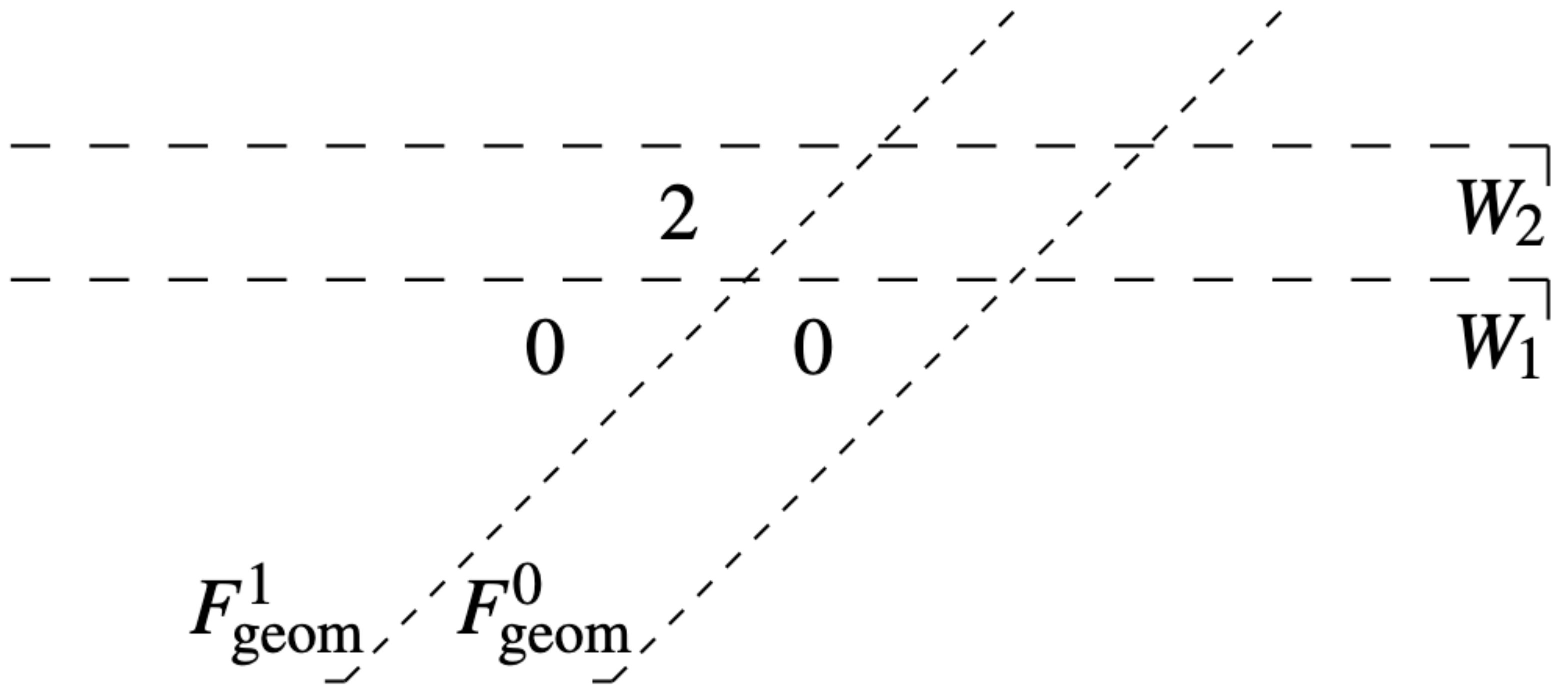


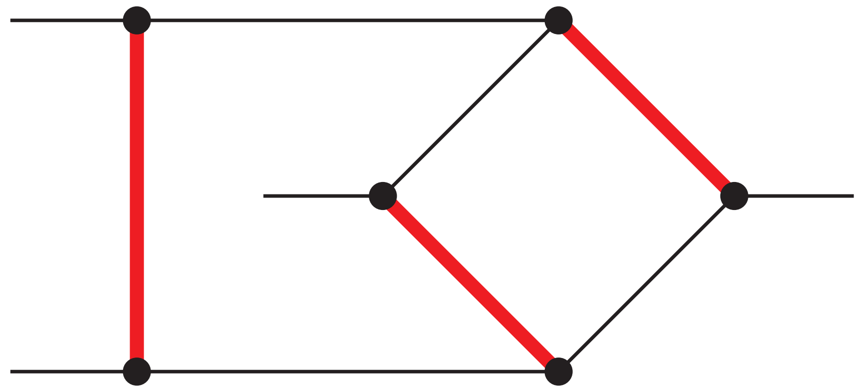
- We can't proceed, and have exhausted all possibilities.

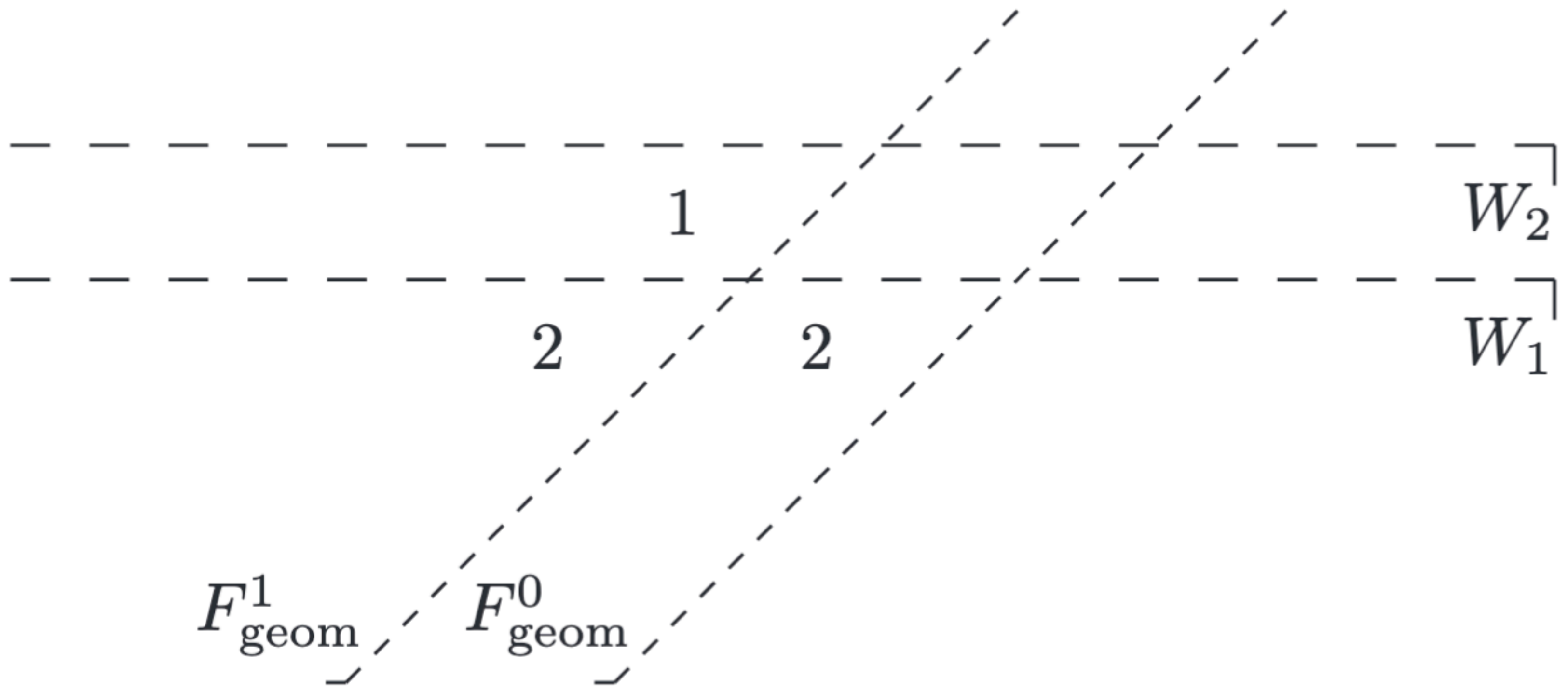
Hodge Diamond ⟶ Feynamn Triangle

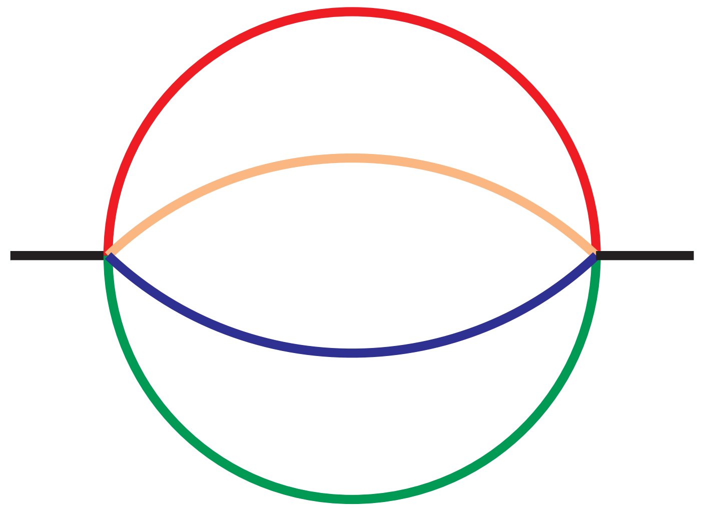
“Feynamn Triangle”: Some Other Exampes

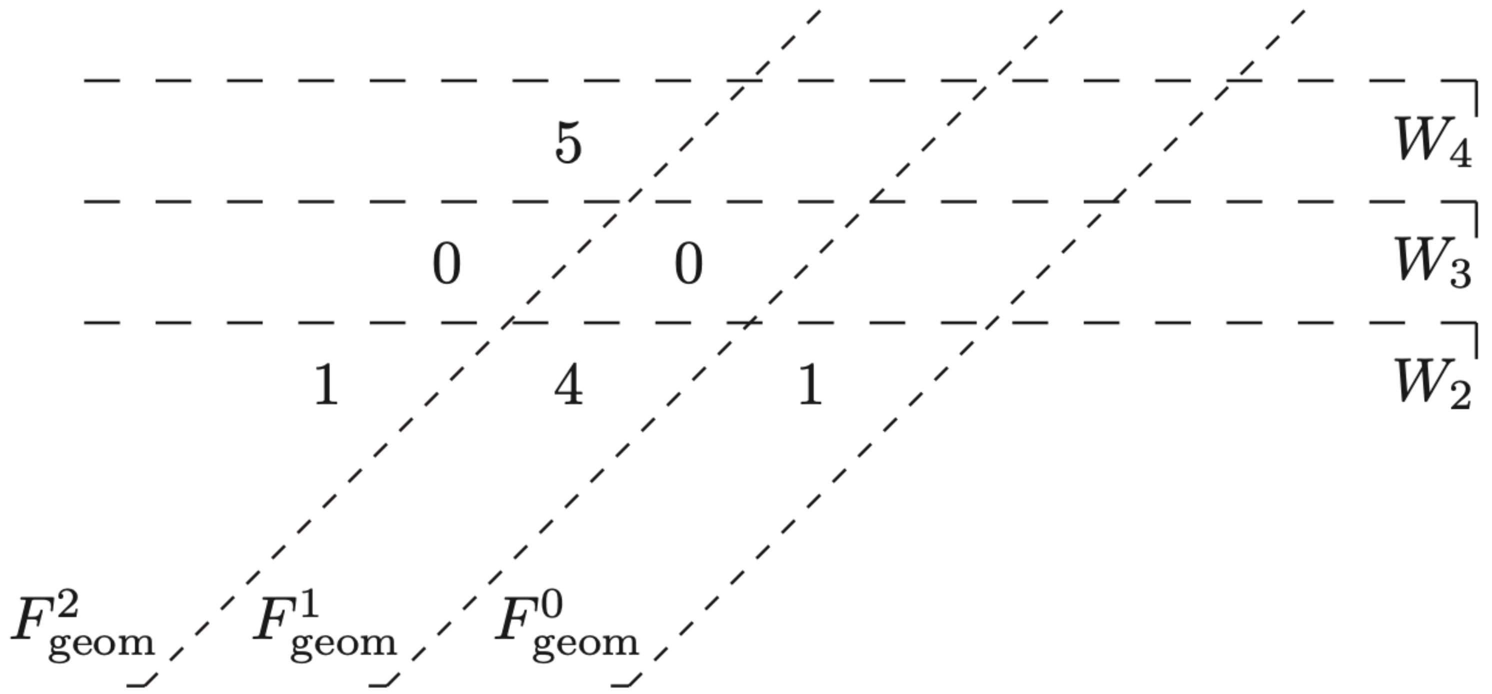












An Example: Layered Structure → Canolicalizing

- Back to integrals: $J_1 = t(1-2\varepsilon)\varepsilon^5 \cdot F_{111111110}$, $J_2 = t(1-2\varepsilon)\varepsilon^4 \cdot F_{111111120}$;
- Its DE takes a nice form, which can be rotated to be canonical easily.

An Example: Layered Structure → Canolicalizing

- Back to integrals: $J_1 = t(1-2\varepsilon)\varepsilon^5 \cdot F_{111111110}$, $J_2 = t(1-2\varepsilon)\varepsilon^4 \cdot F_{111111120}$;
- Its DE takes a nice form, which can be rotated to be canonical easily.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} J_1 \\ J_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{t} & 0 \\ \frac{1}{\varepsilon}\,\frac{1}{t(t+16)} & -\frac{1}{t+16} \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{6}{t(t+16)} & 0 \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$$

$$\mathbf{R}^{(-1)}_{255} \cdot \mathbf{R}^{(0)}_{255} \cdot \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}$$

An Example: Layered Structure → Canolicalizing

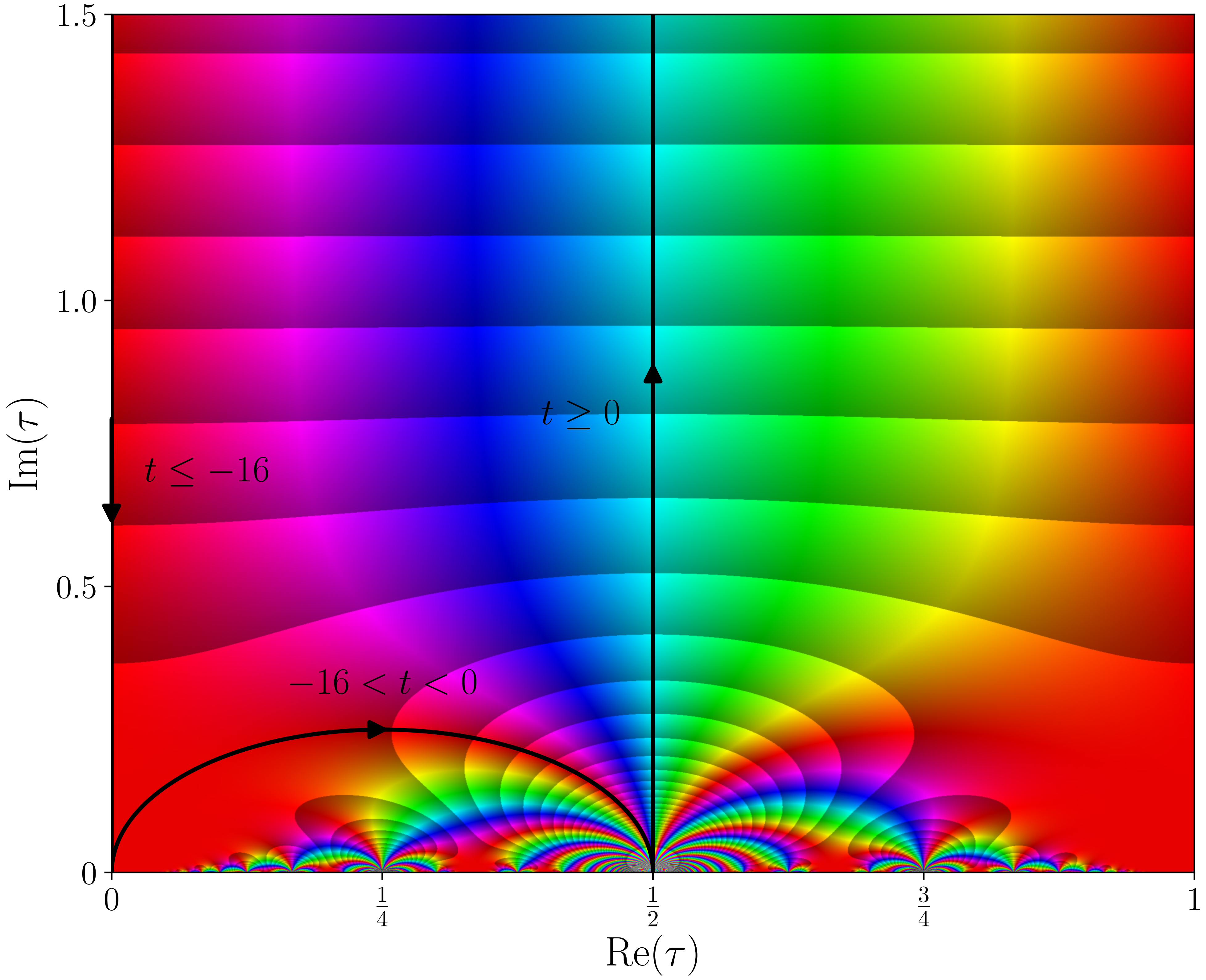
- Back to integrals: $J_1 = t(1-2\varepsilon)\varepsilon^5 \cdot F_{111111110}$, $J_2 = t(1-2\varepsilon)\varepsilon^4 \cdot F_{111111120}$;
- Its DE takes a nice form, which can be rotated to be canonical easily.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} J_1 \\ J_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{t} & 0 \\ \frac{1}{\varepsilon}\frac{1}{t(t+16)} & -\frac{1}{t+16} \end{bmatrix}\begin{bmatrix} J_1 \\ J_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{6}{t(t+16)} & 0 \end{bmatrix}\begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$$

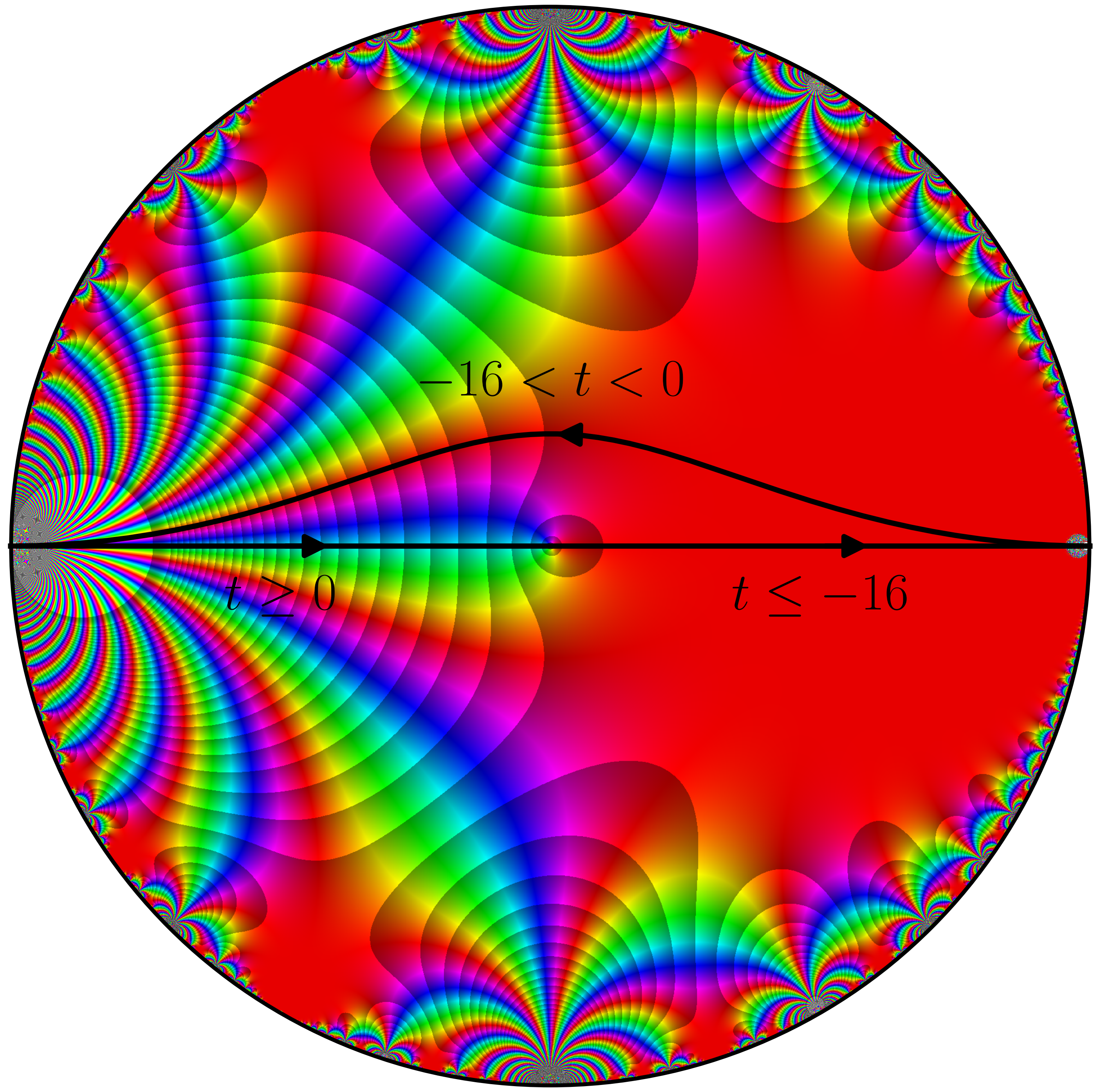
$$\mathbf{R}_{255}^{(-1)} \cdot \mathbf{R}_{255}^{(0)} \cdot \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}$$

- The integrals are elliptic. However, we do not need this knowledge in advance.
  But this knowledge can certainly help to identify the transcendental functions.

Some Extra Information



(a) τ plane



(b) q-disk

$$-\frac{1}{t} = y = \frac{\eta(\tau)^8\eta(4\tau)^{16}}{\eta(2\tau)^{24}} = \frac{1}{16}\lambda(2\tau)$$

Gemini 3.1 Pro generated the Matplotlib code!

Summary

- FIs not only have block structures, but also have **multi-layered structures**, specified by filtrations.
- After filtrations, FIs enjoy a Hodge-like triangle: “**Feynman triangle**”.
- The filtered basis does not have spurious polynomials during IBP, and makes canonicalization always achievable.

The algorithm is universal and cracks the complexity to the minimum.

Thank you for listening.