Factorization scale and scheme dependence in higher-order ISR corrections

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Motivation

- Results of perturbative calculations often depend on factorization and renormalization scale and scheme choices as in QCD as well as in QED (and in EW)
- Existing analytic results in QED provide a unique possibility to check analytically and numerically those dependencies
- QED Drell-Yan-like process $e^+e^- \to \gamma^*, Z \to \mu^+\mu^-$ is the best case for the tests
- Future experiments at e^+e^- colliders demand high precision of theoretical predictions
- Recent progress in QED PDFs (structure functions)
- The talk is based on arXiv:2511.00437 and arXiv:2606.11742 + preliminary results

Drell-Yan process in QED: e^+e^- annihilation

The factorization structure is exactly the same as in QCD:

$$\frac{d\sigma_{e^+e^-\to\gamma^*,Z^*\to\mu^+\mu^-}}{ds'} = \sum_{a,b=e^-,\gamma,e^+} \mathcal{D}_{ae^-} \otimes \mathcal{D}_{be^+} \otimes \hat{\sigma}_{ab\to\gamma^*,Z^*\to\mu^+\mu^-}$$

Complete analytic results as function of the final state invariant mass are known in $\mathcal{O}(\alpha)$, $\mathcal{O}(\alpha^2)$, and in the leading and next-to-leading logarithmic approximations in several higher orders (terms $\sim m_e^2/s$ are neglected). Note, that only initial state radiation is discussed here.

See [J. Ablinger, J. Blümlein, A. De Freitas and K. Schönwald, "Subleading Logarithmic QED Initial State Corrections to $e^+e^- \to \gamma^*/Z^{0*}$ to $O(\alpha^6 L^5)$," NPB 955 (2020) 115045] and [A.A., U.Voznaya, PRD 2024]

Drell-Yan process in QED: ingredients

In the collinear approximation

$$\frac{d\sigma^{\rm NLO}_{\bar{e}e}(s')}{ds'} = \sum_{i,j=e,\bar{e},\gamma} \mathcal{D}_{ie}\left(z_1, \frac{\mu_R^2}{\mu_F^2}\right) \otimes \mathcal{D}_{j\bar{e}}\left(z_2, \frac{\mu_R^2}{\mu_F^2}\right)$$

$$\otimes \left(\sigma_{ij}^{(0)}(sz_1z_2) + \overline{\sigma}_{ij}^{(1)}(sz_1z_2z_3)\right)\delta(s'-sz_1z_2z_3) + \mathcal{O}\left(\frac{m_e^2}{s}\right),$$

- $\sigma^{(0)}$ is the Born cross-section, $\overline{\sigma}_{ij}^{(1)}$ is 1-loop partonic cross section in the MSbar scheme*
- $\mathcal{D}_{ae}$ are electron PDFs, they are solutions of DGLAP evolution eqs.
- $\alpha \equiv \alpha_{\rm QED}(\mu_R^2)$, $\quad s = (p_{e^+} + p_{e^-})^2$, $\quad s' = (p_{\mu^+} + p_{\mu^-})^2 \equiv zs$

Optimization of factorization scale choice was discussed in [A.A., U.Voznaya, A.Sadouski, arXiv:2511.00437; to appear in Chin. Phys. C]

Below we address also the scheme dependence

QED DGLAP evolution equations

$$\mathcal{D}_{ba}\left(x,\frac{\mu_R^2}{\mu_F^2}\right) = \delta_{ab}\delta(1-x)+\frac{\alpha}{2\pi}d_{ba}^{(1)}(x,\mu_R^2/m_e^2)+\dots$$
$$+\sum_{c=e,\gamma,\bar{e}}\int_{\mu_R^2}^{\mu_F^2}\frac{dt}{t}\int_x^1\frac{dy}{y}P_{bc}(y,t)\mathcal{D}_{ca}\left(\frac{x}{y},\frac{\mu_R^2}{t}\right)$$

b, c are massless partons ($\sim e^{\pm}, \gamma$)

μ_F is a factorization (energy) scale

μ_R is a renormalization (energy) scale

$\mathcal{D}_{ba}$ is a parton density function (PDF)

$d_{ba}^{(m)}$ are initial conditions

P_{bc} is a splitting function or kernel of the DGLAP equation

Analytic QED PDFs in the leading and next-to-leading log approximations are known [A.A., U.Voznaya, JPG '2023] see also [S.Frixione et al.]

Drell-Yan process in QED: perturbative expansion

$$\frac{d\sigma_{e^+e^-\to\gamma^*,Z\to\mu^+\mu^-}}{ds'} = \sum_{a,b=e^-,\gamma,e^+} \mathcal{D}_{ae^-} \otimes \mathcal{D}_{be^+} \otimes \hat{\sigma}_{ab\to\gamma^*,Z\to\mu^+\mu^-}$$

- Large logs $L \equiv \ln(\mu_F^2/\mu_R^2)$ come from running α and anomalous dimensions

$$\frac{d\sigma_{\bar{e}e}^{\text{NNLO}}(s')}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left\{\delta(1-z) + \sum_{\substack{k=1 \\ k\geq l\geq k-2}}^{\infty} \left(\frac{\alpha}{2\pi}\right)^k L^l c_{kl}(z)\right\} + \mathcal{O}\left(\frac{m_e^2}{s}\right)$$

where $c_{kl}(z)$ are functions of $z = s'/s$ and they contain

$$\delta(1-z), \quad \zeta(m), \quad N_f, \quad \left[\frac{\ln^m(1-z)}{1-z}\right]_+, \quad \frac{\ln^m(z)}{z}, \quad \textit{HPOLs, etc.}$$

ISR corrections to $e^+e^- \to Z(\gamma^*)$ ($\sqrt{s} = M_Z$)

LO $\mathcal{O}(\alpha^n L^n)$ and NLO $\mathcal{O}(\alpha^n L^{n-1})$ ISR corr. [%] at Z-peak, $z_{\min} = 0.1$

Type / n	1	2	3	4	5
LO γ	−32.7365	4.8843	−0.3776	0.0034	0.0032
NLO γ	2.0017	−0.5952	0.0710	−0.0019	
LO pair	—	−0.3057	0.0875	0.0016	−0.0001
NLO pair	—	0.1585	−0.0460	0.0038	
Σ	−30.7348	4.1419	−0.2651	0.0069	0.0031

Experimental precision tag for uncertainty $\delta \sim 10^{-5} = 10^{-3}\,\%$

Even higher orders ($n > 5$) are relevant numerically $\Longrightarrow$ exponentiation (next talk by Uliana Voznaya)

Exponentiation of the leading logs is straightforward and known [Gribov-Lipatov, Kuraev-Fadin, ...]

NLO exponentiation in the MSbar scheme is ambiguous

$\mathcal{O}(\alpha)$ matching

The expansion of the master formula for ISR gives

$$d\sigma^{(1)}_{e\bar{e}\to\gamma^*} = \frac{\alpha}{2\pi}\left\{2LP^{(0)}_{ee} \otimes d\sigma^{(0)}_{e\bar{e}\to\gamma^*} + 2d^{(1)}_{ee} \otimes d\sigma^{(0)}_{e\bar{e}\to\gamma^*}\right\} + d\,\bar{\sigma}^{(1)}_{e\bar{e}\to\gamma^*}$$

We know the massive $d\sigma^{(1)}$ and massless $d\,\bar{\sigma}^{(1)}$ ($m_e \equiv 0$ with $\overline{\text{MS}}$ subtraction)

$$\frac{d\sigma^{(1)}_{e\bar{e}\to\gamma^*}}{d\sigma^{(0)}_{e\bar{e}\to\gamma^*}} = \frac{\alpha}{\pi}\left\{\left[\frac{1+z^2}{1-z}\left(\ln\frac{s}{m_e^2} - 1\right)\right]_+ + \delta(1-z)\left(2\zeta(2) - \frac{1}{2}\right)\right\}$$

$$\frac{d\bar{\sigma}^{(1)}_{e\bar{e}\to\gamma^*}}{d\sigma^{(0)}_{e\bar{e}\to\gamma^*}} = \frac{\alpha}{\pi}\left\{\left[\frac{1+z^2}{1-z}\left(\ln\frac{s}{\mu_F^2} + 2\ln(1-z)\right)\right]_+ + \delta(1-z)\left(2\zeta(2) - \frac{1}{2}\right)\right\}$$

$$d^{(1)}_{ee} = \left[\frac{1+z^2}{1-z}\left(\ln\frac{\mu_R^2}{m_e^2} - 1 - 2\ln(1-z)\right)\right]_+, \quad P^{(0)}_{ee}(z) = \left[\frac{1+z^2}{1-z}\right]_+, \quad L = \ln\frac{\mu_F^2}{\mu_R^2}$$

Scale and scheme dependence comes from here

Drell-Yan process in QED: the general structure

$$\frac{d\sigma_{e^+e^-\to\gamma^*,Z}}{ds'} = \sum_{a,b=e^-,\gamma,e^+} \mathcal{D}_{ae^-} \otimes \mathcal{D}_{be^+} \otimes \left(\hat{\sigma}^{(0)}_{ab\to\gamma^*,Z} + \hat{\bar{\sigma}}^{(1)}_{ab\to\gamma^*,Z} \right)$$

Schematically

$$\begin{array}{llll} c_{00}\alpha^0 \text{ (Born)} & & & \\ c_{11}\alpha^1 L^1 & c_{10}\alpha^1 L^0 & & \\ c_{22}\alpha^2 L^2 & c_{21}\alpha^2 L^1 & c_{20}\alpha^2 L^0 & \\ c_{33}\alpha^3 L^3 & c_{32}\alpha^3 L^2 & c_{31}\alpha^3 L^1 & c_{30}\alpha^3 L^0 \\ \dots & \dots & \dots & \dots \end{array}$$

where $\alpha = \alpha(\mu_R^2)$, $L = \ln(\mu_F^2/\mu_R^2)$, and coefficients $c_{kl} = c_{kl}(z)$

Columns correspond to LO, NLO, NNLO and so on

LO $\sim (\mathcal{D}\otimes\mathcal{D})^{\text{LO}} \otimes \sigma^{(0)}$

NLO $\sim (\mathcal{D}\otimes\mathcal{D})^{\text{LO}} \otimes \bar{\sigma}^{(1)} + (\mathcal{D}\otimes\mathcal{D})^{\text{NLO}} \otimes \sigma^{(0)}$

PROBLEM: in MSbar $(\mathcal{D}\otimes\mathcal{D})^{\text{NLO}} \otimes \bar{\sigma}^{(1)}$ is very singular at $x \to 1$

Factorization (subtraction) scheme choice

NLO exponentiation in the MSbar scheme is ambiguous:

Explicit solution for $D_{ee}(x)$ in the $\overline{\text{MS}}$ scheme in the limit $x \to 1$ doesn't match the (pure photonic) exact solution by Gribov and Lipatov '1972

$$\mathcal{D}_{ee}^{(\gamma)}(x,\beta)\Big|_{x\to 1} = \frac{\beta}{2}\,\frac{(1-x)^{\beta/2-1}}{\Gamma(1+\beta/2)}\exp\left\{\frac{\beta}{2}\left(\frac{3}{4}-\gamma_E\right)\right\}$$

where $\beta = 2\alpha/\pi(L-1)$ and γ_E is the Euler-Mascheroni const.

See also [A.V. Kotikov et al., "α_s from DIS data with large x resummation," JPG '2025]

We suggest a DIS-like scheme with the following modification of the NLO initial condition

$$d_{ee}^{(1)}\Big|_{\overline{\text{MS}}} = \left[\frac{1+x^2}{1-x}\left(\ln\frac{\mu_R^2}{m_e^2} - 1 - 2\ln(1-x)\right)\right]_+ \to \tilde{d}_{ee}^{(1)} = \left[\frac{1+x^2}{1-x}\ln\frac{\mu_R^2}{m_e^2}\right]_+ \to 0$$

for $m_R = m_e$ with subsequent modification of $\bar{\sigma}^{(1)}$ to preserve the NLO matching. Fixed-order LO and NLO cross-sections remain unchanged.

Drell-Yan process in QED: factorization scale dependence

$$
\begin{array}{llll}
c_{00}\alpha^0 \text{ (Born)} & & & \\
c_{11}\alpha^1 L^1 & c_{10}\alpha^1 L^0 & & \\
c_{22}\alpha^2 L^2 & c_{21}\alpha^2 L^1 & c_{20}\alpha^2 L^0 & \\
c_{33}\alpha^3 L^3 & c_{32}\alpha^3 L^2 & c_{31}\alpha^3 L^1 & c_{30}\alpha^3 L^0 \\
\dots & \dots & \dots & \dots
\end{array}
$$

For a given scheme and μ_R the sum of terms in each line is μ_R-independent, e.g.

$$\left(\frac{\alpha}{2\pi}\right)^2 L^2 c_{22} + \left(\frac{\alpha}{2\pi}\right)^2 L c_{21} + \left(\frac{\alpha}{2\pi}\right)^2 c_{20} = \left(\frac{\alpha}{2\pi}\right)^2 \hat{L}^2 \hat{c}_{22} + \left(\frac{\alpha}{2\pi}\right)^2 \hat{L} \hat{c}_{21} + \left(\frac{\alpha}{2\pi}\right)^2 \hat{c}_{20}$$

where

$$\hat{L} = L + \Delta L, \quad \Delta L = \ln \frac{\hat{\mu}_F^2}{\mu_F^2}$$

Factorization scale dependence appear if lines (calculations) are incomplete

Factorization scale choices

The final result of calculation in all orders in α and L would not depend on μ_F

But a fixed-order result for an observable does depend on μ_F (and μ_R)

Many different methods for choosing μ_F were proposed:

- CSS — Conventional Scale Setting (μ_F = hard momentum transfer)
- FAC — Fastest Apparent Convergence [G. Grunberg; N. Krasnikov]
- PMS — Principle of Minimal Sensitivity [P.M. Stevenson]
- BLM — Brodsky-Lepage-Mackenzie (absorb β_0-dependent terms)
- PMC — Principle of Maximal Conformality [S.Brodsky et al.]
- ...

Factorization scale choice in $e^+e^- \to \gamma^*, Z \to \mu^+\mu^-$

We advocate the FAC-like prescription, i.e., hide the bulk of one-loop corrections into the scale choice

For e^+e^- annihilation in $\mathcal{O}(\alpha^1)$ we have

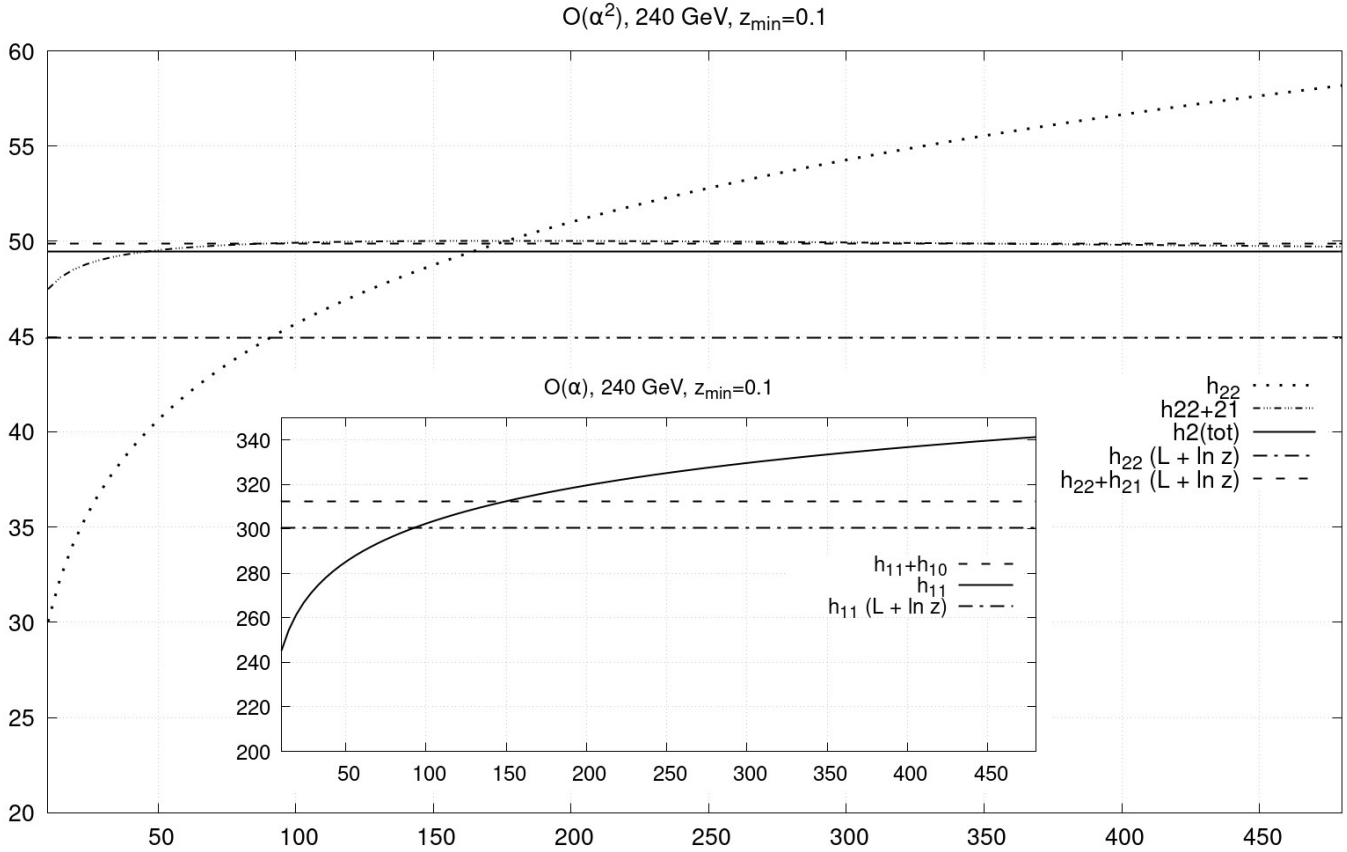
$$\frac{d\sigma^{(1)}_{e\bar{e}\to\gamma^*}}{d\sigma^{(0)}_{e\bar{e}\to\gamma^*}} = \frac{\alpha}{\pi}\left[\frac{1+z^2}{1-z}\right]_+ \left(\ln\frac{s}{m_e^2} - 1\right) + \delta(1-z)\left(2\zeta(2) - \frac{1}{2}\right) \Rightarrow \mu_F^2 = s \text{ or } \frac{s}{e}$$

Remind QCD Drell-Yan processes where we usually take $\mu_F^2 = s' \equiv zs \sim M_Z$, i.e., the energy scale of the hard subprocess (CSS choice)

F.Berends at al. and J. Blümlein et al. used $\mu_F = zs$ without any justification

Numerical results for total cross section at 240 GeV

Compare LO, NLO, and NNLO results as functions of μ_F



Corrections in % vs. factorization scale for $\sqrt{s} = 240$ GeV, $z_{min} = 0.1$

N.B. Comment on π^2-terms

Factorization scale choice: preliminary conclusions

Sensitivity to factorization scale choice is relevant numerically even in QED

Higher-order calculations are required to reduce the uncertainty

The comparison of several concrete schemes shows:

- CSS — Conventional Scale Setting (μ_F = hard momentum transfer) fails
- FAC — Fastest Apparent Convergence looks good
- PMS — Principle of Minimal Sensitivity looks reasonable
- BLM — Brodsky-Lepage-Mackenzie not applicable
- PMC — Principle of Maximal Conformality not applicable

Scale variation test: $\mu_F \to \mu_F/2, \quad \mu_F \times 2$

True (Δ) shifts and the ones estimated (δ) by factorization scale variation by factor 2 in $\mathcal{O}(\alpha^2)$ for $\mu_F = \sqrt{s}$ in the total cross section ($s' \geq s z_{\min}$)

	LO		NLO	
	Δ	δ	Δ	δ
$\sqrt{s} = M_z, \quad z_{min} = 0.1$	0.436	0.524	0.0063	0.0250
$\sqrt{s} = M_z, \quad z_{min} = 0.5$	0.436	0.525	0.0062	0.0250
$\sqrt{s} = 240, \quad z_{min} = 0.1$ GeV	2.468	5.569	0.525	0.148
$\sqrt{s} = 240, \quad z_{min} = 0.5$ GeV	0.114	0.106	0.0092	0.0061

$$\Delta^{\mathrm{LO}} = h_{21}, \qquad \Delta^{\mathrm{NLO}} = h_{20}$$

$$\delta^{\mathrm{LO}} = \frac{|h_{22} - h_{22}(1/2)| + |h_{22} - h_{22}(2)|}{2}$$

$$\delta^{\mathrm{NLO}} = \frac{|h_{22} + h_{21} - (h_{22} + h_{21})(1/2)| + |h_{22} + h_{21} - (h_{22} + h_{21})(2)|}{2}$$

Outlook

- There is a progress in QED PDFs
- Factorization scale and scheme dependencies in QED ISR corrections to cross sections of general high-energy e^+e^- annihilation processes are relevant for future experiments
- The standard MSbar factorization scheme induce unphysical behavior in PDFs at $x \to 1$
- A DIS-like factorization scheme is suggested to cure the problem

Next steps:

- Implementation into ZFITTER (in progress, see the next talk)
- BFKL effects in QED are large
- Polarized NLO QED PDFs
- Monte Carlo applications (within SANC)