Quark-spin Operator:
matching from RI/MOM to $\overline{\mathrm{MS}}$ up to $\mathcal{O}(\alpha_s^5)$

Long Chen (陈龙)

山东大学物理学院
SCHOOL OF PHYSICS, SHANDONG UNIVERSITY

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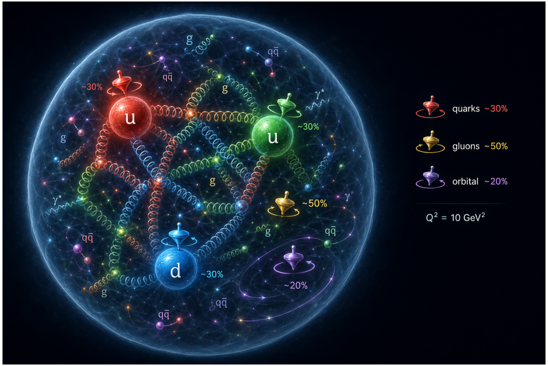
How Much Proton Helicity Comes From Quarks?

Nucleon matrix element of the axial current

$$S_q = \frac{1}{2}\Delta\Sigma(\mu) = \frac{1}{2}\left(g_A^u + g_A^d + g_A^s\right)$$

$$\boxed{\langle P, S| \Big[\sum_f \bar{q}_f \, \gamma^\mu \gamma_5 \, q_f \Big]_R |P, S\rangle = 2M_N \, S^\mu \, \Delta\Sigma(\mu)}$$

[$\rightarrow$See also talks by Liao, Chen, Shu]



Lattice QCD
bare matrix element

matching

Proton
$\Delta\Sigma(\mu)$

Polarized DIS
global col. PDFs fits

Why Singlet Axial Current Operator and Why Special

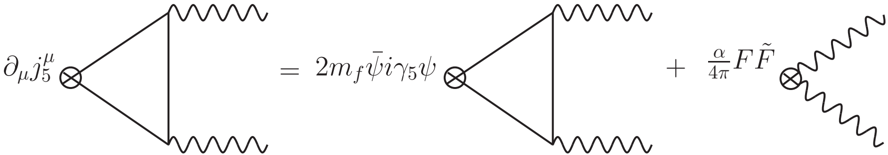
- **Connection to Quark Spin**:

$$\frac{u^\dagger(p,S)\Sigma^\alpha u(p,S)}{u^\dagger(p,S)u(p,S)} = \frac{\bar{u}(p,S)\gamma^\alpha\gamma_5\, u(p,S)}{u^\dagger(p,S)u(p,S)} = \frac{m}{E} S^\alpha$$

Spin is *indifferent* to the flavor (and charge etc) of the quarks $\Rightarrow$ *flavor-singlet* axial current operator

- **Axial anomaly calls for additional operator renormalization**

$$\boxed{\partial_\mu \sum_f \bar{\psi}_f\, \gamma^\mu \gamma_5\, \bar{\psi}_f\, \psi = \sum_f 2m_f \bar{\psi}_f\, i\gamma_5\, \psi_f \;-\; \frac{\alpha}{4\pi} n_f\, T_F\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}}$$

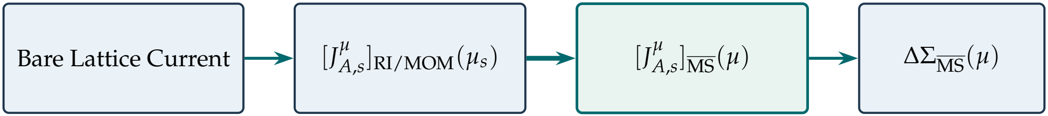


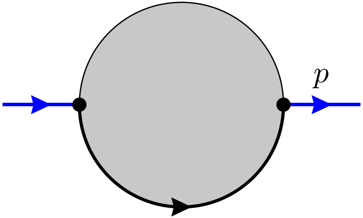
classical mass term + quantum anomaly

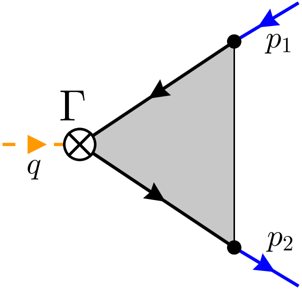
$$\langle P,S|\left[J^\mu_{A,s}\right]_R|P,S\rangle = 2M_N\, S^\mu\, \Delta\Sigma(\mu)$$

Bridge from Lattice RI/MOM to $\overline{\text{MS}}$

Finite Matching Factor $R_A(\mu, \mu_s)$





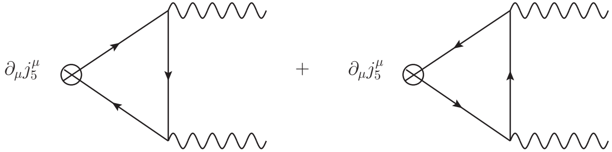


$$Z_{\mathcal{O}}\, \Gamma^{\mu}_{\alpha\beta}\, \hat{\mathcal{P}}^{\alpha\beta}_{\mu}\Big|_{p^2=\mu_s^2} = [\Gamma_{\text{tree}}]^{\mu}_{\alpha\beta}\, \hat{\mathcal{P}}^{\alpha\beta}_{\mu}\Big|_{p^2=\mu_s^2}; \qquad \text{RI/MOM}: \gamma^{\mu}; \quad \text{RI}'/\text{MOM}: \gamma^{\mu} - \not{p}\, p^{\mu}/p^2$$

RIMOM [Martinelli et al. 95] : $\quad p^2 = p_1^2 = p_2^2 = -\mu_s^2 < 0$ and $q = 0$

Proper $\overline{\text{MS}}$ with non-anticommuting γ_5 in DR

- γ_5-issue in DR with Axial Anomaly



"vanishes" with *translational invariant loop integrals* and an *anticommuting* γ_5, due to perturbative pinch identity

$$(\not{p}_1 + \not{p}_2)\,\gamma_5 = \cancel{(\not{l} + \not{p}_1 - m)\,\gamma_5 + \gamma_5\,(\not{l} - \not{p}_2 - m)} + 2\,m\,\gamma_5 + \Delta_5^{ac}(l)$$

- **HV/BM** [72, 79] NAC γ_5 prescription:

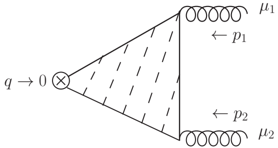
$$\boxed{\gamma_5 = -\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \quad \gamma_\mu\gamma_5 \to \frac{1}{2}\left(\gamma_\mu\gamma_5 - \gamma_5\gamma_\mu\right), \quad \Delta_5^{ac} \equiv \{\gamma^\mu, \gamma_5\} \sim \mathcal{O}(\epsilon) \neq 0\,.}$$

"Symmetrization" needed to have a **hermitian** axial-current [Akyeampong, Delbourgo 77; Fujii, Ohta, Taniguchi 81]

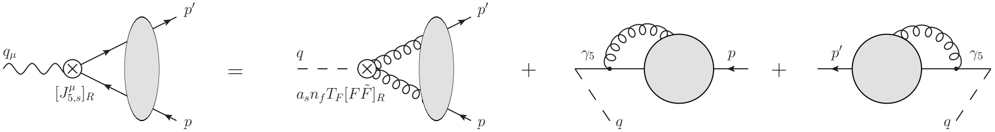
bare $J^\mu_{A,s}$ in $D = 4 - 2\epsilon$ $\xrightarrow{Z^{\overline{\text{MS}}}_{A,s}}$ pole-subtracted current $\xrightarrow{Z^f_{A,s}}$ **proper current** (gauge invariant anomaly preserving)

Proper $\overline{\text{MS}}$ with non-anticommuting γ_5 in DR

- Larin' s variant [Larin, Vermaseren 91; Larin 93] of HVBM prescription $\left[J_5^{\mu}\right]_R \equiv Z_5^f\, Z_5^{ms}\, \bar{\psi}_B\, \frac{-i}{3!}\epsilon^{\mu\nu\rho\sigma}\gamma_\nu\gamma_\rho\gamma_\sigma\, \psi_B$
  - $\epsilon^{\mu\nu\rho\sigma}$ treated outside R-operation formally in D dimensions [Larin, Vermaseren 91; Zijlstra, Neerven 92]
  - Determine $Z_s \equiv Z_s^f\, Z_s^{ms}$ by maintaing ABJ-equation $\boxed{\left[\partial_\mu J_5^{\mu}\right]_R = a_s\, n_f\, \left[F\tilde{F}\right]_R}$ with the standard $\overline{\text{MS}}$-renormalized $\left[F\tilde{F}\right]_R$ and ϵ-independent Z_s^f
  - Larin's original implementation [Larin 93] using AVV-amplitude: Inefficient! (L-integral diagram $\rightarrow Z_s$ @ $\mathcal{O}(\alpha_s^{L-1})$)



- Offshell Axial Ward-Takahashi Identity but at awkward kinematics: **Efficient but inconvenient** [LC, Czakon 21]



A Convenient Prescription for Calculating $Z_s^f \, Z_s^{ms}$

Combining the off-shell Axial WTIs for both singlet and non-singlet currents

$$q_\mu \, \Gamma^\mu_{5,s}(p', p) = a_s \, n_f \, \mathrm{T}_F \, q_\mu \, \Gamma^\mu_K(p', p) + \gamma_5 \, \hat{S}^{-1}(p) + \hat{S}^{-1}(p') \, \gamma_5 \, ,$$
$$q_\mu \, \Gamma^{a,\mu}_{A,ns}(p', p) = t^a \gamma_5 \, \hat{S}^{-1}(p) + \hat{S}^{-1}(p') \, \gamma_5 t^a$$

and by **first differentiating in q and then taking the forward limit**, we now attempt

$$\boxed{\Gamma^\mu_{A,s}(p, p) = a_s \, n_f \, \mathrm{T}_F \, \Gamma^\mu_K(p, p) + \Gamma^\mu_{A,ns}(p, p)}$$

- Makes the determination of the proper $\overline{\mathrm{MS}}$-renormalization $Z_{A,s} = Z^f_{A,s} \, \overline{Z}_{A,s}$ as straightforward as for the non-singlet counterpart (quasi-current-level relation)

- An observation: Only when restricted to the 4-dimensional limit, then follows the same *universal* **gauge-independent** (proper) $\overline{\mathrm{MS}}$-renormalization $Z_{A,s}$ for $J^\mu_{A,s}$

  (beyond which gauge and subtraction-point dependent differences start to appear)

A Short-cut Formula for R_A

Combining the RI/MOM R.C. $Z_{\mathcal{O}}\, \Gamma^{\mu}_{\alpha\,\beta}\, \hat{\mathcal{P}}^{\alpha\,\beta}_{\mu}\Big|_{p^2=\mu_s^2} = \left[\Gamma_{\text{tree}}\right]^{\mu}_{\alpha\,\beta}$ with the previously derived

$$\boxed{\Gamma^{\mu}_{A,s}(p,p) = a_s\, n_f\, \mathrm{T}_F\, \Gamma^{\mu}_{K}(p,p) \,+\, \Gamma^{\mu}_{A,ns}(p,p)}$$

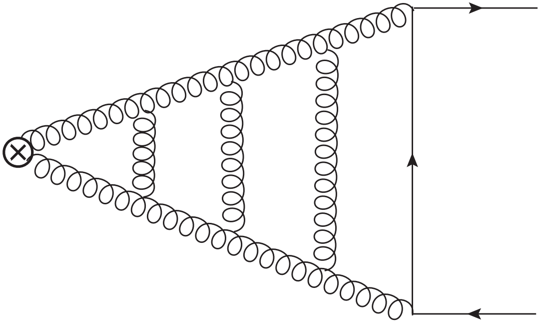
we obtain the short-cut formula for R_A:

$$\boxed{R_A \equiv \frac{\overline{Z}_A}{Z_A^{\text{RI}}}\Big|_{\epsilon\to 0} = 1 + \frac{\left(a_s\, n_f\, \mathrm{T}_F\, Z_q\, \left[\Gamma_K\right]_R\right)^{\mu}_{\alpha\,\beta}\, \hat{\mathcal{P}}^{\alpha\,\beta}_{A,\mu}}{\left[\Gamma_A^{\text{tree}}\right]^{\mu}_{\alpha\,\beta}\, \hat{\mathcal{P}}^{\alpha\,\beta}_{A,\mu}}\Big|_{p^2=\mu_s^2}}$$

- Reduced to the matrix element of the **properly** $\overline{\text{MS}}$**-renormalized** J^{μ}_{A} but with RI/MOM-renormalized external quark fields
- Extract the result at $\mathcal{O}(\alpha_s^{N+1})$ based on N-loop computation of the hardest part
- Feasible to extend to other (such as RI$'$/MOM) but NOT all renormalization schemes

Technical Challenge

A representative Feynmann diagram out of 9622 four-loop diagrams



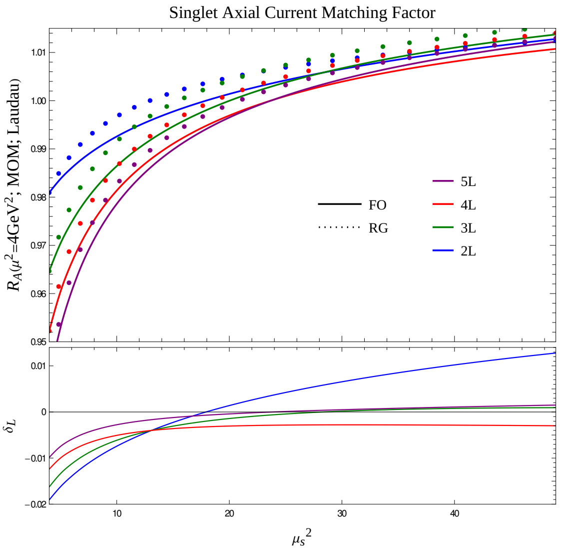
On top of the **multiple triple-gluon vertices, additional combinarical inflation**:

- Generic ξ-gauge (Landau gauge): $2^{11} = 2048$
- $\epsilon^{\mu_1\,\mu_2\,\mu_2\,\mu_4}\,\epsilon^{\nu_1\,\nu_2\,\nu_2\,\nu_4} \longrightarrow 24$ terms
- Lastest FORM 5 with manual tweaks for improving efficiency (thanks to J.Davies)

Results for Matching from RI/MOM to $\overline{\mathrm{MS}}$

- Pertubative convergence(!) through five loops
- Five-loop correction below 1%
- Smaller corrections with increasing μ_s
- Fixed-order and RG approach and agree better at five-loop and more with increasing μ_s

$$R_A(\mu_t/\mu_s) = R_A(\mu_i/\mu_s) \times \exp\left[\int_{\alpha_s(\mu_i)}^{\alpha_s(\mu_t)} \frac{\gamma_s(\alpha_s)}{\beta(\alpha_s)\alpha_s}\, d\alpha_s\right]$$



Results for Matching from RI/MOM to $\overline{\text{MS}}$

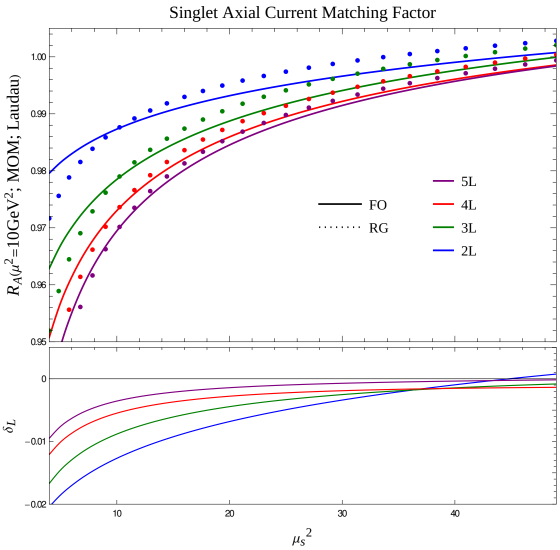
- Pertubative convergence through five loops
- Five-loop correction below 1%
- Smaller corrections with increasing μ_s
- Fixed-order and RG approach and agree better at five-loop and more with increasing μ_s and μ

$\boxed{\mu = 2\,\text{GeV}}$

↓

$\boxed{\mu = \sqrt{10}\,\text{GeV}}$

Convergence further improves



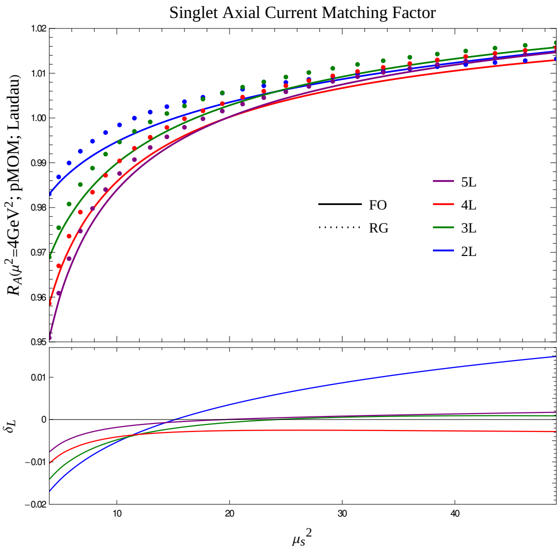
Results for Matching from RI′/MOM to $\overline{\text{MS}}$

- Pertubative convergence through five loops
- Five-loop correction at few per-milles
- Smaller corrections with increasing μ_s
- Fixed-order and RG approach and agree better at five-loop and more with increasing μ_s and μ

Better convergence in RI′/MOM v.s. RI/MOM

Transverse structure projector

$$\mathcal{P}^{\mu}_{V,\perp} = \gamma^{\mu}_{\alpha\,\beta} - \not{p}_{\alpha\,\beta}\, p^{\mu}/p^2$$



Summary and Outlook

The flavor-singlet axial-vector current operator J_A^μ has played a crucial role in determining the $\overline{\text{MS}}$ quark-helicity contribution to the total helicity/spin of the nucleon.

- ☑ An **efficient variant** of Larin's renormalization condition for J_A^μ in DR
  (renders $Z_{A,s} = Z^f_{A,s}\, \overline{Z}_{A,s}$ as straightforward as for the non-singlet counterpart)

- ☑ An observation that $Z_{A,s} = Z^f_{A,s}\, \overline{Z}_{A,s}$ so-determined remains *universal* (e.g. independent of the QCD gauge-fixing ξ) only if the anomalous AWTI is required to hold in **4-dimensional limit** (rather than with full ϵ dependence).

- ☑ State-of-the-art result for the finite matching coefficient R_A at the unprecedented order $\mathcal{O}(\alpha_s^5)$
  - Pertubative convergence through five loops explicitly **demonstrated**, with the pure 5-loop correction generally less than 1% and better in RI′/MOM

- Practical application to improve the accuracy of the Lattice determination of

$$S_q = \frac{1}{2}\Delta\Sigma(\mu) = \frac{1}{2}\left(g_A^u + g_A^d + g_A^s\right)$$

Thank you for listening!