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Postmodern Fermi Liquid

Reporter: Zi-Hao Liu

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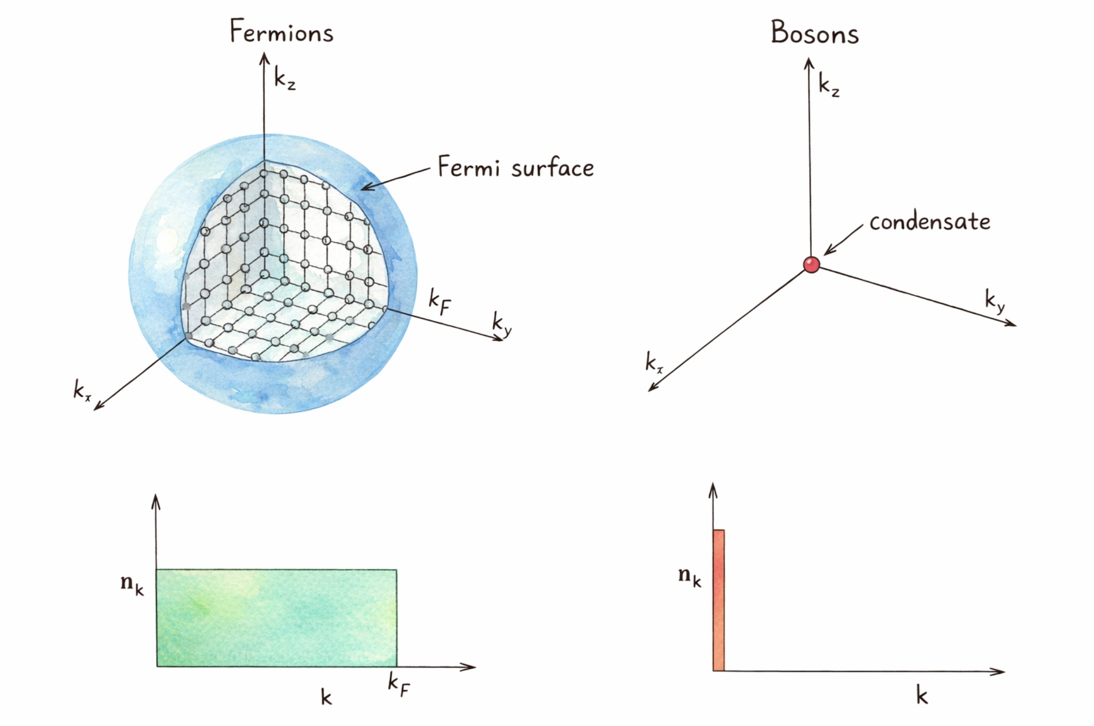
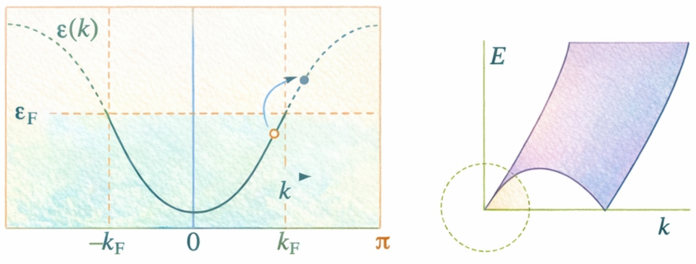
1. Setting the Stage

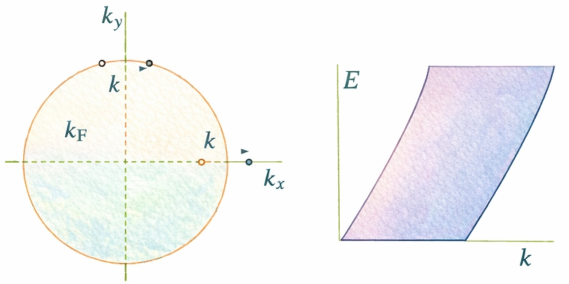
Why “Postmodern” Fermi Liquids?

30-Second Big Picture (What this Work Claims)





Particle-hole excitations in one dimension.



Particle-hole excitations in two dimensions.

- One-Line Goal: Effective Theory from Phase-Space Symmetry
- Key Objects: Canonical Transformations, Coadjoint Orbits
- Roadmap: Algebra → Hamiltonian → Action → Symmetries → Gauging

Motivation: Why "Postmodern"?

- Bottleneck: Lack of a Systematic EFT for Fermi Liquids
- Need: Classify Irrelevant Corrections with Definite Scaling
- Claim: Hidden Geometric Structure Rigidly Constrains EFT
- Bonus: Symmetry Implementation and Gauge Coupling Become Systematic
- Outlook: Stepping Stone toward Perturbative NFLs

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2. From Landau to RG

The Standard Fermi-Liquid Toolkit

Landau Starting Point (For Context)

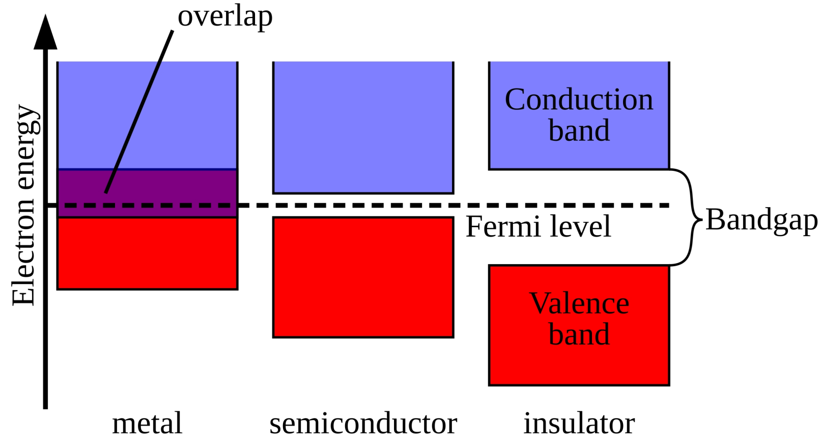
- Ground State Occupation $f_0(p) = \Theta(\epsilon_F - \epsilon(p))$
- Fermi Surface $\epsilon(p) = \epsilon_F \Rightarrow |p| = p_F(\theta)$
- Landau Kinetic Equation (Nonlinear in Interacting Case)
- Equation

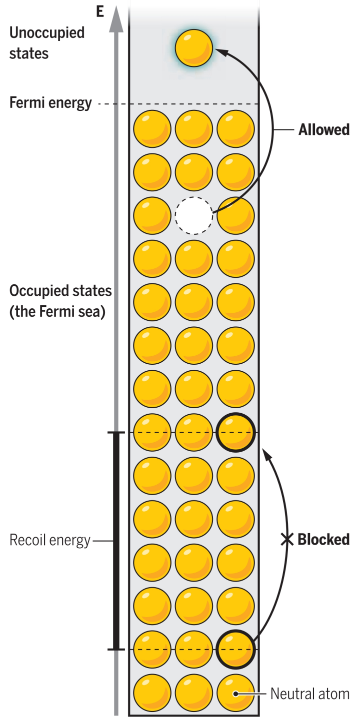
$$\frac{df}{dt} = 0 \qquad (\dot{\mathbf{x}} = \nabla_{\mathbf{p}}\epsilon(\mathbf{p}), \quad \dot{\mathbf{p}} = \mathbf{F}_{\text{ext}})$$

$$\partial_t f + \nabla_{\mathbf{p}}\epsilon(\mathbf{p}) \cdot \nabla_{\mathbf{x}} f + \mathbf{F}_{\text{ext}} \cdot \nabla_{\mathbf{p}} f = 0$$

- Interacting Theory ⇒ Quasiparticle

$$f(t,\mathbf{x},\mathbf{p}) = f_0(\mathbf{p}) + \delta f(t,\mathbf{x},\mathbf{p}) \qquad |\mathbf{p}| - p_F \ll p_F$$





Landau Starting Point (For Context)

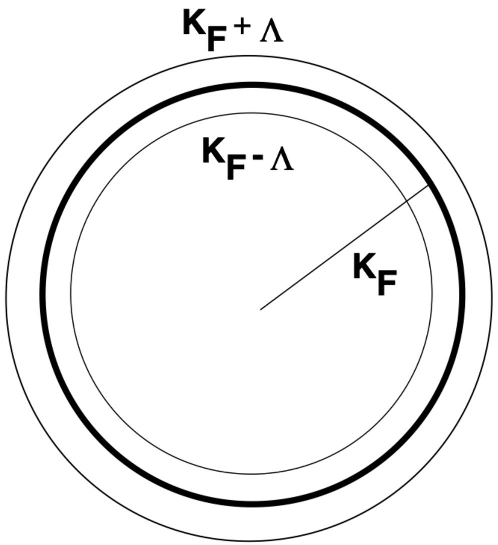
- Dispersion of Quasiparticles $\epsilon_{\mathrm{qp}}(\mathbf{x},\mathbf{p}) = \epsilon(\mathbf{p}) + \int \frac{d^d p'}{(2\pi)^d} F(\mathbf{p},\mathbf{p}') \delta f(\mathbf{x},\mathbf{p}')$
  - Assumption: Any interaction between Quasiparticles is Short-Ranged ⇒ Interaction Term in the Quasiparticle Energy is Local in Space
  - Collisionless Boltzmann Equation / Landau Kinetic Equation

$$\partial_t f + \nabla_{\mathbf{p}}\epsilon(\mathbf{p}) \cdot \nabla_{\mathbf{x}} f + \mathbf{F}_{\mathrm{ext}} \cdot \nabla_{\mathbf{p}} f = 0 \qquad \dot{\mathbf{x}} = \nabla_{\mathbf{p}}\epsilon(\mathbf{p}) \quad \dot{\mathbf{p}} = \mathbf{F}_{\mathrm{ext}}$$

$$\Downarrow$$

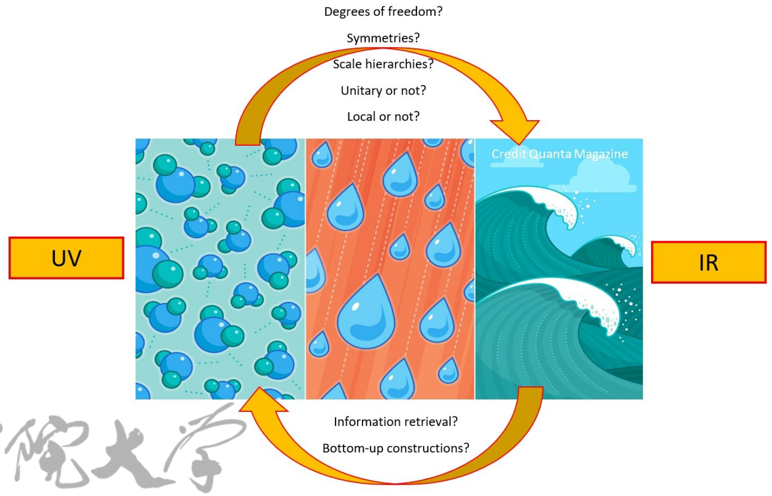
$$\partial_t f + \nabla_{\mathbf{p}}\epsilon_{\mathrm{qp}}[f] \cdot \nabla_{\mathbf{x}} f - (\nabla_{\mathbf{x}}\epsilon_{\mathrm{qp}}[f] - \mathbf{F}_{\mathrm{ext}}) \cdot \nabla_{\mathbf{p}} f = 0 \qquad \dot{\mathbf{x}} = \nabla_{\mathbf{p}}\epsilon_{\mathrm{qp}}[f] \quad \dot{\mathbf{p}} = \mathbf{F}_{\mathrm{ext}} - \nabla_{\mathbf{x}}\epsilon_{\mathrm{qp}}[f]$$

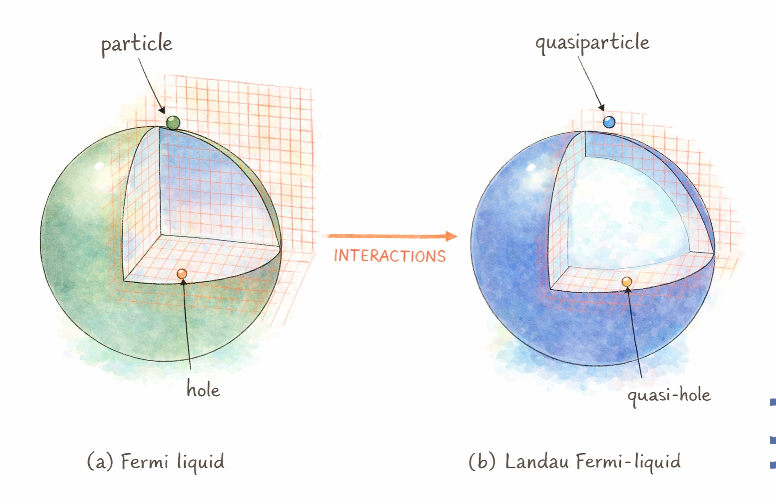
- Landau Parameters Expansion $F(\theta,\theta') \sim \sum_l F_l P_l^{(d)}(\theta,\theta')$



Limitations of Landau's Theory

- No Microscopic Derivation: Landau Theory Introduced Phenomenological Parameters
- No RG or Quantum Corrections: Essentially a Classical (Mean-Field) Framework
- Insensitive to Subleading Corrections: Focuses on Near-Fermi-Surface, Omits Higher Derivatives
- Limited Validity: Cannot Easily Handle Critical Fluctuations or NFL Regimes
- Lacks Gauge Fields Coupling: Difficult to Incorporate Electromagnetic or Long-Range Forces





Modern RG Approach (Shankar–Polchinski)

$\psi^\dagger(\mathbf{p})$:creates a quasiparticle with momentum $\mathbf{p}$

$\psi(\mathbf{p})$:creates a hole at the point $-\mathbf{p}$ (in the Fermi sea)

- Low-Energy Focus
  - Linearize Dispersion around Fermi Surface, Integrate out High-Energy Modes
  - Free Action $S_0 = \int dt \frac{d^d p}{(2\pi)^d} \psi^\dagger(\mathbf{p})\left[i\partial_t - \left(\epsilon(\mathbf{p}) - \epsilon_F\right)\right]\psi(-\mathbf{p}) \qquad \epsilon(\mathbf{p}) - \epsilon_F = \frac{\mathbf{p}^2}{2m} - \frac{\mathbf{p}_F^2}{2m}\,.$
  - Linearize $\mathbf{p} = \mathbf{p}_F + \mathbf{k} \qquad d^d p = d^{\,d-1} p_F \, dk$
    - $\epsilon(\mathbf{p}) = \epsilon(\mathbf{p}_F + \mathbf{k}) \simeq \epsilon(\mathbf{p}_F) + \nabla_\mathbf{p}\epsilon(\mathbf{p})|_{\mathbf{p}_F}\cdot\mathbf{k} + \frac{1}{2}\mathbf{k}^\mathsf{T}\left(\nabla_\mathbf{p}\nabla_\mathbf{p}\epsilon(\mathbf{p})\right)|_{\mathbf{p}_F}\mathbf{k} + \cdots$

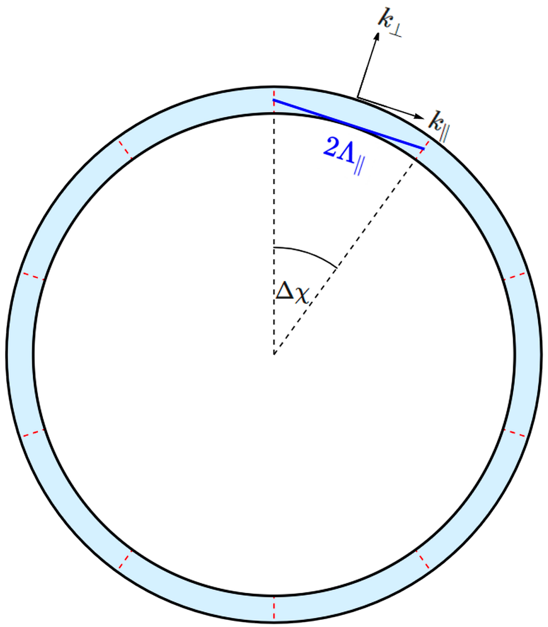
$$\epsilon(\mathbf{p}) \simeq \epsilon_F + \mathbf{k}\cdot\mathbf{v}_F(\mathbf{p}_F) + \mathcal{O}(k^2), \qquad \mathbf{v}_F(\mathbf{p}_F) \equiv \nabla_\mathbf{p}\epsilon(\mathbf{p})|_{\mathbf{p}_F}.$$

$$\Downarrow$$

$$\epsilon(\mathbf{p}) - \epsilon_F = |\mathbf{k}|\,|\mathbf{v}_F(\mathbf{p}_F)| + \mathcal{O}(k^2)\,.$$

- $\hat{\mathbf{n}}(\mathbf{p}_F) = \frac{\mathbf{v}_F(\mathbf{p}_F)}{|\mathbf{v}_F(\mathbf{p}_F)|} \qquad k_\perp \equiv \mathbf{k}\cdot\hat{\mathbf{n}} \qquad \mathbf{k}_\parallel \equiv \mathbf{k} - k_\perp\hat{\mathbf{n}}$

$$\frac{k_\parallel^2}{2m} \lesssim \Lambda \quad \Rightarrow \quad |k_\parallel| \lesssim \sqrt{2m\Lambda} \sim \sqrt{k_F\Lambda} \qquad \psi(\mathbf{k}) \approx \sum_{i=1}^{N(\Lambda)} \psi_i(k_\perp, \omega)$$



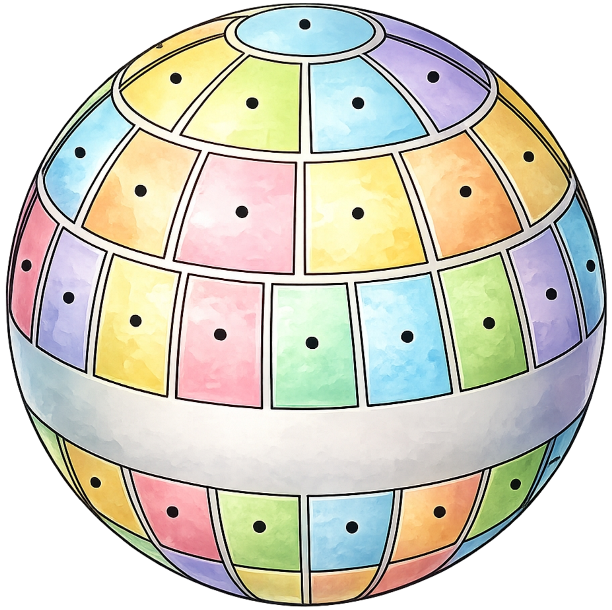
- Scaling Dimension

$$[\partial_t] = [k], \qquad [\psi] = -\frac{1}{2}$$

$$\omega \sim v_F|\mathbf{k}| \;\Rightarrow\; [\omega] = [k] = 1 \qquad [t] = -[\omega] = -1$$

$$[S] = 0 = 2[\psi] - 1 + 1 + 0$$

Modern RG Approach (Shankar–Polchinski)



- Patch Decomposition
  - Divide Fermi Surface into Patches for RG Analysis
- 4-Fermion Interactions

$$S_{\text{int}} = \int dt \int d^d x_1 d^d x_2 d^d x_3 d^d x_4 \mathcal{V}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \psi^\dagger(\mathbf{x}_1) \psi(\mathbf{x}_2) \psi^\dagger(\mathbf{x}_3) \psi(\mathbf{x}_4)$$

$$\psi(\mathbf{x}) = \int_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \psi(\mathbf{p}) \quad \psi^\dagger(\mathbf{x}) = \int_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \psi^\dagger(\mathbf{p})$$

$$\mathcal{V}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \int_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4} V(\mathbf{p}_{F1}, \mathbf{p}_{F2}, \mathbf{p}_{F3}, \mathbf{p}_{F4}) e^{i(\mathbf{p}_1\cdot\mathbf{x}_1 + \mathbf{p}_2\cdot\mathbf{x}_2 + \mathbf{p}_3\cdot\mathbf{x}_3 + \mathbf{p}_4\cdot\mathbf{x}_4)} \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4)$$

$$\Downarrow$$

$$S_{\text{int}} = \int_t \int_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4} V(\mathbf{p}_{F1}, \mathbf{p}_{F2}, \mathbf{p}_{F3}, \mathbf{p}_{F4}) \psi^\dagger(\mathbf{p}_1) \psi(\mathbf{p}_2) \psi^\dagger(\mathbf{p}_3) \psi(\mathbf{p}_4) \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4)$$

$$\delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) = \delta\Big((\mathbf{p}_{F1} + \mathbf{k}_1) + (\mathbf{p}_{F2} + \mathbf{k}_2) + (\mathbf{p}_{F3} + \mathbf{k}_3) + (\mathbf{p}_{F4} + \mathbf{k}_4)\Big) = \delta(\mathbf{p}_F + \mathbf{k})$$

$$\mathbf{p}_F \equiv \mathbf{p}_{F1} + \mathbf{p}_{F2} + \mathbf{p}_{F3} + \mathbf{p}_{F4} \qquad \mathbf{k} \equiv \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4$$

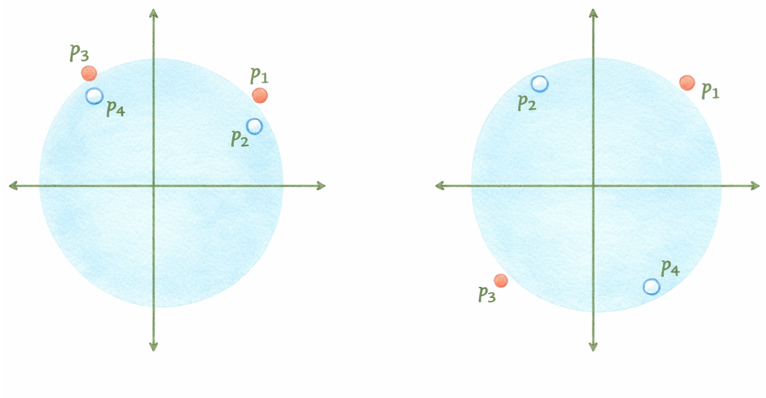
Modern RG Approach (Shankar–Polchinski)

- 4-Fermion Interactions
  - Scaling Dimension of Delta Function

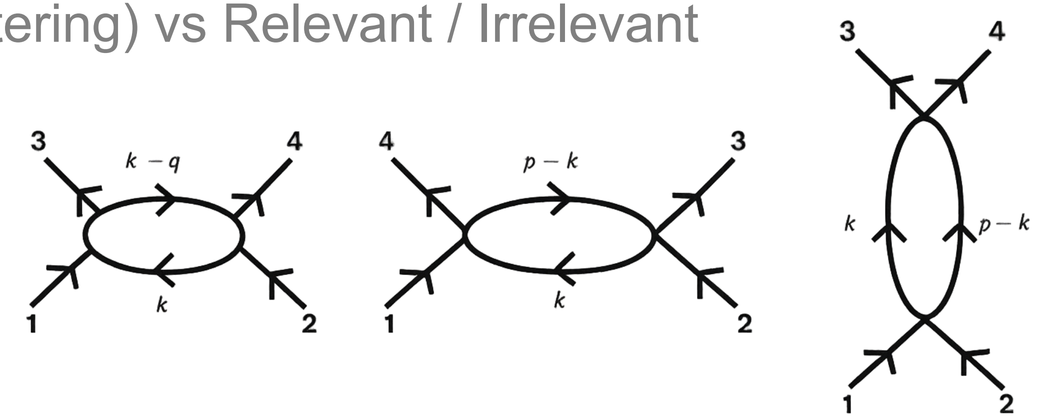
$$\mathbf{p}_F = \mathbf{0}: \ \delta(\mathbf{p}_F + \mathbf{k}) = \delta(\mathbf{0} + \mathbf{k}) \ \Rightarrow \ \delta(s\mathbf{k}) = s^{-d}\delta(\mathbf{k}) \ \Rightarrow \ [\delta] = -d$$

$$\mathbf{p}_F \neq \mathbf{0}: \ \delta(\mathbf{p}_F + \mathbf{k}) \simeq \delta(\mathbf{p}_F) \quad |\mathbf{p}_F| \gg |\mathbf{k}| \ \Rightarrow \ [\delta] = 0$$

  - Classify into Marginal (e.g. Forward Scattering) vs Relevant / Irrelevant



(a) Forward scattering (b) BCS channel



t : Particle–Hole Channel u : Exchange/Crossed Particle Hole Channel s : BCS/Cooper (Particle–Particle) Channel

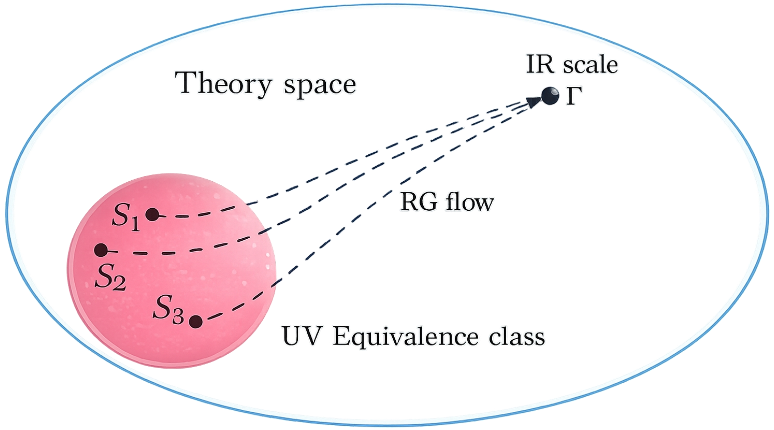
  - Form Factor

$$F(\mathbf{p}_{F1}, \mathbf{p}_{F3}) = V(\mathbf{p}_{F1}, -\mathbf{p}_{F1}, \mathbf{p}_{F3}, -\mathbf{p}_{F3}) \qquad g(\mathbf{p}_{F1}, \mathbf{p}_{F2}) = V(\mathbf{p}_{F1}, \mathbf{p}_{F2}, -\mathbf{p}_{F1}, -\mathbf{p}_{F2})$$

$$F(\mathbf{p}_F, -\mathbf{p}_F) = g(\mathbf{p}_F, -\mathbf{p}_F)$$

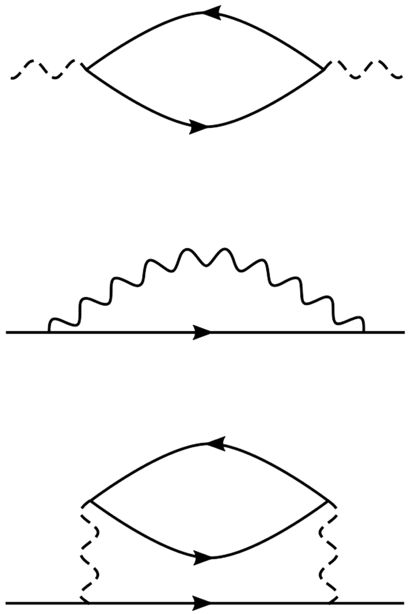
Modern RG Approach (Shankar–Polchinski)

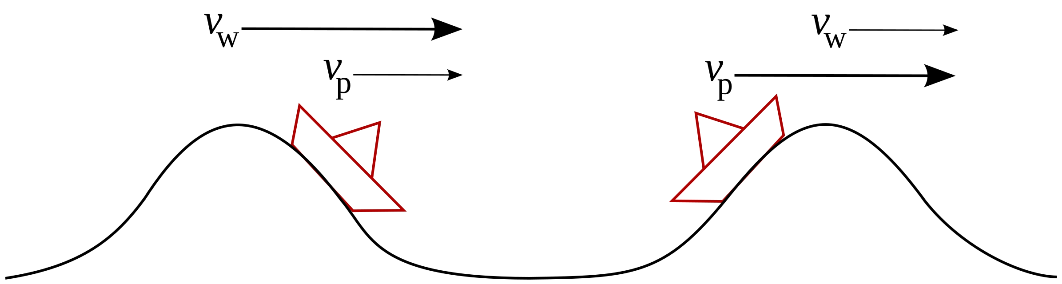
- Marginal Part $\int_{\mathbf{p}_1\mathbf{p}_3} F\left(\mathbf{p}_{F1},\mathbf{p}_{F3}\right)\left[\psi^\dagger\psi\psi^\dagger\psi\right]_{\text{ph}}\left(\mathbf{p}_1,\mathbf{p}_3\right)+\int_{\mathbf{p}_1\mathbf{p}_2} g\left(\mathbf{p}_{F1},\mathbf{p}_{F2}\right)\left[\psi^\dagger\psi\psi^\dagger\psi\right]_{\text{BCS}}\left(\mathbf{p}_1,\mathbf{p}_2\right)$
- Flow of Couplings
  - Landau Parameters Emerge as Fixed-Point Values of Running Couplings
- Systematic
  - Derives Fermi Liquid as a Stable Fixed Point (under RG) with Controlled Corrections



Limitations of RG Approach

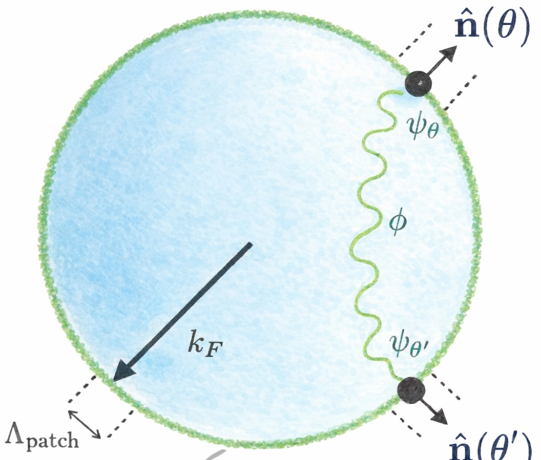
- Artificial Momentum Cutoffs: Patching Scheme Introduces an Arbitrary Scale
- Incomplete for Currents: Difficulty Handling Finite Momentum q Interactions, Coupling to A_μ
- Scaling Issues: Unclear Treatment of Certain Marginal Operators (e.g. BCS Channel needs Special Handling)
- Gauge Fields: Integrating out can Miss Singular Effects (e.g. Landau Damping of Gauge Bosons)
- Complex for Higher Corrections: Hard to Include Higher-Derivative Terms beyond Leading Order

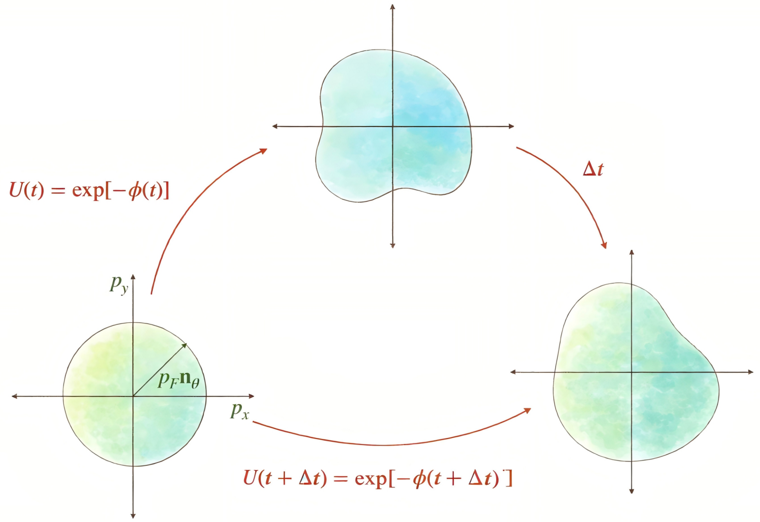




Patch Bosonization (Contemporary Approach)

- Multidimensional Bosonization: Treat Fermi Surface as Many 1D Slices (Patches)
- Patch Fermions: $\psi_i(\mathrm{patch})$ Represented by Bosonic Density Waves ϕ_i
- Chiral Boson Modes: Map Particle-Hole Excitations to Bosonic Fields on each Patch
- Captures Collective Modes: Zero Sound and Particle-Hole Continuum via Bosons
- Examples: 2D Fermi Surface Bosonization (Haldane's approach), "Fermi Surface Hydrodynamics"





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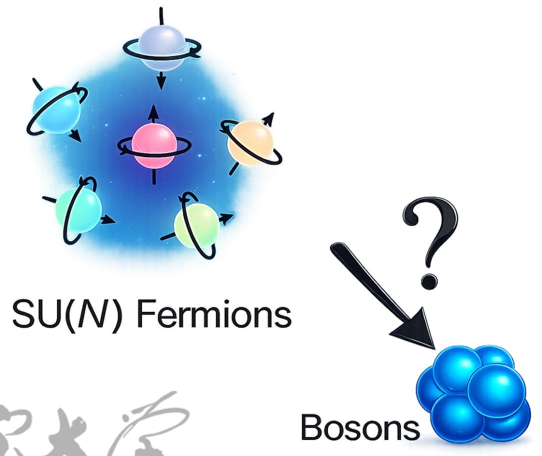

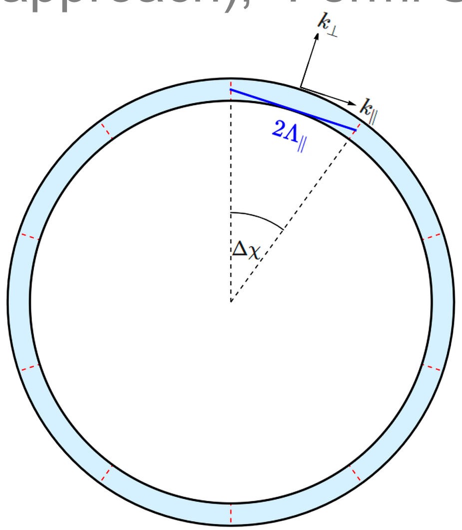

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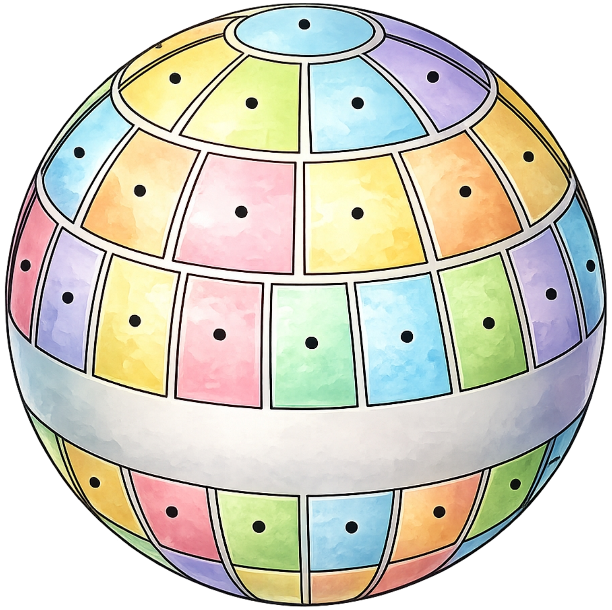
Issues with Patch Bosonization



- Fermion–Boson Mismatch: Requires Klein Factors to Enforce Anticommutation (ad hoc Fix)
- Non-Abelian Internal Symmetry: Spinful Fermions need Extra WZW Terms (not in Patch Schemes)
- Patch Overlaps: Ensuring Consistency between Patches is Non-Trivial (Gauge of Patch Choice)
- Limited Systematics: Lacks a Clear Power-Counting; Difficult to Go beyond RPA-Type Approximations
- Partial Success: Describes some Low-Energy Physics, but not a Complete Effective Theory for Fermi Liquids

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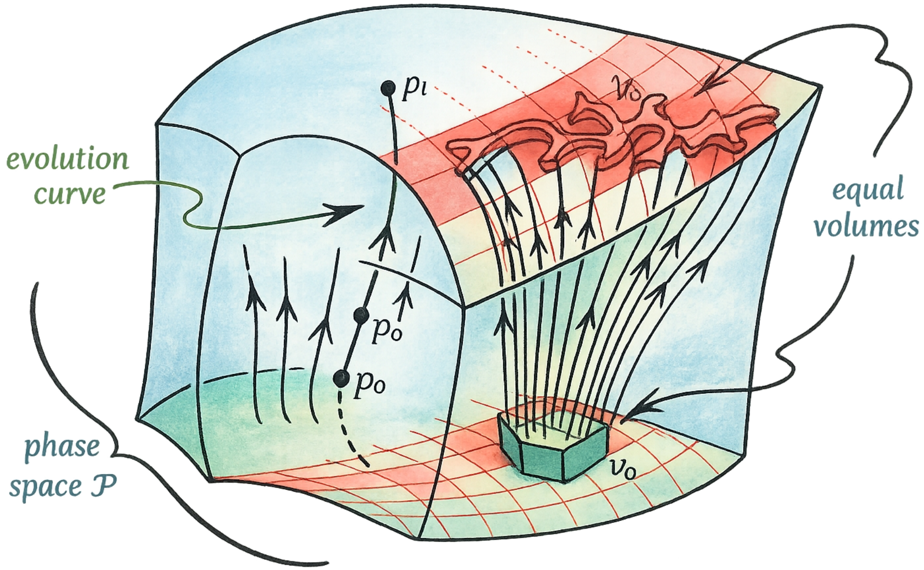
3. Postmodern Fermi Liquids: The Big Picture

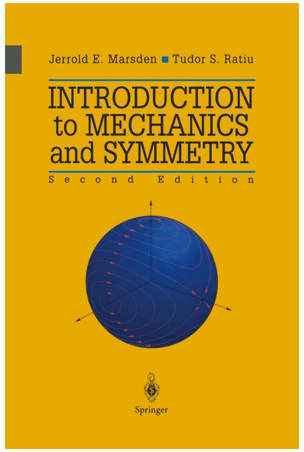
Postmodern Fermi Liquid Formalism – Concept

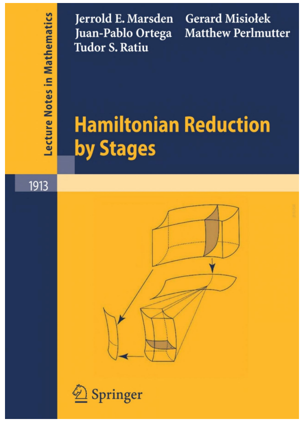
- New Framework: Exploit an Infinite-Dimensional Symmetry underlying Fermi Liquids
- Canonical Transformations Group: All Volume-Preserving Phase-Space Maps as Symmetry
- Lie Algebraic Structure: Build Theory from Algebra of Fermion Bilinears (Density Operators)
- Nonlinear Bosonization: Construct an Action for Fermi Surface Fluctuations (Bosonic Fields)
- Recover Landau Theory: with Quantum/Systematic Corrections



Conceptual Shift: From “Quasiparticles” To “Group Geometry”

- Start from Microscopic Fermions $\psi(x)$
- Focus on Closed Subalgebra: Charge-0 Fermion Bilinears
- Generator (Schematic): $T(x,y) \sim i\psi^\dagger(x)\psi(y)$
- Use Coadjoint Orbit Method as Organizing Principle
- Physical Payoff: Systematic Corrections + Symmetry + Gauging





Lie Algebra of Fermion Bilinears

费米子双线性算符生成的李代数 $\mathfrak{g}$ 同构于
" 相空间函数在 Moyal 括号下的代数 "
（量子 w_∞/W_∞ 结构）

- Generators: Fermion bilinear operators $T(\boldsymbol{x},\boldsymbol{y})$ form a basis
- Weyl–Wigner correspondence (Operator – Symbol Correspondence)
  - Linear Conbination $O_F \equiv \int d^dx\, d^dy\, F(\mathbf{x},\mathbf{y})\, T(\mathbf{x},\mathbf{y}) \sim i \int d^dx\, d^dy\, F(\mathbf{x},\mathbf{y})\, \psi^\dagger(\mathbf{x})\psi(\mathbf{y})$
  - Wigner Transform $T(\mathbf{x},\mathbf{p}) \equiv \int d^dy\, T\Big(\mathbf{x}+\frac{\mathbf{y}}{2},\, \mathbf{x}-\frac{\mathbf{y}}{2}\Big) e^{i\mathbf{p}\cdot\mathbf{y}}$
  - Elements of the Lie Algebra Expressed in $\boldsymbol{x}, \boldsymbol{p}$

$$O_F = \int d^dx_1\, d^dx_2\, F(\mathbf{x}_1,\mathbf{x}_2)\, T(\mathbf{x}_1,\mathbf{x}_2)$$

$$= \int d^dx\, d^dy \left|\frac{\partial(\mathbf{x}_1,\mathbf{x}_2)}{\partial(\mathbf{x},\mathbf{y})}\right| F\Big(\mathbf{x}+\frac{\mathbf{y}}{2},\, \mathbf{x}-\frac{\mathbf{y}}{2}\Big) T\Big(\mathbf{x}+\frac{\mathbf{y}}{2},\, \mathbf{x}-\frac{\mathbf{y}}{2}\Big) \qquad \mathbf{x}_1 = \mathbf{x}+\frac{\mathbf{y}}{2} \qquad \mathbf{x}_2 = \mathbf{x}-\frac{\mathbf{y}}{2}$$

$$= \int d^dx\, d^dy\, F\Big(\mathbf{x}+\frac{\mathbf{y}}{2},\, \mathbf{x}-\frac{\mathbf{y}}{2}\Big) T\Big(\mathbf{x}+\frac{\mathbf{y}}{2},\, \mathbf{x}-\frac{\mathbf{y}}{2}\Big) = \int d^dx\, d^dy \left(\int \frac{d^dp}{(2\pi)^d} F(\mathbf{x},\mathbf{p})\, e^{i\mathbf{p}\cdot\mathbf{y}}\right)\left(\int \frac{d^dq}{(2\pi)^d} T(\mathbf{x},\mathbf{q})\, e^{-i\mathbf{q}\cdot\mathbf{y}}\right)$$

$$= \int d^dx \int \frac{d^dp\, d^dq}{(2\pi)^{2d}} F(\mathbf{x},\mathbf{p})\, T(\mathbf{x},\mathbf{q}) \int d^dy\, e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{y}} = \int d^dx \int \frac{d^dp\, d^dq}{(2\pi)^{2d}} F(\mathbf{x},\mathbf{p})\, T(\mathbf{x},\mathbf{q})\, (2\pi)^d \delta^{(d)}(\mathbf{p}-\mathbf{q})$$

$$= \int d^dx \int \frac{d^dp}{(2\pi)^d} F(\mathbf{x},\mathbf{p})\, T(\mathbf{x},\mathbf{p}).$$

Lie Algebra of Fermion Bilinears

- Weyl–Wigner correspondence (Operator – Symbol Correspondence)
  - Commutator Yields Moyal Bracket (Quantum Phase-Space)

$$[O_F, O_G] = O_{\{\{F,G\}\}} \qquad \{\{F,G\}\}(\mathbf{x},\mathbf{p}) \equiv 2F(\mathbf{x},\mathbf{p}) \sin\left(\frac{\overleftarrow{\nabla}_{\mathbf{x}} \cdot \overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{x}}}{2}\right) G(\mathbf{x},\mathbf{p})$$

    - Commutation Relation

$$[\psi^\dagger(\mathbf{x}_1)\psi(\mathbf{x}_2),\ \psi^\dagger(\mathbf{x}_3)\psi(\mathbf{x}_4)] = \psi^\dagger(\mathbf{x}_1)\{\psi(\mathbf{x}_2), \psi^\dagger(\mathbf{x}_3)\}\psi(\mathbf{x}_4) - \psi^\dagger(\mathbf{x}_3)\{\psi(\mathbf{x}_4), \psi^\dagger(\mathbf{x}_1)\}\psi(\mathbf{x}_2)$$

$$= \Big(\psi^\dagger(\mathbf{x}_1)\psi(\mathbf{x}_4)\,\delta(\mathbf{x}_3 - \mathbf{x}_2) - \psi^\dagger(\mathbf{x}_1)\psi^\dagger(\mathbf{x}_3)\psi(\mathbf{x}_2)\psi(\mathbf{x}_4)\Big) - \Big(\psi^\dagger(\mathbf{x}_3)\psi(\mathbf{x}_2)\,\delta(\mathbf{x}_1 - \mathbf{x}_4) - \psi^\dagger(\mathbf{x}_1)\psi^\dagger(\mathbf{x}_3)\psi(\mathbf{x}_2)\psi(\mathbf{x}_4)\Big)$$

$$= \psi^\dagger(\mathbf{x}_1)\psi(\mathbf{x}_4)\,\delta(\mathbf{x}_3 - \mathbf{x}_2) - \psi^\dagger(\mathbf{x}_3)\psi(\mathbf{x}_2)\,\delta(\mathbf{x}_1 - \mathbf{x}_4)$$

$$[O_F, O_G] = \int d^dx_1\, d^dx_2\, d^dx_3\, d^dx_4\ F(\mathbf{x}_1,\mathbf{x}_2)\, G(\mathbf{x}_3,\mathbf{x}_4)\, [T(\mathbf{x}_1,\mathbf{x}_2), T(\mathbf{x}_3,\mathbf{x}_4)] \sim \int d^dx_1\, d^dx_2\, d^dx_3\, d^dx_4\ F(\mathbf{x}_1,\mathbf{x}_2)\, G(\mathbf{x}_3,\mathbf{x}_4)\, [\psi^\dagger(\mathbf{x}_1)\psi(\mathbf{x}_2), \psi^\dagger(\mathbf{x}_3)\psi(\mathbf{x}_4)]$$

$$= \int d^dx_1\, d^dx_2\, d^dx_3\, d^dx_4\ F(\mathbf{x}_1,\mathbf{x}_2)\, G(\mathbf{x}_3,\mathbf{x}_4) \Big[\psi^\dagger(\mathbf{x}_1)\psi(\mathbf{x}_4)\delta(\mathbf{x}_3 - \mathbf{x}_2) - \psi^\dagger(\mathbf{x}_3)\psi(\mathbf{x}_2)\delta(\mathbf{x}_1 - \mathbf{x}_4)\Big] = \int d^dx_1\, d^dx_2\, d^dy\, \Big[F(\mathbf{x}_1,\mathbf{y})G(\mathbf{y},\mathbf{x}_2) - G(\mathbf{x}_1,\mathbf{y})F(\mathbf{y},\mathbf{x}_2)\Big]\, \psi^\dagger(\mathbf{x}_1)\psi(\mathbf{x}_2)$$

$$\Rightarrow \quad [O_F, O_G] \sim \int d^dx_1\, d^dx_2\, (F \circ G - G \circ F)(\mathbf{x}_1,\mathbf{x}_2)\, \psi^\dagger(\mathbf{x}_1)\psi(\mathbf{x}_2) \sim O_{\mathcal{H}}$$

$$\mathcal{H}(\mathbf{x}_1,\mathbf{x}_2) = (F \circ G - G \circ F)(\mathbf{x}_1,\mathbf{x}_2) \qquad (F \circ G)(\mathbf{x}_1,\mathbf{x}_2) = \int d^dy\, F(\mathbf{x}_1,\mathbf{y})G(\mathbf{y},\mathbf{x}_2)$$

Lie Algebra of Fermion Bilinears

- Weyl–Wigner correspondence (Operator – Symbol Correspondence)
  - Commutator Yields Moyal Bracket (Quantum Phase-Space)

$$[O_F, O_G] = O_{\{\{F,G\}\}} \qquad \{\{F,G\}\}(\mathbf{x},\mathbf{p}) \equiv 2F(\mathbf{x},\mathbf{p}) \sin\left(\frac{\overleftarrow{\nabla}_{\mathbf{x}}\cdot\overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}}\cdot\overrightarrow{\nabla}_{\mathbf{x}}}{2}\right) G(\mathbf{x},\mathbf{p})$$

    - Wigner Transform & Inverse Transform

$$T_W(\mathbf{x},\mathbf{p}) = \int d^d y\, T\Big(\mathbf{x}+\frac{\mathbf{y}}{2}, \mathbf{x}-\frac{\mathbf{y}}{2}\Big) e^{i\mathbf{p}\cdot\mathbf{y}} \quad \Longleftrightarrow \quad T\Big(\mathbf{x}+\frac{\mathbf{y}}{2}, \mathbf{x}-\frac{\mathbf{y}}{2}\Big) = \int \frac{d^d p}{(2\pi)^d}\, T_W(\mathbf{x},\mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{y}}$$

    - Convolution

$$(F\circ G)_W(\mathbf{x},\mathbf{p}) = \int d^d y\, e^{i\mathbf{p}\cdot\mathbf{y}}\, (F\circ G)\Big(\mathbf{x}+\frac{\mathbf{y}}{2}, \mathbf{x}-\frac{\mathbf{y}}{2}\Big) = \int d^d y\, e^{i\mathbf{p}\cdot\mathbf{y}} \int d^d z\, F\Big(\mathbf{x}+\frac{\mathbf{y}}{2}, \mathbf{z}\Big) G\Big(\mathbf{z}, \mathbf{x}-\frac{\mathbf{y}}{2}\Big)$$

$$= \int d^d y\, e^{i\mathbf{p}\cdot\mathbf{y}} \int d^d z \left(\int \frac{d^d p_1}{(2\pi)^d} F_W\Big(\frac{\mathbf{x}+\mathbf{z}}{2}+\frac{\mathbf{y}}{4}, \mathbf{p}_1\Big) e^{-i\mathbf{p}_1\cdot(\mathbf{x}-\mathbf{z}+\frac{\mathbf{y}}{2})}\right)\cdot\left(\int \frac{d^d p_2}{(2\pi)^d} G_W\Big(\frac{\mathbf{x}+\mathbf{z}}{2}-\frac{\mathbf{y}}{4}, \mathbf{p}_2\Big) e^{-i\mathbf{p}_2\cdot(\mathbf{z}-\mathbf{x}+\frac{\mathbf{y}}{2})}\right)$$

$$= \int \frac{d^d y d^d z d^d p_1 d^d p_2}{(2\pi)^{2d}} F_W\Big(\frac{\mathbf{x}+\mathbf{z}}{2}+\frac{\mathbf{y}}{4}, \mathbf{p}_1\Big) G_W\Big(\frac{\mathbf{x}+\mathbf{z}}{2}-\frac{\mathbf{y}}{4}, \mathbf{p}_2\Big) \exp\Big(-i(\mathbf{p}_1-\mathbf{p}_2)\cdot(\mathbf{x}-\mathbf{x}_3) + i\Big(\mathbf{p}-\frac{\mathbf{p}_1+\mathbf{p}_2}{2}\Big)\cdot\mathbf{y}\Big)$$

Lie Algebra of Fermion Bilinears

- Weyl–Wigner correspondence (Operator – Symbol Correspondence)
  - Commutator Yields Moyal Bracket (Quantum Phase-Space)

$$[O_F, O_G] = O_{\{\{F,G\}\}} \qquad \{\{F,G\}\}(\mathbf{x},\mathbf{p}) \equiv 2F(\mathbf{x},\mathbf{p}) \sin\left(\frac{\overleftarrow{\nabla}_{\mathbf{x}} \cdot \overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{x}}}{2}\right) G(\mathbf{x},\mathbf{p})$$

    - To Integrate out $\boldsymbol{y}, \boldsymbol{z}$, we need an Identity

$$\int d^d x \; e^{i\mathbf{q}\cdot\mathbf{x}} \, e^{\mathbf{x}\cdot\mathbf{A}} = (2\pi)^d \, e^{-i\mathbf{A}\cdot\nabla_{\mathbf{q}}} \, \delta(\mathbf{q})$$

$$e^{\mathbf{x}\cdot\mathbf{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{x}\cdot\mathbf{A})^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1,\ldots,i_n} x_{i_1} \cdots x_{i_n} A_{i_1} \cdots A_{i_n}$$

$$\int d^d x \; e^{i\mathbf{q}\cdot\mathbf{x}} \, x_{i_1} \cdots x_{i_n} = (2\pi)^d \, (-i)^n \, \partial_{q_{i_1}} \cdots \partial_{q_{i_n}} \delta(\mathbf{q})$$

$$\Rightarrow \int d^d x \; e^{i\mathbf{q}\cdot\mathbf{x}} \, e^{\mathbf{x}\cdot\mathbf{A}} = (2\pi)^d \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} (\mathbf{A}\cdot\nabla_{\mathbf{q}})^n \delta(\mathbf{q}) = (2\pi)^d \, e^{-i\mathbf{A}\cdot\nabla_{\mathbf{q}}} \, \delta(\mathbf{q}).$$

Lie Algebra of Fermion Bilinears

- Weyl–Wigner correspondence (Operator – Symbol Correspondence)
  - Commutator Yields Moyal Bracket (Quantum Phase-Space)

$$[O_F, O_G] = O_{\{\{F,G\}\}} \qquad \{\{F,G\}\}(\mathbf{x},\mathbf{p}) \equiv 2F(\mathbf{x},\mathbf{p}) \sin\left(\frac{\overleftarrow{\nabla}_{\mathbf{x}}\cdot\overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}}\cdot\overrightarrow{\nabla}_{\mathbf{x}}}{2}\right) G(\mathbf{x},\mathbf{p})$$

    - Insert into the Convolution

$$(F\circ G)_W(\mathbf{x},\mathbf{p}) = \int d^d p_1\, d^d p_2\, F_W(\mathbf{x},\mathbf{p}_1)\left\{\exp\left[-\frac{i}{2}\left(\overleftarrow{\nabla}_{\mathbf{x}}^F + \overrightarrow{\nabla}_{\mathbf{x}}^G\right)\cdot\nabla_{\mathbf{p}_1-\mathbf{p}_2}\right]\delta(\mathbf{p}_1-\mathbf{p}_2)\right\}$$
$$\cdot\left\{\exp\left[-\frac{i}{4}\left(\overleftarrow{\nabla}_{\mathbf{x}}^F - \overrightarrow{\nabla}_{\mathbf{x}}^G\right)\cdot\nabla_{\mathbf{p}-\frac{\mathbf{p}_1+\mathbf{p}_2}{2}}\right]\delta\left(\mathbf{p}-\frac{\mathbf{p}_1+\mathbf{p}_2}{2}\right)\right\} G_W(\mathbf{x},\mathbf{p}_2)$$

    - Replace Variable $\mathbf{q}_1 = \mathbf{p}_1 - \mathbf{p}_2, \quad \mathbf{q}_2 = \frac{\mathbf{p}_1+\mathbf{p}_2}{2} \quad \Rightarrow \quad \mathbf{p}_1 = \mathbf{q}_2 + \frac{\mathbf{q}_1}{2}, \quad \mathbf{p}_2 = \mathbf{q}_2 - \frac{\mathbf{q}_1}{2}$
    - Chain Rule $\mathbf{p}_1 = \mathbf{q}_2 + \frac{\mathbf{q}_1}{2} \quad \mathbf{p}_2 = \mathbf{q}_2 - \frac{\mathbf{q}_1}{2} \quad \Rightarrow \quad \nabla_{\mathbf{q}_1} = \frac{1}{2}(\nabla_{\mathbf{p}_1} - \nabla_{\mathbf{p}_2}) \qquad \nabla_{\mathbf{q}_2} = \nabla_{\mathbf{p}_1} + \nabla_{\mathbf{p}_2}$

Lie Algebra of Fermion Bilinears

- Weyl–Wigner correspondence (Operator – Symbol Correspondence)
  - Commutator Yields Moyal Bracket (Quantum Phase-Space)

$$[O_F, O_G] = O_{\{\{F,G\}\}} \qquad \{\{F,G\}\}(\mathbf{x},\mathbf{p}) \equiv 2F(\mathbf{x},\mathbf{p}) \sin\left(\frac{\overleftarrow{\nabla}_{\mathbf{x}} \cdot \overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{x}}}{2}\right) G(\mathbf{x},\mathbf{p})$$

    - Simplified Convolution

$$(F \circ G)_W(\mathbf{x},\mathbf{p}) = F_W(\mathbf{x},\mathbf{p}) \exp\left[\frac{i}{2}\left(\overleftarrow{\nabla}_{\mathbf{x}} \cdot \overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{x}}\right)\right] G_W(\mathbf{x},\mathbf{p}) \equiv F_W \star G_W$$

$$\Downarrow$$

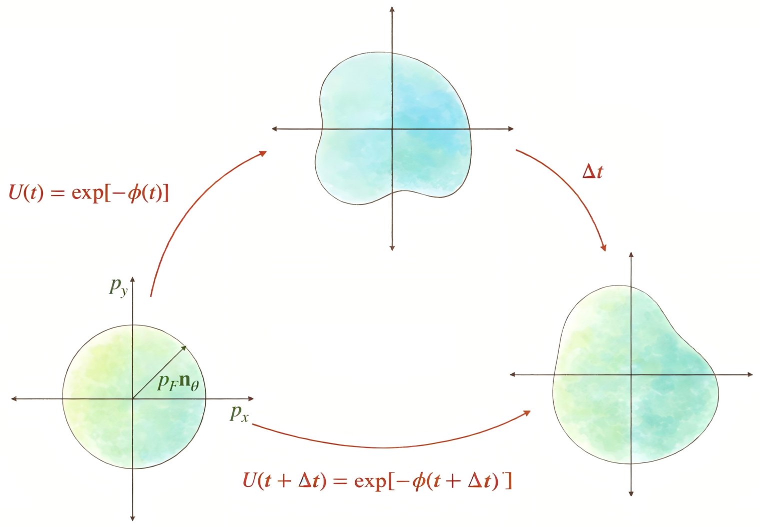
$$\mathcal{H}_W(\mathbf{x},\mathbf{p}) = \int d^d y\, \mathcal{H}\left(\mathbf{x} + \frac{\mathbf{y}}{2}, \mathbf{x} - \frac{\mathbf{y}}{2}\right) e^{i\mathbf{p}\cdot\mathbf{y}} = (F \circ G - G \circ F)_W = F_W * G_W - G_W * F_W$$

$$= F_W(\mathbf{x},\mathbf{p}) \left[\exp\left(\frac{i}{2}\Lambda\right) - \exp\left(-\frac{i}{2}\Lambda\right)\right] G_W(\mathbf{x},\mathbf{p}) = i\, F_W(\mathbf{x},\mathbf{p})\, 2\sin\left(\frac{1}{2}\Lambda\right) G_W(\mathbf{x},\mathbf{p})$$

$$\Lambda \equiv \overleftarrow{\nabla}_{\mathbf{x}} \cdot \overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{x}}$$

Lie Algebra of Fermion Bilinears

- Infinite Algebra: All functions on Single-Particle Phase space can Serve as Parameters $F(x, p)$ $\mathfrak{g}_{\mathrm{Moyal}} \equiv (\{F(\mathbf{x}, \mathbf{p})\}; \{\{\cdot, \cdot\}\})$
- Lie Group: Exponentiating $U = \exp(O_F)$
  - Canonical Transformations in Phase Space
  - Formal Action Exactly Describing Fermi Surfaces



Lie Algebra of Fermion Bilinears

- Leading Order(Moyal Algebra ⇒ Poisson Algebra)

$$\{\{F,G\} = \{F,G\} + \mathcal{O}(\nabla_{\mathbf{x}},\nabla_{\mathbf{p}})^3 \qquad \{F,G\} \equiv \nabla_{\mathbf{x}}F\cdot\nabla_{\mathbf{p}}G - \nabla_{\mathbf{p}}F\cdot\nabla_{\mathbf{x}}G \quad \mathfrak{g} = (\{F(\mathbf{x},\mathbf{p})\};\{.\,,.\})$$

$$\mathfrak{g}_{\mathrm{Moyal}} \equiv (\{F(\mathbf{x},\mathbf{p})\};\{\{\cdot,\cdot\}\}) \quad \Rightarrow \quad \mathfrak{g}_{\mathrm{Poisson}} \equiv (\{F(\mathbf{x},\mathbf{p})\};\{\cdot,\cdot\})$$

  - Canonical Pairs

$$\delta k = \{F,k\} \quad \begin{aligned} \delta x_i &= \{F,x_i\} = \nabla_{\mathbf{x}}F\cdot\nabla_{\mathbf{p}}x_i - \nabla_{\mathbf{p}}F\cdot\nabla_{\mathbf{x}}x_i = -\frac{\partial F}{\partial p_i} = -(\nabla_{\mathbf{p}}F)_i \\ \delta p_i &= \{F,p_i\} = \nabla_{\mathbf{x}}F\cdot\nabla_{\mathbf{p}}p_i - \nabla_{\mathbf{p}}F\cdot\nabla_{\mathbf{x}}p_i = \frac{\partial F}{\partial x_i} = (\nabla_{\mathbf{x}}F)_i \end{aligned} \qquad \mathbf{x}' = \mathbf{x} - \nabla_{\mathbf{p}}F \qquad \mathbf{p}' = \mathbf{p} + \nabla_{\mathbf{x}}F$$

$$\{x_i',p_j'\} = \left\{x_i - \frac{\partial F}{\partial p_i}, p_j + \frac{\partial F}{\partial x_j}\right\} = \{x_i,p_j\} + \left\{x_i, \frac{\partial F}{\partial x_j}\right\} - \left\{\frac{\partial F}{\partial p_i}, p_j\right\} - \left\{\frac{\partial F}{\partial p_i}, \frac{\partial F}{\partial x_j}\right\}$$

$$\{x_i,p_j\} = \delta_{ij} \qquad \left\{x_i, \frac{\partial F}{\partial x_j}\right\} = \frac{\partial}{\partial p_i}\left(\frac{\partial F}{\partial x_j}\right) = \frac{\partial^2 F}{\partial p_i\,\partial x_j} \left\{\frac{\partial F}{\partial p_i}, p_j\right\} = \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial p_i}\right) = \frac{\partial^2 F}{\partial x_j\,\partial p_i}$$

$$\left\{\frac{\partial F}{\partial p_i}, \frac{\partial F}{\partial x_j}\right\} = \mathcal{O}(F^2) \quad \{x_i',p_j'\} = \delta_{ij} + \mathcal{O}(F^2) \quad \{x_i',x_j'\} = \mathcal{O}(F^2) \qquad \{p_i',p_j'\} = \mathcal{O}(F^2)$$

Lie Algebra of Fermion Bilinears

- Leading Order(Moyal Algebra ⇒ Poisson Algebra)

$$\{\{F,G\} = \{F,G\} + \mathcal{O}(\nabla_{\mathbf{x}}, \nabla_{\mathbf{p}})^3 \qquad \{F,G\} \equiv \nabla_{\mathbf{x}} F \cdot \nabla_{\mathbf{p}} G - \nabla_{\mathbf{p}} F \cdot \nabla_{\mathbf{x}} G \quad \mathfrak{g} = (\{F(\mathbf{x},\mathbf{p})\}; \{.\,,.\})$$

  - Vector Fields

$$X_F = \nabla_{\mathbf{x}} F \cdot \nabla_{\mathbf{p}} - \nabla_{\mathbf{p}} F \cdot \nabla_{\mathbf{x}}, = \{F, \cdot\}$$

$$[X_F, X_G] \cdot K(\mathbf{x},\mathbf{p}) = X_{\{F,G\}} \cdot K(\mathbf{x},\mathbf{p}) \quad \forall K(\mathbf{x},\mathbf{p})$$

  - Subalgebra × Truncation of the Lie Bracket ✓
  - Deformation Quantization(Deformation of the Associative Algebra)

$$A_0 = C^\infty(M) \;\to\; A_\hbar = C^\infty(M)[[\hbar]]$$

$$\mu_0(f,g) = fg \;\mapsto\; \mu_\hbar(f,g) = f \star g = fg + \hbar C_1(f,g) + \hbar^2 C_2(f,g) + \cdots$$

$$\begin{array}{ccc} A_\hbar \otimes A_\hbar & \xrightarrow{b_\star} & A_\hbar \\ {\scriptstyle \sigma\otimes\sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ A_0 \otimes A_0 & \xrightarrow{\{\cdot,\cdot\}} & A_0 \end{array}$$

Moyal Algebra to Poisson Algebra (Semi-Classical Limit)

- Weyl Quantization
  - Moyal Algebra = Quantum Deformation of Phase-Space Algebra (Star-Product)
- Truncation: Take $\hbar \to 0$ $\Rightarrow$ Poisson Bracket Emerges as Leading Term
- Consistent Truncation: Poisson Bracket is the only Truncation Preserving the Jacobi Identity (Lie Algebra Structure Stays Intact)
- Interpretation: Poisson Algebra = Lie Algebra of Infinitesimal Canonical Transformations (Classical Phase-Space Diffeomorphisms)

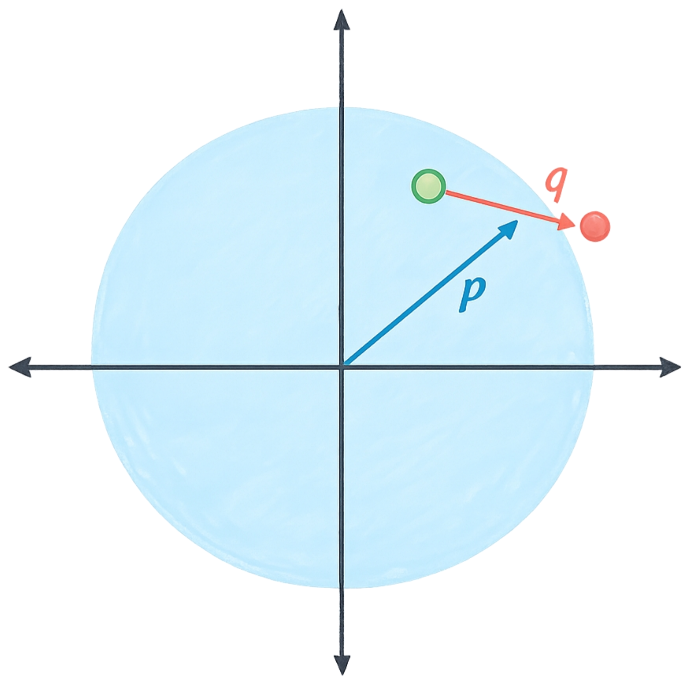
Moyal Algebra to Poisson Algebra (Semi-Classical Limit)

- Moyal Bracket is Extremely Tedious
  - Truncation $\{\{F,G\}\} = \{F,G\} + \mathcal{O}(\nabla_{\mathbf{x}}, \nabla_{\mathbf{p}})^3$ $\quad \{F,G\} \equiv \nabla_{\mathbf{x}} F \cdot \nabla_{\mathbf{p}} G - \nabla_{\mathbf{p}} F \cdot \nabla_{\mathbf{x}} G$
  - Good Approximation for

$$\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{p}} \ll 1 \quad \Leftrightarrow \quad \mathbf{q} \cdot \mathbf{y} \ll 1 \quad \Leftrightarrow \quad \nabla_{\mathbf{x}} \cdot \mathbf{y} \ll 1 \quad \Leftrightarrow \quad \mathbf{q} \cdot \nabla_{\mathbf{p}} \ll 1$$

    - $(\boldsymbol{x}, \boldsymbol{q})$ Long Distance or IR Scale
    - $(\boldsymbol{y}, \boldsymbol{p})$ Short Distance or UV Scale
    - Truncation Suppress Higher Derivatives

$$|\mathbf{p}| \sim p_F \qquad |\mathbf{q}| \ll p_F \;\Rightarrow\; \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{p}} \sim \frac{|\nabla_{\mathbf{x}}|}{p_F} \ll 1$$



Space of States – Dual to Algebra

- States via Density Matrix: Quantum Many-Body State $\rho \rightarrow \langle O\rangle = \mathrm{Tr}(\rho O)$
  - "States" in Phase Space / Statistical Mechanics instead of States in Quantum Mechanics
  - Equivalent Class: Identical Expectation Values of all Fermion Bilinears

$$[\rho] = \left\{\rho' \in \mathcal{S} \;\middle|\; \mathrm{Tr}\big(\rho\, O_F\big) = \mathrm{Tr}\big(\rho'\, O_F\big),\ \forall\, F\right\}$$

  - $W(\boldsymbol{x}, \boldsymbol{p})$: Dual Basis of $T(\boldsymbol{x}, \boldsymbol{p})$

$$\mathrm{Tr}\left(W\left(\mathbf{x}', \mathbf{p}'\right) T(\mathbf{x}, \mathbf{p})\right) = \delta\left(\mathbf{x} - \mathbf{x}'\right)(2\pi)^d \delta\left(\mathbf{p} - \mathbf{p}'\right)$$

- Dual Space $\mathfrak{g}^*$: Space of Linear Functionals on the Algebra $\mathfrak{g}$ (Represented by Phase-Space Distributions $f(\boldsymbol{x}, \boldsymbol{p})$)

$$O_F = \int \frac{d^d\mathbf{x}\, d^d\mathbf{p}}{(2\pi)^d} F(\mathbf{x}, \mathbf{p})\, T(\mathbf{x}, \mathbf{p}) \qquad \rho_f = \int \frac{d^d\mathbf{x}'\, d^d\mathbf{p}'}{(2\pi)^d} f(\mathbf{x}', \mathbf{p}')\, W(\mathbf{x}', \mathbf{p}')$$

Space of States – Dual to Algebra

- Dual Space $\mathfrak{g}^*$: Space of Linear Functionals on the Algebra $\mathfrak{g}$ (Represented by Phase-Space Distributions $f(\boldsymbol{x}, \boldsymbol{p})$)

$$O_F = \int \frac{d^d\mathbf{x}\, d^d\mathbf{p}}{(2\pi)^d}\, F(\mathbf{x},\mathbf{p})\, T(\mathbf{x},\mathbf{p}) \qquad \rho_f = \int \frac{d^d\mathbf{x}'\, d^d\mathbf{p}'}{(2\pi)^d}\, f(\mathbf{x}',\mathbf{p}')\, W(\mathbf{x}',\mathbf{p}')$$

  - Representatives of the Equivalence Class of States can be Expanded in this Dual Basis with the 'Coefficients' Given by a Function of $\boldsymbol{x}, \boldsymbol{p}$

$$\mathrm{Tr}(\rho_f O_F) = \mathrm{Tr}\Big[\int \frac{d^d\mathbf{x}'\, d^d\mathbf{p}'}{(2\pi)^d}\, f(\mathbf{x}',\mathbf{p}')\, W(\mathbf{x}',\mathbf{p}') \int \frac{d^d\mathbf{x}\, d^d\mathbf{p}}{(2\pi)^d}\, F(\mathbf{x},\mathbf{p})\, T(\mathbf{x},\mathbf{p})\Big]$$

$$= \int \frac{d^d\mathbf{x}\, d^d\mathbf{p}\, d^d\mathbf{x}'\, d^d\mathbf{p}'}{(2\pi)^{2d}}\, f(\mathbf{x}',\mathbf{p}')\, F(\mathbf{x},\mathbf{p})\ \mathrm{Tr}\big(W(\mathbf{x}',\mathbf{p}')\, T(\mathbf{x},\mathbf{p})\big) = \int \frac{d^d\mathbf{x}\, d^d\mathbf{p}\, d^d\mathbf{x}'\, d^d\mathbf{p}'}{(2\pi)^{2d}}\, f(\mathbf{x}',\mathbf{p}')\, F(\mathbf{x},\mathbf{p})\, (2\pi)^d\, \delta^{(d)}(\mathbf{x}-\mathbf{x}')\, \delta^{(d)}(\mathbf{p}-\mathbf{p}')$$

$$= \int \frac{d^d\mathbf{x}\, d^d\mathbf{p}}{(2\pi)^d}\, f(\mathbf{x},\mathbf{p})\, F(\mathbf{x},\mathbf{p}).$$

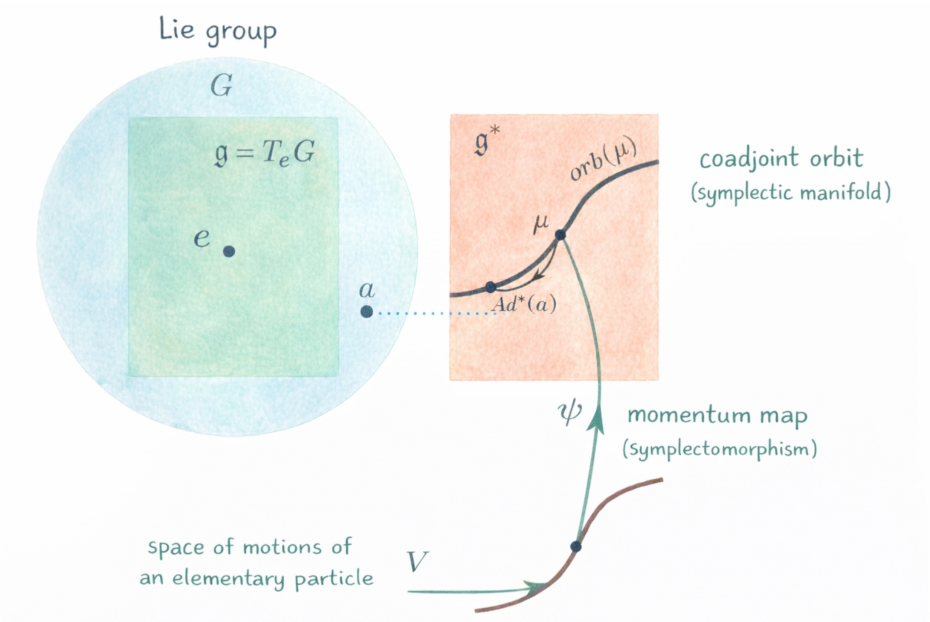
$$\mathfrak{g}^* \equiv \{f(\mathbf{x},\mathbf{p})\} \qquad f[F] \equiv \langle f, F\rangle \equiv \int \frac{d^dx d^dp}{(2\pi)^d} f(\mathbf{x},\mathbf{p}) F(\mathbf{x},\mathbf{p})$$

Space of States – Dual to Algebra

- Phase-Space Distribution $f(\boldsymbol{x}, \boldsymbol{p})$ Plays Role of Expectation Value of $n(\mathbf{x}, \mathbf{p}) = \psi^\dagger \psi$
- Zero-Temperature Fermi Sea: Reference Distribution $f_0(\boldsymbol{p}) = \Theta(p_F - |\boldsymbol{p}|)$
- Particle-Hole Excitations is Described by Small Deviations $\delta f(\boldsymbol{x}, \boldsymbol{p})$ in the Dual Space (Fluctuations of Fermi Surface Shape)

Coadjoint Orbits – Fermi Surface as an Orbit

- Coadjoint Orbit: Set of States Obtained by Group Action on a Reference State in $\mathfrak{g}^*$
- Reference State f_0: Spherical Fermi Sea (Uniform, Isotropic Fermi Surface)
- Group Action: Apply Canonical Transform $U \in G$ on f_0: New State $f = \mathrm{Ad}_U^* f_0$
- General Fermi Surface: $f_F(\boldsymbol{x}, \boldsymbol{p}) = \Theta(p_F(\boldsymbol{x}, \theta) - |\boldsymbol{p}|)$ Deformed Boundary $\rightarrow p_F(\boldsymbol{x}, \theta)$
- State Space Restricted: Physical Fermi Liquid States = Orbits of f_0 (0/1 Distributions Separated by a Closed Fermi Surface)



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4. Operator Algebra to Lie-Poisson Dynamics

Detailed Algebra of Fermion Bilinears

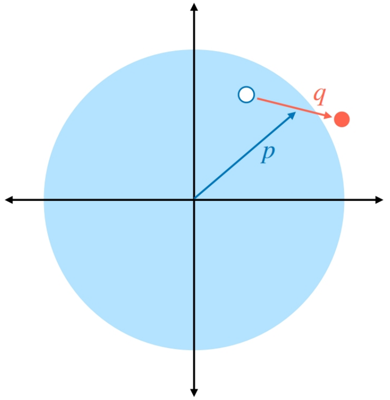
- Generators in Center of Mass and Relative Coordinates

$$T(\mathbf{x},\mathbf{y}) \sim i\psi^\dagger(\mathbf{x})\psi(\mathbf{y}) \qquad \left[\psi(\mathbf{x}),\psi^\dagger(\mathbf{y})\right]_+ = \delta(\mathbf{x}-\mathbf{y})$$

$$\Downarrow$$

$$T(\mathbf{x},\mathbf{y}) \equiv \frac{i}{2}\left[\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right] \quad T^\dagger(\mathbf{x},\mathbf{y}) = -T(\mathbf{x},-\mathbf{y})$$

- Bilinears in Momentum Space



$$T(\mathbf{x},\mathbf{p}) = \int d^d\mathbf{y}\, T(\mathbf{x},\mathbf{y})\, e^{i\mathbf{p}\cdot\mathbf{y}} \quad \psi\left(\mathbf{x}\pm\frac{\mathbf{y}}{2}\right) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\, e^{-i\mathbf{k}\cdot\left(\mathbf{x}\pm\frac{\mathbf{y}}{2}\right)}\,\psi(\mathbf{k})$$

$$\mathbf{q} = \mathbf{k}+\mathbf{k}' \quad \mathbf{p} = \frac{\mathbf{k}-\mathbf{k}'}{2} \quad \Longrightarrow \quad \mathbf{k} = \mathbf{p}+\frac{\mathbf{q}}{2} \quad \mathbf{k}' = -\mathbf{p}+\frac{\mathbf{q}}{2}$$

$$T(\mathbf{q},\mathbf{p}) = \int d^d\mathbf{x}\, T(\mathbf{x},\mathbf{p})\, e^{i\mathbf{q}\cdot\mathbf{x}} = \int d^d\mathbf{x}\, d^d\mathbf{y}\, T(\mathbf{x},\mathbf{y})\, e^{i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}}$$

$$T(\mathbf{q},\mathbf{p}) = \int d^d\mathbf{x}\, d^d\mathbf{y}\, \frac{1}{2}\left[\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\psi^\dagger\left(\mathbf{x}\frac{\mathbf{y}}{2}\right)\right] e^{i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}}$$

$$= \frac{1}{2}\left[\psi^\dagger(\mathbf{k})\psi(\mathbf{k}')-\psi(\mathbf{k}')\psi^\dagger(\mathbf{k})\right] = \frac{1}{2}\left[\psi^\dagger\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right)\psi\left(-\mathbf{p}+\frac{\mathbf{q}}{2}\right)-\psi\left(-\mathbf{p}+\frac{\mathbf{q}}{2}\right)\psi^\dagger\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right)\right]$$

Detailed Algebra of Fermion Bilinears

- Useful Identities

$$\{\psi^\dagger(\mathbf{k}),\psi(\mathbf{k}')\} = \int d^d\mathbf{x}\, d^d\mathbf{x}'\ \{\psi^\dagger(\mathbf{x}),\psi(\mathbf{x}')\}\, e^{-i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{x}'} = \int d^d\mathbf{x}\ e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = (2\pi)^d\,\delta^{(d)}(\mathbf{k}-\mathbf{k}')$$

$$\left[\psi^\dagger(\mathbf{k})\psi(\mathbf{k}'),\psi^\dagger(\mathbf{q})\psi(\mathbf{q}')\right] = (2\pi)^d\,\delta^{(d)}(\mathbf{k}'-\mathbf{q})\,\psi^\dagger(\mathbf{k})\psi(\mathbf{q}') - (2\pi)^d\,\delta^{(d)}(\mathbf{k}-\mathbf{q}')\,\psi^\dagger(\mathbf{q})\psi(\mathbf{k}')$$

- Various Fourier Transform

$$T(\mathbf{x},\mathbf{y}) \equiv \frac{i}{2}\left[\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right) - \psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right]$$

$$T(\mathbf{q},\mathbf{p}) \equiv \frac{i}{2}\left[\psi^\dagger\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)\psi\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right) - \psi\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)\psi^\dagger\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)\right]$$

$$T(\mathbf{x},\mathbf{p}) \equiv \int_{\mathbf{y}} T(\mathbf{x},\mathbf{y})e^{i\mathbf{p}\cdot\mathbf{y}} = \int_{\mathbf{q}} T(\mathbf{q},\mathbf{p})e^{-i\mathbf{q}\cdot\mathbf{x}}$$

$$T(\mathbf{q},\mathbf{y}) \equiv \int_{\mathbf{x},\mathbf{p}} T(\mathbf{x},\mathbf{p})e^{i\mathbf{q}\cdot\mathbf{x}}e^{-i\mathbf{p}\cdot\mathbf{y}} = \int_{\mathbf{x}} T(\mathbf{x},\mathbf{y})e^{i\mathbf{q}\cdot\mathbf{x}} = \int_{\mathbf{p}} T(\mathbf{q},\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{y}}$$

Detailed Algebra of Fermion Bilinears

- Anti-Hermitian

$$T(\mathbf{x},\mathbf{y})=\frac{1}{2}\left[\psi^{\dagger}\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\psi^{\dagger}\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right]$$

$$T^{\dagger}(\mathbf{x},\mathbf{y})=-\frac{1}{2}\left[\psi^{\dagger}\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)-\psi\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi^{\dagger}\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right]\;=\;-T(\mathbf{x},-\mathbf{y})$$

$$T(\mathbf{x},\mathbf{p})=\int d^{d}\mathbf{y}\;T(\mathbf{x},\mathbf{y})\,e^{i\mathbf{p}\cdot\mathbf{y}}$$

$$T^{\dagger}(\mathbf{x},\mathbf{p})=\int d^{d}\mathbf{y}\;T^{\dagger}(\mathbf{x},\mathbf{y})\left(e^{i\mathbf{p}\cdot\mathbf{y}}\right)^{*}=\int d^{d}\mathbf{y}\;T^{\dagger}(\mathbf{x},\mathbf{y})\,e^{-i\mathbf{p}\cdot\mathbf{y}}=\int d^{d}\mathbf{y}\;\left[-T(\mathbf{x},-\mathbf{y})\right]e^{-i\mathbf{p}\cdot\mathbf{y}}=-\int d^{d}\mathbf{y}\;T(\mathbf{x},\mathbf{y})\,e^{i\mathbf{p}\cdot\mathbf{y}}=-T(\mathbf{x},\mathbf{p})$$

- Commutator of Generators

$$\left[T(\mathbf{q},\mathbf{y}),T\left(\mathbf{q}',\mathbf{y}'\right)\right]=2\sin\left(\frac{\mathbf{q}'\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{y}'}{2}\right)T\left(\mathbf{q}+\mathbf{q}',\mathbf{y}+\mathbf{y}'\right)$$

$$\left[T(\mathbf{x},\mathbf{p}),T\left(\mathbf{x}',\mathbf{p}'\right)\right]=2\sin\left(\frac{\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{p}'}-\nabla_{\mathbf{x}'}\cdot\nabla_{\mathbf{p}}}{2}\right)\left[\delta\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(\mathbf{p}-\mathbf{p}'\right)T(\mathbf{x},\mathbf{p})\right]$$

Detailed Algebra of Fermion Bilinears

- Commutator of Generators
  - Proof
    - Useful Substitution Form $i\,\psi^\dagger(\mathbf{u})\psi(\mathbf{v}) = T\left(\frac{\mathbf{u}+\mathbf{v}}{2},\, \mathbf{u}-\mathbf{v}\right) - \frac{1}{2}\,\delta(\mathbf{u}-\mathbf{v})$

$$T(\mathbf{x},\mathbf{y}) \equiv \frac{i}{2}\left[\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right) - \psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right]$$

$$= \frac{i}{2}\left[\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right) - \left\{\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right), \psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right\} + \psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right]$$

$$= i\,\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right) + \frac{1}{2}\,\delta(\mathbf{y})$$

Detailed Algebra of Fermion Bilinears

- Commutator of Generators
  - Proof

$$\left[T(\mathbf{x},\mathbf{y}),T(\mathbf{x}',\mathbf{y}')\right] = \left[i\,\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)+\frac{1}{2}\delta(\mathbf{y}),\ i\,\psi^\dagger\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\psi\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right)+\frac{1}{2}\delta(\mathbf{y}')\right]$$

$$= -\left[\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right),\ \psi^\dagger\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\psi\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right)\right]$$

$$= \psi^\dagger\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\psi\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right)\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\psi^\dagger\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\psi\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right)$$

$$= \psi^\dagger\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\left[i\,\delta\left(\mathbf{x}-\mathbf{x}'+\frac{\mathbf{y}+\mathbf{y}'}{2}\right)-\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right)\right]\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)$$

$$-\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\left[i\,\delta\left(\mathbf{x}-\mathbf{x}'-\frac{\mathbf{y}+\mathbf{y}'}{2}\right)-\psi^\dagger\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right]\psi\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right)$$

$$= i\,\psi^\dagger\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\delta\left(\mathbf{x}-\mathbf{x}'+\frac{\mathbf{y}+\mathbf{y}'}{2}\right)-i\,\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right)\delta\left(\mathbf{x}-\mathbf{x}'-\frac{\mathbf{y}+\mathbf{y}'}{2}\right)$$

Detailed Algebra of Fermion Bilinears

- Commutator of Generators
  - Proof

$$i\,\psi^\dagger(\mathbf{u})\psi(\mathbf{v}) = T\left(\frac{\mathbf{u}+\mathbf{v}}{2}, \mathbf{u}-\mathbf{v}\right) - \frac{1}{2}\,\delta(\mathbf{u}-\mathbf{v})$$

$$\Downarrow$$

$$\begin{aligned}
[T(\mathbf{x},\mathbf{y}), T(\mathbf{x}',\mathbf{y}')] = &\left[T\left(\frac{\mathbf{x}+\mathbf{x}'}{2} + \frac{\mathbf{y}'-\mathbf{y}}{4}, \mathbf{x}'-\mathbf{x}+\frac{\mathbf{y}+\mathbf{y}'}{2}\right) - \frac{1}{2}\,\delta\left(\mathbf{x}'-\mathbf{x}+\frac{\mathbf{y}+\mathbf{y}'}{2}\right)\right]\delta\left(\mathbf{x}-\mathbf{x}'+\frac{\mathbf{y}+\mathbf{y}'}{2}\right) \\
&-\left[T\left(\frac{\mathbf{x}+\mathbf{x}'}{2} - \frac{\mathbf{y}'-\mathbf{y}}{4}, \mathbf{x}-\mathbf{x}'+\frac{\mathbf{y}+\mathbf{y}'}{2}\right) - \frac{1}{2}\,\delta\left(\mathbf{x}-\mathbf{x}'+\frac{\mathbf{y}+\mathbf{y}'}{2}\right)\right]\delta\left(\mathbf{x}-\mathbf{x}'-\frac{\mathbf{y}+\mathbf{y}'}{2}\right)
\end{aligned}$$

$$= T\left(\frac{\mathbf{x}+\mathbf{x}'}{2} + \frac{\mathbf{y}'-\mathbf{y}}{4}, \mathbf{x}'-\mathbf{x}+\frac{\mathbf{y}+\mathbf{y}'}{2}\right)\delta\left(\mathbf{x}-\mathbf{x}'+\frac{\mathbf{y}+\mathbf{y}'}{2}\right) - T\left(\frac{\mathbf{x}+\mathbf{x}'}{2} - \frac{\mathbf{y}'-\mathbf{y}}{4}, \mathbf{x}-\mathbf{x}'+\frac{\mathbf{y}+\mathbf{y}'}{2}\right)\delta\left(\mathbf{x}-\mathbf{x}'-\frac{\mathbf{y}+\mathbf{y}'}{2}\right)$$

Detailed Algebra of Fermion Bilinears

- Commutator of Generators
  - Proof
    - Fourier Transform to Derive other Commutators in Momentum Space

$$[T(\mathbf{q},\mathbf{y}),T(\mathbf{q}',\mathbf{y}')] = \int d^d\mathbf{x}\, d^d\mathbf{x}'\ [T(\mathbf{x},\mathbf{y}),T(\mathbf{x}',\mathbf{y}')]\, e^{i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{q}'\cdot\mathbf{x}'}$$

$$= T(\mathbf{q}+\mathbf{q}',\mathbf{y}+\mathbf{y}')\left(e^{\,i(\mathbf{q}'\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{y}')/2} - e^{-i(\mathbf{q}'\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{y}')/2}\right)(-i)$$

$$= 2\sin\left(\frac{\mathbf{q}'\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{y}'}{2}\right) T(\mathbf{q}+\mathbf{q}',\mathbf{y}+\mathbf{y}')$$

$$T(\mathbf{x},\mathbf{p}) = \int d^d\mathbf{q}\, d^d\mathbf{y}\, T(\mathbf{q},\mathbf{y}) e^{-i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}}$$

$$\Downarrow$$

$$[T(\mathbf{x},\mathbf{p}),T(\mathbf{x}',\mathbf{p}')] = \int d^d\mathbf{q}\, d^d\mathbf{y}\, d^d\mathbf{q}'\, d^d\mathbf{y}'\ [T(\mathbf{q},\mathbf{y}),T(\mathbf{q}',\mathbf{y}')]\, e^{-i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}} e^{-i\mathbf{q}'\cdot\mathbf{x}'} e^{i\mathbf{p}'\cdot\mathbf{y}'}$$

$$= \int d^d\mathbf{q}\, d^d\mathbf{y}\, d^d\mathbf{q}'\, d^d\mathbf{y}'\ 2\sin\left(\frac{\mathbf{q}'\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{y}'}{2}\right) T(\mathbf{q}+\mathbf{q}',\mathbf{y}+\mathbf{y}') e^{-i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}} e^{-i\mathbf{q}'\cdot\mathbf{x}'} e^{i\mathbf{p}'\cdot\mathbf{y}'}$$

Detailed Algebra of Fermion Bilinears

- Commutator of Generators
  - Proof
    - Fourier Transform to Derive other Commutators in Momentum Space

$$\nabla_{\mathbf{p}}\, e^{-i\mathbf{q}\cdot\mathbf{x}-i\mathbf{q}'\cdot\mathbf{x}'+i\mathbf{p}\cdot\mathbf{y}+i\mathbf{p}'\cdot\mathbf{y}'} = i\mathbf{y}\, e^{-i\mathbf{q}\cdot\mathbf{x}-i\mathbf{q}'\cdot\mathbf{x}'+i\mathbf{p}\cdot\mathbf{y}+i\mathbf{p}'\cdot\mathbf{y}'} \qquad \nabla_{\mathbf{x}}\, e^{-i\mathbf{q}\cdot\mathbf{x}-i\mathbf{q}'\cdot\mathbf{x}'+i\mathbf{p}\cdot\mathbf{y}+i\mathbf{p}'\cdot\mathbf{y}'} = -i\mathbf{q}\, e^{-i\mathbf{q}\cdot\mathbf{x}-i\mathbf{q}'\cdot\mathbf{x}'+i\mathbf{p}\cdot\mathbf{y}+i\mathbf{p}'\cdot\mathbf{y}'}$$

$$\nabla_{\mathbf{p}'}\, e^{-i\mathbf{q}\cdot\mathbf{x}-i\mathbf{q}'\cdot\mathbf{x}'+i\mathbf{p}\cdot\mathbf{y}+i\mathbf{p}'\cdot\mathbf{y}'} = i\mathbf{y}'\, e^{-i\mathbf{q}\cdot\mathbf{x}-i\mathbf{q}'\cdot\mathbf{x}'+i\mathbf{p}\cdot\mathbf{y}+i\mathbf{p}'\cdot\mathbf{y}'} \qquad \nabla_{\mathbf{x}'}\, e^{-i\mathbf{q}\cdot\mathbf{x}-i\mathbf{q}'\cdot\mathbf{x}'+i\mathbf{p}\cdot\mathbf{y}+i\mathbf{p}'\cdot\mathbf{y}'} = -i\mathbf{q}'\, e^{-i\mathbf{q}\cdot\mathbf{x}-i\mathbf{q}'\cdot\mathbf{x}'+i\mathbf{p}\cdot\mathbf{y}+i\mathbf{p}'\cdot\mathbf{y}'}$$

$$\Downarrow$$

$$\left[T(\mathbf{x},\mathbf{p}), T(\mathbf{x}',\mathbf{p}')\right] = 2\sin\left(\frac{\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{p}'} - \nabla_{\mathbf{x}'}\cdot\nabla_{\mathbf{p}}}{2}\right)\left[\delta(\mathbf{x}-\mathbf{x}')\,\delta(\mathbf{p}-\mathbf{p}')\,T(\mathbf{x},\mathbf{p})\right]$$

- "Structure Constants" of $\mathfrak{g}_{\mathrm{Moyal}}$ in Basis $T(\boldsymbol{x},\boldsymbol{p})$

$$\left[T(\mathbf{q},\mathbf{y}), T\left(\mathbf{q}',\mathbf{y}'\right)\right] = 2\sin\left(\frac{\mathbf{q}'\cdot\mathbf{y} - \mathbf{q}\cdot\mathbf{y}'}{2}\right) T\left(\mathbf{q}+\mathbf{q}', \mathbf{y}+\mathbf{y}'\right)$$

$$\left[T(\mathbf{x},\mathbf{p}), T\left(\mathbf{x}',\mathbf{p}'\right)\right] = 2\sin\left(\frac{\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{p}'} - \nabla_{\mathbf{x}'}\cdot\nabla_{\mathbf{p}}}{2}\right)\left[\delta\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(\mathbf{p}-\mathbf{p}'\right)T(\mathbf{x},\mathbf{p})\right]$$

Detailed Algebra of Fermion Bilinears

- Orthogonality Relations

$$\mathrm{Tr}\left[T(\mathbf{x},\mathbf{p})T\left(\mathbf{x}',\mathbf{p}'\right)\right]=2\delta\left(\mathbf{x}-\mathbf{x}'\right)(2\pi)^{d}\delta\left(\mathbf{p}-\mathbf{p}'\right)\qquad \mathrm{Tr}\left[T(\mathbf{q},\mathbf{y})T\left(\mathbf{q}',\mathbf{y}'\right)\right]=2(2\pi)^{d}\delta\left(\mathbf{q}+\mathbf{q}'\right)\delta\left(\mathbf{y}+\mathbf{y}'\right)$$

- Moyal Algebra
  - Charge-0 Bosonic Operators Forms an Infinite Dimensional Lie Algebra
  - Expansion in Generators

$$H_{\text{micro}}=\int_{\mathbf{p}}\epsilon(\mathbf{p})\psi^{\dagger}(\mathbf{p})\psi(-\mathbf{p})+\int_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4}V\left(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4\right)\psi^{\dagger}\left(\mathbf{p}_1\right)\psi\left(\mathbf{p}_2\right)\psi^{\dagger}\left(\mathbf{p}_3\right)\psi\left(\mathbf{p}_4\right)\delta\left(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4\right)+\mathcal{O}\left(\psi^{\dagger}\psi\right)^{3}$$

$$\langle O_F\rangle_{\rho_f}=\mathrm{Tr}(\rho_f O_F)=\int_{\mathbf{x},\mathbf{p}}F(\mathbf{x},\mathbf{p})f(\mathbf{x},\mathbf{p})$$

$$H_2\simeq -i\int_{\mathbf{x},\mathbf{p}}\epsilon(\mathbf{p})T(\mathbf{x},\mathbf{p})\qquad \langle H_2\rangle_{\rho_f}\simeq -i\int_{\mathbf{x},\mathbf{p}}\epsilon(\mathbf{p})\,f(\mathbf{x},\mathbf{p})\quad (+\text{Contact Terms})$$

Detailed Algebra of Fermion Bilinears

- Moyal Algebra
  - Expansion in Generators

$$T(\mathbf{q},\mathbf{p}) \equiv \frac{i}{2}\left[\psi^\dagger\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)\psi\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)-\psi\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)\psi^\dagger\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)\right] \qquad T(\mathbf{x},\mathbf{p}) = \int_{\mathbf{q}} T(\mathbf{q},\mathbf{p})\, e^{-i\mathbf{q}\cdot\mathbf{x}} \qquad T(\mathbf{0},\mathbf{p}) = \int_{\mathbf{x}} T(\mathbf{x},\mathbf{p})$$

$$T(\mathbf{0},\mathbf{p}) = \frac{i}{2}\left[\psi^\dagger(\mathbf{p})\psi(-\mathbf{p})-\psi(-\mathbf{p})\psi^\dagger(\mathbf{p})\right] \qquad \psi(-\mathbf{p})\psi^\dagger(\mathbf{p}) = \{\psi(-\mathbf{p}),\psi^\dagger(\mathbf{p})\}-\psi^\dagger(\mathbf{p})\psi(-\mathbf{p})$$

$$T(\mathbf{0},\mathbf{p}) = i\,\psi^\dagger(\mathbf{p})\psi(-\mathbf{p})-\frac{i}{2}\{\psi(-\mathbf{p}),\psi^\dagger(\mathbf{p})\} \qquad \psi^\dagger(\mathbf{p})\psi(-\mathbf{p}) = -\,i\,T(\mathbf{0},\mathbf{p})+\frac{1}{2}\{\psi(-\mathbf{p}),\psi^\dagger(\mathbf{p})\}$$

$$\int_{\mathbf{p}} \epsilon(\mathbf{p})\,\psi^\dagger(\mathbf{p})\psi(-\mathbf{p}) = -\,i\int_{\mathbf{p}}\epsilon(\mathbf{p})\,T(\mathbf{0},\mathbf{p})+\frac{1}{2}\int_{\mathbf{p}}\epsilon(\mathbf{p})\,\{\psi(-\mathbf{p}),\psi^\dagger(\mathbf{p})\} \simeq -\,i\int_{\mathbf{p}}\epsilon(\mathbf{p})\,T(\mathbf{0},\mathbf{p}) = -\,i\int_{\mathbf{x},\mathbf{p}}\epsilon(\mathbf{p})\,T(\mathbf{x},\mathbf{p})$$

Detailed Algebra of Fermion Bilinears

- Moyal Algebra
  - Expansion in Generators

$$H^{(4)}_{\text{micro}} = \int_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4} V(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\,\psi^\dagger(\mathbf{p}_1)\psi(\mathbf{p}_2)\psi^\dagger(\mathbf{p}_3)\psi(\mathbf{p}_4)\,\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)$$

$$\mathbf{p}_1 = \frac{\mathbf{q}}{2}+\mathbf{p} \quad \mathbf{p}_2 = \frac{\mathbf{q}}{2}-\mathbf{p} \quad \mathbf{p}_3 = \frac{\mathbf{q}'}{2}+\mathbf{p}' \quad \mathbf{p}_4 = \frac{\mathbf{q}'}{2}-\mathbf{p}' \qquad \mathbf{p}_1+\mathbf{p}_2=\mathbf{q} \qquad \mathbf{p}_3+\mathbf{p}_4=\mathbf{q}'$$

$$\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4) = \delta(\mathbf{q}+\mathbf{q}') \quad V(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}') \equiv V\left(\frac{\mathbf{q}}{2}+\mathbf{p},\frac{\mathbf{q}}{2}-\mathbf{p},\frac{\mathbf{q}'}{2}+\mathbf{p}',\frac{\mathbf{q}'}{2}-\mathbf{p}'\right)$$

$$H^{(4)}_{\text{micro}} = \int_{\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}'} V(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')\,\psi^\dagger\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)\psi\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)\psi^\dagger\left(\frac{\mathbf{q}'}{2}+\mathbf{p}'\right)\psi\left(\frac{\mathbf{q}'}{2}-\mathbf{p}'\right)\delta(\mathbf{q}+\mathbf{q}')$$

$$T(\mathbf{q},\mathbf{p}) = i\,\psi^\dagger\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)\psi\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right) - \frac{i}{2}\left\{\psi\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right),\psi^\dagger\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)\right\}$$

$$\psi^\dagger\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)\psi\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right) = -\,i\,T(\mathbf{q},\mathbf{p}) + \frac{1}{2}\left\{\psi\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right),\psi^\dagger\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)\right\}$$

$$H^{(4)}_{\text{micro}} \simeq \int_{\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}'} V(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')\,T(\mathbf{q},\mathbf{p})T(\mathbf{q}',\mathbf{p}')\,\delta(\mathbf{q}+\mathbf{q}')$$

Constructing the Hamiltonian Formalism

$$\mathfrak{g}^* \equiv \{f(\mathbf{x},\mathbf{p})\} \qquad f[F] \equiv \langle f, F\rangle \equiv \int \frac{d^d x d^d p}{(2\pi)^d} f(\mathbf{x},\mathbf{p}) F(\mathbf{x},\mathbf{p})$$

- Adjoint and Coadjoint Representations
  - Adjoint Representation
    - Adjoint Action of a Lie Algebra $\mathrm{ad}_F : \mathfrak{g} \to \mathfrak{g}$ $\qquad \mathrm{ad}_F\, G \equiv \{F, G\}$
    - Corresponding Lie Group

$$\mathrm{Ad}_U : \mathfrak{g} \to \mathfrak{g} \qquad \mathrm{Ad}_{U=\exp F}\, G \equiv U G U^{-1} \equiv e^{\mathrm{ad}_F} G = G + \{F, G\} + \frac{1}{2!}\{F, \{F, G\}\} + \ldots$$

  - Coadjoint Representation

$$\mathrm{ad}^*_F, \mathrm{Ad}^*_U : \mathfrak{g}^* \to \mathfrak{g}^*$$

$$\mathrm{ad}^*_F\, f \equiv \{F, f\} \qquad \mathrm{Ad}^*_{U=\exp F} f \equiv U f U^{-1} \equiv e^{\mathrm{ad}^*_F} f = f + \{F, f\} + \frac{1}{2!}\{F, \{F, f\}\} + \ldots$$

  - Define Coadjoint Action of Lie Algebra and Group as

$$\langle \mathrm{ad}^*_F\, f, G\rangle = -\langle f, \mathrm{ad}_F\, G\rangle \qquad \langle \mathrm{Ad}^*_U f,\ G\rangle = \langle f,\ \mathrm{Ad}_{U^{-1}} G\rangle$$

Constructing the Hamiltonian Formalism

$$\mathfrak{g}^* \equiv \{f(\mathbf{x},\mathbf{p})\} \quad f[F] \equiv \langle f, F\rangle \equiv \int \frac{d^dx d^dp}{(2\pi)^d} f(\mathbf{x},\mathbf{p})F(\mathbf{x},\mathbf{p})$$

- Adjoint and Coadjoint Representations
  - Hadamard Lemma $\mathrm{Ad}_{\exp(F)} = e^{\mathrm{ad}_F}$
    - Connect Coadjoint Representations with Adjoint Representations
    - We have Known $\langle \mathrm{ad}^*_F f, G\rangle = -\langle f, \mathrm{ad}_F G\rangle \quad \langle \mathrm{Ad}^*_U f, G\rangle = \langle f, \mathrm{Ad}_{U^{-1}} G\rangle$
    - We need to Prove $\mathrm{Ad}_{\exp(F)} = e^{\mathrm{ad}_F}$

Conduct Taylor expansion on both sides of $\langle \mathrm{Ad}^*_U f,\ G\rangle = \langle f,\ \mathrm{Ad}_{U^{-1}} G\rangle$

$$\text{R.H.S.} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle f,\ (\mathrm{ad}_F)^n G\rangle = \left\langle f,\ G + (-1)^1\{F,G\} + \frac{(-1)^2}{2!}\{F,\{F,G\}\} + \cdots \right\rangle$$

$$\text{L.H.S.} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle (\mathrm{ad}^*_F)^n f,\ G\rangle = \left\langle f + \{F,f\} + \frac{1}{2!}\{F,\{F,f\}\} + \cdots,\ G \right\rangle$$

We prove $\langle (\mathrm{ad}^*_F)^n f,\ G\rangle = (-1)^n \langle f,\ (\mathrm{ad}_F)^n G\rangle$ by induction.

Constructing the Hamiltonian Formalism

- Adjoint and Coadjoint Representations

$$\mathfrak{g}^* \equiv \{f(\mathbf{x},\mathbf{p})\} \quad f[F] \equiv \langle f, F\rangle \equiv \int \frac{d^dx d^dp}{(2\pi)^d} f(\mathbf{x},\mathbf{p})F(\mathbf{x},\mathbf{p})$$

  - Hadamard Lemma

$$k=0: \quad \langle (\mathrm{ad}_F^*)^0 f,\ G\rangle = \langle f,\ G\rangle \qquad \text{trivial}$$

$$k=1: \quad \langle \mathrm{ad}_F^* f,\ G\rangle = -\langle f,\ \mathrm{ad}_F G\rangle = -\langle f,\ \{F,G\}\rangle$$

$$\text{Assume } k=n: \quad \langle (\mathrm{ad}_F^*)^n f,\ G\rangle = (-1)^n\langle f,\ (\mathrm{ad}_F)^n G\rangle$$

$$\text{For } k=n+1:$$

$$\langle (\mathrm{ad}_F^*)^{n+1} f,\ G\rangle = \langle (\mathrm{ad}_F^*)^n f,\ \mathrm{ad}_F G\rangle = \langle (\mathrm{ad}_F^*)^n f,\ \{F,G\}\rangle$$

$$= (-1)^n\langle f,\ (\mathrm{ad}_F)^n\{F,G\}\rangle = (-1)^n\langle f,\ \{(\mathrm{ad}_F)^n G,\ F\}\rangle = (-1)^{n+1}\langle f,\ (\mathrm{ad}_F)^{n+1} G\rangle$$

- We have Used an Identity (Still Proved by Induction) $(\mathrm{ad}_F)^n\{G,F\} = \{(\mathrm{ad}_F)^n G,\ F\}$

$$k=0: \quad \{F,\{G,F\}\} + \{G,\{F,F\}\} + \{F,\{F,G\}\} = 0$$

$$\Rightarrow \quad \mathrm{ad}_F\{G,F\} + 0 + \{\mathrm{ad}_F G, F\} = 0 \ \Rightarrow\ \mathrm{ad}_F\{G,\ F\} = \{\mathrm{ad}_F G,\ F\}$$

$$\text{Assume } k=n: \quad (\mathrm{ad}_F)^n\{G,F\} = \{(\mathrm{ad}_F)^n G,\ F\}$$

$$\text{For } n+1: \quad (\mathrm{ad}_F)^{n+1}\{G,F\} = \{F,\ (\mathrm{ad}_F)^n\{G,F\}\} = \{F,\ \{(\mathrm{ad}_F)^n G,\ F\}\}$$

$$= \{\{F,\ (\mathrm{ad}_F)^n G\},\ F\} = \{(\mathrm{ad}_F)^{n+1} G,\ F\}$$

Constructing the Hamiltonian Formalism

- Adjoint and Coadjoint Representations
  - Hadamard Lemma $\mathrm{Ad}_{\exp(F)} = e^{\mathrm{ad}_F}$
    - Physical Implication: Expectation Value won't Change in Heisenberg or Schrodinger Picture

$$\mathrm{Tr}\left(\rho_f\left(U_F^{-1} O_G U_F\right)\right) = \mathrm{Tr}\left(\rho_f\left(e^{-\,\mathrm{ad}_{O_F}} O_G\right)\right) = \mathrm{Tr}\left(\rho_f O_{\left(e^{\mathrm{ad}_{-F}}\right)}\right)$$

$$\Updownarrow$$

$$\mathrm{Tr}\left(\left(U_F \rho_f U_F^{-1}\right) O_G\right) = \mathrm{Tr}\left(\left(e^{\mathrm{ad}^*_{O_F}} \rho_f\right) O_G\right) = \mathrm{Tr}\left(\rho_{(\mathrm{ad}^*_F f)} O_G\right)$$

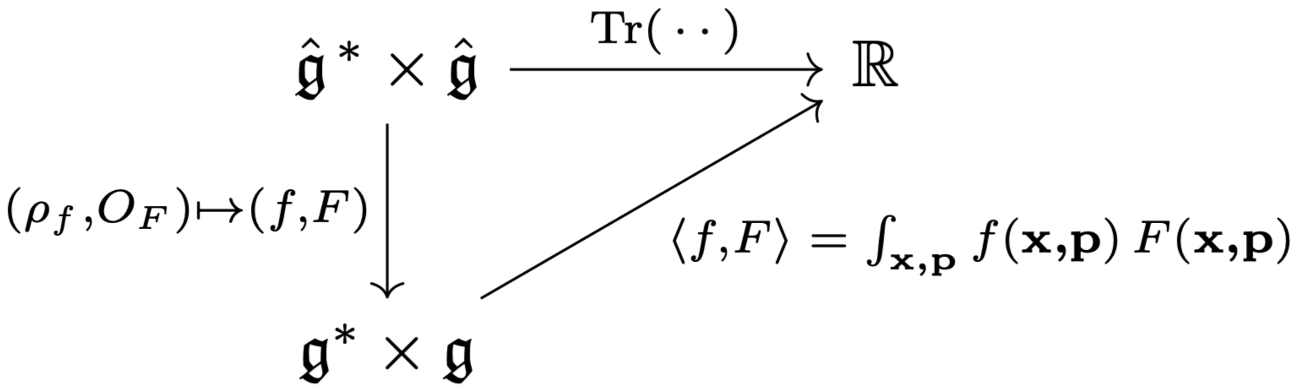
    - Intuitively: I.B.P. of the Lie Bracket Version (First Order)

$$\langle \mathrm{ad}^*_F f, G\rangle = -\langle f, \mathrm{ad}_F G\rangle \;\Leftrightarrow\; \langle \{F, f\}, G\rangle = -\langle f, \{F, G\}\rangle$$

Lie-Poisson Stucture and Hamiltonian

- Poisson Manifold
  - LFLT from a Geometrized Systematic EFT View
  - Lie Algebra

$$\mathfrak{g} = \{ F(\mathbf{x},\mathbf{p}) \} \qquad \mathfrak{g}^* = \{ f(\mathbf{x},\mathbf{p}) \mid \langle f,\cdot\rangle : \mathfrak{g} \to \mathbb{R},\ F \mapsto \langle f,F\rangle = \int_{\mathbf{x},\mathbf{p}} f(\mathbf{x},\mathbf{p})\, F(\mathbf{x},\mathbf{p}) \}$$



Lie-Poisson Stucture and Hamiltonian

- Poisson Manifold
  - Vector Fields Give Dynamics

$$\Phi_{t-t_0}: M \to M \quad \Phi_{t-t_0}(\gamma(t_0)) = \gamma(t)$$

$$\frac{d}{dt}\Phi_{t-t_0}(\gamma(t_0)) = X\Big(\Phi_{t-t_0}(\gamma(t_0))\Big)$$

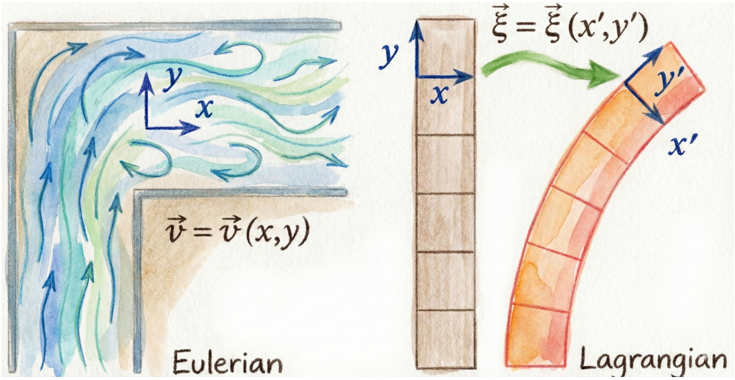
$$\gamma(t_0) = (\boldsymbol{x_0}, \boldsymbol{p_0}) \quad \Phi_0 = \mathrm{id} \quad \gamma(t) = \Phi_{t-t_0}(\gamma(t_0)) = (\boldsymbol{x}, \boldsymbol{p})$$

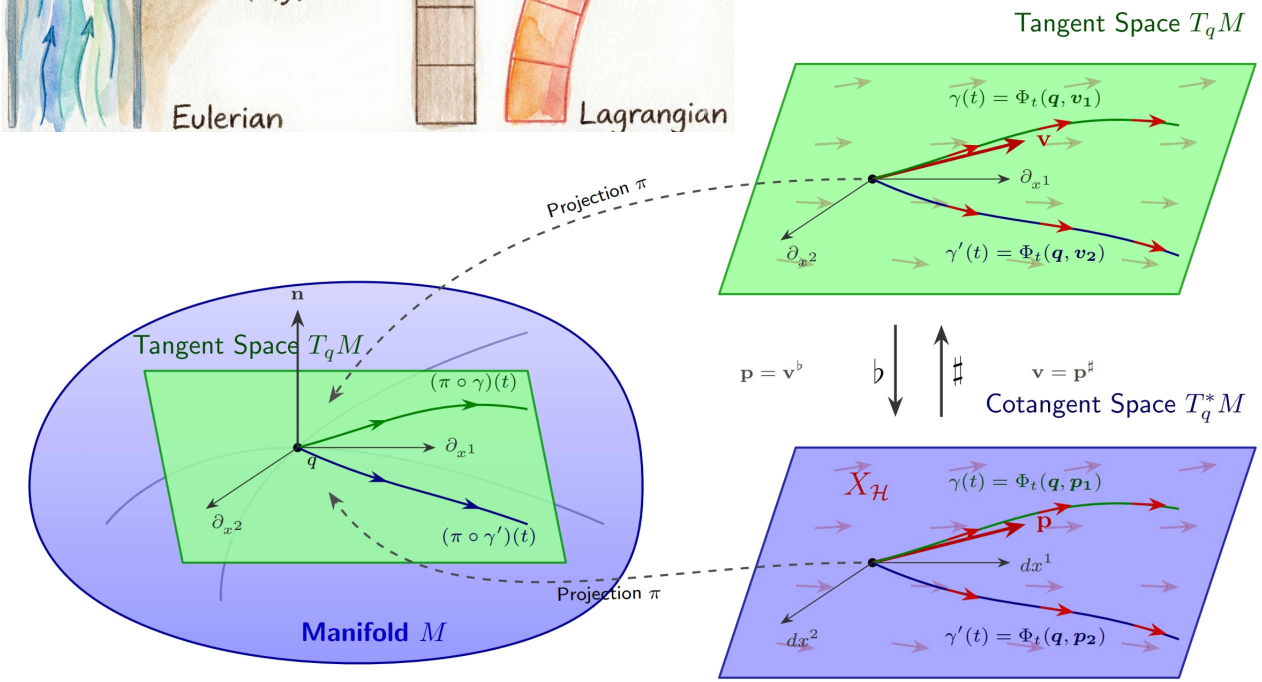
  - Pullback (Eulerian Specification)

$$f(t, \boldsymbol{x}, \boldsymbol{p}) = \Phi^*_{-(t-t_0)} f(t_0, \boldsymbol{x}, \boldsymbol{p}) = f\Big(t_0, \Phi_{-(t-t_0)}(\boldsymbol{x}, \boldsymbol{p})\Big)$$

$$\Updownarrow$$

$$f(t_0, \boldsymbol{x_0}, \boldsymbol{p_0}) = \Phi^*_{t-t_0} f(t, \boldsymbol{x_0}, \boldsymbol{p_0}) = f\Big(t, \Phi_{t-t_0}(\boldsymbol{x_0}, \boldsymbol{p_0})\Big)$$





$$0 = \frac{d}{dt} f(t, \gamma(t)) = \partial_t f(t, \gamma(t)) + \underbrace{\dot{\gamma}(t)}_{=X(\gamma(t))} \cdot (\nabla f)(t, \gamma(t)) = \partial_t f + X(f) \quad \Rightarrow \quad \partial_t f(t, \boldsymbol{x}, \boldsymbol{p}) + X_{\mathscr{F}}\Big(f(t, \boldsymbol{x}, \boldsymbol{p})\Big) = 0$$

Lie-Poisson Stucture and Hamiltonian

- Poisson Manifold
  - Vector Fields Give Dynamics

$$\Phi_{t-t_0}: M \to M \quad \Phi_{t-t_0}(\gamma(t_0)) = \gamma(t)$$

$$\frac{d}{dt}\Phi_{t-t_0}(\gamma(t_0)) = X\Big(\Phi_{t-t_0}(\gamma(t_0))\Big)$$

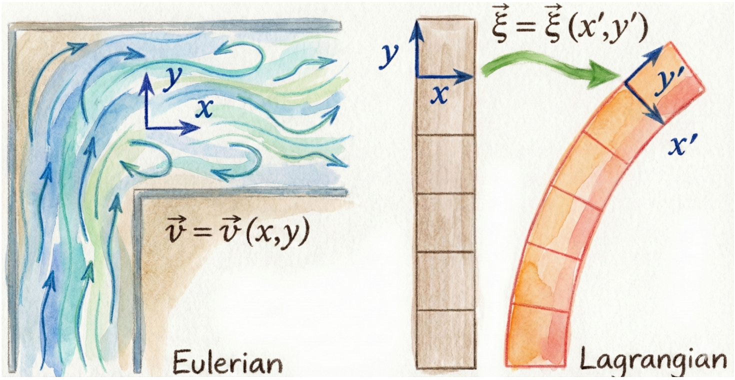
$$\gamma(t_0) = (\boldsymbol{x_0}, \boldsymbol{p_0}) \quad \Phi_0 = \mathrm{id} \quad \gamma(t) = \Phi_{t-t_0}(\gamma(t_0)) = (\boldsymbol{x}, \boldsymbol{p})$$

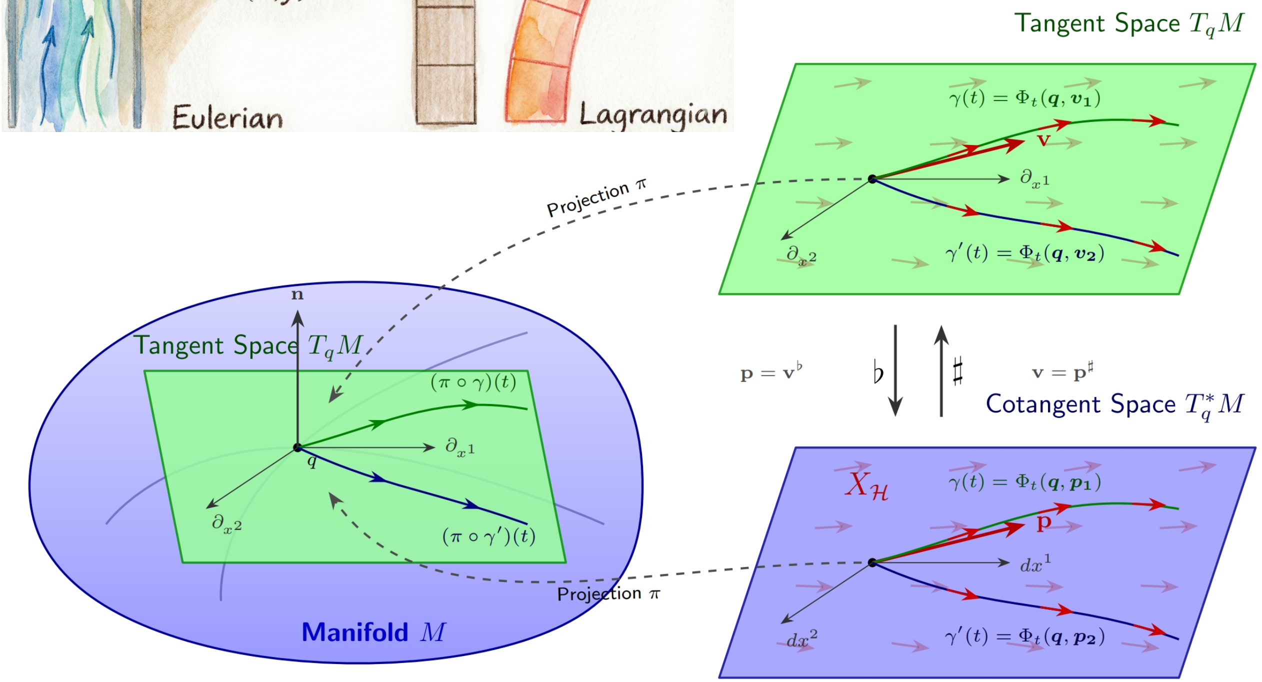
  - Pullback (Lagrangian Specification)

$$f(t, \boldsymbol{x}, \boldsymbol{p}) = \Phi^*_{-(t-t_0)} f(t_0, \boldsymbol{x}, \boldsymbol{p}) = f\Big(t_0, \Phi_{-(t-t_0)}(\boldsymbol{x}, \boldsymbol{p})\Big)$$

$$\Updownarrow$$

$$f(t_0, \boldsymbol{x_0}, \boldsymbol{p_0}) = \Phi^*_{t-t_0} f(t, \boldsymbol{x_0}, \boldsymbol{p_0}) = f\Big(t, \Phi_{t-t_0}(\boldsymbol{x_0}, \boldsymbol{p_0})\Big)$$





$$\partial_t f(t, \boldsymbol{x}, \boldsymbol{p}) = \frac{\partial}{\partial t} f\Big(t_0, \Phi_{t-t_0}(\boldsymbol{x}, \boldsymbol{p})\Big) = \nabla f_0\Big(\Phi_{t-t_0}(\boldsymbol{x}, \boldsymbol{p})\Big) \cdot \underbrace{\frac{d}{dt}\Phi_{t-t_0}(\boldsymbol{x}, \boldsymbol{p})}_{=X(\gamma(t))} = X(f)(t, \boldsymbol{x}, \boldsymbol{p}) \quad \Rightarrow \quad \partial_t f(t, \boldsymbol{x}, \boldsymbol{p}) = X_{\mathscr{F}}\Big(f(t, \boldsymbol{x}, \boldsymbol{p})\Big)$$

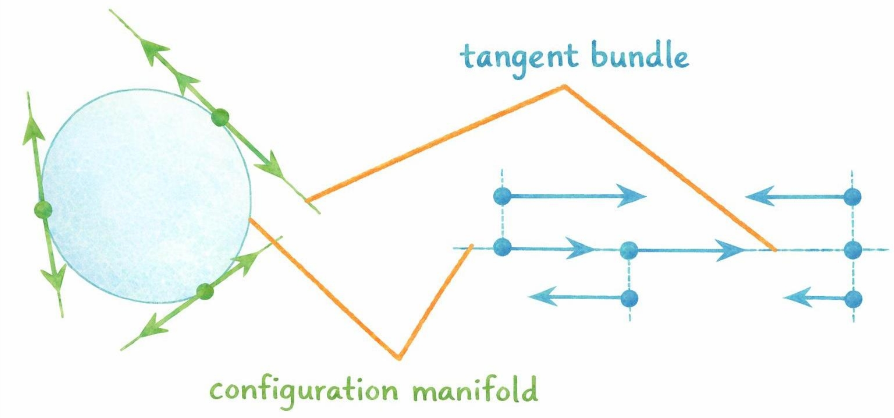
Lie-Poisson Stucture and Hamiltonian

- Poisson Manifold
  - Hamiltonian Vector Fields are Tangent Vectors Generated by Coadjoint Action Naturally

$$\sharp_f : T_f^* g^* \cong g \longrightarrow T_f g^* \cong g^* \qquad X_{\mathscr{F}}(f) \equiv \sharp_f \left(d\mathscr{F}\big|_f = \frac{\delta \mathscr{F}}{\delta f}\bigg|_f \right) = -\operatorname{ad}^*_{\frac{\delta\mathscr{F}}{\delta f}\big|_f} f$$

$$\operatorname{ad}^*_{d\mathscr{F}|f}(\cdot) \ : \ \mathfrak{g}^* \to \mathfrak{g}^* \qquad \operatorname{ad}^*_{(\cdot)} f \ : \ \mathfrak{g} \to \mathfrak{g}^*$$

  - Tangent Spaces and Bundles



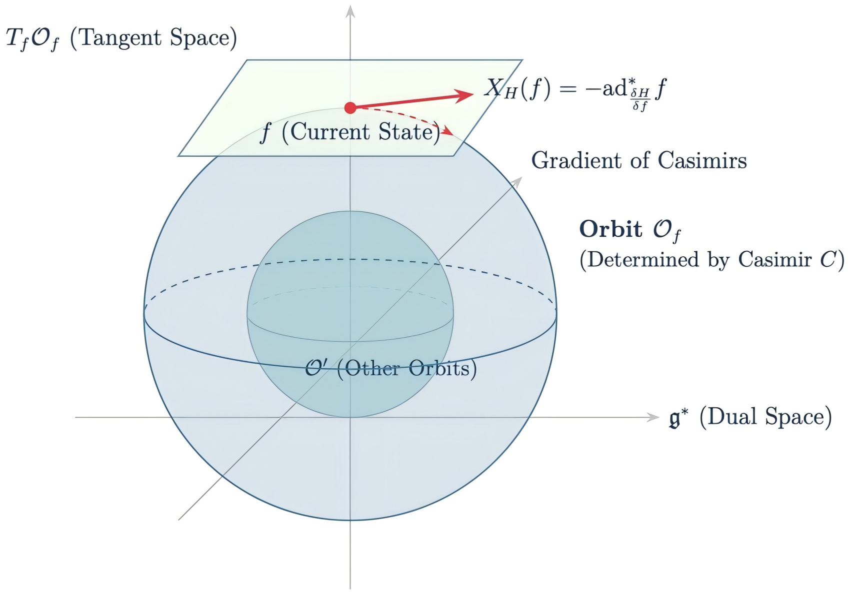
$$T_f\mathfrak{g}^* = \left\{ \delta f(\mathbf{x},\mathbf{p}) \,\middle|\, \delta f = \frac{d}{d\varepsilon}(f+\varepsilon h)\bigg|_{\varepsilon=0},\ h \in \mathfrak{g}^* \right\} \cong \mathfrak{g}^* \times \mathfrak{g}^*$$

$$T\mathfrak{g}^* = \bigsqcup_{f\in\mathfrak{g}^*} T_f\mathfrak{g}^* = \left\{ (f(\mathbf{x},\mathbf{p}), \delta f) \,\middle|\, \delta f \in T_f\mathfrak{g}^*,\ f \in \mathfrak{g}^* \right\} \cong \mathfrak{g}^* \times \mathfrak{g}^*$$

$$T_f^*\mathfrak{g}^* = (T_f\mathfrak{g}^*)^* = \left\{ F(\mathbf{x},\mathbf{p}) \,\middle|\, \langle \cdot, F\rangle : T_f\mathfrak{g}^* \to \mathbb{R},\ \delta f \mapsto \langle \delta f, F\rangle = \int_{\mathbf{x},\mathbf{p}} \delta f(\mathbf{x},\mathbf{p})\, F(\mathbf{x},\mathbf{p}) \right\} \cong \mathfrak{g}^* \times \mathfrak{g}^{**}$$

$$T^*\mathfrak{g}^* = \bigsqcup_{f\in\mathfrak{g}^*} T_f^*\mathfrak{g}^* = \left\{ (f(\mathbf{x},\mathbf{p}), F) \,\middle|\, F \in T_f^*\mathfrak{g}^*,\ f \in \mathfrak{g}^* \right\} \cong \mathfrak{g}^* \times \mathfrak{g}^{**}$$

Lie-Poisson Stucture and Hamiltonian

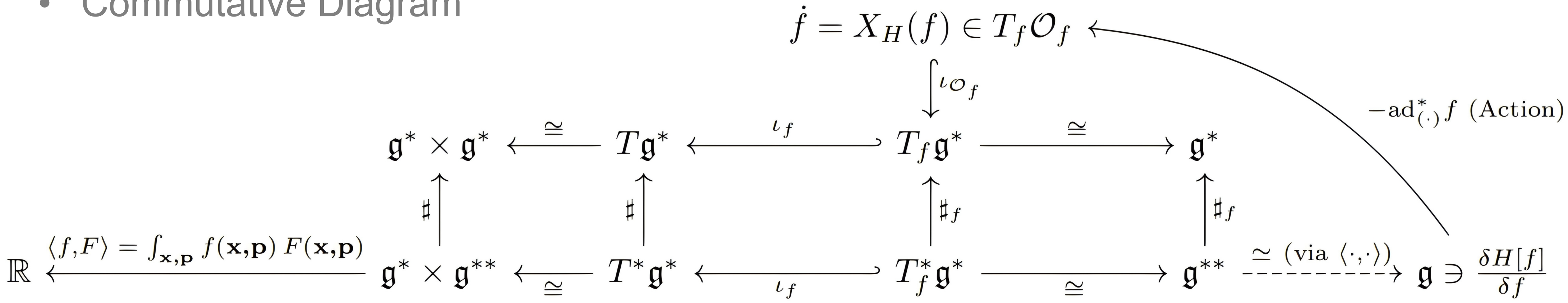


- Poisson Manifold
  - One-Form / Cotangent Vector ($T_f\,\mathfrak{g}^* \to \mathbb{R}$)

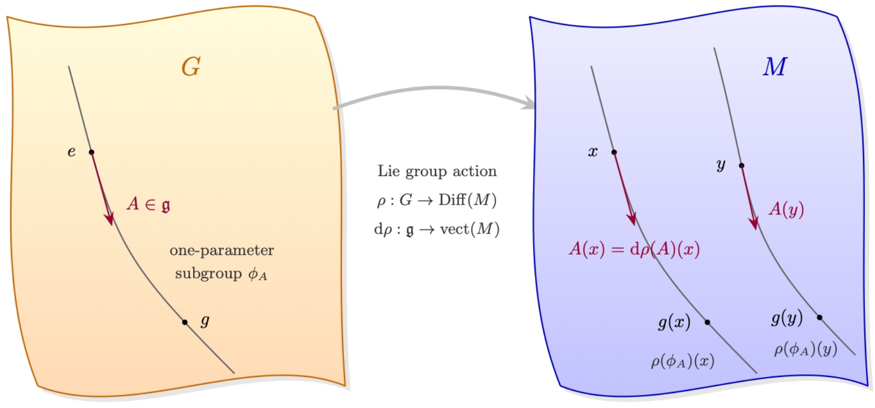
$$\delta f \in T_f\mathfrak{g}^* \quad \mathscr{F}[f] \in \mathfrak{g}^* \quad d\mathscr{F}\big|_f = \frac{\delta \mathscr{F}}{\delta f}\bigg|_f \in T_f^*\mathfrak{g}^*$$

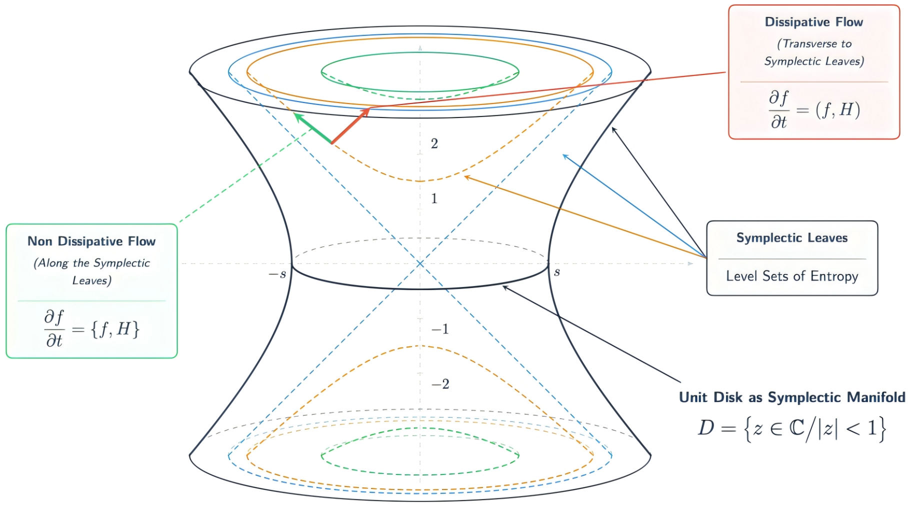
$$d\mathscr{F}\big|_f(\delta f) \equiv \left\langle \delta f, \frac{\delta \mathscr{F}}{\delta f}\bigg|_f \right\rangle \equiv \int_{\mathbf{x},\mathbf{p}} \delta f(\mathbf{x},\mathbf{p}) \frac{\delta \mathscr{F}}{\delta f(\mathbf{x},\mathbf{p})}$$

  - Commutative Diagram



Lie-Poisson Stucture and Hamiltonian





- Poisson Manifold
  - Evolution over Certain Parameter (Eulerian)

$$\dot{f} = X_{\mathscr{G}}(f) = -\mathrm{ad}^*_{\frac{\delta\mathscr{G}}{\delta f}} f \qquad \frac{\partial}{\partial t}\mathscr{F}[f(t)] = d\mathscr{F}|_f(\dot{f}) = d\mathscr{F}|_f\left(-\mathrm{ad}^*_{\frac{\delta\mathscr{G}}{\delta f}} f\right)$$

$$d\mathscr{G}|_f(\delta f) = \left\langle \delta f, \left.\frac{\delta\mathscr{G}}{\delta f}\right|_f \right\rangle \qquad d\mathscr{F}|_f(\dot{f}) = \left\langle -\mathrm{ad}^*_{\frac{\delta\mathscr{G}}{\delta f}} f, \left.\frac{\delta\mathscr{F}}{\delta f}\right|_f \right\rangle$$

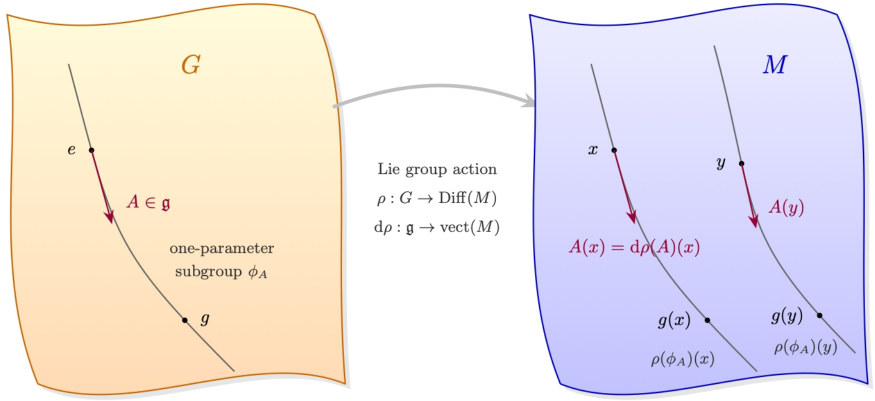
$$\left\langle -\mathrm{ad}^*_{\frac{\delta\mathscr{G}}{\delta f}} f, \left.\frac{\delta\mathscr{F}}{\delta f}\right|_f \right\rangle = \left\langle f, \left\{ \left.\frac{\delta\mathscr{G}}{\delta f}\right|_f, \left.\frac{\delta\mathscr{F}}{\delta f}\right|_f \right\}_{\text{Poisson}} \right\rangle = -\left\langle f, \left\{ \left.\frac{\delta\mathscr{F}}{\delta f}\right|_f, \left.\frac{\delta\mathscr{G}}{\delta f}\right|_f \right\}_{\text{Poisson}} \right\rangle$$

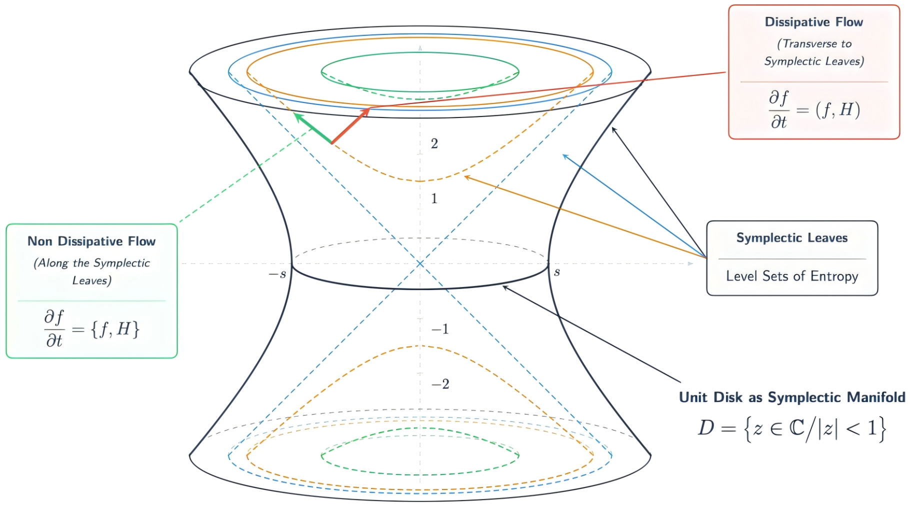
$$\frac{\partial}{\partial t}\mathscr{F}[f(t)] + \left\langle f, \left\{ \left.\frac{\delta\mathscr{F}}{\delta f}\right|_f, \left.\frac{\delta\mathscr{G}}{\delta f}\right|_f \right\}_{\text{Poisson}} \right\rangle \equiv \frac{\partial}{\partial t}\mathscr{F}[f(t)] + \{\mathscr{F}, \mathscr{G}\}_{\text{LP}} = 0$$

  - Lie-Poisson Bracket

$$\{\mathscr{F}, \mathscr{G}\}_{\text{LP}}[f] \equiv \left\langle f, \left\{ \left.\frac{\delta\mathscr{F}}{\delta f}\right|_f, \left.\frac{\delta\mathscr{G}}{\delta f}\right|_f \right\}_{\text{Poisson}} \right\rangle$$

Lie-Poisson Stucture and Hamiltonian





- Poisson Manifold
  - Evolution over Certain Parameter (Lagrangian)

$$\dot{f} = -X_{\mathscr{G}}(f) = +\mathrm{ad}^*_{\frac{\delta \mathscr{G}}{\delta f}} f \qquad \frac{\partial}{\partial t}\mathscr{F}[f(t)] = d\mathscr{F}|_f(\dot{f}) = d\mathscr{F}|_f\left(+\mathrm{ad}^*_{\frac{\delta \mathscr{G}}{\delta f}} f\right)$$

$$d\mathscr{G}|_f(\delta f) = \left\langle \delta f, \left.\frac{\delta \mathscr{G}}{\delta f}\right|_f \right\rangle \qquad d\mathscr{F}|_f(\dot{f}) = \left\langle +\mathrm{ad}^*_{\frac{\delta \mathscr{G}}{\delta f}} f, \left.\frac{\delta \mathscr{F}}{\delta f}\right|_f \right\rangle$$

$$\left\langle +\mathrm{ad}^*_{\frac{\delta \mathscr{G}}{\delta f}} f, \left.\frac{\delta \mathscr{F}}{\delta f}\right|_f \right\rangle = -\left\langle f, \left\{ \left.\frac{\delta \mathscr{G}}{\delta f}\right|_f, \left.\frac{\delta \mathscr{F}}{\delta f}\right|_f \right\}_{\text{Poisson}} \right\rangle = \left\langle f, \left\{ \left.\frac{\delta \mathscr{F}}{\delta f}\right|_f, \left.\frac{\delta \mathscr{G}}{\delta f}\right|_f \right\}_{\text{Poisson}} \right\rangle$$

$$\frac{\partial}{\partial t}\mathscr{F}[f(t)] - \left\langle f, \left\{ \left.\frac{\delta \mathscr{F}}{\delta f}\right|_f, \left.\frac{\delta \mathscr{G}}{\delta f}\right|_f \right\}_{\text{Poisson}} \right\rangle \equiv \frac{\partial}{\partial t}\mathscr{F}[f(t)] - \{\mathscr{F}, \mathscr{G}\}_{\text{LP}} = 0$$

  - Lie-Poisson Bracket

$$\{\mathscr{F}, \mathscr{G}\}_{\text{LP}}[f] \equiv \left\langle f, \left\{ \left.\frac{\delta \mathscr{F}}{\delta f}\right|_f, \left.\frac{\delta \mathscr{G}}{\delta f}\right|_f \right\}_{\text{Poisson}} \right\rangle$$

Lie-Poisson Stucture and Hamiltonian

- Poisson Manifold
  - Lie-Poisson Bracket is Still a Poisson Algebra
    - Product Rule

$$\{\mathscr{F}\mathscr{G},\mathscr{H}\}_{\mathrm{LP}}[f]=\mathscr{F}[f]\{\mathscr{G},\mathscr{H}\}_{\mathrm{LP}}[f]+\mathscr{G}[f]\{\mathscr{F},\mathscr{H}\}_{\mathrm{LP}}[f]$$

$$\{\mathscr{F}\mathscr{G},\mathscr{H}\}_{\mathrm{LP}}[f]=\left\langle f,\left\{\frac{\delta(\mathscr{F}\mathscr{G})}{\delta f},\frac{\delta\mathscr{H}}{\delta f}\right\}_{\text{Poisson}}\right\rangle=\left\langle f,\left\{\left(\frac{\delta\mathscr{F}}{\delta f}\right)\mathscr{G}+\mathscr{F}\left(\frac{\delta\mathscr{G}}{\delta f}\right),\frac{\delta\mathscr{H}}{\delta f}\right\}_{\text{Poisson}}\right\rangle$$

$$=\mathscr{G}[f]\left\langle f,\left\{\frac{\delta\mathscr{F}}{\delta f},\frac{\delta\mathscr{H}}{\delta f}\right\}_{\text{Poisson}}\right\rangle+\mathscr{F}[f]\left\langle f,\left\{\frac{\delta\mathscr{G}}{\delta f},\frac{\delta\mathscr{H}}{\delta f}\right\}_{\text{Poisson}}\right\rangle=\left(\mathscr{G}\{\mathscr{F},\mathscr{H}\}_{\mathrm{LP}}+\mathscr{F}\{\mathscr{G},\mathscr{H}\}_{\mathrm{LP}}\right)[f]$$

Lie-Poisson Stucture and Hamiltonian

- Poisson Manifold
  - Lie-Poisson Bracket is Still a Poisson Algebra
    - Jacobi Identity

Jacobi identity of Poisson brackets

$\Downarrow$

Jacobi identity of Lie--Poisson brackets.

$$\{\mathscr{F},\{\mathscr{G},\mathscr{H}\}_{\mathrm{LP}}\}_{\mathrm{LP}}[f]+\{\mathscr{G},\{\mathscr{H},\mathscr{F}\}_{\mathrm{LP}}\}_{\mathrm{LP}}[f]+\{\mathscr{H},\{\mathscr{F},\mathscr{G}\}_{\mathrm{LP}}\}_{\mathrm{LP}}[f]=0$$

$$\{\mathscr{F},\{\mathscr{G},\mathscr{H}\}_{\mathrm{LP}}\}_{\mathrm{LP}}[f]=\left\langle f,\ \left\{\left.\frac{\delta\mathscr{F}}{\delta f}\right|_f,\ \left.\frac{\delta}{\delta f}\left(\{\mathscr{G},\mathscr{H}\}_{\mathrm{LP}}[f]\right)\right|_f\right\}_{\mathrm{Poisson}}\right\rangle$$

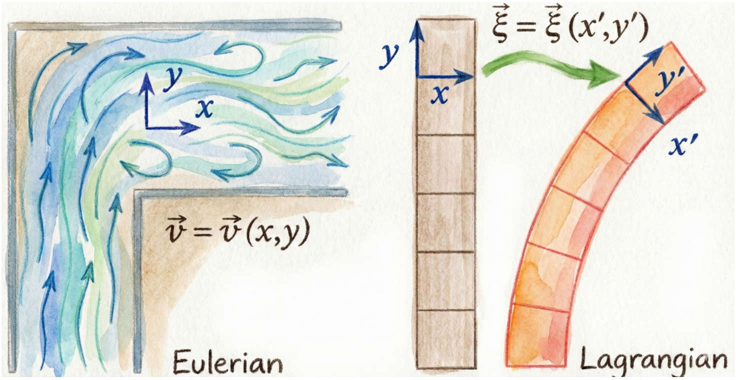
$$=\int_{\mathbf{x},\mathbf{p}} f(\mathbf{x},\mathbf{p})\left\{\left.\frac{\delta\mathscr{F}}{\delta f}\right|_f(\mathbf{x},\mathbf{p}),\ \frac{\delta}{\delta f}\left(\int_{\mathbf{x}',\mathbf{p}'} f(\mathbf{x}',\mathbf{p}')\left.\left\{\frac{\delta\mathscr{G}}{\delta f},\ \frac{\delta\mathscr{H}}{\delta f}\right\}_{\mathrm{Poisson}}\right|_f(\mathbf{x}',\mathbf{p}')\right)\right\}_{\mathrm{Poisson}}$$

$$=\int_{\mathbf{x},\mathbf{p}} f(\mathbf{x},\mathbf{p})\left\{\left.\frac{\delta\mathscr{F}}{\delta f}\right|_f(\mathbf{x},\mathbf{p}),\ \left\{\left.\frac{\delta\mathscr{G}}{\delta f}\right|_f(\mathbf{x},\mathbf{p}),\ \left.\frac{\delta\mathscr{H}}{\delta f}\right|_f(\mathbf{x},\mathbf{p})\right\}_{\mathrm{Poisson}}\right\}_{\mathrm{Poisson}}$$

$$+\int_{\mathbf{x},\mathbf{p}}\int_{\mathbf{x}',\mathbf{p}'} f(\mathbf{x},\mathbf{p})\,f(\mathbf{x}',\mathbf{p}')\left\{\left.\frac{\delta\mathscr{F}}{\delta f}\right|_f(\mathbf{x},\mathbf{p}),\ \left.\frac{\delta}{\delta f}\left(\left\{\frac{\delta\mathscr{G}}{\delta f},\ \frac{\delta\mathscr{H}}{\delta f}\right\}_{\mathrm{Poisson}}\right)\right|_f(\mathbf{x}',\mathbf{p}')\right\}_{\mathrm{Poisson}}$$

$$\Rightarrow\quad \{\mathscr{F},\{\mathscr{G},\mathscr{H}\}_{\mathrm{LP}}\}_{\mathrm{LP}}[f]=\int_{\mathbf{x},\mathbf{p}} f(\mathbf{x},\mathbf{p})\left\{\left.\frac{\delta\mathscr{F}}{\delta f}\right|_f,\ \left\{\left.\frac{\delta\mathscr{G}}{\delta f}\right|_f,\ \left.\frac{\delta\mathscr{H}}{\delta f}\right|_f\right\}_{\mathrm{Poisson}}\right\}_{\mathrm{Poisson}}+\ 0$$

Equation of Motion



- Write down E.o.M. on $\mathfrak{g}^*$ (Eulerian Specification)

$$\partial_t f(t,\mathbf{x},\mathbf{p}) + \{f, H\}_{\mathrm{LP}}[f(t,\mathbf{x},\mathbf{p})]$$

$$= \partial_t f(t,\mathbf{x},\mathbf{p}) + \int_{\mathbf{x}',\mathbf{p}'} f(t,\mathbf{x}',\mathbf{p}')\left\{\frac{\delta f(t,\mathbf{x},\mathbf{p})}{\delta f(t,\mathbf{x}',\mathbf{p}')}, \frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')}\right\}_{\text{Poisson}} (t,\mathbf{x}',\mathbf{p}')$$

$$= \partial_t f(t,\mathbf{x},\mathbf{p}) + \int_{\mathbf{x}',\mathbf{p}'} f(t,\mathbf{x}',\mathbf{p}')\left\{\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{p}-\mathbf{p}'), \frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')}\right\}_{\text{Poisson}} (t,\mathbf{x}',\mathbf{p}')$$

$$= \partial_t f(t,\mathbf{x},\mathbf{p}) + \int_{\mathbf{x}',\mathbf{p}'} f(t,\mathbf{x}',\mathbf{p}')\left[\nabla_{\mathbf{x}'}[\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{p}-\mathbf{p}')]\nabla_{\mathbf{p}'}\left[\frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')}\right] - \nabla_{\mathbf{x}'}\left[\frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')}\right]\nabla_{\mathbf{p}'}[\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{p}-\mathbf{p}')]\right]$$

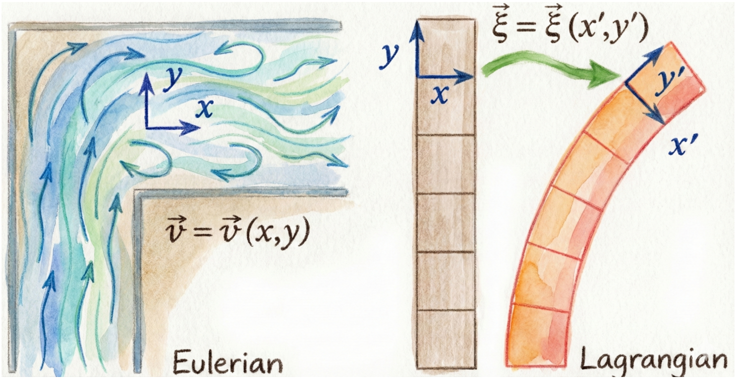
$$\overset{\text{I.B.P.}}{=} \partial_t f(t,\mathbf{x},\mathbf{p}) + \int_{\mathbf{x}',\mathbf{p}'} \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{p}-\mathbf{p}')\left[-\nabla_{\mathbf{x}'}\left(f(t,\mathbf{x}',\mathbf{p}')\nabla_{\mathbf{p}'}\left[\frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')}\right]\right) + \nabla_{\mathbf{p}'}\left(f(t,\mathbf{x}',\mathbf{p}')\nabla_{\mathbf{x}'}\left[\frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')}\right]\right)\right]$$

$$= \partial_t f(t,\mathbf{x},\mathbf{p}) + \int_{\mathbf{x}',\mathbf{p}'} \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{p}-\mathbf{p}')\left\{\frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')}, f(t,\mathbf{x}',\mathbf{p}')\right\}_{\text{Poisson}}$$

$$= \partial_t f(t,\mathbf{x},\mathbf{p}) - \left\{f(t,\mathbf{x},\mathbf{p}), \frac{\delta H}{\delta f(t,\mathbf{x},\mathbf{p})}\right\}_{\text{Poisson}} = 0$$

Equation of Motion

- Write down E.o.M. on $\mathfrak{g}^*$ (Lagrangian Specification)



$$
\begin{aligned}
&\partial_t f(t,\mathbf{x},\mathbf{p}) - \{f, H\}_{\text{LP}}[f(t,\mathbf{x},\mathbf{p})] \\
&= \partial_t f(t,\mathbf{x},\mathbf{p}) - \int_{\mathbf{x}',\mathbf{p}'} f(t,\mathbf{x}',\mathbf{p}') \left\{ \frac{\delta f(t,\mathbf{x},\mathbf{p})}{\delta f(t,\mathbf{x}',\mathbf{p}')}, \frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')} \right\}_{\text{Poisson}} (t,\mathbf{x}',\mathbf{p}') \\
&= \partial_t f(t,\mathbf{x},\mathbf{p}) - \int_{\mathbf{x}',\mathbf{p}'} f(t,\mathbf{x}',\mathbf{p}') \left\{ \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{p}-\mathbf{p}'), \frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')} \right\}_{\text{Poisson}} (t,\mathbf{x}',\mathbf{p}') \\
&= \partial_t f(t,\mathbf{x},\mathbf{p}) - \int_{\mathbf{x}',\mathbf{p}'} f(t,\mathbf{x}',\mathbf{p}') \left[\nabla_{\mathbf{x}'}[\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{p}-\mathbf{p}')] \nabla_{\mathbf{p}'} \left[\frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')}\right] - \nabla_{\mathbf{x}'} \left[\frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')}\right] \nabla_{\mathbf{p}'}[\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{p}-\mathbf{p}')] \right] \\
&\overset{\text{I.B.P.}}{=} \partial_t f(t,\mathbf{x},\mathbf{p}) - \int_{\mathbf{x}',\mathbf{p}'} \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{p}-\mathbf{p}') \left[-\nabla_{\mathbf{x}'} \left(f(t,\mathbf{x}',\mathbf{p}') \nabla_{\mathbf{p}'} \left[\frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')}\right] \right) + \nabla_{\mathbf{p}'} \left(f(t,\mathbf{x}',\mathbf{p}') \nabla_{\mathbf{x}'} \left[\frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')}\right] \right) \right] \\
&= \partial_t f(t,\mathbf{x},\mathbf{p}) - \int_{\mathbf{x}',\mathbf{p}'} \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{p}-\mathbf{p}') \left\{ \frac{\delta H[f(t,\mathbf{x},\mathbf{p})]}{\delta f(t,\mathbf{x}',\mathbf{p}')}, f(t,\mathbf{x}',\mathbf{p}') \right\}_{\text{Poisson}} \\
&= \partial_t f(t,\mathbf{x},\mathbf{p}) + \left\{ f(t,\mathbf{x},\mathbf{p}), \frac{\delta H}{\delta f(t,\mathbf{x},\mathbf{p})} \right\}_{\text{Poisson}} = 0
\end{aligned}
$$

Equation of Motion

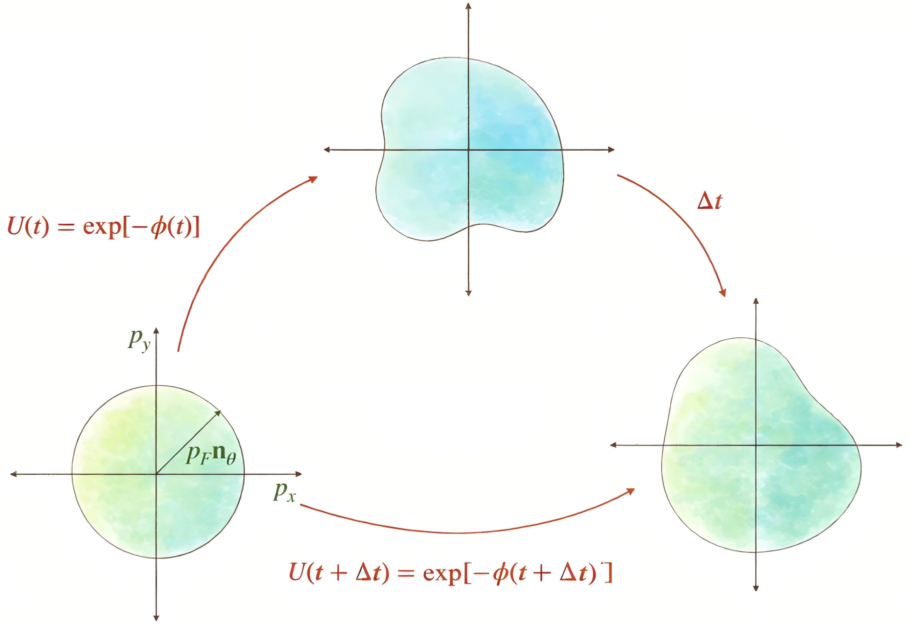
- Construct Hamiltonian from EFT
  - Fluctuations around the Reference State

$$f_0(\mathbf{p}) = \Theta\left(p_F - |\mathbf{p}|\right) \qquad \delta f(\mathbf{x},\mathbf{p}) \equiv f(\mathbf{x},\mathbf{p}) - f_0(\mathbf{p})$$

  - General Hamiltonian

$$\begin{aligned} H[f] = & \int_{\mathbf{xp}} \epsilon(\mathbf{p}) f(\mathbf{x},\mathbf{p}) \\ & + \frac{1}{2}\int_{\mathbf{xpp}'} F^{(2,0)}\left(\mathbf{p},\mathbf{p}'\right)\delta f(\mathbf{x},\mathbf{p})\delta f\left(\mathbf{x},\mathbf{p}'\right) + \mathbf{F}^{(2,1)}\left(\mathbf{p},\mathbf{p}'\right)\cdot\left(\frac{\nabla_{\mathbf{x}}}{p_F}\delta f(\mathbf{x},\mathbf{p})\right)\delta f\left(\mathbf{x},\mathbf{p}'\right) + \ldots \\ & + \frac{1}{3}\int_{\mathbf{xppp'p''}} F^{(3,0)}\left(\mathbf{p},\mathbf{p}',\mathbf{p}''\right)\delta f(\mathbf{x},\mathbf{p})\delta f\left(\mathbf{x},\mathbf{p}'\right)\delta f\left(\mathbf{x},\mathbf{p}''\right) + \ldots \\ & + \ldots \end{aligned}$$

    - $F^{(m,n)}$: Index $m \to$ Nonlinearity of the Action   Index n -> # of $\boldsymbol{x}$-Derivatives in Coupling



Equation of Motion

- Quasiparticle Dispersion Relation : Variation of the Hamiltonian

$$\partial_t f(t,\mathbf{x},\mathbf{p}) + \left\{ f(t,\mathbf{x},\mathbf{p}), \frac{\delta H}{\delta f(t,\mathbf{x},\mathbf{p})} \right\}_{\text{Poisson}} = 0$$

$$\epsilon_{\text{qp}}[f] \equiv \frac{\delta H}{\delta f} = \epsilon(\mathbf{p}) + \int_{\mathbf{p}'} F^{(2,0)}(\mathbf{p},\mathbf{p}')\delta f(t,\mathbf{x},\mathbf{p}') + \cdots$$

$$\Downarrow$$

$$\partial_t f + \nabla_{\mathbf{p}}\epsilon_{\text{qp}}[f] \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}}\epsilon_{\text{qp}}[f] \cdot \nabla_{\mathbf{p}} f = 0$$

  - $F^{(2,0)}(\boldsymbol{p},\boldsymbol{p}')$: Landau's Interaction Function
  - Infinite Series of Higher Order Corrections to Quasiparticle Energy
  - Physics of FL is in the Dispersion Relation

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5. Coadjoint Orbits and Emergent Effective Action

Effective Action from the Coadjoint Orbit Method

- Phase Space Manifold Γ with Poisson Brackets

$$\{F, G\} = \Pi^{IJ} \partial_I F \partial_J G$$

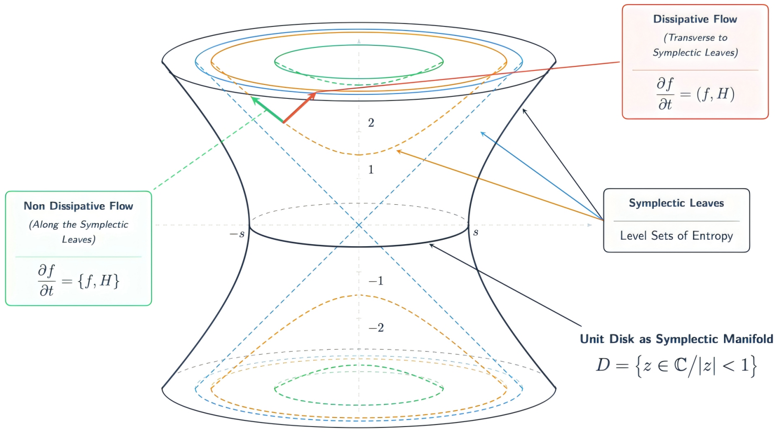
  - π^{IJ} : Poisson Bi-Vector
  - ω_{IJ} : Inverse of π^{IJ} $\rightarrow$ Closed, Anti-Symmetric, Non-Degenerate Symplectic Form

$$\omega = \Pi^{-1} \qquad \omega_{IJ} \Pi^{JK} = \delta_I^K$$

  - Legendre Transformation : Hamiltonian to Lagrangian

$$\int dt \int_0^1 ds \omega \left(\partial_t \phi, \partial_s \phi\right) = \int dt \int_0^1 ds \omega_{IJ} \partial_t \phi^I \partial_s \phi^J$$

$$S = \int dt \int_0^1 ds \omega \left(\partial_t \phi, \partial_s \phi\right) - \int dt H[\phi]$$



  - Invertibility is not Guaranteed by the Definition of the Poisson Bracket (Symplectic Leaves)

Effective Action from the Coadjoint Orbit Method

- Phase Space Manifold Γ with Poisson Brackets $\omega = \Pi^{-1}$ $\quad \omega_{IJ}\Pi^{JK} = \delta_I^K$
  - Legendre Transformation : Proof
    - 2-Form ω is Exact Locally $\Rightarrow$ 1-Form Symplectic Potential θ

$$\omega = d\theta \qquad \omega_{IJ} = \partial_I \theta_J - \partial_J \theta_I$$

    - Poincaré–Cartan Form Action $S[\phi] = \int dt \left(\theta_I(\phi)\dot{\phi}^I - H(\phi)\right)$
    - Variation over ϕ and I.B.P.

$$\delta S = \int dt \left[(\partial_J \theta_I)\delta\phi^J \dot{\phi}^I + \theta_I \delta\dot{\phi}^I - \partial_I H \delta\phi^I\right]$$

$$\int dt \theta_I \delta\dot{\phi}^I = \left[\theta_I \delta\phi^I\right]_{t_i}^{t_f} - \int dt \dot{\theta}_I \delta\phi^I = 0 - \int dt \left[(\partial_J \theta_I)\dot{\phi}^J\right]\delta\phi^I$$

$$\Downarrow$$

$$\delta S = \int dt \left[(\partial_J \theta_I)\dot{\phi}^I - (\partial_I \theta_J)\dot{\phi}^J - \partial_J H\right]\delta\phi^J = \int dt \left[\omega_{JI}\dot{\phi}^I - \partial_J H\right]\delta\phi^J$$

Effective Action from the Coadjoint Orbit Method

- Phase Space Manifold Γ with Poisson Brackets $\omega = \Pi^{-1} \qquad \omega_{IJ}\Pi^{JK} = \delta_I^K$
  - Legendre Transformation : Proof
    - E.o.M. (Same as Hamiltonian Equation)

$$\delta S = \int dt \left[(\partial_J \theta_I)\dot{\phi}^I - (\partial_I \theta_J)\dot{\phi}^J - \partial_J H\right]\delta\phi^J = \int dt \left[\omega_{JI}\dot{\phi}^I - \partial_J H\right]\delta\phi^J$$

$$\Downarrow$$

$$\omega_{JI}\dot{\phi}^I = \partial_J H \quad \Rightarrow \quad \dot{\phi}^K = \Pi^{KJ}\partial_J H$$

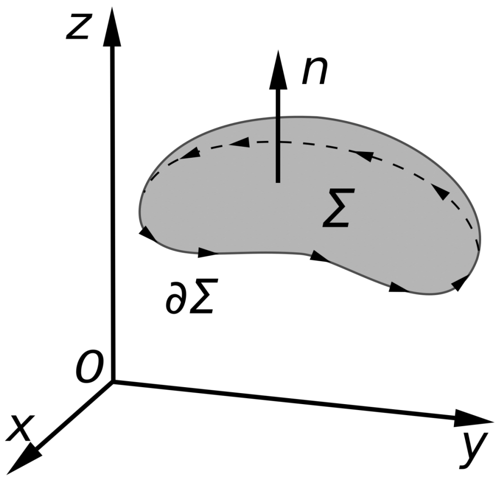
    - Action on the Phase Space
  - ω : Kirillov-Kostant-Souriau(KKS) Symplectic Form

Fermi Surface States and Their Excitations

- KKS Form is Closed
  - Independent of the Bulk

$$\int_{\Sigma} \omega = \int_{\Sigma} d\theta = \int_{\partial\Sigma} \theta$$

  - Prove it's Closed
    - To Satisfy Jacobi Identity of the Poisson Bracket



$$\{\{F,G\},H\} = \{\Pi^{IJ}\partial_I F\partial_J G, H\} = \Pi^{KL}\partial_K\left(\Pi^{IJ}\partial_I F\partial_J G\right)\partial_L H$$
$$= \Pi^{KL}\partial_K\Pi^{IJ}\partial_I F\partial_J G\partial_L H + \Pi^{KL}\Pi^{IJ}\left[\partial_K\left(\partial_I F\partial_J G\partial_L H\right) + 0\right]$$

$$\{\{F,G\},H\} + \{\{G,H\},F\} + \{\{H,F\},G\}$$
$$= \left(\Pi^{KL}\partial_K\Pi^{IJ} + \Pi^{KI}\partial_K\Pi^{JL} + \Pi^{KJ}\partial_K\Pi^{LI}\right)\partial_I F\partial_J G\partial_L H$$
$$+ \left(\Pi^{KL}\Pi^{IJ} + \Pi^{KI}\Pi^{JL} + \Pi^{KJ}\Pi^{LI}\right)\partial_K\left(\partial_I F\partial_J G\partial_L H\right)$$

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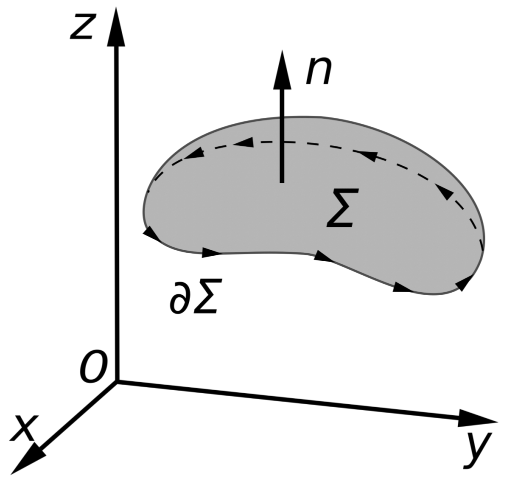
Fermi Surface States and Their Excitations

- KKS Form is Closed
  - Independent of the Bulk
  - Prove it's Closed
    - Exterior Derivative

$$\int_\Sigma \omega = \int_\Sigma d\theta = \int_{\partial\Sigma} \theta$$



$$\omega = \frac{1}{2}\omega_{IJ} d\phi^I \wedge d\phi^J$$

$$d\omega = \frac{1}{2}\partial_K \omega_{IJ} d\phi^K \wedge d\phi^I \wedge d\phi^J = \frac{1}{3!}(\partial_K\omega_{IJ} + \partial_I\omega_{JK} + \partial_J\omega_{KI}) d\phi^K \wedge d\phi^I \wedge d\phi^J$$

- Mutually Inverse ⇒ Solve for $\partial_K \omega_{IJ}$

$$\partial_L\left(\delta_I^K\right) = 0 = \partial_L\left(\omega_{IJ}\Pi^{JK}\right) = (\partial_L\omega_{IJ})\Pi^{JK} + \omega_{IJ}\partial_L\Pi^{JK} \quad \Rightarrow \quad \partial_L\omega_{IJ}\Pi^{JK}\omega_{KM} + \omega_{IJ}\partial_L\Pi^{JK}\omega_{KM}$$

$$\Rightarrow \partial_L\omega_{IJ} = -\omega_{IM}\partial_L\Pi^{MN}\left(\Pi^{-1}\right)_{NJ} \Rightarrow \partial_K\omega_{IJ} = -\omega_{IM}\partial_K\Pi^{MN}\omega_{NJ}$$

$$\Rightarrow d\omega = \frac{1}{3!}\left[-\omega_{IM}\partial_K\Pi^{MN}\omega_{NJ} - \omega_{JM}\partial_I\Pi^{MN}\omega_{NK} - \omega_{KM}\partial_J\Pi^{MN}\omega_{NI}\right] d\phi^K \wedge d\phi^I \wedge d\phi^J$$

$$= -\frac{1}{2}\omega_{IM}\partial_K\Pi^{MN}\omega_{NJ} d\phi^K \wedge d\phi^I \wedge d\phi^J$$

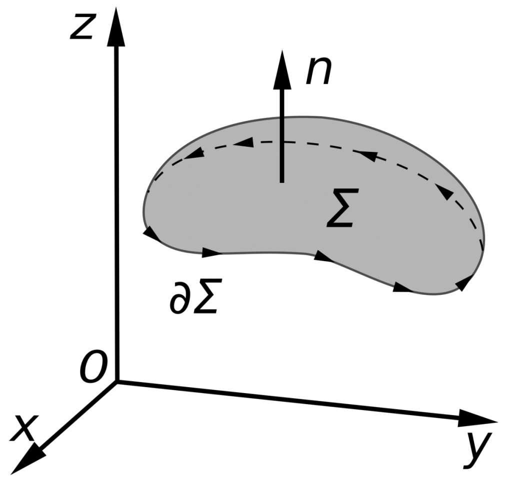
$$(d\omega)_{IJK} = -\left(\omega_{IM}\partial_K\Pi^{MN}\omega_{NJ} + \omega_{JM}\partial_I\Pi^{MN}\omega_{NK} + \omega_{KM}\partial_J\Pi^{MN}\omega_{NI}\right) \equiv \omega_{IP}\omega_{JQ}\omega_{KR}S^{PQR}$$

Fermi Surface States and Their Excitations

- KKS Form is Closed
  - Independent of the Bulk

$$\int_\Sigma \omega = \int_\Sigma d\theta = \int_{\partial\Sigma} \theta$$

  - Prove it's Closed
    - Exterior Derivative



$$S^{PQR} \equiv \Pi^{PI}\Pi^{QJ}\Pi^{RK}(d\omega)_{IJK} = -\left(\delta^P_M \Pi^{RK}\partial_K\Pi^{MN}\delta^Q_N + \delta^Q_M\Pi^{PI}\partial_I\Pi^{MN}\delta^R_N + \delta^R_M\Pi^{QJ}\partial_J\Pi^{MN}\delta^P_N\right)$$

$$= -\left(\Pi^{RK}\partial_K\Pi^{PQ} + \Pi^{PI}\partial_I\Pi^{QR} + \Pi^{QJ}\partial_J\Pi^{RP}\right)$$

- Set $F, G, H = \phi$ (No 2nd Variation)

$$\left\{\{F,G\},H\right\} + \left\{\{G,H\},F\right\} + \left\{\{H,F\},G\right\}$$

$$= \left(\Pi^{KL}\partial_K\Pi^{IJ} + \Pi^{KI}\partial_K\Pi^{JL} + \Pi^{KJ}\partial_K\Pi^{LI}\right)\partial_I F\partial_J G\partial_L H + \left(\Pi^{KL}\Pi^{IJ} + \Pi^{KI}\Pi^{JL} + \Pi^{KJ}\Pi^{LI}\right)\partial_K\left(\partial_I F\partial_J G\partial_L H\right)$$

$$\Downarrow$$

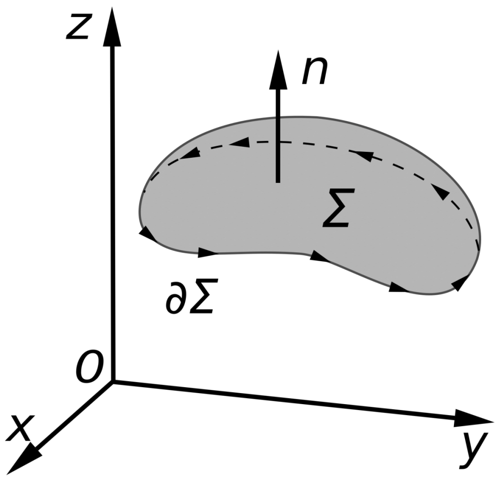
$$\left\{\{\phi^A,\phi^B\},\phi^C\right\} + \left\{\{\phi^B,\phi^C\},\phi^A\right\} + \left\{\{\phi^C,\phi^A\},\phi^B\right\}$$

$$= \left(\Pi^{KL}\partial_K\Pi^{IJ} + \Pi^{KI}\partial_K\Pi^{JL} + \Pi^{KJ}\partial_K\Pi^{LI}\right)\delta^A_I\delta^B_J\delta^C_L + 0$$

$$= \Pi^{KC}\partial_K\Pi^{AB} + \Pi^{KA}\partial_K\Pi^{BC} + \Pi^{KB}\partial_K\Pi^{CA} = S^{PQR} = 0$$

Fermi Surface States and Their Excitations

- KKS Form is Closed
  - Independent of the Bulk $$\int_\Sigma \omega = \int_\Sigma d\theta = \int_{\partial\Sigma} \theta$$
  - Prove it's Closed
    - *Π* is Non-Degenerate $$(d\omega)_{IJK} = \omega_{IP}\omega_{JQ}\omega_{KR}S^{PQR} = 0$$
    - Q.E.D.



- Recap : Typical Representative of the Equivalent Classes of Density Matrices

$$\rho_f = \int_{\mathbf{x},\mathbf{p}} f(\mathbf{x},\mathbf{p})W(\mathbf{x},\mathbf{p})$$

$$\mathrm{Tr}\left[W(\mathbf{x},\mathbf{p})T\left(\mathbf{x}',\mathbf{p}'\right)\right] = \delta\left(\mathbf{x}-\mathbf{x}'\right)(2\pi)^d\delta\left(\mathbf{p}-\mathbf{p}'\right)$$

$$\langle O_F\rangle_{\rho_f} = \mathrm{Tr}\left[\rho_f O_F\right] = \int_{\mathbf{x},\mathbf{p}} F(\mathbf{x},\mathbf{p})f(\mathbf{x},\mathbf{p}) = \langle f, F\rangle$$

Fermi Surface States and Their Excitations

- Stratonovich-Weyl Kernel $T(\boldsymbol{x},\boldsymbol{p})$ $\quad f(\mathbf{x},\mathbf{p})=\langle T(\mathbf{x},\mathbf{p})\rangle_\rho$
- State of the Spherical Fermi Surface

$$|\mathrm{FS}\rangle=\prod_{|\mathbf{k}|\le p_F}\psi^\dagger(\mathbf{k})|0\rangle \qquad f_0(\mathbf{p})=\langle\mathrm{FS}|T(\mathbf{x},\mathbf{p})|\mathrm{FS}\rangle=\frac{i}{2}\,\mathrm{sign}\,(p_F-|\mathbf{p}|)$$

  - Fermion of Certain Momentum $\langle\mathrm{FS}\,|\psi^\dagger(\mathbf{k})\psi(\mathbf{k}')|\,\mathrm{FS}\rangle=(2\pi)^d\delta(\mathbf{k}-\mathbf{k}')n(\mathbf{k})$
  - Discrete Case $\quad \mathbf{k}\in\frac{2\pi}{L}\mathbb{Z}^d \quad \{c_\mathbf{k},c^\dagger_{\mathbf{k}'}\}=\delta_{\mathbf{k},\mathbf{k}'} \quad |\,\mathrm{FS}\rangle=\prod_{|\mathbf{k}|\le p_F}c^\dagger_\mathbf{k}|0\rangle$
  - (i)Out of the Fermi Surface

$$|\mathbf{k}'|>p_F \quad c_{\mathbf{k}'}|\,\mathrm{FS}\rangle=0 \implies \langle\mathrm{FS}\,|c^\dagger_\mathbf{k}c_{\mathbf{k}'}|\,\mathrm{FS}\rangle=0$$

  - (ii)In the Fermi Surface (Pauli Exclusion Principle: 1 Electron with Momentum $\boldsymbol{k}$)

$$|\mathbf{k}'|\le p_F \quad \langle\mathrm{FS}\,|c^\dagger_\mathbf{k}c_{\mathbf{k}'}|\,\mathrm{FS}\rangle=\delta_{\mathbf{k},\mathbf{k}'} \implies \langle\mathrm{FS}\,|c^\dagger_\mathbf{k}c_{\mathbf{k}'}|\,\mathrm{FS}\rangle=\delta_{\mathbf{k},\mathbf{k}'}n(\mathbf{k}) \quad n(\mathbf{k})=\Theta(p_F-|\mathbf{k}|)$$

Fermi Surface States and Their Excitations

- Stratonovich-Weyl Kernel $T(\boldsymbol{x}, \boldsymbol{p})$ $\quad f(\mathbf{x},\mathbf{p}) = \langle T(\mathbf{x},\mathbf{p})\rangle_\rho$
- State of the Spherical Fermi Surface

$$|\mathrm{FS}\rangle = \prod_{|\mathbf{k}|\le p_F} \psi^\dagger(\mathbf{k})|0\rangle \qquad f_0(\mathbf{p}) = \langle \mathrm{FS}|T(\mathbf{x},\mathbf{p})|\mathrm{FS}\rangle = \frac{i}{2}\,\mathrm{sign}\,(p_F - |\mathbf{p}|)$$

  - Continuum Limit

$$\psi(\mathbf{k}) = \sqrt{V}c_{\mathbf{k}} \quad \{\psi(\mathbf{k}),\psi^\dagger(\mathbf{k}')\} = V\delta_{\mathbf{k},\mathbf{k}'} \xrightarrow{V\to\infty} \{\psi(\mathbf{k}),\psi^\dagger(\mathbf{k}')\} = (2\pi)^d\delta(\mathbf{k}-\mathbf{k}')$$

$$\Downarrow$$

$$\langle \mathrm{FS}\,|\psi^\dagger(\mathbf{k})\psi(\mathbf{k}')|\,\mathrm{FS}\rangle = V\delta_{\mathbf{k},\mathbf{k}'}n(\mathbf{k}) \xrightarrow{V\to\infty} \langle \mathrm{FS}\,|\psi^\dagger(\mathbf{k})\psi(\mathbf{k}')|\,\mathrm{FS}\rangle = (2\pi)^d\delta(\mathbf{k}-\mathbf{k}')n(\mathbf{k})$$

$$\langle \mathrm{FS}\,|\psi(\mathbf{k})\psi^\dagger(\mathbf{k}')|\,\mathrm{FS}\rangle = \langle \mathrm{FS}\,|\{\psi(\mathbf{k}),\psi^\dagger(\mathbf{k}')\} - \psi^\dagger(\mathbf{k}')\psi(\mathbf{k})|\,\mathrm{FS}\rangle = (2\pi)^d\delta(\mathbf{k}-\mathbf{k}')(1-n(\mathbf{k}))$$

  - Recall : In Position Space

$$T(\mathbf{x},\mathbf{y}) = \tfrac{i}{2}\left[\psi^\dagger(\mathbf{x}+\tfrac{\mathbf{y}}{2})\psi(\mathbf{x}-\tfrac{\mathbf{y}}{2}) - \psi(\mathbf{x}-\tfrac{\mathbf{y}}{2})\psi^\dagger(\mathbf{x}+\tfrac{\mathbf{y}}{2})\right]$$

Fermi Surface States and Their Excitations

- Stratonovich-Weyl Kernel $T(\boldsymbol{x},\boldsymbol{p})$ $\quad f(\mathbf{x},\mathbf{p})=\langle T(\mathbf{x},\mathbf{p})\rangle_\rho$
- State of the Spherical Fermi Surface

$$|\mathrm{FS}\rangle=\prod_{|\mathbf{k}|\leq p_F}\psi^\dagger(\mathbf{k})|0\rangle \qquad f_0(\mathbf{p})=\langle\mathrm{FS}|T(\mathbf{x},\mathbf{p})|\mathrm{FS}\rangle=\frac{i}{2}\,\mathrm{sign}\,(p_F-|\mathbf{p}|)$$

  - Expectation Value of the Bilinear under the State

$$\langle\mathrm{FS}\,|\psi^\dagger(\mathbf{x}+\frac{\mathbf{y}}{2})\psi(\mathbf{x}-\frac{\mathbf{y}}{2})|\,\mathrm{FS}\rangle=\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{d^d\mathbf{k}'}{(2\pi)^d}e^{i(\mathbf{x}+\frac{\mathbf{y}}{2})\cdot\mathbf{k}}e^{-i(\mathbf{x}-\frac{\mathbf{y}}{2})\cdot\mathbf{k}'}\langle\mathrm{FS}\,|\psi^\dagger(\mathbf{k})\psi(\mathbf{k}')|\,\mathrm{FS}\rangle$$
$$=\int\frac{d^d\mathbf{k}\,d^d\mathbf{k}'}{(2\pi)^{2d}}e^{i(\mathbf{x}+\frac{\mathbf{y}}{2})\cdot\mathbf{k}}e^{-i(\mathbf{x}-\frac{\mathbf{y}}{2})\cdot\mathbf{k}'}(2\pi)^d\delta(\mathbf{k}-\mathbf{k}')n(\mathbf{k})=\int\frac{d^d\mathbf{k}}{(2\pi)^d}e^{i\mathbf{y}\cdot\mathbf{k}}n(\mathbf{k})$$

$$\langle\mathrm{FS}\,|\psi(\mathbf{x}-\frac{\mathbf{y}}{2})\psi^\dagger(\mathbf{x}+\frac{\mathbf{y}}{2})|\,\mathrm{FS}\rangle=\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{d^d\mathbf{k}'}{(2\pi)^d}e^{i(\mathbf{x}+\frac{\mathbf{y}}{2})\cdot\mathbf{k}'}e^{-i(\mathbf{x}-\frac{\mathbf{y}}{2})\cdot\mathbf{k}}\langle\mathrm{FS}\,|\psi(\mathbf{k})\psi^\dagger(\mathbf{k}')|\,\mathrm{FS}\rangle$$
$$=\int\frac{d^d\mathbf{k}\,d^d\mathbf{k}'}{(2\pi)^{2d}}e^{i(\mathbf{x}+\frac{\mathbf{y}}{2})\cdot\mathbf{k}'}e^{-i(\mathbf{x}-\frac{\mathbf{y}}{2})\cdot\mathbf{k}}(2\pi)^d\delta(\mathbf{k}-\mathbf{k}')(1-n(\mathbf{k}))=\int\frac{d^d\mathbf{k}}{(2\pi)^d}e^{i\mathbf{y}\cdot\mathbf{k}}(1-n(\mathbf{k}))$$

$$\langle\mathrm{FS}\,|T(\mathbf{x},\mathbf{y})|\,\mathrm{FS}\rangle=\frac{i}{2}\left[\langle\mathrm{FS}\,|\psi^\dagger(\mathbf{x}+\frac{\mathbf{y}}{2})\psi(\mathbf{x}-\frac{\mathbf{y}}{2})|\,\mathrm{FS}\rangle-\langle\mathrm{FS}\,|\psi(\mathbf{x}-\frac{\mathbf{y}}{2})\psi^\dagger(\mathbf{x}+\frac{\mathbf{y}}{2})|\,\mathrm{FS}\rangle\right]=\frac{i}{2}\int\frac{d^d\mathbf{k}}{(2\pi)^d}e^{i\mathbf{y}\cdot\mathbf{k}}\,[2n(\mathbf{k})-1]$$

Fermi Surface States and Their Excitations

- Stratonovich-Weyl Kernel $T(\boldsymbol{x},\boldsymbol{p})$ $\quad f(\mathbf{x},\mathbf{p}) = \langle T(\mathbf{x},\mathbf{p})\rangle_\rho$
- State of the Spherical Fermi Surface

$$|\mathrm{FS}\rangle = \prod_{|\mathbf{k}|\le p_F} \psi^\dagger(\mathbf{k})|0\rangle \qquad f_0(\mathbf{p}) = \langle \mathrm{FS}|T(\mathbf{x},\mathbf{p})|\mathrm{FS}\rangle = \frac{i}{2}\,\mathrm{sign}\left(p_F - |\mathbf{p}|\right)$$

  - Expectation Value of the Bilinear under the State

$$f_0(\mathbf{p}) = \langle \mathrm{FS}\,|T(\mathbf{x},\mathbf{p})|\,\mathrm{FS}\rangle = \int d^d\mathbf{y}\langle \mathrm{FS}\,|T(\mathbf{x},\mathbf{y})|\,\mathrm{FS}\rangle e^{-i\mathbf{p}\cdot\mathbf{y}} = \frac{i}{2}\int \frac{d^d\mathbf{k}}{(2\pi)^d}\int d^d\mathbf{y} e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{y}}\left[2n(\mathbf{k})-1\right]$$

$$= \frac{i}{2}\int \frac{d^d\mathbf{k}}{(2\pi)^d}(2\pi)^d\delta(\mathbf{p}-\mathbf{k})\left[2n(\mathbf{k})-1\right] = \frac{i}{2}\left[2n(\mathbf{p})-1\right]$$

$$2n(\mathbf{p})-1 = 2\Theta(p_F-|\mathbf{p}|)-1 = \mathrm{sign}(p_F-|\mathbf{p}|) \implies f_0(\mathbf{p}) = \frac{i}{2}\mathrm{sign}(p_F-|\mathbf{p}|)$$

Fermi Surface States and Their Excitations

- Excitations around the Fermi Surface: Particle-Hole Pairs $|\mathbf{k}_1;\mathbf{k}_2\rangle \equiv \psi^\dagger(\mathbf{k}_1)\psi(-\mathbf{k}_2)|\mathrm{FS}\rangle$
- Local Basis v.s. Global Basis (More Hydrodynamic / Coarse Graining)

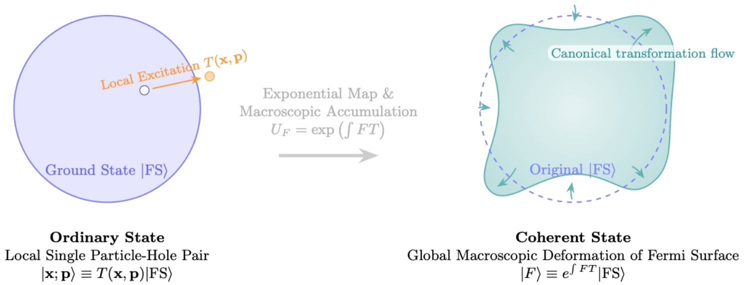
$$|\mathbf{x};\mathbf{p}\rangle \equiv T(\mathbf{x},\mathbf{p})|\mathrm{FS}\rangle \quad \longrightarrow \quad |F(\mathbf{x},\mathbf{p})\rangle \equiv e^{\int_{\mathbf{xp}} F(\mathbf{x},\mathbf{p})T(\mathbf{x},\mathbf{p})}|\mathrm{FS}\rangle$$

- Distribution Function in the Coherent State

$$f_F(\mathbf{x},\mathbf{p}) = f_0(\mathbf{p}) + \{\{F, f_0\}\} + \frac{1}{2!}\left\{\left\{F, \{\{F, f_0\}\}\right\}\right\} + \dots$$

  - Proof
    - Recall $$\mathrm{Tr}(\rho_f O_F) = \int \frac{d^d\mathbf{x}d^d\mathbf{p}}{(2\pi)^d} f(\mathbf{x},\mathbf{p})F(\mathbf{x},\mathbf{p}) = \langle f, F\rangle$$
    - Pure Coherent State $$\rho_F = |F(\mathbf{x},\mathbf{p})\rangle\langle F(\mathbf{x},\mathbf{p})|$$

Comparison of Physical Picture: Ordinary State vs Coherent State



Fermi Surface States and Their Excitations

- Distribution Function in the Coherent State

$$f_F(\mathbf{x},\mathbf{p}) = f_0(\mathbf{p}) + \{\{F, f_0\}\} + \frac{1}{2!}\left\{\left\{F, \{\{F, f_0\}\}\right\}\right\} + \dots$$

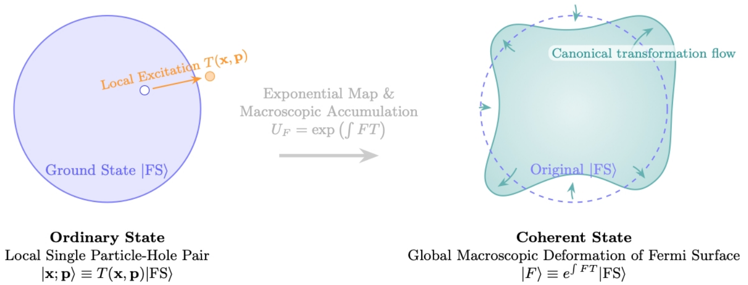
  - Proof
    - Choose Operator $O_G = \int_{\mathbf{x}',\mathbf{p}'} \delta(\mathbf{x}'-\mathbf{x})(2\pi)^d \delta(\mathbf{p}'-\mathbf{p}) T(\mathbf{x}',\mathbf{p}') = T(\mathbf{x},\mathbf{p})$
    - Distribution Function

$$f_F(\mathbf{x},\mathbf{p}) = \langle T(\mathbf{x},\mathbf{p})\rangle_\rho = \mathrm{Tr}[\rho_F O_G]$$

$$= \mathrm{Tr}\left[|F(\mathbf{x},\mathbf{p})\rangle\langle F(\mathbf{x},\mathbf{p})| \int_{\mathbf{x}',\mathbf{p}'} \delta(\mathbf{x}'-\mathbf{x})(2\pi)^d \delta(\mathbf{p}'-\mathbf{p}) T(\mathbf{x}',\mathbf{p}')\right]$$

$$= \langle F(\mathbf{x},\mathbf{p})|T(\mathbf{x},\mathbf{p})|F(\mathbf{x},\mathbf{p})\rangle = \langle \mathrm{FS}|U_F^{-1} T(\mathbf{x},\mathbf{p}) U_F|\mathrm{FS}\rangle$$

Comparison of Physical Picture: Ordinary State vs Coherent State



Fermi Surface States and Their Excitations

- Distribution Function in the Coherent State

$$f_F(\mathbf{x},\mathbf{p}) = f_0(\mathbf{p}) + \{\{F, f_0\}\} + \frac{1}{2!}\left\{\left\{F, \{\{F, f_0\}\}\right\}\right\} + \dots$$

  - Proof
    - Use BCH Formula (Operators ⇒ Commutators as Lie Brackets)

$$f_F(\mathbf{x},\mathbf{p}) = \langle \mathrm{FS}|U_F^{-1}T(\mathbf{x},\mathbf{p})U_F|\mathrm{FS}\rangle = \langle \mathrm{FS}|O_G + (-1)^1[O_F, O_G] + \frac{(-1)^2}{2!}\Big[O_F, [O_F, O_G]\Big] + \dots|\mathrm{FS}\rangle$$

$$\overset{(3.5)}{=} \langle \mathrm{FS}|O_G + (-1)^1 O_{\{\{F,G\}\}} + \frac{(-1)^2}{2!} O_{\left\{\left\{F,\{\{F,G\}\}\right\}\right\}} + \dots|\mathrm{FS}\rangle$$

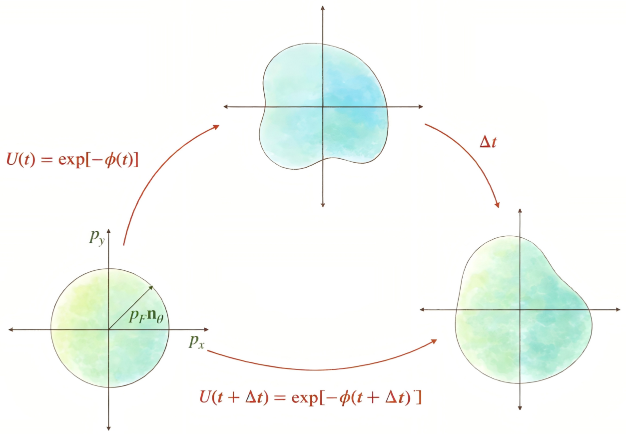
$$= \int_{\mathbf{x}',\mathbf{p}'} \left[G(\mathbf{x}',\mathbf{p}') + (-1)^1\{\{F,G\}\}(\mathbf{x}',\mathbf{p}') + \frac{(-1)^2}{2!}\left\{\left\{F,\{\{F,G\}\}\right\}\right\}(\mathbf{x}',\mathbf{p}') + \dots\right]\langle \mathrm{FS}|T(\mathbf{x}',\mathbf{p}')|\mathrm{FS}\rangle$$

$$= \int_{\mathbf{x}',\mathbf{p}'} f_0(\mathbf{x}',\mathbf{p}')\left[G(\mathbf{x}',\mathbf{p}') + (-1)^1\{\{F,G\}\}(\mathbf{x}',\mathbf{p}') + \frac{(-1)^2}{2!}\left\{\left\{F,\{\{F,G\}\}\right\}\right\}(\mathbf{x}',\mathbf{p}') + \dots\right] \quad f_0(\mathbf{x}',\mathbf{p}') = \langle \mathrm{FS}|T(\mathbf{x}',\mathbf{p}')|\mathrm{FS}\rangle$$

$$\overset{(4.22)}{=} \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\langle f_0, (\mathrm{ad}_F)^n G\rangle$$

Fermi Surface States and Their Excitations

- Distribution Function in the Coherent State



$$f_F(\mathbf{x},\mathbf{p}) = f_0(\mathbf{p}) + \{\{F, f_0\}\} + \frac{1}{2!}\left\{\left\{F, \{\{F, f_0\}\}\right\}\right\} + \dots$$

  - Proof
    - "I.B.P." of the Moyal Bracket $-\left\langle f, \{\{F, G\}\}\right\rangle = \left\langle \{\{F, f\}\}, G\right\rangle$

$$f_F(\mathbf{x},\mathbf{p}) \overset{(4.24)}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\langle f_0, (\mathrm{ad}_F)^n G\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}\langle (\mathrm{ad}_F^*)^n f_0, G\rangle$$

$$= \int_{\mathbf{x}',\mathbf{p}'} \left[f_0(\mathbf{x}',\mathbf{p}') + \{\{F, f_0\}\}(\mathbf{x}',\mathbf{p}') + \frac{1}{2!}\left\{\left\{F, \{\{F, f_0\}\}\right\}\right\}(\mathbf{x}',\mathbf{p}') + \dots \right] \delta(\mathbf{x}'-\mathbf{x})(2\pi)^d \delta(\mathbf{p}'-\mathbf{p})$$

$$= f_0(\mathbf{x},\mathbf{p}) + \{\{F, f_0\}\}(\mathbf{x},\mathbf{p}) + \frac{1}{2!}\left\{\left\{F, \{\{F, f_0\}\}\right\}\right\}(\mathbf{x},\mathbf{p}) + \cdots$$

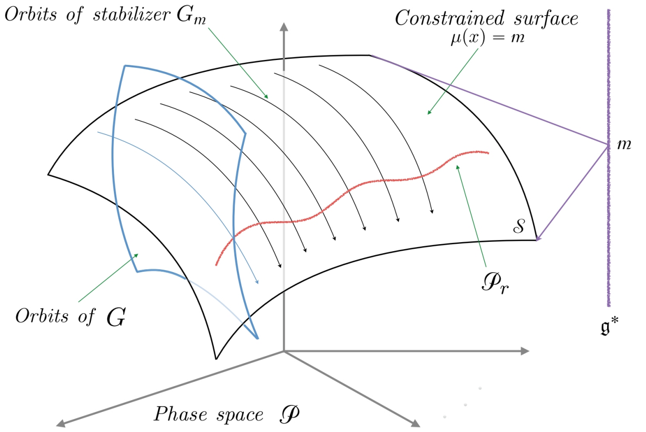
  - Semi-Classical Limit

$$f_F = \mathrm{Ad}^*_{\exp F} f_0 = f_0 + \{F, f_0\} + \frac{1}{2!}\left\{F, \{F, f_0\}\right\} + \cdots$$

Fermi Surface States and Their Excitations

- Coadjoint Orbits and the Kirillov-Kostant-Souriau Form
  - All Possible Canonical Transformation on f_0 $\Rightarrow$ Coadjoint Orbit of f_0

$$\mathcal{O}_{f_0} \equiv \{f = \mathrm{Ad}^*_U f_0 \in \mathfrak{g}^* \mid U \in \mathcal{G}\}$$



  - Stablizer Subgroup of f_0

$$\mathcal{H} \equiv \{V \in \mathcal{G} \mid \mathrm{Ad}^*_V f_0 = f_0\} = \{V = \exp \alpha \mid \alpha \in \mathfrak{g}, \mathrm{ad}^*_\alpha f_0 = 0\}$$

    - Derive the Same State

$$\mathrm{Ad}^*_{UV} f_0 = UV f_0 (UV)^{-1} = U \left(V f_0 V^{-1}\right) U^{-1} = \mathrm{Ad}^*_U f_0$$

    - Each State f in the Coadjoint Orbit is Represented by a Left Coset $U\mathcal{H}$
    - Coadjoint Orbit Expressed in Coset $\mathcal{O}_{f_0} \cong \mathcal{G}/\mathcal{H}$
    - No Dissipation $\Leftrightarrow$ States Evolve in the Same Coadjoint Orbit

$$f(t+\delta t) = f(t) + \{\delta t H, f(t)\} \quad \delta H\big|_f \in \mathfrak{g} \quad f(t) = \mathrm{Ad}^*_{e^{tH}} f(0)$$

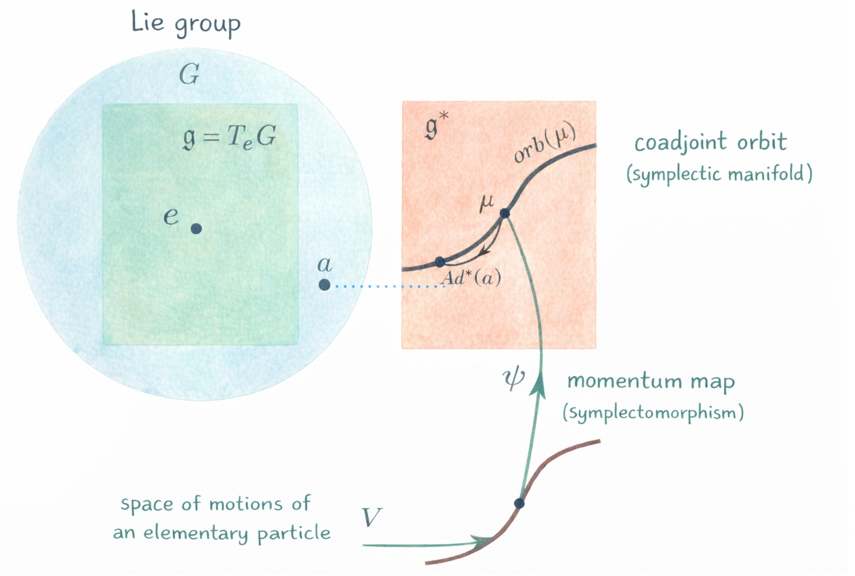
Fermi Surface States and Their Excitations

- Coadjoint Orbits and the Kirillov-Kostant-Souriau Form
  - Poisson Bracket as Lie Bracket (on Poisson Manifold) [Marsden, et al. (2007) Hamiltonian Reduction by Stages.]

$$\omega^{\mp}_{\mathcal{O}_f}(f)\,(\mathrm{ad}^*_G f, \mathrm{ad}^*_K f) = \mp\langle f, [G, K]\rangle$$

$$\Downarrow$$

$$\omega_{\mathrm{KKS}}(g, k) = \langle f, \{G, K\}_{\mathrm{Poisson}}\rangle \quad \mathrm{ad}^*_G f = g \quad \mathrm{ad}^*_K f = k$$



  - Verify it's indeed a Symplectic Form
    - Well-Defined (Independent of Representatives)

$$\mathrm{ad}^*_G f = \mathrm{ad}^*_{G'} f = g \quad \Longrightarrow \quad \mathrm{ad}^*_{\Delta G} f = 0 \quad \Delta G = G - G'$$

$$\Delta\omega = \langle f, \{G, K\}_{\mathrm{Poisson}}\rangle - \langle f, \{G', K\}_{\mathrm{Poisson}}\rangle = \langle f, \{\Delta G, K\}_{\mathrm{Poisson}}\rangle$$

$$= \langle f, \mathrm{ad}_{\Delta G} K\rangle = -\langle \mathrm{ad}^*_{\Delta G} f, K\rangle$$

$$\Delta G \in \mathcal{H} \implies \mathrm{ad}^*_{\Delta G} f = 0 \implies \Delta\omega = 0$$

Fermi Surface States and Their Excitations

- Coadjoint Orbits and the Kirillov-Kostant-Souriau Form
  - Verify it's indeed a Symplectic Form
    - Closed
      - Lie Algebra Cohomology (Chevalley-Eilenberg Cohomology Differential Formula on the Lie Algebra)

$$(d\omega)(x_1,\dots,x_{n+1}) = \sum_i (-1)^{i+1} x_i \cdot \omega(x_1,\dots,\hat{x}_i,\dots,x_{n+1}) + \sum_{i<j} (-1)^{i+j} \omega([x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{n+1})$$

$$\Longrightarrow (d\omega)(g,k,l) = g\cdot(\omega(k,l)) - k\cdot(\omega(g,l)) + l\cdot(\omega(g,k)) - \omega([g,k],l) + \omega([g,l],k) - \omega([k,l],g)$$

- $g\cdot\omega(k,l)$: directional derivative of $\omega(k,l)$ along vector field g
- $[g,k]$: the commutator of vector fields g, k

$$g\cdot(\omega(k,l)) = \frac{d}{dt}\bigg|_{t=0} \langle \mathrm{Ad}^*_{e^{tG}} f, \{K,L\}_{\text{Poisson}}\rangle = \langle \mathrm{ad}^*_G f, \{K,L\}_{\text{Poisson}}\rangle = -\langle f, \mathrm{ad}_G\{K,L\}_{\text{Poisson}}\rangle = -\langle f, \{G,\{K,L\}\}\rangle$$

Fermi Surface States and Their Excitations

- Coadjoint Orbits and the Kirillov-Kostant-Souriau Form
  - Verify it's indeed a Symplectic Form
    - Closed (Cancelled out by Jacobi Identity)

$$[g,k] = [\mathrm{ad}^*_G, \mathrm{ad}^*_K] f = \mathrm{ad}^*_{\{G,K\}_{\text{Poisson}}} f \implies \omega([g,k],l) = \langle f, \{\{G,K\},L\}\rangle$$

$$\implies (d\omega)(g,k,l) = -\langle f,\{G,\{K,L\}\}\rangle + \langle f,\{K,\{G,L\}\}\rangle - \langle f,\{L,\{G,K\}\}\rangle$$

$$-\langle f,\{\{G,K\},L\}\rangle + \langle f,\{\{G,L\},K\}\rangle - \langle f,\{\{K,L\},G\}\rangle = 0$$

    - Non-Degenerate

$$g \in T_f\mathcal{O}_{f_0} \quad g = 0 \iff \omega_{\text{KKS}}(g,k) = \langle f, \{G,K\}_{\text{Poisson}}\rangle = 0, \quad \forall k \in T_f\mathcal{O}_{f_0}$$

" $\implies$ ": Obvious " $\impliedby$ ": Assume $\omega_{\text{KKS}}(g,k) = \langle f, \{G,K\}_{\text{Poisson}}\rangle = 0, \quad \forall K \in \mathfrak{g}$

$$\omega_{\text{KKS}}(g,k) = -\langle \mathrm{ad}^*_G f, K\rangle = 0 \quad \mathrm{ad}^*_G f = 0 \implies G \in \mathcal{H} \implies g = \mathrm{ad}^*_G f = 0 \in T_f\mathcal{O}_{f_0}.$$

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Wess-Zumino-Witten Term and Effective Action

- Coadjoint Orbits and the Kirillov-Kostant-Souriau Form
  - Wess-Zumino-Witten Term in the KKS Form

$$S_{\mathrm{FL}}[f] = S_{\mathrm{WZW}}[f] - \int dt H[f] \quad S_{\mathrm{WZW}}[f] = \int dt \int_0^1 ds \omega_{\mathrm{KKS}} \left(\partial_t f, \partial_s f\right)$$

$$f(t, s=1) = f(t) \qquad f(t, s=0) = 0$$

  - Parametrize the Elements of Coadjoint Orbits

$$f_\phi(\mathbf{x}, \mathbf{p}) = \mathrm{Ad}^*_{\exp(-\phi)} f_0 = f_0 + \{-\phi, f_0\} + \frac{1}{2!}\Big\{ -\phi, \{-\phi, f_0\}\Big\} + \ldots = \Theta\left(p_F - |\mathbf{p}|\right) + (\mathbf{n}_\theta \cdot \nabla_{\mathbf{x}} \phi)\delta\left(|\mathbf{p}| - p_F\right) + \ldots$$

$$\{\phi, f_0\} = \nabla_{\mathbf{x}}\phi(\mathbf{x},\mathbf{p}) \cdot \nabla_{\mathbf{p}} f_0(\mathbf{p}) - \nabla_{\mathbf{p}}\phi(\mathbf{x},\mathbf{p}) \cdot \nabla_{\mathbf{x}} f_0(\mathbf{p}) = \nabla_{\mathbf{x}}\phi(\mathbf{x},\mathbf{p}) \cdot \nabla_{\mathbf{p}}\Theta(p_F - |\mathbf{p}|)$$

$$= \nabla_{\mathbf{x}}\phi(\mathbf{x},\mathbf{p}) \cdot \Big(\nabla_{\mathbf{p}}|\mathbf{p}|\Big) \frac{d}{d|\mathbf{p}|}\Theta(p_F - |\mathbf{p}|) = \nabla_{\mathbf{x}}\phi(\mathbf{x},\mathbf{p}) \cdot \mathbf{n}_\theta \left(-\delta(p_F - |\mathbf{p}|)\right) = -(\mathbf{n}_\theta \cdot \nabla_{\mathbf{x}}\phi)\, \delta(|\mathbf{p}| - p_F) \qquad \mathbf{n}_\theta \equiv \frac{\mathbf{p}}{|\mathbf{p}|}$$

  - Stablizer of f_0 (Corresponding Canonical Transformation Leaves f_0 Invariant)

$$\mathrm{ad}^*_\alpha f_0 = \{\alpha, f_0\} = 0 \quad \Longrightarrow \quad (\mathbf{n}_\theta \cdot \nabla_{\mathbf{x}}\alpha)\big|_{|\mathbf{p}|=p_F} = 0 \quad \alpha(\mathbf{x},\mathbf{p}) \in \mathfrak{h} \subset \mathfrak{g}$$

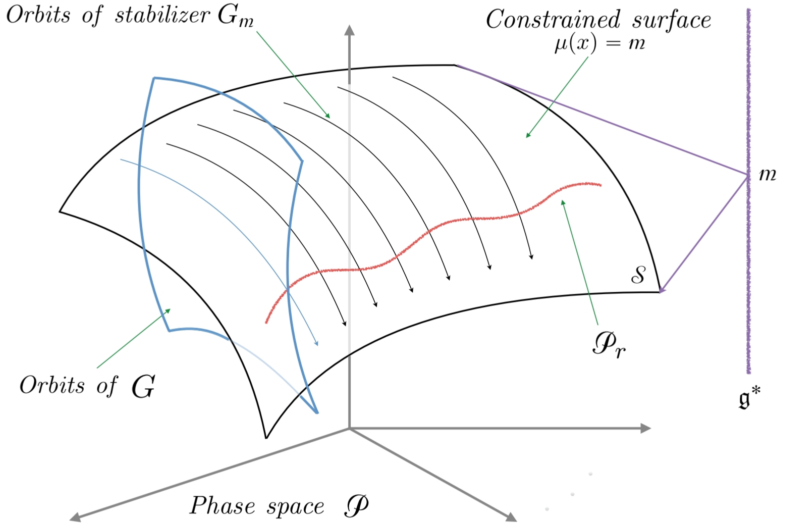
$$\Longrightarrow \quad \mathrm{Ad}^*_{\exp\alpha} f_0 = e^{\mathrm{ad}^*_\alpha} f_0 = f_0$$

Wess-Zumino-Witten Term and Effective Action

- Coadjoint Orbits and the Kirillov-Kostant-Souriau Form
  - Real Physics ⇒ Modulo Stablizer (Gauge Redundancy of Radial Momentum)

$$U \simeq UV \quad (U \in \mathcal{G}, V \in \mathcal{H}) \qquad \mathcal{O}_{f_0} \cong \mathcal{G}/\mathcal{H} \qquad f_U \equiv U f_0 U^{-1} \in \mathcal{O}_{f_0}$$

- BCH Formula: $\log\left(e^X e^Y\right) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\Big[X,[X,Y]\Big] - \frac{1}{12}\Big[Y,[X,Y]\Big] + \cdots$



$$\exp\Big(-\phi(\mathbf{x},\mathbf{p})\Big) \simeq \exp\Big(-\phi(\mathbf{x},\mathbf{p})\Big)\exp\Big(\alpha(\mathbf{x},\mathbf{p})\Big)$$

$$\Longrightarrow \phi(\mathbf{x},\theta) = \phi(\mathbf{x},\mathbf{p}) - \alpha(\mathbf{x},\mathbf{p}) + \frac{1}{2}\Big\{\phi(\mathbf{x},\mathbf{p}), \alpha(\mathbf{x},\mathbf{p})\Big\} + \ldots$$

$$\Longrightarrow \alpha_\phi(\mathbf{x},\mathbf{p}) \approx \phi(\mathbf{x},\mathbf{p}) - \phi(\mathbf{x},\theta)\Big|_{|\mathbf{p}|=p_F}$$

$$\{\alpha_\phi, f_0\} = \nabla_{\mathbf{x}}\alpha_\phi \cdot \nabla_{\mathbf{p}} f_0(\mathbf{p}) - 0 = -(\mathbf{n}_\theta \cdot \nabla_{\mathbf{x}}\alpha_\phi)\,\delta(|\mathbf{p}| - p_F)$$

$$= \mathbf{n}_\theta \cdot \nabla_{\mathbf{x}}\Big(\phi(\mathbf{x},\theta)\Big|_{|\mathbf{p}=p_F} - \phi(\mathbf{x},\mathbf{p})\Big)\,\delta(|\mathbf{p}| - p_F) = 0$$

Maurer-Cartan (MC) Form & EFT

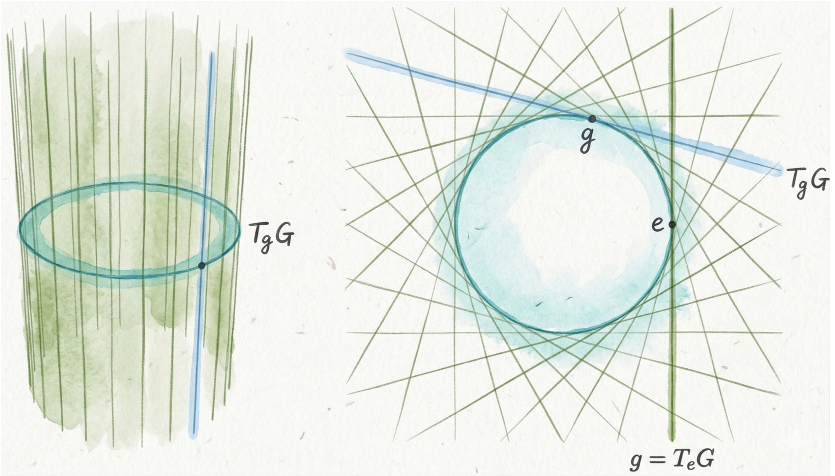
- Why the Maurer-Cartan Form Keeps Appearing
  - Central Observation: both Goldstone EFT (CCWZ) and Coadjoint Orbit Quantization Rely on the Same Geometric Object — the MC Form on a Lie group $\mathcal{G}$
  - Root Cause: both Contexts Involve Homogeneous Spaces $\mathcal{G}/\mathcal{H}$, where the MC form is the **Universal** Differential-Geometric Tool
  - Roadmap of the Part: General Framework → EFT (Brief Review) → Coadjoint Orbits → WZW / Legendre Transform Term → Comparison

Maurer-Cartan (MC) Form & EFT

- Maurer-Cartan Form on a Lie Group
  - Definition: $\theta = g^{-1}\mathrm{d}g \in \Omega^1(\mathcal{G}, \mathfrak{g})$, the Unique Left-Invariant $\mathfrak{g}$-valued 1-form on $\mathcal{G}$
  - Structural Equation (MC Equation)

$$d\theta + \frac{1}{2}[\theta, \theta] = 0$$

  - Geometric Meaning: θ Trivializes the Tangent Bundle $T\mathcal{G}$ by left translation; it Converts any Tangent Vector back to the Lie Algebra



Maurer-Cartan (MC) Form & EFT

- Universality: Every Flat $\mathfrak{g}$-Connection Comes from θ
  - Theorem (Universal Property): if M is simply connected and $\omega \in \Omega^1(M, \mathfrak{g})$ satisfies $d\omega + \frac{1}{2}[\omega, \omega] = 0$, then there Exists $f: M \to G$ such that $\omega = f^*\theta$, Unique up to left Multiplication by a Constant
  - Meaning: the MC Form is the "Mother of All Flat $\mathfrak{g}$-Valued 1-Forms" — every such Form is a Pullback of θ
  - This is why MC form is **Inevitable** whenever one Does Differential Geometry on $\mathcal{G}$ or $\mathcal{G}/\mathcal{H}$

Maurer-Cartan (MC) Form & EFT

- Principal Bundle Structure of $G \to \mathcal{G}/\mathcal{H}$
  - For a Closed Subgroup as a Principal $\mathcal{H}$-Bundle $H \hookrightarrow G \xrightarrow{\pi} G/H$
  - - Reductive Decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$
  - - MC Form Splits Accordingly: $\theta = g^{-1}dg = \theta_{\mathfrak{h}} + \theta_{\mathfrak{m}}$
    - $\theta_{\mathfrak{h}}$: Vertical Part → **Connection** on the Bundle
    - $\theta_{\mathfrak{m}}$: Horizontal Part → Vielbein (Soldering Form) on $\mathcal{G}/\mathcal{H}$
  - A **Section** $\sigma: \mathcal{G}/\mathcal{H} \to \mathcal{G}$ Pulls back the MC Form to $\mathcal{G}/\mathcal{H}$:

$$\sigma^{-1}d\sigma = V + e, \quad V \in \mathfrak{h}, e \in \mathfrak{m}$$

Maurer-Cartan (MC) Form & EFT

- MC Form in Symmetry Breaking EFT (Review)
  - Physical setup: $\mathcal{G}$ Spontaneously Broken to $\mathcal{H}$; Goldstone Fields Live on $\mathcal{G}/\mathcal{H}$
  - CCWZ Procedure: Lift Goldstone Field $U(x) \in \mathcal{G}/\mathcal{H}$ via Section $\xi(x) = \sigma(U(x))$, then Decompose $$\xi^{-1}\partial_\mu \xi = V_\mu + e_\mu$$
    - e_μ is the Covariant "Velocity" of the Goldstone Field
    - V_μ is the $\mathcal{H}$-Connection (Gauges the Residual Symmetry)
  - Leading-Order EFT Lagrangian: $\mathcal{L} = \frac{f^2}{4}\mathrm{tr}\left(e_\mu e^\mu\right) + \cdots$
  - Key Point: the **Metric** (Riemannian) Structure on $\mathcal{G}/\mathcal{H}$ is Used; the Lagrangian is Built from the **Symmetric 2-Tensor** $\mathrm{tr}(e_\mu e^\mu)$

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Maurer-Cartan (MC) Form & EFT

- Coadjoint Orbits as Homogeneous Spaces
  - Coadjoint action: $\mathrm{Ad}_g^*: \mathfrak{g}^* \to \mathfrak{g}^*$
  - Coadjoint Orbit through $\mu \in \mathfrak{g}^*$: $\mathcal{O}_\mu = \{\mathrm{Ad}_g^* \mu \mid g \in G\}$
  - Orbit-Stabilizer Theorem $\mathcal{O}_\mu \cong \mathcal{G}/\mathcal{G}_\mu$ $\quad \mathcal{G}_\mu = \{g \in \mathcal{G} \mid \mathrm{Ad}_g^* \mu = \mu\}$
  - So $\mathcal{O}_\mu$ is a Homogeneous Space
    - **The Same Mathematical Setting as CCWZ**
    - $\mathcal{G}/\mathcal{H} = \mathcal{G}_\mu$
  - Crucial Extra Structure: Coadjoint Orbits Carry a **Natural Symplectic Form**
    - Not just a Metric

Maurer-Cartan (MC) Form & EFT

- MC Form on Coadjoint Orbits: Symplectic Potential
  - Choose a Section $\sigma: \mathcal{G}/\mathcal{H} \to \mathcal{G}$ and Pull back the MC Form $\sigma^{-1}d\sigma \in \Omega^1(\mathcal{G}/\mathcal{G}_\mu, \mathfrak{g})$
  - Pair with the Base Point $\mu \in \mathfrak{g}^*$ to Define the **Symplectic Potential** (1-Form on $\mathcal{O}_\mu$)

$$\alpha = \langle \mu, \sigma^{-1}d\sigma \rangle$$

  - Compare with CCWZ: in EFT we **Project** $\sigma^{-1}d\sigma$ onto $\mathfrak{m}$ to get the Vielbein; here we **Pair** it with μ to Get a Scalar-Valued 1-Form
  - Information Used
    - EFT: $\mathfrak{h} \oplus \mathfrak{m}$ Splitting
    - Coadjoint Orbits: Element $\mu \in \mathfrak{g}^*$ (which Itself Determines $\mathcal{G}_\mu$ and hence the Splitting)

Maurer-Cartan (MC) Form & EFT

- Kirillov-Kostant-Souriau Symplectic Form
  - Take Exterior Derivative of the Symplectic Potential, Using the MC Structural Equation

$$\omega = d\alpha = -\frac{1}{2}\langle \mu, [\sigma^{-1}d\sigma, \sigma^{-1}d\sigma] \rangle$$

  - Intrinsic expression at a point $\nu = \mathrm{Ad}_g^* \mu$:

$$\omega_\nu(\xi_{\mathcal{O}}, \eta_{\mathcal{O}}) = \langle \nu, [\xi, \eta] \rangle$$

  - Key Properties: Closed ($d\omega = 0$), Non-degenerate, $\mathcal{G}$-Invariant
  - Compare with EFT
    - EFT Kinetic Term $\mathrm{tr}(e \wedge * e)$ is a **Symmetric** Bilinear in e_μ
    - KKS Form is Antisymmetric Bilinear
    - Reflects the Difference between Riemannian and Symplectic Geometry on $\mathcal{G}/\mathcal{H}$

Maurer-Cartan (MC) Form & EFT

- Symplectic Symplectic Potential as WZW Term and Legendre Transform
  - In the path integral on coadjoint orbits (à la Alekseev-Faddeev-Shatashvili), the action contains $$S = \int dt \left\langle \mu, g^{-1}\dot{g} \right\rangle - \int dt H(g)$$
  - First Term: $\left\langle \mu, g^{-1}\dot{g} \right\rangle$
    - Pullback of the Symplectic Potential α to the Worldline (**WZW-Type Topological Term**)
    - Plays the Role of $p\dot{q}$ in the Phase-Space Path Integral (**Legendre Transform**)
    - No Metric is Needed for this Term — Purely **Symplectic/Topological**, Determined Entirely by μ and MC Form
    - Connection to Berry Phase: in the Adiabatic Limit, this Term Becomes the **Geometric (Berry) Phase** Associated with the Coadjoint Orbit

Maurer-Cartan (MC) Form & EFT

- Comparison: EFT vs. Coadjoint Orbits

	Symmetry Breaking EFT	**Coadjoint Orbit**
Space	Vacuum manifold G/H	$\mathcal{O}_\mu \cong G/G_\mu$
MC form usage	Project onto $\mathfrak{m}$ → vielbein e_μ	Pair with μ → symplectic potential α
Geometry used	**Riemannian** (metric on G/H)	**Symplectic** (KKS form)
Physical output	Kinetic Lagrangian $\mathrm{tr}(e_\mu e^\mu)$	Phase-space action $\langle \mu, g^{-1}\dot{g} \rangle$
Role of the term	Dynamics (equations of motion)	Kinematics (symplectic structure / Berry phase)
Derivative counting	2-derivative leading order	1-derivative (first order in time)

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6. Doing Physics with the Action: Scaling, Response, and Damping

Nonlinear Bosonized Action

- From Hamiltonian to Action: Perform Legendre transform to get action S on the coadjoint orbit
- Field Parametrization: Introduce bosonic field $\phi(t,x,p)$ such that $f = \mathrm{Ad}^*_{\exp(-\phi)} f_0$ (generates a phase-space diffeomorphism)
- Action Structure: $S[\phi] = \int dt, \langle f, \dot{\phi}\rangle - H[f]$ plus possible WZW integral (from symplectic form)
- WZW Term: Arises from phase-space Berry phase (ensures correct Poisson brackets and quantization)
- Bosonic Effective Action: Nonlinear in ϕ; reproduces Landau’s theory at tree level, and upon quantization yields an EFT expansion for Fermi liquids

Symmetry and Gauging

- Huge Symmetry Group: Invariant under canonical transformations (volume-preserving diffeomorphisms in phase space)
- Conventional Symmetries: Contains global $U(1)$ (particle number conservation) and momentum-space rotations, etc.
- Phase-Space Gauge Fields: Can couple to background $A_\mu(x)$; gauge symmetry in phase space becomes noncommutative $U(1)_*$ (Moyal brackets in gauge algebra)
- Ward Identities: Generalized continuity equation in phase space (conservation law modified by $\{J, A\}$ terms) governs response to external fields
- Emergent Invariants: Though not a simple global charge, the canonical transform symmetry organizes infinite conservation laws (e.g. for free Fermi gas, each momentum mode density conserved)

Spinful Extension of Formalism

- Internal $SU(2)$ Symmetry: Incorporate spin-$\frac{1}{2}$ degrees by extending the algebra with spin indices
- Extended Algebra: $g_{\text{spin-Moyal}} ;\cong; (\mathbb{C},\oplus, su(2)) ;\otimes; g_{\text{Moyal}}$ (include spin operator basis)
- Generators: Density operators now carry spin index ($\psi^\dagger \sigma^a \psi$) splitting into charge (singlet) and spin (triplet) sectors
- Spin-Poisson Algebra: Poisson brackets act on spatial and spin variables (spin rotations + phase-space shifts unified)
- Coadjoint Orbit Action: Yields coupled charge and spin dynamics – new terms (spin WZW-like term) cause spin fluctuations to scale differently, hinting at spin–charge separation phenomena (to be explored)

BCS (Superconductivity) Extension

- Charged Bilinears: Extend operator set to include pairing operators (Cooper pairs $\psi\psi$, charge $2e$)
- Off-Diagonal Long-Range Order: Allows describing superconducting order parameter within the group formalism
- Broken $U(1)$ Handling: Enlarged group accommodates gauge symmetry breaking (superfluid phase invariants)
- BCS Channel in EFT: Pairing interactions can be treated on equal footing with Landau forward-scattering terms
- Toward SC Phases: This paves a path to incorporate superconducting and magnetic phases into the geometric Fermi liquid framework (ongoing work)

Two-Point Correlation Function

- Density–Density Correlator: Calculated using the bosonized quadratic action (Gaussian approximation)
- Result: Recovers known Fermi liquid response (e.g. compressibility, plasmons) in each spatial dimension d
- Dimensional Dependence: 2D vs 3D show expected behavior (agreement with Landau–Silin theory and RPA results)
- Minimal Action Suffices: The free (Gaussian) part of the action already captures the two-point function exactly
- Validation: Confirms that the formalism reproduces standard results for linear response of Fermi liquids (serves as a consistency check)

Three-Point and Higher Correlators

- Three-Point Function: Computed including interaction vertices (using cubic terms in action)
- Diagrammatic Contributions: Different Feynman-like diagrams (loop corrections) evaluated within orbit action framework
- Interactions Enter: Landau parameter $F(p,p')$ appears in three-point vertex, affecting collective mode coupling
- General n-Point Structure: Derived scaling forms for arbitrary n-point density correlations (systematic power counting of Landau parameters and momenta)
- Insight: Formalism makes higher-order correlation scaling more transparent, organizing vertex corrections by symmetry and algebraic structure

Non-Fermi-Liquid Example – Critical Boson Coupling

- Case Study: Fermi liquid coupled to a gapless bosonic mode (e.g. gauge field or critical order parameter)
- Bosonized Action Extension: Add boson field Φ interacting with Fermi surface fluctuations (Gaussian FL action + Φ coupling)
- RG Analysis: Examine scaling of the coupled system (similar to quantum critical metal scenarios)
- Specific Heat Scaling: Finds $c_V \propto T^{2/3}$, a non-Fermi-liquid temperature dependence
- Consistency: The $T^{2/3}$ law matches results from other methods (e.g. Hertz–Millis or holographic models), demonstrating the EFT’s capability to handle NFL behavior (at least at Gaussian level)

Advantages of Geometric Formalism

- Systematic Expansion: Organizes low-energy physics in a controlled expansion (powers of momenta $/ p_F$)
- Transparent Power Counting: Easy to track relevance of nonlinear interactions and gradients (symmetry-imposed structure)
- Simplified Calculations: Bosonized variables turn fermionic loops into tree-level diagrams, easing perturbative computations
- Emergent Symmetries: Geometric view reveals hidden conservation laws and symmetry principles (e.g. momentum-space conservation, analogies to fluid dynamics)
- Unified Effective Theory: Places Fermi liquid theory into the symmetry-based EFT paradigm (akin to how rotational symmetry governs magnons, etc.), bridging condensed matter and high-energy effective theory techniques

Limitations and Challenges

- Semi-Classical Truncation: Poisson approximation required for tractability – neglects some $\hbar$-dependent (quantum) corrections of Moyal bracket
- UV/IR Mixing: High-momentum (UV) contributions entangled with low-energy (IR) behavior, leading to unusual divergences (needs careful regularization)
- Validity Regime: Assumes well-defined Fermi surface and quasiparticles – if Landau paradigm fails (strong NFL), foundation of formalism is strained
- Complexity: Infinite-dimensional Lie algebra is mathematically rich but challenging for explicit calculations beyond leading order
- Open Technical Issues: E.g., how to systematically resum the Moyal expansion, handle a preferred cutoff (pallette of allowed deformations) and extend to lattice or discrete systems

Classical vs. Modern vs. Postmodern – Comparison

- Landau Fermi Liquid (Classical): Phenomenological quasiparticle picture; introduces Landau parameters by hand; no inherent RG or systematic expansion
- RG-Based Theory (Modern): Perturbative renormalization around Fermi surface; identifies marginal interactions; requires patching or momentum cutoffs (artificial scales)
- Patch Bosonization (Contemporary): Divides Fermi surface into 1D-like segments; bosonizes density fluctuations; captures some collective modes but needs ad hoc fixes (Klein factors, etc.)
- Orbit Formalism (Postmodern): Symmetry-driven approach using Lie algebra of densities; derives an action from first principles (coadjoint orbit); no patching needed and naturally includes higher-order corrections
- Evolution: Progression from phenomenological models to RG improvements to a fully geometric, symmetry-governed EFT signifies a unification of Fermi liquid theory with broader theoretical frameworks

Future Directions

- Non-Fermi-Liquid Regimes: Extend the formalism to handle cases with no well-defined quasiparticles (e.g. strange metals, critical Fermi surfaces)
- Non-Perturbative Effects: Develop methods to include the full Moyal bracket (quantum corrections) or resum beyond leading order (perhaps via numerical or analytic resummation)
- UV/IR Issue Resolution: Investigate regularization schemes to deal with UV/IR mixing and remove dependence on ad hoc cutoffs (ensure predictive power for physical observables)
- Incorporate More Symmetries: Adapt the approach to systems with lattice symmetry, disorder, or reduced dimensions (explore how diffeomorphism-based EFT works in those contexts)
- Broader Applications: Utilize infinite-dimension group techniques in other many-body problems (e.g. Bose liquids, fractional quantum Hall, spin systems) to uncover new emergent symmetries and conservation laws

Conclusion and Outlook

- Conceptual Shift: Fermi liquid theory is reinterpreted as a symmetry-based effective field theory governed by an infinite-dimensional group (canonical phase-space diffeomorphisms)
- Unified Framework: The postmodern formalism provides a geometric EFT with a rigid structure fixed by Fermi surface geometry and symmetry (no need for ad hoc patching or parameters)
- Recovers & Extends Landau: Reproduces Landau's kinetic theory in the appropriate limit, while systematically incorporating higher-order corrections and quantum effects
- Benefits: Reveals emergent conservation laws, clarifies scaling and power-counting, and simplifies calculations by bosonizing Fermi surface dynamics
- Looking Forward: Bridges condensed matter and high-energy viewpoints – opens new avenues by leveraging group theory (coadjoint orbits, diffeomorphisms) to tackle long-standing problems in many-body physics

Thanks for Your Listening

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