S-Matrix Theory

Proposal Presentation: Analyticity, Unitarity, and the Froissart Bound

Members: Zimu Li, Zhen Yang

Outline

Section 1

Research Background and Significance

Origins of the S-Matrix

Background

- In the mid-20th century, quantum electrodynamics (QED) achieved great success.
- The electromagnetic interaction is weak; perturbation theory yields $e^2 \approx \frac{1}{137}$.

Challenges

- However, the strong interaction coupling is large: $g^2 \sim 15$.
- The perturbation expansion does not converge.
- Existing theories lose calculability for strong interactions.

Main Idea

- Bypass the detailed mechanism and study observable scattering outcomes directly.

Outline

Section 2

Research Object and Methods

What the S-Matrix Describes

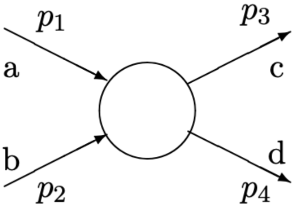
S-Matrix

The complete scattering information for a physical system evolving from free incoming states in the distant past to free outgoing states in the distant future.

$$S_{fi} = \langle f|S|i\rangle$$

S-Matrix Program

Instead of assuming a specific field-theory model in advance, impose general physical principles directly on the scattering amplitude and derive the structure of collisions from them.



Basic Requirements of the S-Matrix

Superposition: Quantum mechanics requires the state space to be linear.

Relativistic invariance: Descriptions in different inertial frames must be consistent.

Unitarity: Probability is conserved; the sum over all possible final-state probabilities must equal 1.

Short-range: The strong interaction is short-range.

Causality: This leads to analyticity of the scattering amplitude.

Outline

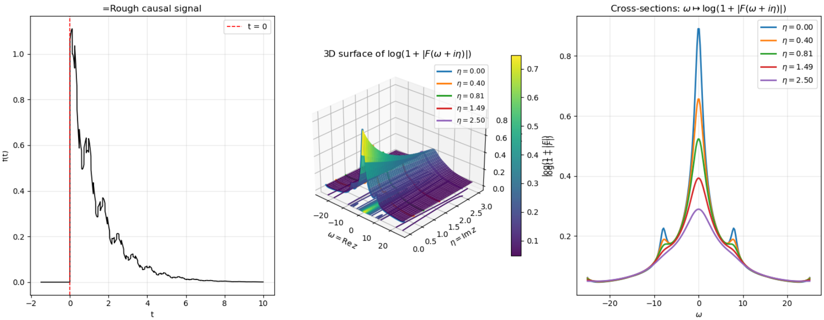
Section 3

Research Questions

Question 1: Causality and Amplitude Analyticity

Fourier transform: If a signal vanishes identically for $t < 0$, then its Fourier transform in frequency space ω is analytic in the upper half-plane of the complex ω-plane (Cauchy integral theorem!).

Physical meaning: Causality turns the time-ordering constraint in the time domain into analyticity in the frequency domain.



Question 2: Partial-Wave Unitarity: $|a_\ell(s)| \leq 1$

1. Rotational Symmetry and Diagonalization

In a spherically symmetric potential, angular momentum conservation implies $[\hat{S}, \hat{L}^2] = 0$. Hence the S matrix is **diagonal** in the **angular-momentum basis**.

$$S_\ell = e^{2i\delta_\ell}$$

2. Physical Intuition: Phase Shift δ_ℓ

δ_ℓ measures the wave deformation caused by the interaction:

- $\delta_\ell > 0$: Attractive potential (crests advance)
- $\delta_\ell < 0$: Repulsive potential (crests lag)

3. Building the Scattering Amplitude

Expand the scattering amplitude in Legendre polynomials $P_\ell(\cos\theta)$:

$$F(\theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) f_\ell P_\ell(\cos\theta)$$

Define the partial-wave amplitude a_ℓ as the key dimensionless quantity:

$$S_\ell = 1 + 2iqf_\ell$$
$$a_\ell = qf_\ell = e^{i\delta_\ell} \sin\delta_\ell$$

Note: a_ℓ removes the trivial no-scattering term and isolates the interaction itself.

Unitarity of the Partial-Wave Amplitude: $|a_\ell(s)| \leq 1$

1. Necessity of Probability Conservation

Unitarity is equivalent to conservation of probability flux. The outgoing intensity cannot exceed the incoming:

$$|S_\ell|^2 \leq 1$$

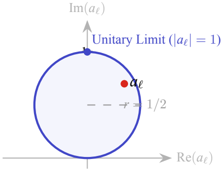
2. Using the Relation

$$S_\ell = 1 + 2ia_\ell$$

3. Geometric Constraint: Unitarity Circle

Substitute into $|S_\ell|^2 \leq 1$:

$$|a_\ell|^2 \leq \mathrm{Im}(a_\ell)$$



Conclusion: the maximum scattering strength is bounded by

$$|a_\ell|_{max} = 1$$

Question 3: The Froissart Theorem

Object of Study	High-energy hadron-hadron scattering
Total Cross Section	The total cross section $\sigma_{\text{tot}}(s)$ for elastic + inelastic scattering
Core Question	As $s \to \infty$, what is the maximal possible growth rate of $\sigma_{\text{tot}}(s)$?
Froissart Bound	$\sigma_{\text{tot}}(s) \lesssim \frac{\pi}{m_\pi^2} \ln^2 \frac{s}{s_0}$

Kinematics, Partial Waves, and the Optical Theorem

- **Result** $$\sigma_{\text{tot}}(s) \lesssim \frac{\pi}{m_\pi^2} \ln^2 \frac{s}{s_0}$$
- **Mandelstam Variables** $$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$
- **Mass Relation** $$s + t + u = \sum_i m_i^2 \qquad (\text{for equal masses } 4\mu^2)$$
- **Forward Limit** $$t = 0$$
- **Partial-Wave Series** $$A(s,t) = \sum_{\ell=0}^{\infty} (2\ell + 1) f_\ell(s) P_\ell(z)$$
- **Optical Theorem** $$\operatorname{Im} A(s,0) = v\, s\, \sigma_{\text{tot}}(s)$$
- **Physical Meaning** The total cross section is determined by the imaginary part of the forward amplitude

Outline of the Derivation

Goal

$$\operatorname{Im} A(s,0) = \sum_{\ell} (2\ell+1) \operatorname{Im} f_\ell(s)$$

$$\operatorname{Im} f_\ell(s) \le \text{constant} \sim O(1)$$

- **Constraint 1**
- **Constraint 2**
- **Constraint 3**

$$f_\ell(s) \lesssim c(s)\, \exp\left(-\frac{2\mu}{k_s}\ell\right)$$

$$c(s) \lesssim \left(\frac{s}{s_0}\right)^N$$

After Combining

$$\operatorname{Im} f_\ell(s) \lesssim \left(\frac{s}{s_0}\right)^N \exp\left(-\frac{2\mu}{k_s}\ell\right)$$

Key Quantity

Define an effective maximum angular momentum $L(s)$ to truncate the dominant contribution

Sum Estimate and Final Result

Split the Sum	$\ell \leq L:$ each term is at most $O(1)$, $\quad \ell > L:$ exponential tail, subdominant
Dominant Part	$\operatorname{Im} A(s,0) \lesssim \sum_{\ell \leq L} (2\ell+1) \sim L^2 \sim \frac{k_s^2}{\mu^2} \ln^2 \frac{s}{s_0}$
High-Energy Limit	$k_s^2 \sim \frac{s}{4} \quad \Longrightarrow \quad \operatorname{Im} A(s,0) \lesssim s \ln^2 \frac{s}{s_0}$
Optical Theorem	$\operatorname{Im} A(s,0) = vs\, \sigma_{\text{tot}}(s)$
Final Result	$\sigma_{\text{tot}}(s) \lesssim \ln^2 \frac{s}{s_0}$
Conclusion	**Hence proved**

Outline

Section 4

Challenges, Research Plan, and Expected Outcomes

Challenges

- The causal-to-analyticity picture is clear, but a rigorous proof is difficult and requires:

  1. the conflict between time localization and energy precision;

  2. a rigorous determination of the complex domains where the scattering amplitude $F(s,t)$ is analytic.

- We need to fill in steps omitted or stated without proof in the literature, such as the exponential decay of high-ℓ partial waves.

- Significant background knowledge must be added.

Research Plan and Expected Outcomes

Research Plan

1. Continue reading *The Analytic S-Matrix*; after forming an intuitive picture, consult other references and work toward a more rigorous proof.
2. Continue reading *Quantum Field Theory* to clarify the derivation and details of partial-wave amplitudes.
3. Continue reading *The Theory of Complex Angular Momenta* to trace the origin of the assumed axioms and build a more complete derivation.

Expected Outcomes

Final presentation　　Course paper　　Derivation notes

Summary

- This talk introduced the background, content, and significance of S-matrix theory, including **analyticity, unitarity, growth bounds**, as well as **partial-wave expansion** and the **Froissart bound**.
- We outlined the direction of future work and set a research plan and division of tasks.

References

[1] Eden, Landshoff, Olive and Polkinghorne, *The Analytic S-Matrix*.

[2] Itzykson and Zuber, *Quantum Field Theory*, Section 5-3.

[3] V.N. Gribov, *The Theory of Complex Angular Momenta*, Section 1.

Thank you