The Four-Jet Production in Electron-Positron Annihilation at Order α_s^4

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Phys.Rev.Lett. 136 (2026) 251901

The 2026 international workshop on the CEPC

Shanghai, 23 October 2026

NNLO JET

山东大学 SHANDONG UNIVERSITY

Outline

I. **Motivation**

II. **Setup of Four-Jet Production**

- Matrix elements
- Infrared Subtraction Scheme
- Validation

III. **Results**

- Four-Jet Rate@NNLO
- Four-Jet Event Shape@NNLO
  - ➤ Monte Carlo Model
  - ➤ Renormalization Scale Choice

IV. **Summary & Outlook**

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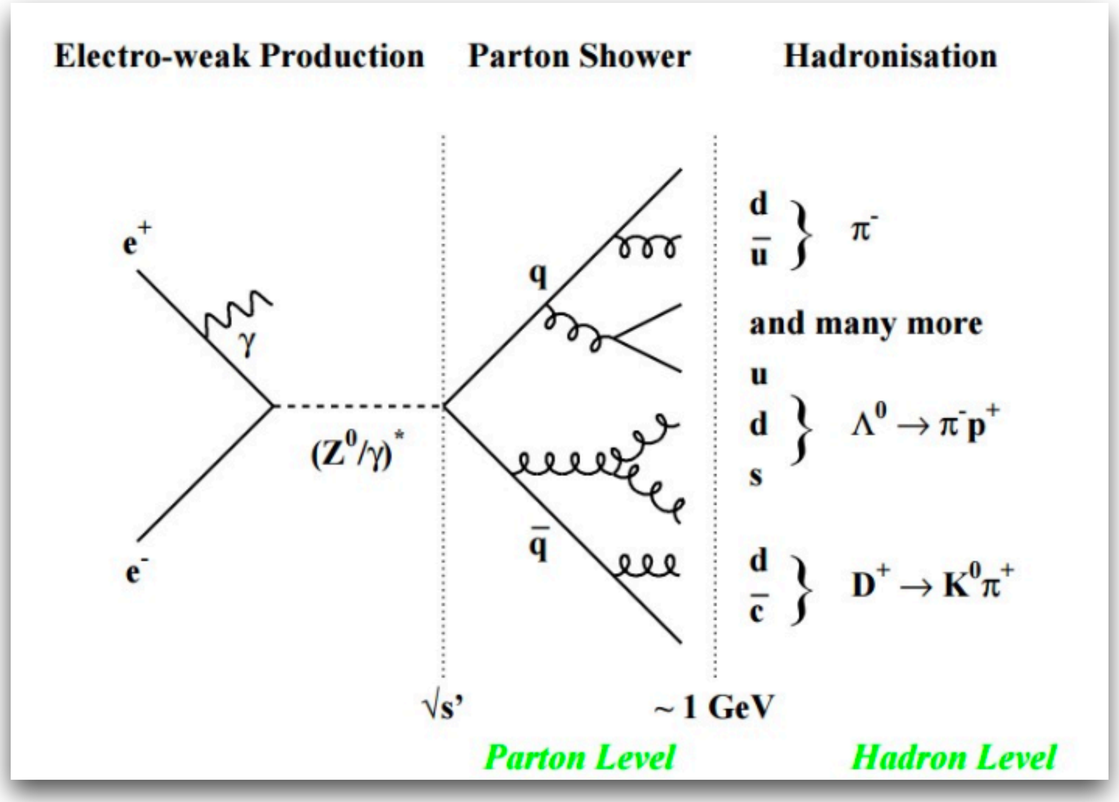
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IV. **Summary & Outlook**

Motivation

➤ e^+e^- collisions provide a clean environment to test QCD



[Kluth, hep-ex/0603011]

- QED initial-state with known kinematics
- No parton distributions/hadron remnant
- Non-perturbative effects only from final states

Motivation

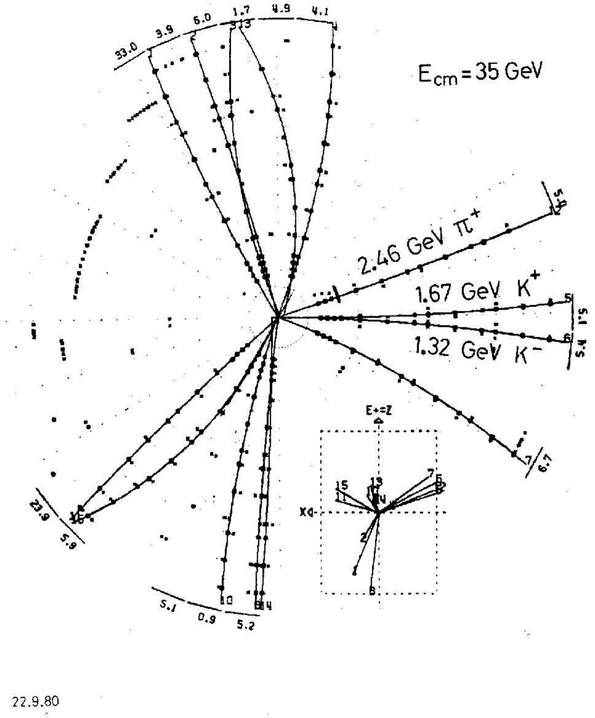
➤ Multi-jet production

Jets are extremely useful way to describe the production of hadrons in e^+e^- annihilation

Three-jet events

$e^+e^- \to q\bar{q}g$

Established the existence of the gluon



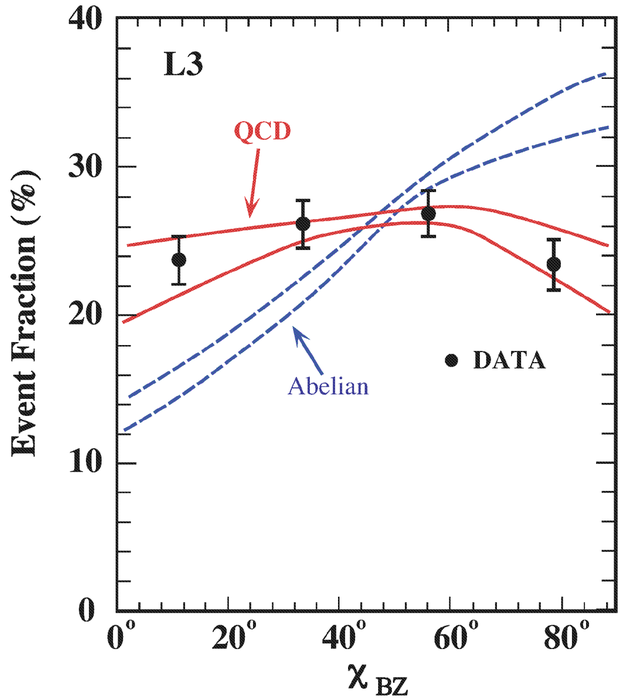
TASSO/PETRA event, 35 GeV (1980)

[Brandelik et al. (TASSO), Phys. Lett. B86 (1979) 243]

Four-jet angular correlations

$e^+e^- \to q\bar{q}gg, q\bar{q}Q\bar{Q}$

Tested gluon self-coupling and non-Abelian QCD



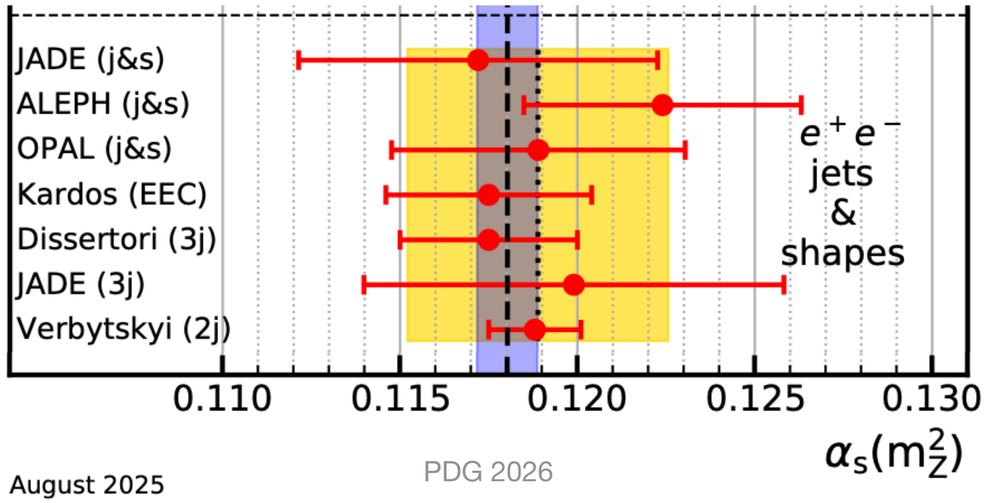
[Adeva et al. (L3), Phys. Lett. B248 (1990) 227]

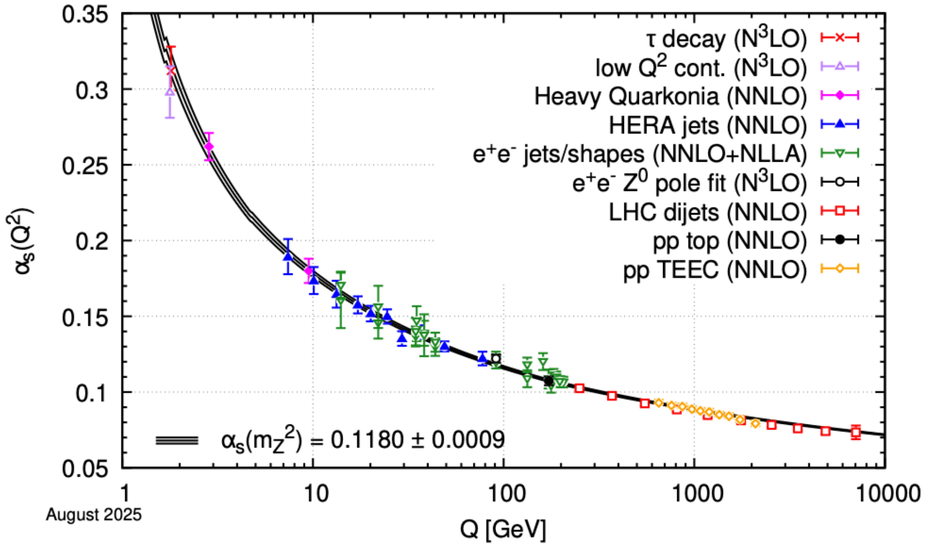
[Bethke, Phys. Rept. 403–404 (2004) 203]

Motivation

➤ Precision α_s extraction

- Strong coupling constant α_s is perhaps the most important parameter of QCD
- Can be measured from a number of precision observables at e^+e^- collider
  - **Exclusive observables**: event shape and jet rates
  - **Inclusive observables**: $R_{e^+e^-} = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$, $R_\tau = \Gamma(\tau \to \text{hadrons})/\Gamma(\tau \to \text{leptons})$ ...





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Setup of Four-Jet Production: Matrix Elements

- $e^+e^- \to \gamma^*/Z \to jjjj$@NLO = LO + Real + Virtual

https://nnlojet.github.io/

[NNLOJET Collaboration, A. Huss et al. SciPost Phys.Codeb. 69 (2026) 1]

LO	$e^+e^- \to q\bar{q}gg$	tree-level
	$e^+e^- \to q\bar{q}Q\bar{Q}$	tree-level
	$e^+e^- \to q\bar{q}q\bar{q}$	tree-level
NLO **Virtual**	$e^+e^- \to q\bar{q}gg$	one-loop
	$e^+e^- \to q\bar{q}Q\bar{Q}$	one-loop
	$e^+e^- \to q\bar{q}q\bar{q}$	one-loop
Real	$e^+e^- \to q\bar{q}ggg$	tree-level
	$e^+e^- \to q\bar{q}Q\bar{Q}g$	tree-level
	$e^+e^- \to q\bar{q}q\bar{q}g$	tree-level

Already implemented in **NNLOJET**

- Analytic helicity amplitudes
- Reused in $e^+e^- \to \gamma^*/Z \to$ 3j@NNLO and 2j@N3LO

[Gehrmann-De Ridder, Gehrmann, Glover, Heinrich, JHEP 11 (2007) 058]
[Chen, Jakubčík, Marcoli, Stagnitto, Phys. Lett. B869 (2025) 139804]

- $e^+e^- \to \gamma^*/Z \to jjjj$@NNLO_only = Real-Real + Real-Virtual + Virtual-Virtual

[Buccioni, Lang, Lindert, Maierhöfer, Pozzorini, Zhang, Zoller 1907.13071]

Real-Virtual	$e^+e^- \to q\bar{q}ggg$	one-loop
	$e^+e^- \to q\bar{q}Q\bar{Q}g$	one-loop
	$e^+e^- \to q\bar{q}q\bar{q}g$	one-loop
Real-Real	$e^+e^- \to q\bar{q}gggg$	tree-level
	$e^+e^- \to q\bar{q}Q\bar{Q}gg$	tree-level
	$e^+e^- \to q\bar{q}q\bar{q}gg$	tree-level
	$e^+e^- \to q\bar{q}Q\bar{Q}R\bar{R}$	tree-level
	$e^+e^- \to q\bar{q}q\bar{q}Q\bar{Q}$	tree-level
	$e^+e^- \to q\bar{q}q\bar{q}q\bar{q}$	tree-level

New RR and RV ingredients from **OpenLoops2**

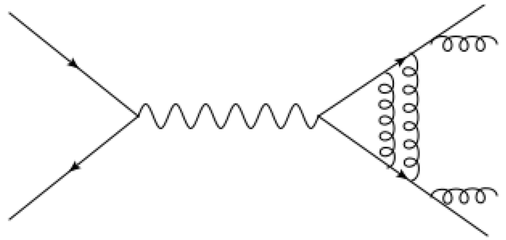
- Efficient amplitude evaluation with numerical stability control
- Access to tree-level amplitudes in a colour basis
  - ***RR***: fully decomposed into colour factors ⟶ improve computational efficiency
  - ***RV***: without colour decomposition since different convension

Setup of Four-Jet Production: Matrix Elements

- $e^+e^- \to \gamma^*/Z \to jjjj$@NNLO_only = Real-Real + Real-Virtual + Virtual-Virtual

$$VV \propto 2\mathrm{Re}(\mathrm{M}^{(0)}\mathrm{M}^{(2)}) + \mathrm{M}^{(1)}\mathrm{M}^{(1)}$$

$e^+e^- \to q\bar{q}gg$	two-loop
$e^+e^- \to q\bar{q}Q\bar{Q}$	two-loop
$e^+e^- \to q\bar{q}q\bar{q}$	two-loop



➤ **Two-loop five-point with one external massive leg amplitudes**

- ☑ Planar two-loop amplitudes for pp →Vjj in the leading-colour approximation
- ☑ One-mass pentagon functions in the production region $\{f_j^{\mathrm{prod}}\}$
- ☑ The two-loop finite remainders are defined in Catani's scheme

matrices of rational numbers

$$R^{\vec{a}}(\vec{p}) = \sum_{i,j} r_i(\vec{p}) M^{\vec{a}}_{ij} f_j(\vec{p})$$

rational coefficients — Bases of transcendental special functions: one-mass pentagon functions

[Abreu, Febres Corder, Ita, Klinkert, Page, Sotnikov 2110.07541]
[De Laurentis, Ita, Page, Sotnikov 2503.10595]

[Chicherin, Sotnikov, Zoia, JHEP 01 (2022) 096]
[Abreu et al., Phys. Rev. Lett. 132 (2024) 141601]

NEW!

FivePointAmplitudes++: https://gitlab.com/five-point-amplitudes/FivePointAmplitudes-cpp.git

- **Planar** two-loop finite remainders in the **leading-colour approximation** (LCA): $N_c \to \infty$, with N_f/N_c kept constant
- Complete set of **decay pentagon functions** $\{f_j^{\mathrm{decay}}\}$ PentagonFunctions++ public library: https://gitlab.com/pentagon-functions/PentagonFunctions-cpp
- The two-loop finite remainders are defined in Catani's scheme [Catani 9802439]

Numerical integration takes little time compared with the rest of the calculation

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Setup of Four-Jet Production: Infrared Subtraction Scheme

• Subtraction Method

➤ NLO

$$d\sigma_{\mathrm{NLO}} = \int_{d\Phi_{m+1}} \left(d\sigma^{\mathrm{R}}_{\mathrm{NLO}} - d\sigma^{\mathrm{S}}_{\mathrm{NLO}} \right) \longrightarrow \text{Finite in single-unresolved limits}$$

$$+ \left[\int_{d\Phi_{m+1}} d\sigma^{\mathrm{S}}_{\mathrm{NLO}} + \int_{d\Phi_m} d\sigma^{\mathrm{V}}_{\mathrm{NLO}} \right] \longrightarrow \text{No explicit poles: } 1/\epsilon^2,\ 1/\epsilon$$

➤ NNLO

$$d\sigma_{\mathrm{NNLO}} = \int_{d\Phi_{m+2}} \left(d\sigma^{\mathrm{R}}_{\mathrm{NNLO}} - d\sigma^{\mathrm{S}}_{\mathrm{NNLO}} \right) \longrightarrow \text{Finite in single and double-unresolved limits}$$

$$+ \int_{d\Phi_{m+1}} \left(d\sigma^{\mathrm{V,1}}_{\mathrm{NNLO}} - d\sigma^{\mathrm{VS,1}}_{\mathrm{NNLO}} \right) \longrightarrow \begin{array}{l} \text{No explicit poles: } 1/\epsilon^2,\ 1/\epsilon \\ \text{Finite in single-unresolved limits} \end{array}$$

$$+ \int_{d\Phi_m} d\sigma^{\mathrm{V,2}}_{\mathrm{NNLO}} + \int_{d\Phi_{m+2}} d\sigma^{\mathrm{S}}_{\mathrm{NNLO}} + \int_{d\Phi_{m+1}} d\sigma^{\mathrm{VS,1}}_{\mathrm{NNLO}} \longrightarrow \text{No explicit poles: } 1/\epsilon^4 \ldots 1/\epsilon$$

Setup of Four-Jet Production: Infrared Subtraction Scheme

• Subtraction Method

➤ NLO

$$d\sigma_{\mathrm{NLO}} = \int_{d\Phi_{m+1}} \left(d\sigma^{\mathrm{R}}_{\mathrm{NLO}} - d\sigma^{\mathrm{S}}_{\mathrm{NLO}} \right) \longrightarrow \text{Finite in single-unresolved limits}$$

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➤ NNLO

$$d\sigma_{\mathrm{NNLO}} = \int_{d\Phi_{m+2}} \left(d\sigma^{\mathrm{R}}_{\mathrm{NNLO}} - d\sigma^{\mathrm{S}}_{\mathrm{NNLO}} \right) \longrightarrow \text{Finite in single and double-unresolved limits}$$

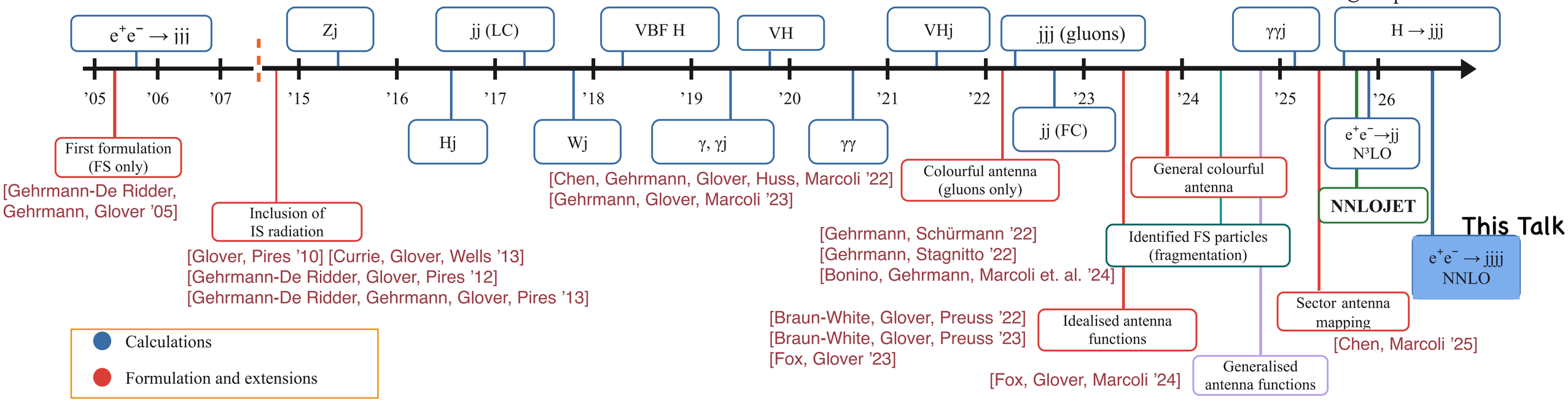
$$+ \int_{d\Phi_{m+1}} \left(d\sigma^{\mathrm{V,1}}_{\mathrm{NNLO}} - d\sigma^{\mathrm{VS,1}}_{\mathrm{NNLO}} \right) \longrightarrow \text{No explicit poles: } 1/\epsilon^2,\ 1/\epsilon \text{; Finite in single-unresolved limits}$$

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• Antenna Subtraction Method

NNLOJET

Based on Marcoli's slide@Loop Summit 2



Setup of Four-Jet Production: Infrared Subtraction Scheme

• Traditional Antenna Subtraction

[hep-ph/0505111] [0710.0346]

➤ Subtraction terms

mapped reduced matrix element $\mathscr{M}^{loops}_{hard\ partons}$

$$\mathrm{d}\sigma^{S}_{\mathrm{NLO}} = \mathscr{N} \sum_{m+1} \mathrm{d}\Phi_{m+1}(p_1,\ldots,p_{m+1};q)\frac{1}{S_{m+1}} \times \sum_{j} X^{0}_{ijk} \left| \mathscr{M}^{0}_{m}(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1}) \right|^2 J^{(m)}_{m}(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1})$$

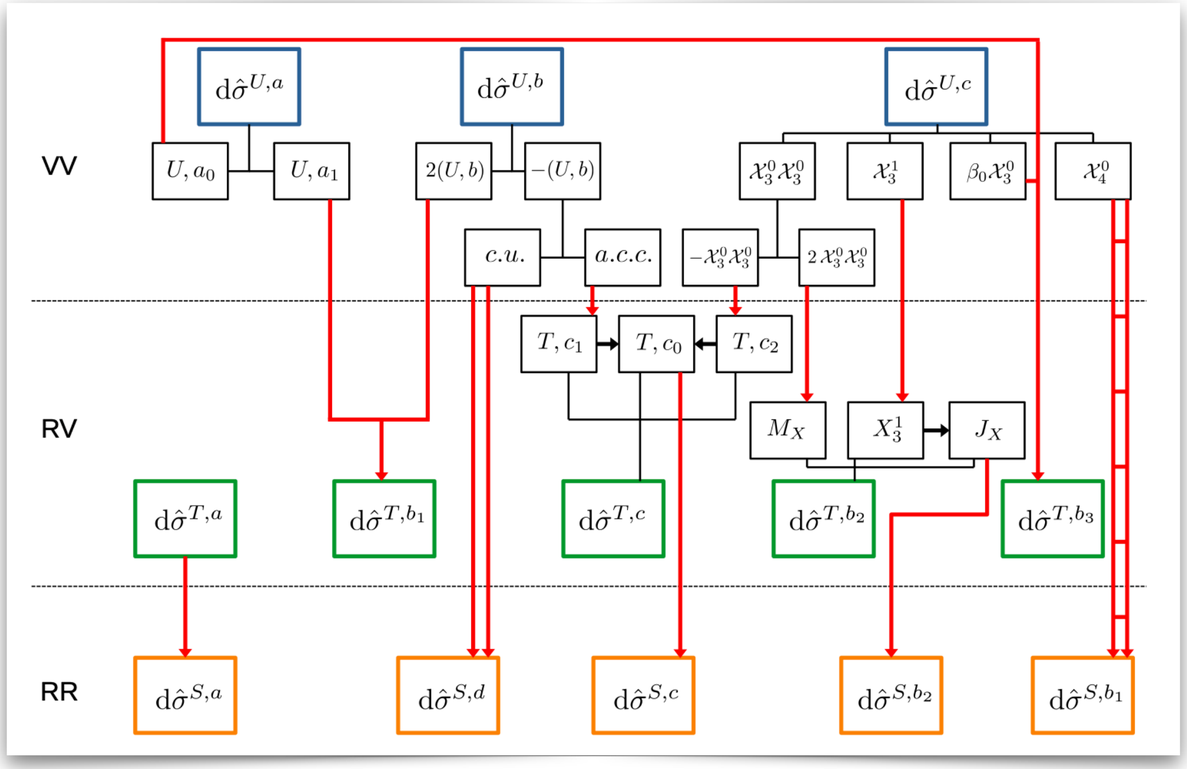
antenna function $X^{loops}_{emissions+2}$ — jet function

$\mathscr{N}$: QCD-independent factors and α_s^n

$\sum_{m+1}$: sum over all configurations with n-parton

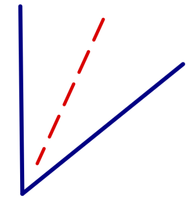
$\sum_{j}$: allowed unresolved partons

$\frac{1}{S_{m+1}}$: symmetry factor for identical partons in the final state

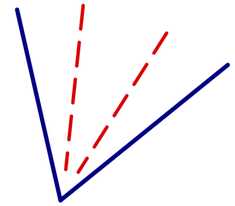


[Chen, Gehrmann, Glover, Huss, Marcoli JHEP 10 (2022) 099]
[Gehrmann, Glover, Marcoli JHEP 03 (2024) 114]

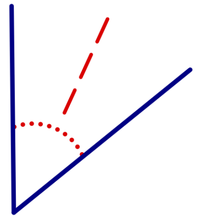
$$X^0_3(i,j,k) = \frac{|\mathscr{M}^0_3(i,j,k)|^2}{|\mathscr{M}^0_2(\tilde{I},\tilde{K})|^2}$$



$$X^0_4(i,j,k,l) = \frac{|\mathscr{M}^0_4(i,j,k,l)|^2}{|\mathscr{M}^0_2(\widetilde{ijk},\widetilde{jkl})|^2}$$



$$X^1_3(i,j,k) = \frac{|\mathscr{M}^1_3(i,j,k)|^2}{|\mathscr{M}^0_2(\tilde{I},\tilde{K})|^2} - X^0_3\frac{|\mathscr{M}^1_2(\tilde{I},\tilde{K})|^2}{|\mathscr{M}^0_2(\tilde{I},\tilde{K})|^2}$$



Setup of Four-Jet Production: Infrared Subtraction Scheme

• Traditional Antenna Subtraction

[hep-ph/0505111] [0710.0346]

➤ Subtraction terms

$$\mathrm{d}\sigma^{S}_{\mathrm{NLO}} = \mathcal{N}\sum_{m+1}\mathrm{d}\Phi_{m+1}(p_1,\ldots,p_{m+1};q)\frac{1}{S_{m+1}}\times\sum_{j}X^{0}_{ijk}\left|\mathscr{M}^{0}_{m}(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1})\right|^{2}J^{(m)}_{m}(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1})$$

mapped reduced matrix element $\mathscr{M}^{loops}_{hard\ partons}$

antenna function $X^{loops}_{emissions+2}$

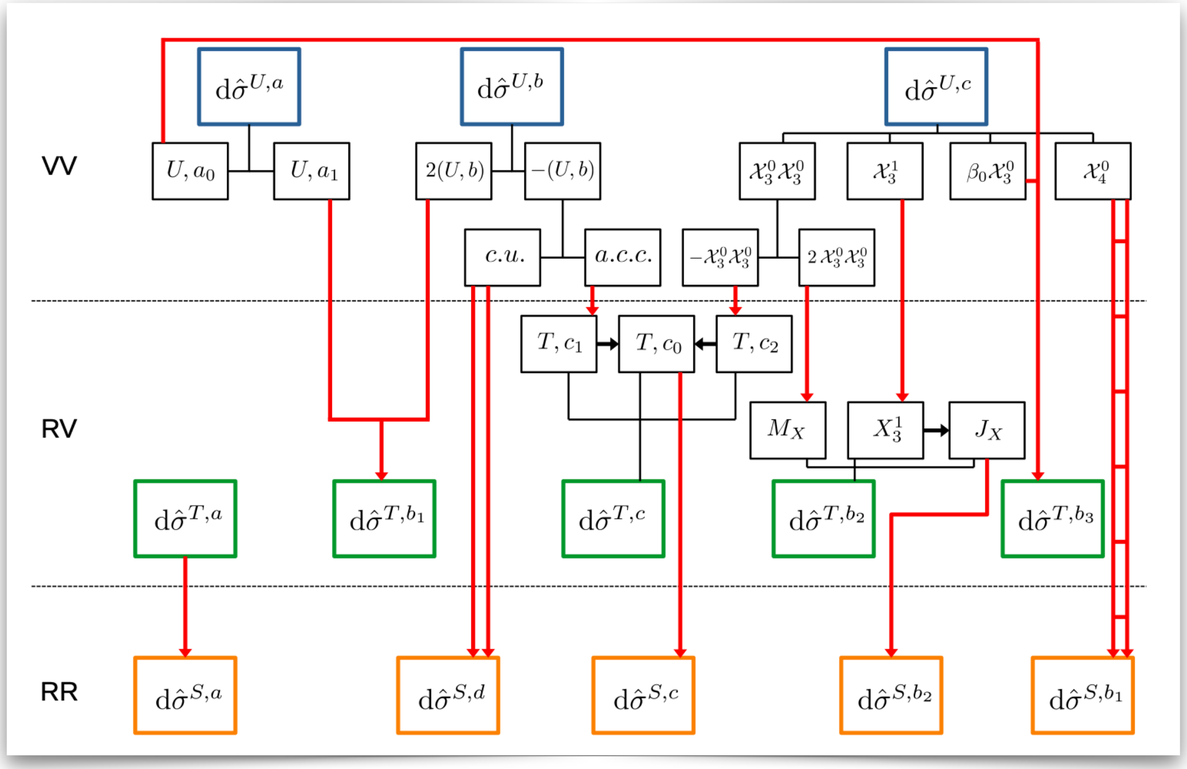
jet function

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$\sum_{m+1}$: sum over all configurations with n-parton

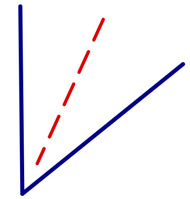
$\sum_{j}$: Allowed unresolved partons

$\frac{1}{S_{m+1}}$: symmetry factor for identical partons in the final state

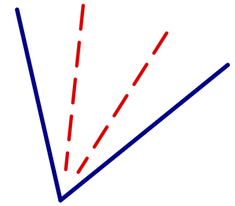


[Chen, Gehrmann, Glover, Huss, Marcoli JHEP 10 (2022) 099]
[Gehrmann, Glover, Marcoli JHEP 03 (2024) 114]

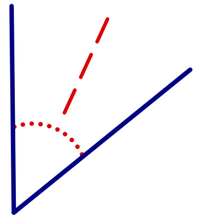
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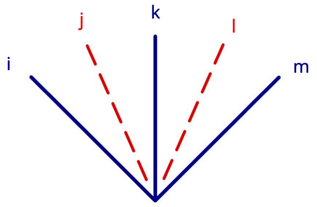


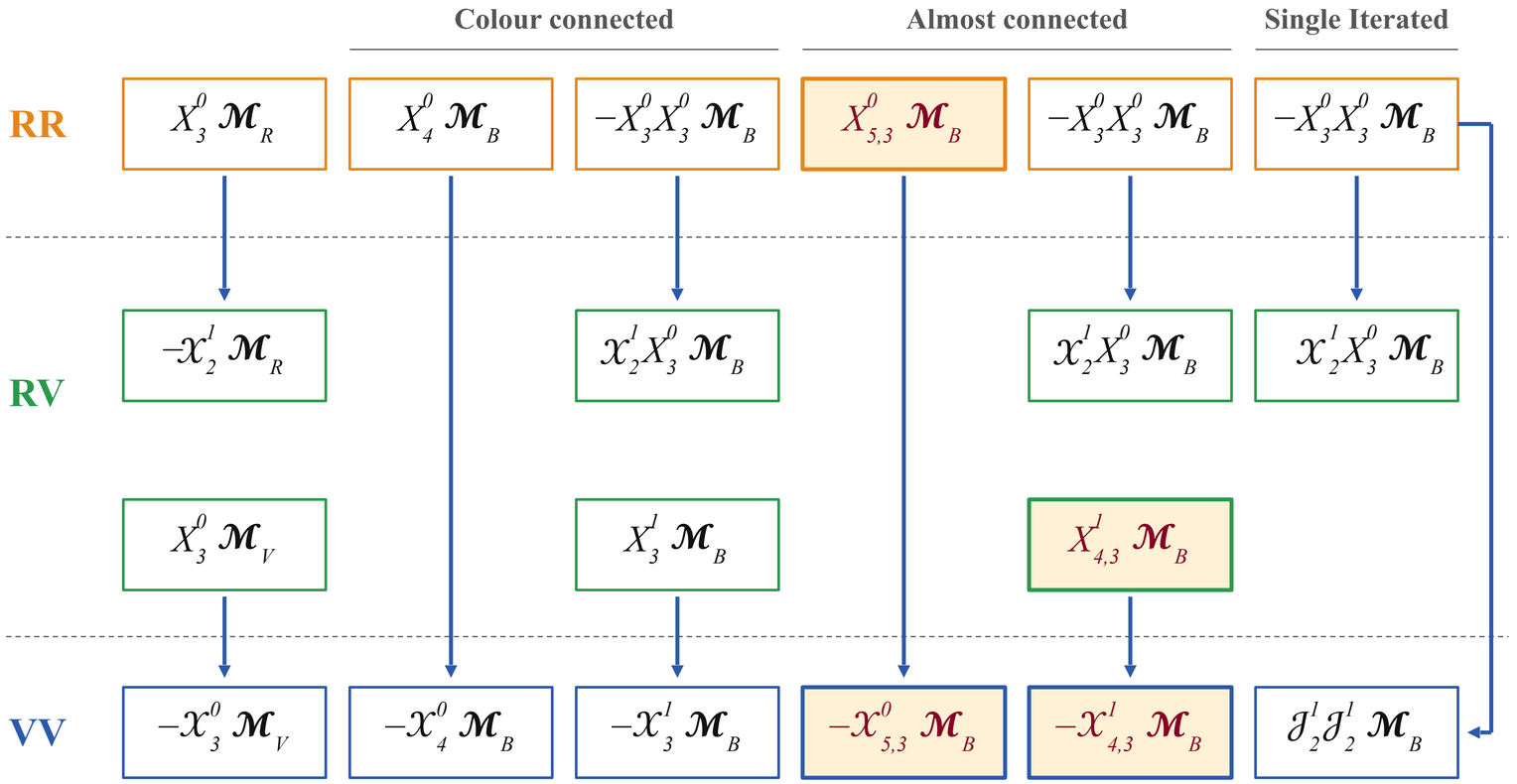
➤ Direct extension to N3LO is highly none-trivial
➤ Can we simplify the subtraction picture at NNLO ?
➤ Generalized antenna

Setup of Four-Jet Production: Infrared Subtraction Scheme

• Generalized Antenna

Almost-colour-connected configurations can involve three hard radiators, while traditional antennae have two.
The traditional treatment needs iterated two-radiator antennae, large-angle soft terms and sums over colour orderings.





$\mathcal{M}_{B/R/V}$: reduced Born/real/virtual terms; $\mathcal{J}_2^1$: integrated dipole; $\mathcal{X}$: integrated antenna

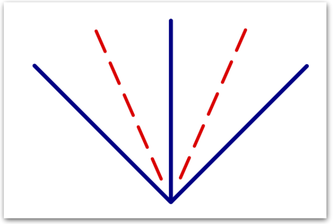
[Fox, Glover, Marcoli, JHEP 12 (2024) 225]

Based on Xuan Chen's slide@Loops and Legs 2026

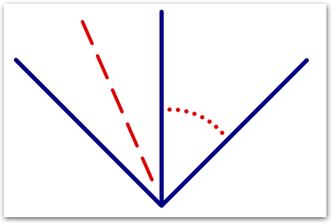
New three-radiator antennae

Constructed from specified soft and collinear limits:
designer antenna algorithm [Braun-White,Nigel,Preuss '22,'23]

$X_{5,3}^0$



$X_{4,3}^1$



- **Much shorter subtraction terms**
- **Large improvement in numerical performance (5–10× faster for $e^+e^- \to 3j$ @NNLO)**
- **Employed in $e^+e^- \to 4j$ @NNLO**

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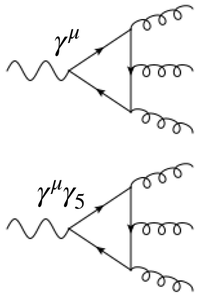
Setup of Four-Jet Production: Validation

• Parameter

- $M_Z = 91.2$ GeV, $\alpha_s(M_Z) = 0.118$, $N_F = 5$ massless quark flavours
- $\mu_r = \sqrt{s}$, estimate scale uncertainties by varying $\mu_r \in [\sqrt{s}/2, 2\sqrt{s}]$

Approximation:

- No top-loop contributions: expected to be negligible
- No singlet contribution (one-loop/two-loop): expected to be small
- No axial-singlet: neglect the top-bottom mass splitting



[Knihel, Kuhn Phys.Lett.B 224 (1989) 229-232]
[Hagiwara, Kuruma, Yamada Nucl.Phys.B 358 (1991) 80-96]
[Dixon, Signer 9609460, 9706285]
[Frederix, Frixione, Melnikov, Zanderighi 1008.5313]

• Validation with SHERPA

- Total cross section: 4jet@NLO, 5jet@NLO, 6jet@LO (Durham algorithm with different jet resolution parameter y_{cut})
- Differential distributions of 4-jet event shape@NLO

Finding good agreement between NNLOJET and SHERPA

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Jet Rate

$$R_n(y_{\text{cut}}) = \frac{\sigma_n(y_{\text{cut}})}{\sigma_{\text{inc.}}}$$

The ratio of the exclusive n-jet cross section to the total hadronic cross section

➤ Four-jet rate R_4

Durham algorithm $y_{ij} = \dfrac{2\min(E_i^2, E_j^2)(1-\cos\theta_{ij})}{s}$

[Catani, Dokshitzer, Olsson, Turnock, Webber, Phys. Lett. B269 (1991) 491]

- σ_{inc} with matching α_s order: up to α_s^2 at LO, α_s^3 at NLO, α_s^4 at NNLO
- Ensure that different jet-multiplicity rates add up to 1 at any order in α_s

$$R_4(y_{cut}) = \frac{\sigma_4(y_{cut})}{\sigma_{inc.}} = \frac{1}{\sigma_{inc.}}\left(\int_{y_{cut}}^{1}\frac{d\sigma}{dy_{34}}dy_{34} - \int_{y_{cut}}^{1}\frac{d\sigma}{dy_{45}}dy_{45}\right)$$

inclusive four-jet - inclusive five-jet = exclusive four-jet

The four-jet resolution y_{34} : the minimum y_{cut} that produces three jets

The five-jet resolution y_{45}: the minimum y_{cut} that produces four jets

$R_n^{(\alpha_s)}$	α_s^0	α_s^1	α_s^2	α_s^3	α_s^4	
n=2		1977	2004, 2004,	2008, 2025		FO
n=3		1976	1980, 1981,	2008, 2008,		N4LO
n=4			1979, 1980	1996, 1997,		N3LO
n=5				1989, 1989,	2010, 2011	NNLO
n=6					1999	NLO
						LO

From Xuan Chen's slide@Loops and Legs 2026

[Physics Briefing Book: de Blas et al. 2511.03883]

Table 4.2: Wish-list for calculations of missing higher-order perturbative QCD $\mathcal{O}(\alpha_s^n)$ and/or EW $\mathcal{O}(\alpha^n)$ to match the expected experimental uncertainty at future e^+e^- and ep colliders.

Observable	Missing higher-order & power-suppressed corrections
Hadronic Z width	$\mathcal{O}(\alpha_s^5)$, $\mathcal{O}(\alpha_s^6)$, $\mathcal{O}(\alpha^3)$, $\mathcal{O}(\alpha_s\alpha^3)$, $\mathcal{O}(\alpha_s^2\alpha^2)$
Hadronic W width	$\mathcal{O}(\alpha_s^5)$, $\mathcal{O}(\alpha^2)$, $\mathcal{O}(\alpha_s^2\alpha)$
Hadronic τ width	$\mathcal{O}(\alpha_s^5)$
Hadronic event shapes (Z, W, H decays)	N^{3}LO differential, N^{3,4}LL resummation, power corrections
Inclusive jet rates	3-jet cross-sections at N^{3}LO, 4-jets at N^{2}LO, 5-jets at NLO
Lattice QCD results	$\mathcal{O}(\alpha_s^6)$ β-function; $\mathcal{O}(\alpha_s^5)$ heavy quark decoupling; $\mathcal{O}(\alpha_s^4)$ static potential
(α_s extr.; quark masses m_c, m_b)	$\mathcal{O}(\alpha_s^3)$ lattice perturbation theory matching (lattice coupling to $\alpha_s^{\overline{MS}}$ etc.)
$\sigma(e^+e^- \to W^+W^-)$ vs. $\sqrt{s}$	EW N^{2}LO: $\mathcal{O}(\alpha^2)$, Mixed EW-QCD: $\mathcal{O}(\alpha_s\alpha^2)$, $\mathcal{O}(\alpha_s^2\alpha)$
$\sigma(e^+e^- \to t\bar{t})$ vs. $\sqrt{s}$	NRQCD: $\mathcal{O}(\alpha_s^5)$, Non-resonant: $\mathcal{O}(\alpha_s^5)$, $\mathcal{O}(\alpha_s^3)$ differential; QED: $\mathcal{O}(\alpha^3)$ at NNLL
$H \to b\bar{b}$ width	N^{4}LO ($m_b \neq 0$); N^{4}LO differential ($m_b = 0$)
$H \to gg$ width	N^{5}LO (heavy-top limit), N^{4}LO ($m_t \neq 0$); N^{4}LO differential, N^{3}LO differential ($m_t \neq 0$)
MC simulations for $e^+e^- \to X$ processes	N^{2,3}LO matched to N^{2,3}LL PS. Permille control of non-pQCD effects (hadronization, CR, …)
$ep \to$ hadrons (PDF and α_s determ.)	N^{3,4}LO evolution equations and inclusive cross-sections
$ep \to$ jets (α_s determ.)	N^{3}LO cross-sections

Jet Rate

$$R_n(y_{\text{cut}}) = \frac{\sigma_n(y_{\text{cut}})}{\sigma_{\text{inc.}}}$$

The ratio of the exclusive n-jet cross section to the total hadronic cross section

➤ Four-jet rate R_4

Durham algorithm $y_{ij} = \dfrac{2\min(E_i^2, E_j^2)(1-\cos\theta_{ij})}{s}$

[Catani, Dokshitzer, Olsson, Turnock, Webber, Phys. Lett. B269 (1991) 491]

- σ_{inc} with matching α_s order: up to α_s^2 at LO, α_s^3 at NLO, α_s^4 at NNLO
- Ensure that different jet-multiplicity rates add up to 1 at any order in α_s

$$R_4(y_{cut}) = \frac{\sigma_4(y_{cut})}{\sigma_{inc.}} = \frac{1}{\sigma_{inc.}}\left(\int_{y_{cut}}^{1}\frac{d\sigma}{dy_{34}}dy_{34} - \int_{y_{cut}}^{1}\frac{d\sigma}{dy_{45}}dy_{45}\right)$$

inclusive four-jet - inclusive five-jet = exclusive four-jet

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n=4			1979, 1980	1996, 1997,	2026 This work	N3LO
n=5				1989, 1989,	2010, 2011	NNLO
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						LO

From Xuan Chen's slide@Loops and Legs 2026

This Talk

[Physics Briefing Book: de Blas et al. 2511.03883]

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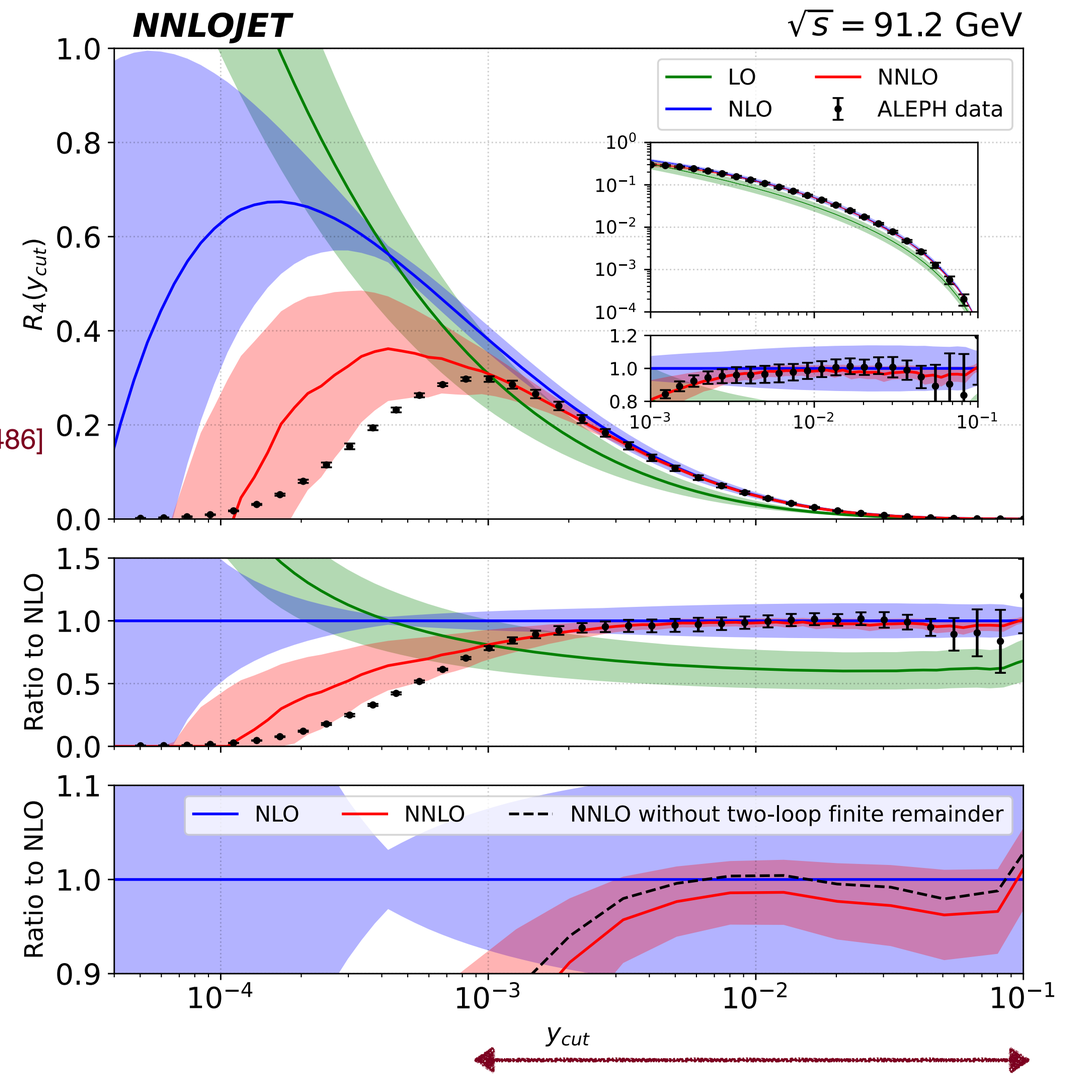
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$ep \to$ jets (α_s determ.)	N^{3}LO cross-sections

Results: Four-Jet Rate@NNLO

$$R_4(y_{cut}) = \frac{\sigma_4(y_{cut})}{\sigma_{inc.}} = \frac{1}{\sigma_{inc.}}\left(\int_{y_{cut}}^{1} \frac{d\sigma}{dy_{34}} dy_{34} - \int_{y_{cut}}^{1} \frac{d\sigma}{dy_{45}} dy_{45}\right)$$ Durham algorithm

[Chen, Chicherin, Fox, Glover, Marcoli, Sotnikov, HTS, Zhang, Zoia, Phys.Rev.Lett. 136 (2026) 251901]

- Perturbative region: $10^{-3} \leq y_{\text{cut}} \leq 10^{-1}$
  - ➤ Remarkable convergence: scale uncertainty substantially reduced from $\pm 15\,\%$ at NLO to $\pm(3\% - 5\%)$ at NNLO (symmetrized), which smaller than ALEPH systematic errors
  - ➤ Small NNLO correction, amounts to -2% ~ -5%
  - ➤ Excellent agreement with ALEPH data [Heister et al. Eur.Phys.J.C 35 (2004) 457-486]
  - ➤ Two-loop finite reminder (LCA) ~ -2% ~ -5%
  - ➤ Missing subleading-colour contributions ~ $1/N_c^2$ suppressed

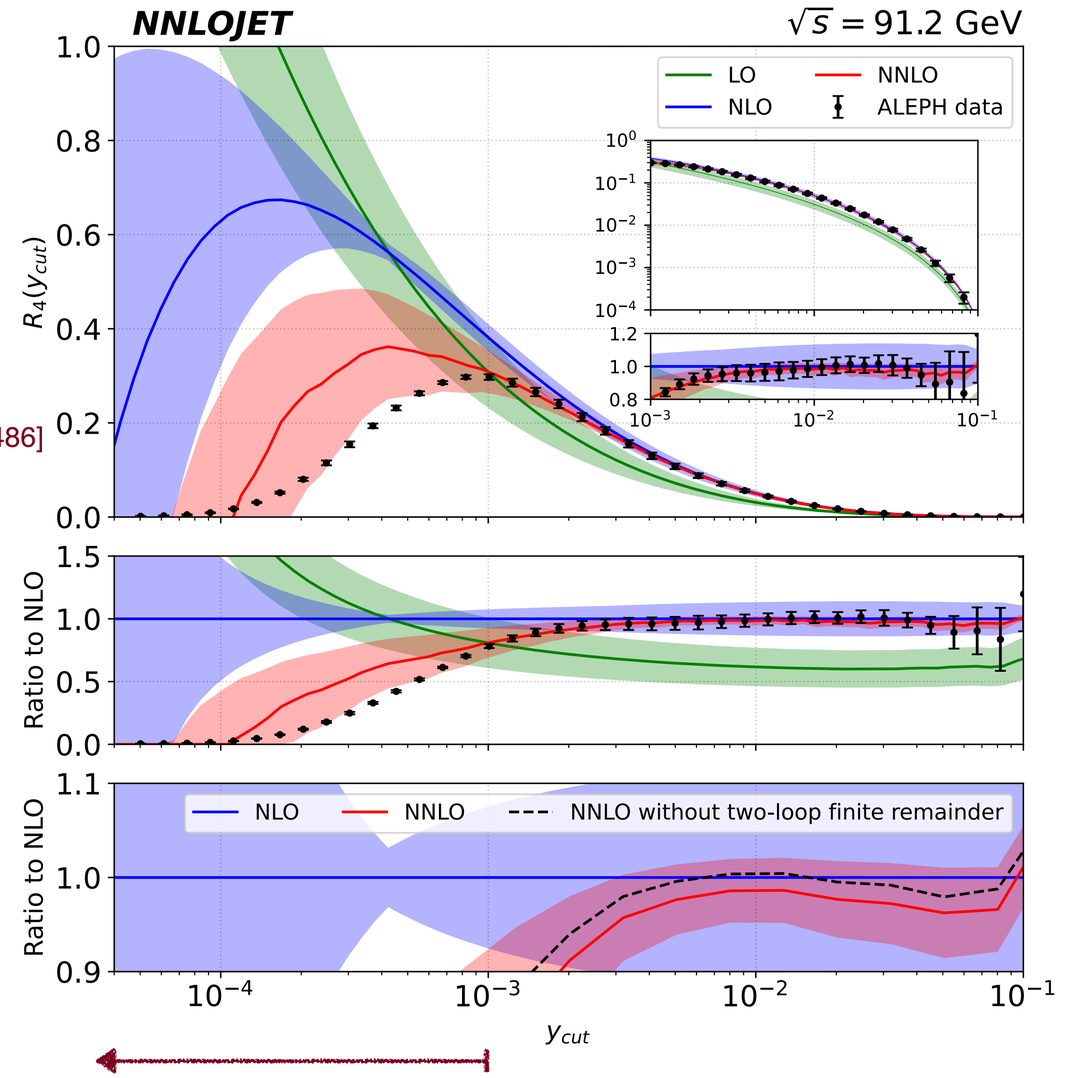


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  - ➤ Two-loop finite reminder (LCA) ~ -2% ~ -5%
  - ➤ Missing subleading-colour contributions $\sim 1/N_c^2$ suppressed
- Resummation and non-perturbative region: $y_{\text{cut}} \leq 10^{-3}$
  - ➤ Predictions turn negative, larger scale uncertainties, non-overlapping scale bands
  - ➤ NNLO peak moves to larger y_{cut} value than NLO
  - ➤ Resummation and non-perturbative correction are needed to improve the prediction



Results: Four-Jet Rate@NNLO

$$R_4(y_{cut}) = \frac{\sigma_4(y_{cut})}{\sigma_{inc.}} = \frac{1}{\sigma_{inc.}}\left(\int_{y_{cut}}^{1} \frac{d\sigma}{dy_{34}} dy_{34} - \int_{y_{cut}}^{1} \frac{d\sigma}{dy_{45}} dy_{45}\right)$$ Durham algorithm

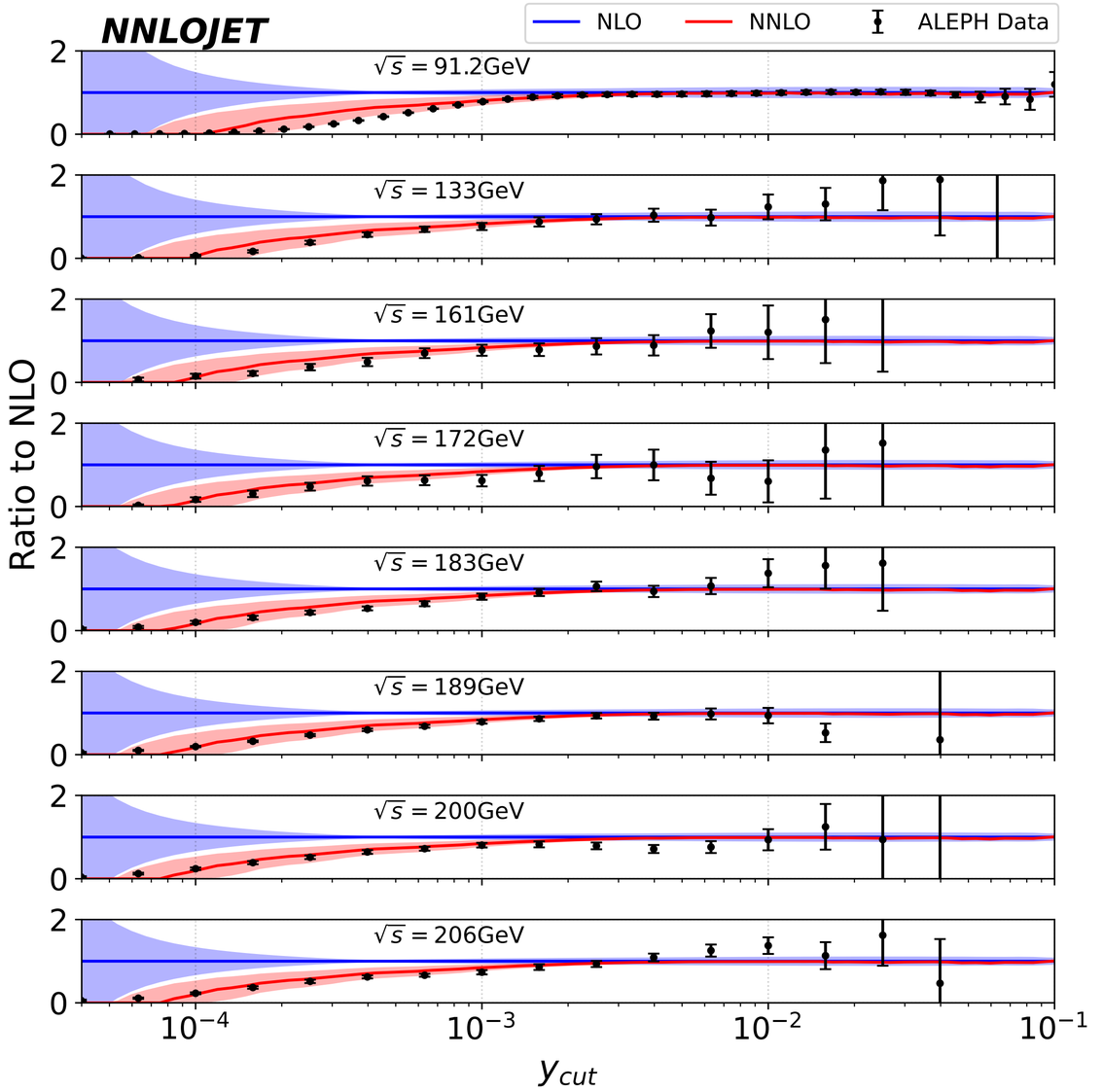
[Chen, Chicherin, Fox, Glover, Marcoli, Sotnikov, HTS, Zhang, Zoia, Phys.Rev.Lett. 136 (2026) 251901]

Energy scan

The $\sqrt{s}$ dependence enters only through the running of $\alpha_s(\mu_r)$
(With the approximations mentioned above)

- Perturbative region
  - ➤ Features similar to those at 91.2 GeV
  - ➤ Perturbative breakdown is pushed to lower values of y_{cut}
  - ➤ Two-loop finite remainder is largely insensitive to $\sqrt{s}$

- Resummation and non-perturbative region
  - ➤ Need resummation and non-perturbative correction to improve the fixed-order prediction



Outline

I. Motivation

II. Setup of Four-Jet Production

- Matrix elements
- Infrared Subtraction Scheme
- Validation

III. Results

- Four-Jet Rate@NNLO
- Four-Jet Event Shape@NNLO
  - ➤ Monte Carlo Model
  - ➤ Renormalization Scale Choice

IV. Summary & Outlook

Event Shape

- **IRC-safe measures of hadronic event geometry**

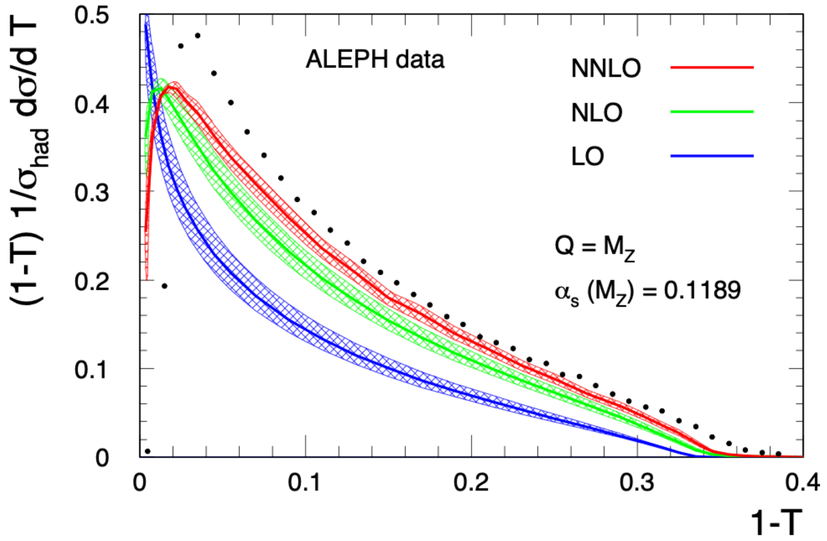
n-jet-like: zero for fewer than n-parton final states

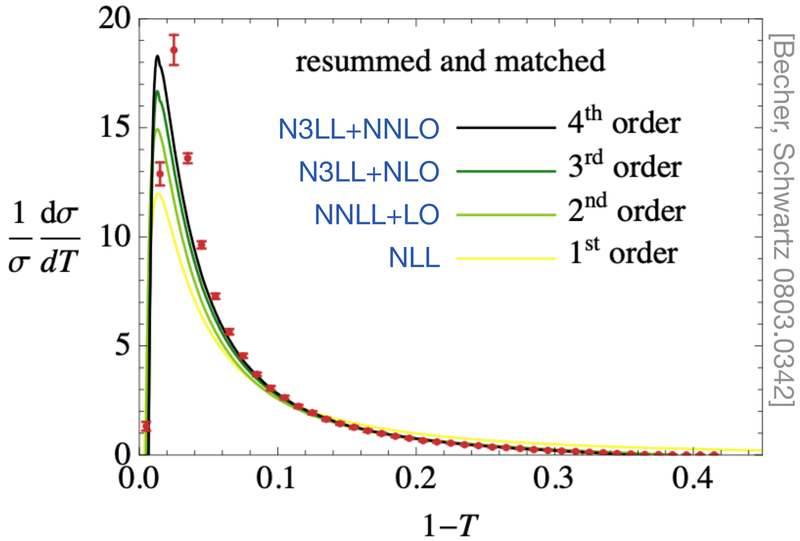
➤ Three-jet event shape

thrust, heavy jet mass,
wide and total jet broadening,
C-parameter, jet resolution y_{23}

NNLO

[Gehrmann-De Ridder et al., JHEP 0712 (2007) 094]
[S. Weinzierl, JHEP 0906 (2009) 041]
[Del Duca, et al., 1606.03453]





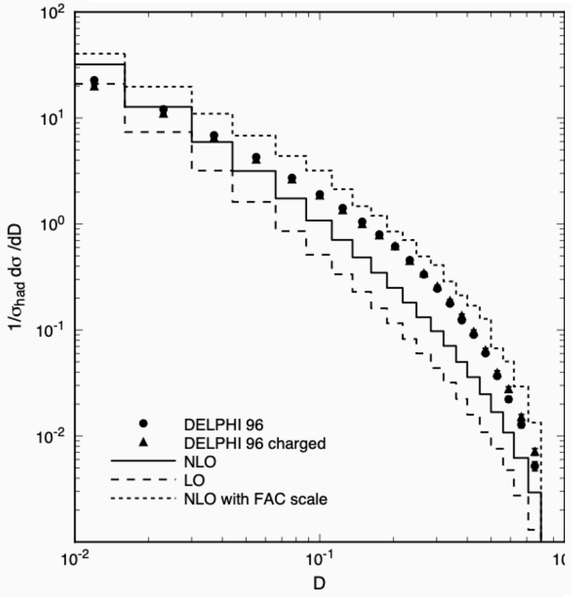
[Becher, Schwartz 0803.0342]

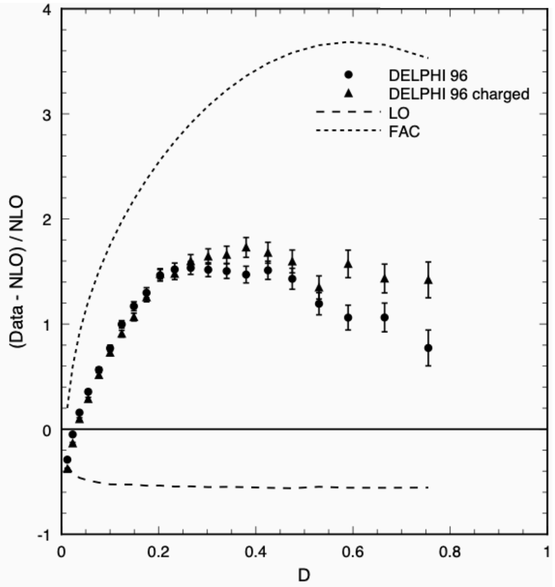
➤ Four-jet event shape

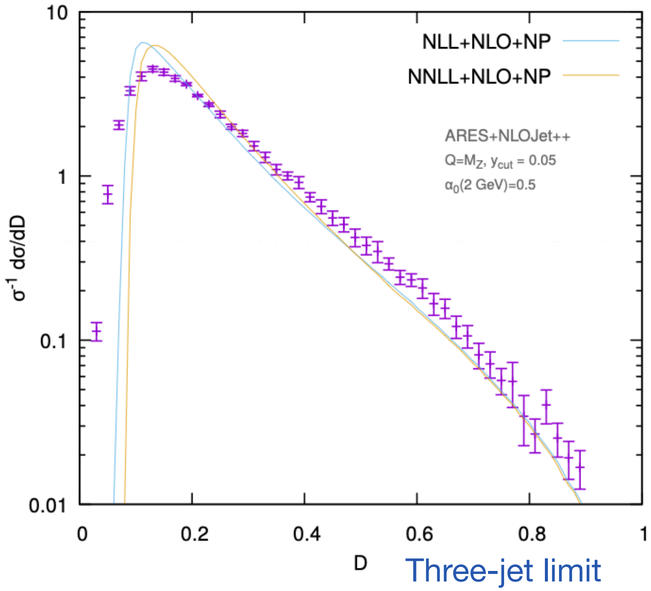
thrust minor, aplanarity
light jet mass, narrow jet broadening,
D parameter, jet resolution y_{34}

NLO

[Nagy, Trócsányi 9708344, 9707309, 9806317]
[Campbell, Cullen, Glover 9809429]





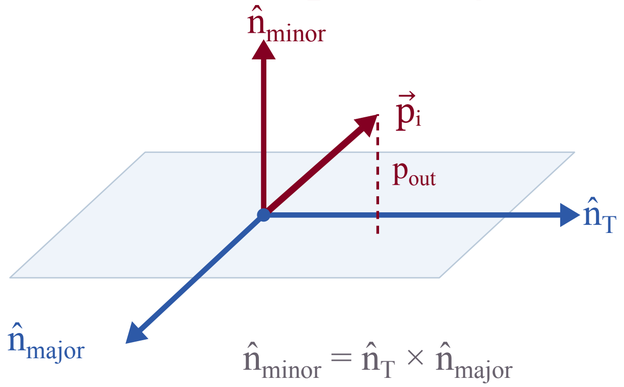


[Arpino, Banfi, El-Menoufi, 1912.09341]

Four-Jet Event Shape

First non-zero for a generic four-parton configuration

Non-planarity



$\hat{n}_{minor} = \hat{n}_T \times \hat{n}_{major}$

➤ Thrust minor

$$T_{minor} = \frac{\sum_i \left|\vec{p}_i \cdot \vec{n}_{minor}\right|}{\sum_i \left|\vec{p}_i\right|}$$

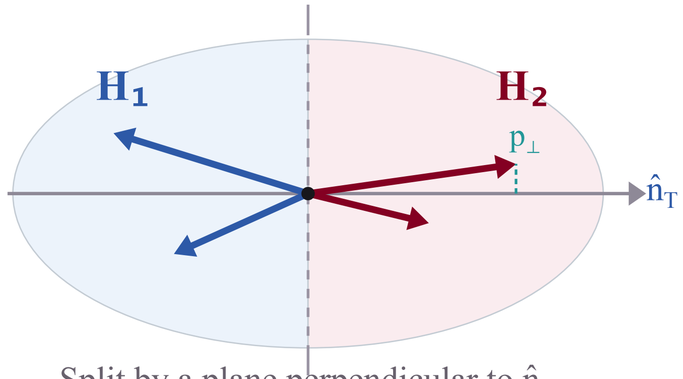
➤ D-parameter

linearised momentum tensor

$D = 27\,\lambda_1\,\lambda_2\,\lambda_3$ $\quad \Theta^{\alpha\beta} = \frac{1}{\sum_i |\vec{p}_i|} \sum_i \frac{p_i^\alpha p_i^\beta}{|\vec{p}_i|}$, $\alpha, \beta = 1,2,3$

Planar event: $\lambda_3 = 0 \rightarrow D = 0$

Hemisphere structure



Split by a plane perpendicular to $\hat{n}_T$

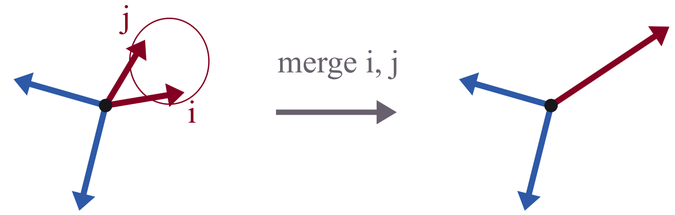
➤ Light jet mass

$\rho_L = \min(M_1^2, M_2^2) / Q^2$ $\quad M_i^2 = \left(\sum_{k \in H_i} p_k\right)^2$, $i = 1,2$

➤ Narrow jet broadening

$B_N = \min(B_1, B_2)$ $\quad B_i = \frac{\sum_{k \in H_i} \left|\vec{p}_k \times \vec{n}_T\right|}{2 \sum_j \left|\vec{p}_j\right|}$, $i = 1,2$

Four-jet resolution



4 jets $\qquad$ 3 jets

➤ Durham resolution parameter

$$y_{ij}^D = \frac{2\min(E_i^2, E_j^2)(1 - \cos\theta_{ij})}{Q^2}$$

y_{34}: smallest $y_{i,j}$ at the $4 \rightarrow 3$ step

Recombine four-momenta: $p_{i,j} = p_i + p_j$ (E-scheme)

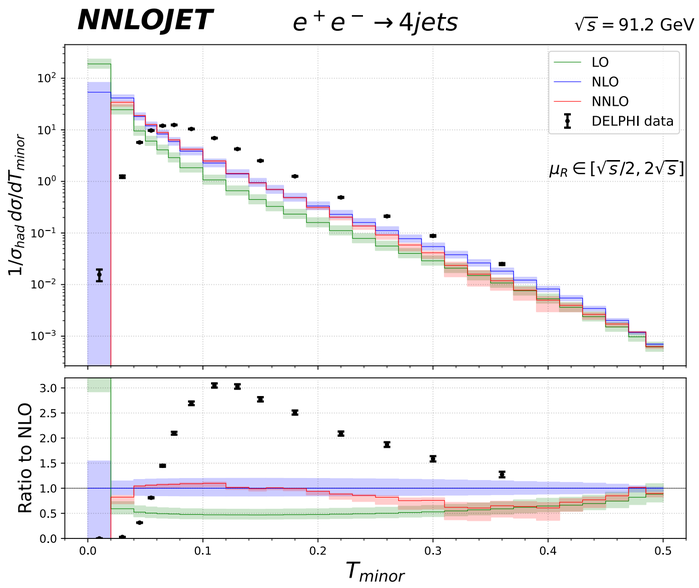
$Q = \sum_i |\vec{p}_i| = E_{vis}$ (massless final state)

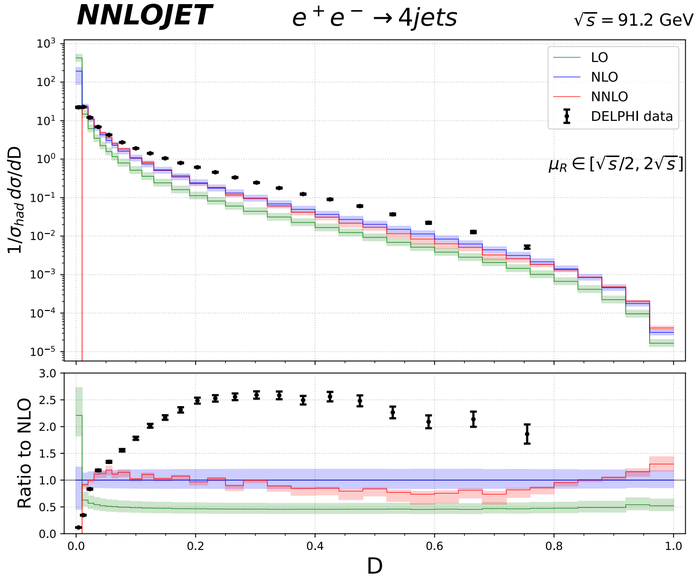
Small event shape (denote as y) region Large ln(1/y) terms require resummation as $y \rightarrow 0$

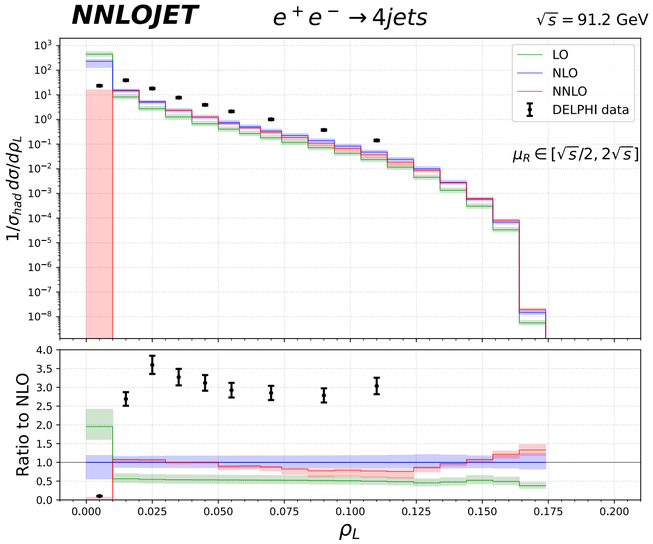
For the fixed-order study, restrict distributions with event shape cut $y > y_{min}$

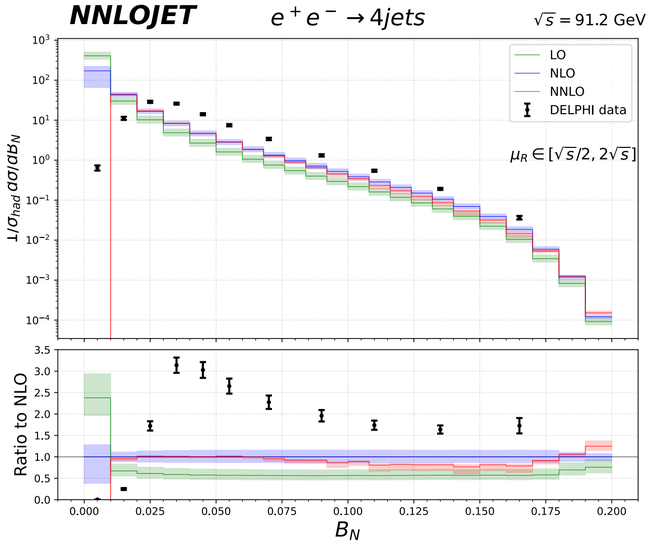
Four-Jet Event Shape@NNLO

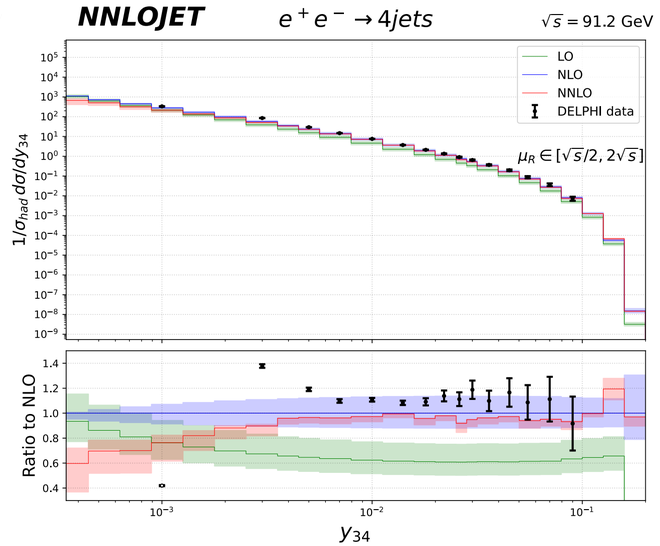
- **Intermediate region** improved perturbative convergence
  - ➤ Scale uncertainty substantially reduced from NLO to NNLO, overlapping scale bands
  - ➤ **Significant discrepancy with data**
- **Tail region** large NNLO corrections
  - ➤ Larger and asymmetric NNLO scale uncertainty
  - ➤ NNLO corrections turn **negative** and become **large,** enhance the discrepancy with data
- **Resummation and non-perturbative region**
  - ➤ Prediction turns negative and becomes large
  - ➤ Larger scale uncertainties
  - ➤ All-order resummation and non-perturbative correction are required











Monte Carlo Generator: SHERPA 3.0.4

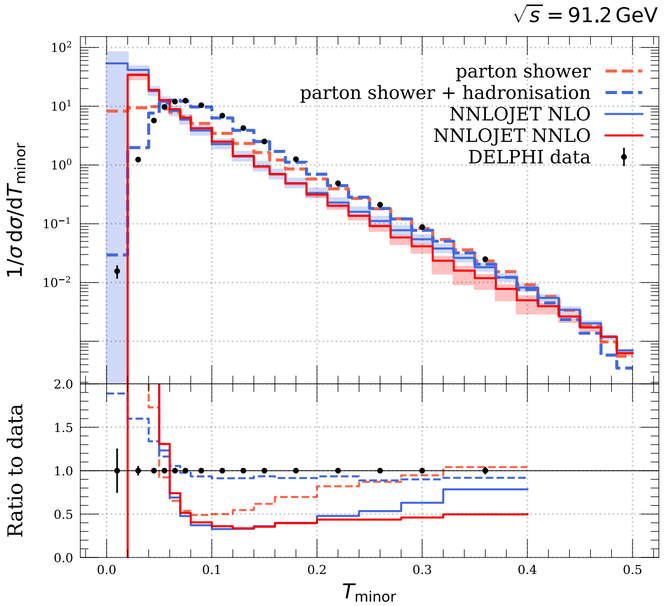
- Matrix Elements generator: AMEGIC COMIX OpenLoops

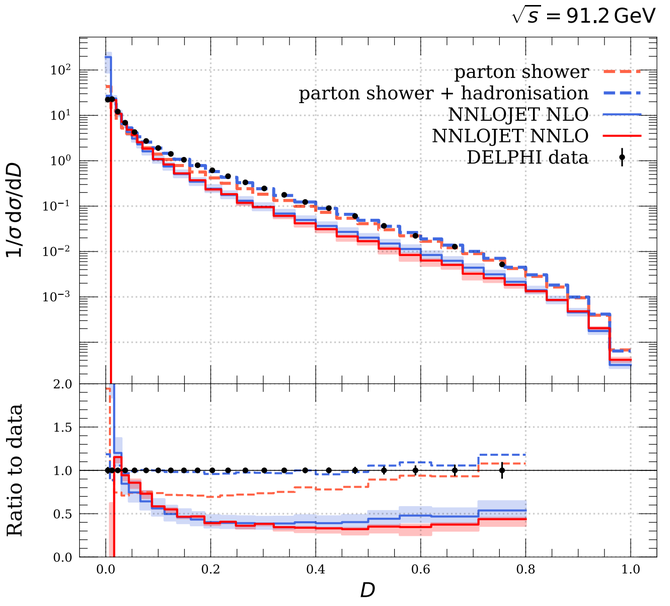
  $e^+e^- \to \gamma^*/Z \to$ **2, 3, 4-parton@NLO and 5-parton@LO** [0709.1027]

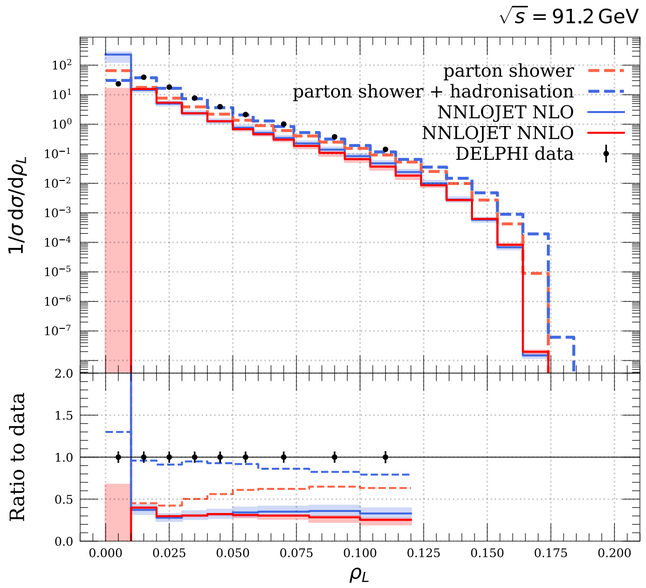
- Parton shower (leading logarithmic) CSSHOWER++

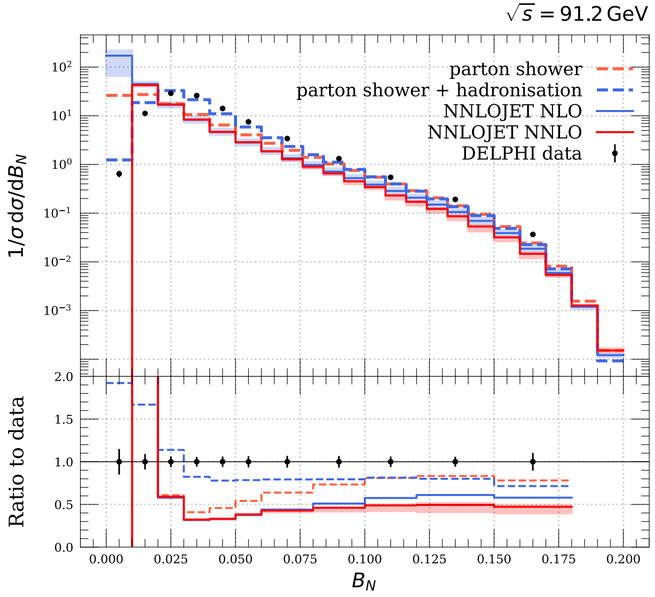
- Hadronisation Cluster fragmentation model [2203.11385]

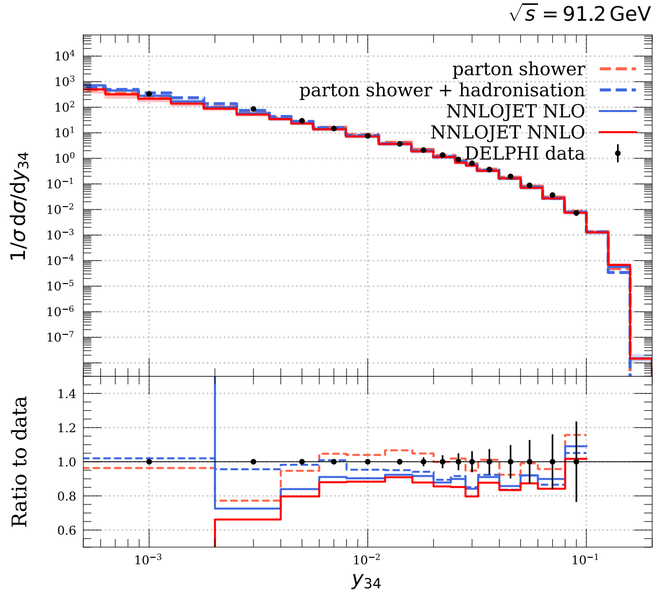
- Hadronisation improves T_{minor} and D over much of the range
- Differences remain for ρ_L and B_N; smaller hadronisation effects for y_{34}











Renormalization Scale Choice

SHERPA: dynamic scale μ_R

Fixed-order predictions: fixed scale $\mu_R = Q = \sqrt{s}$

Reconstructs a shower-like history; its branching and core scales determine the matrix elements scale

- Different branchings probe different momentum scales, often below Q
- The same history provides consistent scales for ME couplings, shower evolution and Sudakov factors.

➤ Durham clustering $$y_{ij} = \frac{2\min(E_i^2, E_j^2)(1-\cos\theta_{ij})}{Q^2}$$

➤ At each reconstructed vertex i, branching scale q_i is the relative transverse momentum of that clustering $q_i = k_{T,i} = Q\sqrt{y_{i-1,i}}$

5 partons — y_{45} — 4 jets — y_{34} — 3 jets — y_{23} — 2-jet core

Branching scale: $q_5 = Q\sqrt{y_{45}}$ $q_4 = Q\sqrt{y_{34}}$ $q_3 = Q\sqrt{y_{23}}$

The strong coupling in ME $\alpha_s(q_5^2)$ $\alpha_s(q_4^2)$ $\alpha_s(q_3^2)$ $\alpha_s(\mu_R^2)^n \to \prod_{i=1}^{n}\alpha_s(q_i^2)$

[Catani et al., JHEP 11 (2001) 063] [Gehrmann et al., JHEP 01 (2013) 144]

In process: study the clustering-based dynamic scale at fixed order

Summary & Outlook

Summary

We performed the NNLO calculation of the process $e^+e^- \to \gamma^*/Z \to jjjj$ within NNLOJET

- **The NNLO prediction for the four-jet rate**
  - The NNLO correction leads to improved agreement with LEP data
  - The theory uncertainty is substantially reduced from NLO to NNLO
  - All-order resummation and non-perturbative correction are required in the infrared region
- **The NNLO predictions for the four-jet event shape distributions**
  - Even after including NNLO correction, sizable differences from the LEP data remain
  - Effects (resummation, hadronisation ...) beyond fixed order are crucial to explain the data

Outlook

Four-jet event shape:

- Study the renormalization scale choice
- Match NNLO prediction with resummation
- Assess hadronisation corrections using analytic models

Thank you for your attention !